A PROOF OF THE EXISTENCE OF MINIMAL R-ALGEBRA RESOLUTIONS

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Throughout this note R denotes a local, Noetherian ring with maximal ideal $\mathfrak{m}$ and residue field $K = R/\mathfrak{m}$. It is well known that K as an R-module has a *minimal* resolution X, i.e. $dX \subset \mathfrak{m}X$. It was shown by Tate [4, Theorem 1] that K has a free resolution which is a differential skew-commutative algebra, briefly called an R-algebra.

In the present note we prove that K always has a minimal resolution which is an R-algebra. This settles a question raised by Tate, see footnote in [4, p. 23].

The existence of minimal R-algebra resolutions simplifies the study of the R-algebra $\operatorname{Tor}^R(K, K)$, cf. [4, § 5]. In particular one immediately obtains generalizations of known results on the Betti-numbers of R; see [1, §§ 2, 4].

Notations and definitions

The term "R-algebra" will be used in the sense of [4] i.e. an associative, graded, differential, strictly skew-commutative, algebra X over R, with unit element 1, such that the homogeneous components X_q are finitely generated modules over R. We require that

$$X_0 = 1 \cdot R \quad \text{and} \quad X_q = 0 \ \text{for} \ q < 0.$$

R is considered as an R-algebra with trivial grading and differential.

We shall use the symbol

$$X\langle S \rangle;\, dS = s$$

to denote the R-algebra obtained from an R-algebra X "by the adjunction of a variable" S which kills a cycle s. Cf. [4, § 2].

Let $\langle \ldots, S_i, \ldots \rangle$ be a set of variables indexed by an initial part of the natural numbers, which may be empty or infinite. If these S_i are adjoined successively to an R-algebra X

to kill cycles s_i, there results a natural direct system of R-algebras and inclusion maps. We denote the direct limit of this system by

$$X\langle \ldots, S_i, \ldots \rangle;\ dS_i = s_i$$

The degree of a homogeneous map j or a homogeneous element x will be denoted by $\deg j$ and $\deg x$ respectively. R-algebras and elements are indexed by superscripts and subscripts respectively.

The vectorspace dimensions $\dim_K \operatorname{Tor}_p^R(K, K)$ are called the Betti-numbers of R. They are denoted by $b_p(R)$. The Betti-series of R is the power series

$$B(R) = \sum_{p=0}^{\infty} b_p(R) Z^p$$

Definition. Let X be an R-algebra with differential d. A *derivation* j on X is an R-linear homogeneous map $j: X \to X$ satisfying

(i) $dj = jd$

(ii) $j(xy) = (-1)^{w \cdot q} j(x)y + xj(y)$,

where $w = \deg j$ and $y \in X_q$.

LEMMA. *Let j be a derivation on an R-algebra X, and s a cycle in X. Put $Y = X\langle S \rangle$; $dS = s$. Then j can be extended to a derivation j' on Y if and only if*

$$j(s) \in B(Y). \tag{1}$$

Proof. If j can be extended, (1) is satisfied because $j(s) = j(dS) = dj'(S)$. On the other hand, if (1) is satisfied, choose an element $G \in Y$ with the property

$$dG = j(s).$$

We treat the cases $\deg S$ odd and $\deg S$ even separately. If $\deg S$ is odd, we have

$$Y = X \oplus XS.$$

For $x_0, x_1 \in X$ define

$$j'(x_0 + x_1 S) = j(x_0) + (-1)^{\deg j} j(x_1) S + x_1 G. \tag{2}$$

If $\deg S$ is even, we have

$$Y = \coprod_{i=0}^{\infty} XS^{(i)}.$$

For $x_0, \ldots, x_m \in X$ define

$$j' \sum_{i=0}^{m} x_i S^{(i)} = \sum_{i=0}^{m} j(x_i) S^{(i)} + \sum_{i=1}^{m} x_i S^{(i-1)} G. \tag{3}$$

It is a straightforward matter to check that in both cases j' becomes a derivation on Y.

THEOREM. *Let R be a local Noetherian ring with maximal ideal $\mathfrak{m}$. There exists an R-algebra X which is an R-free resolution of $R/\mathfrak{m}$ with the property*

(i) $dX \subset \mathfrak{m}X$

d being the differential on X.

In fact every R-algebra satisfying (ii)–(v) *below has the property* (i).

(ii) $H_p(X) = 0$ *for* $p \neq 0$. $H_0(X) = R/\mathfrak{m}$.

(iii) *X has the form* $X = R\langle ..., S_i, ...\rangle$; $dS_i = s_i$

(iv) $\deg S_{i+1} \geqslant \deg S_i$ *for all* $i \geqslant 1$.

(v) *The cycles s_α of degree* 0 *form a minimal system of generators for m. If* $\deg s_\alpha \geqslant 1$ *then s_α is not a boundary in* $R\langle S_1, ..., S_{\alpha-1}\rangle$; $dS_i = s_i$.

Proof. In [4] Tate showed that there exists an R-algebra X satisfying (ii)–(v) above. Let X be such an R-algebra. We are going to show (i). We assume that $\mathfrak{m} \neq 0$, otherwise it follows from (v) that $X = R$. We also assume that the set of all adjoined variables is infinite. Only trivial modifications must be carried out if this set is finite.

Let X^0 denote the R-algebra R. Define inductively

$$X^\alpha = X^{\alpha-1}\langle S_\alpha\rangle;\ dS_\alpha = s_\alpha \quad \text{for} \quad \alpha \geqslant 1.$$

Let i^α denote the natural inclusion map $i^\alpha : X^{\alpha-1} \to X^\alpha$. We have

$$X = \varinjlim X^\alpha.$$

For each $\alpha \geqslant 1$ define a derivation

$$j^\alpha : X^\alpha \to X^\alpha$$

in the following way. Let $j = 0$ be the trivial derivation on $X^{\,-}$. Put $G = 1$ and let j^α be the extension of j given by (2) resp. (3). Then

$$\deg j^\alpha = -\deg S_\alpha.$$

First we show that for all $\alpha \geqslant 1$, j^α can be extended to a derivation J^α on X which is of negative degree. By passing to a direct limit it clearly suffices to show the following: If $\alpha \leqslant \gamma$ and $j^{\alpha,\gamma}$ is a derivation on X^γ which is an extension of j^α, then $j^{\alpha,\gamma}$ can be extended to a derivation $j^{\alpha,\gamma+1}$ on $X^{\gamma+1}$.

Now let $j^{\alpha,\gamma}$ be a derivation on X^γ which extends j^α. We will prove that $j^{\alpha,\gamma}$ can be extended to a derivation on $X^{\gamma+1}$. By the lemma it suffices to show that

$$j^{\alpha,\gamma}(s_{\gamma+1}) \in B(X^\gamma). \tag{4}$$

To prove (4) we consider two cases. First assume that $\deg S_\alpha \neq \deg s_{\gamma+1}$. This yields

$$0 \neq \deg j^{\alpha,\gamma}(s_{\gamma+1}) < \deg s_{\gamma+1}$$

However, it follows from (ii) and (iv) that

$$H_p(X^\gamma) = 0 \quad \text{for} \quad 0 \neq p < \deg s_{\gamma+1}.$$

Hence in this case (4) follows. Next assume that $\deg S_\alpha = \deg s_{\gamma+1}$. Then $j^{\alpha,\gamma}(s_{\gamma+1}) \in X_0$. Let $S_\mu, \ldots, S_{\mu+\nu}$ be all the adjoined variables of degree $\deg s_{\gamma+1}$. Then there exist elements $x \in X^{\mu-1}$ and $r_\mu, \ldots, r_{\mu+\nu} \in R$ such that

$$s_{\gamma+1} = x + \sum_{i=\mu}^{\mu+\nu} r_i S_i. \tag{5}$$

Differentiation yields

$$\sum_{i=\mu}^{\mu+\nu} r_i s_i \in B(X^{\mu-1}).$$

It follows from (v) that $r_i \in \mathfrak{m}$ for $i = \mu, \ldots, \mu+\nu$. Since $\mu - 1 < \alpha$ we have

$$j^{\alpha,\gamma}(x) = j^\alpha(x) = 0.$$

Hence applying $j^{\alpha,\gamma}$ to (5) one deduces

$$j^{\alpha,\gamma}(s_{\gamma+1}) \in \mathfrak{m} X_0.$$

However, $\deg s_{\gamma+1} = \deg S_\alpha \geqslant 1$ so $\mathfrak{m} X_0$ is already killed. Again (4) follows.

In the rest of the proof we consider the underlying complexes of the respective R-algebras. For each $\alpha \geqslant 1$, j^α leads to an exact sequence of complexes

$$0 \to X^{\alpha-1} \xrightarrow{i^\alpha} X^\alpha \xrightarrow{j^\alpha} X^\alpha \tag{6}$$

which splits as a sequence of R-modules, cf. [4, p. 17–18]. Consider the functor $X \mapsto \bar{X}$, where $\bar{X} = X/\mathfrak{m}X$. For $\alpha \geqslant 0$ let I^α denote the natural inclusion map $I^\alpha\colon X^\alpha \to X$. It follows that I^α is direct, hence we may identify $\bar{X}^\alpha$ with its image in $\bar{X}$. From (6) we deduce a commutative diagram

$$\begin{array}{ccccccc} 0 \to \bar{X}^{\alpha-1} \to & \bar{X}^\alpha & \to & \bar{X}^\alpha & & \alpha \geqslant 1 \\ & \downarrow \bar{I}^\alpha & & \downarrow \bar{I}^\alpha & & \\ & \bar{X} & \xrightarrow{\bar{j}^\alpha} & \bar{X} & & \end{array} \tag{7}$$

in which the upper row is exact. This yields

$$\bigcap_{\gamma \geqslant 1} \ker \bar{J}^\gamma \subset \bar{X}^0. \tag{8}$$

Indeed let $x \in \bigcap_{\gamma \geqslant 1} \ker \bar{J}^{\gamma}$. Let $\alpha \geqslant 1$ and suppose that $x \in \bar{X}^{\alpha}$. It follows from (7) that $x \in \bar{X}^{\alpha-1}$. Repeating this it follows that $x \in \bar{X}^{0}$.

By induction on q we are going to show that

$$B_q(\bar{X}) = 0. \tag{9}$$

For $q=0$ this is clear by (v). Let $r \geqslant 1$ and assume that (9) has been established for $q<r$. For every $\gamma \geqslant 1$, $\bar{J}^{\gamma}$ is of negative degree and commutes with the differential on $\bar{X}$. Hence

$$\bar{J}^{\gamma}(B_r(\bar{X})) \subseteq \coprod_{q<r} B_q(\bar{X}) = 0 \text{ for } \gamma \geqslant 1.$$

It follows from (8) that $B_r(\bar{X}) \subseteq B_r(\bar{X}) \cap \bar{X}^{0} = 0$.

Since $B(\bar{X}) = 0$ we have $B(X) \subset \mathfrak{m}X$. Q.E.D.

Let X be a minimal R-algebra resolution of K as described in the theorem (ii)–(v). There is an isomorphism of R-algebras, cf. [4, § 5]:

$$\operatorname{Tor}^{R}(K, K) \approx H(X \underset{R}{\otimes} K) = X \underset{R}{\otimes} K.$$

This yields the following generalization of a result due to Assmus [1, § 4]:

Corollary 1. *The Betti-series of R may be written in the form*

$$B(R) = \frac{(1+Z)^{n_1}(1+Z^3)^{n_3}\dots}{(1-Z^2)^{n_2}(1-Z^4)^{n_4}\dots},$$

where n_q $q=1, 2, \dots$ is the number of adjoined variables of degree q in a minimal R-algebra resolution.

Corollary 2. *The Betti-numbers $\{b_p(R)\}$ of a non-regular local ring R form a non-decreasing sequence. Cf.* [2].

Proof. In the above notation we have

$$n_1 = \dim_K \mathfrak{m}/\mathfrak{m}^2$$

and

$$n_2 = \dim_K H_1(X^{n_1}).$$

Let R be non-regular. It follows from the Eilenberg characterization of regularity that $n_2 \neq 0$, cf. [4, Lemma 5].(1) Since also $n_1 \neq 0$, $B(R)$ contains a factor $1/(1-Z)$. Hence $\{b_p(R)\}$ is non-decreasing. Q.E.D.

(1) Tate has requested me to point out that his "outline of proof" of Lemma 5 in [4] is neither correct, nor due to Zariski, and that a correct proof can be obtained (for example) by using Prop. 3 on page IV–5 of Serre [3], with $M = A$, together with Cor. 2, page IV–35, and the characterization of regular local rings as those which are Noetherian of finite homological dimension.

References

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