ON THE DIRICHLET PROBLEM OF THE CHOQUET BOUNDARY

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The present paper is a study of the Dirichlet problem of the Choquet boundary ∂X of a real or complex sup-norm space L over a compact Hausdorff space X. It is proved that a continuous and bounded function f on ∂X can be extended to a function of the class L on X iff f is annihilated by every L-orthogonal boundary measure and every limit point of ∂X is a non-singular Shilov point for f. (Precise definitions follow). The case of a metrizable compact space X was treated in [2], and it was observed independently by A. Lazar [8] and E. Effros [7] that the metrizability could be avoided in the case where representing boundary measures are unique ("simplicial case"). We have found it convenient to state and prove the general theorem in the "analytic" setting. A "geometric" version of the theorem is presented as a corollary.

We shall assume that X is an arbitrary, but fixed compact Hausdorff space. A subset L of $C_R(X)$, or $C_C(X)$, in said to be *a real, or complex, sup-norm space* if:

(i) L is a linear subspace,
(ii) L contains the constant functions,
(iii) L separates points,
(iv) L is closed in uniform norm.

The σ-field $\mathcal{B}_0$ of *Baire subsets* of X is generated by the sets $f^{-1}(O)$, $f \in C_R(X)$, O open in X. It is the smallest σ-field rendering measurable every $f \in C_R(X)$. A measure m on a σ-field $\mathcal{F} \supset \mathcal{B}_0$ is said to *represent a point* $x \in X$ (relatively to L) if

$$a(x) = \int a\, dm, \quad \text{all } a \in L. \tag{1}$$

More generally, m is said to *represent a linear function* q on L if

$$q(a) = \int a\,dm, \quad \text{all } a \in L. \tag{2}$$

A subset Y of X is said to be a *representing boundary* for L if there exists a σ-field $\mathcal{F} \supset \mathcal{B}_0$ such that $Y \in \mathcal{F}$, and such that every point $x \in X$ admits a representing probability measure m on $\mathcal{F}$ with $m(Y) = 1$.

The *Choquet boundary* ∂X of L consists of all points $x \in X$ such that the one-point measure ε_y is the only probability measure on $\mathcal{B}_0$ which represents x. Clearly ∂X is contained in any representing boundary. On the other hand, ∂X is itself a representing boundary since every point $x \in X$ admits a probability measure m on the σ-field $\mathcal{F}_0$ generated by $\mathcal{B}_0 \cup \{\partial X\}$, such that m represents x and $m(\partial X) = 1$. (Choquet–Bishop–de Leeuw Theorem [4], [5], cf. also [9]). Thus *the Choquet boundary is the smallest representing boundary*, and it appears that it is an appropriate set for prescription of boundary values.

The closure of the Choquet boundary of a sup-norm space L is the *Shilov-boundary* of L, i.e. it is the smallest closed subset of X on which every function of class L assumes its maximum modulus (cf. e.g. [3]). The Dirichlet problem of the Shilov boundary is quite well understood (cf. e.g. [2], [3]). If we denote by $L^{\perp}(\overline{\partial X})$ the set of all *L-orthogonal Baire measures supported by* $\overline{\partial X}$, i.e. the set of all real (complex) measures μ on $\mathcal{B}_0$ such that

$$|\mu|(X \setminus \overline{\partial X}) = 0, \quad \int a\,d\mu = 0 \quad \text{all } a \in L, \tag{3}$$

then we can state a necessary and sufficient condition that a continuous function f on $\overline{\partial X}$ be extendable to a function of class L, as follows:

$$\int_{\overline{\partial X}} f\,d\mu = 0, \quad \text{all } \mu \in L^{\perp}(\overline{\partial X}). \tag{4}$$

Similarly we denote by $L^{\perp}(\partial X)$ the set of all *L-orthogonal boundary measures*, i.e. the set of all real (complex) measures m on $\mathcal{F}_0$ such that

$$|m|(X \setminus \partial X) = 0, \quad \int a\,dm = 0 \quad \text{all } a \in L. \tag{5}$$

(Note that this concept of "boundary measure" is formally distinct from that of [1], since the domains of definition are different. However, there is a canonical isomorphism between the two spaces of "boundary measures", cf. e.g. [9].)

Clearly a necessary condition that a function f on ∂X be extendable to a function of class L, is uniform continuity of f and

$$\int_{\partial X} f\,dm = 0, \quad \text{all } m \in L^{\perp}(\partial X). \tag{6}$$

(Note that a uniformly continuous function f on ∂X is extendable to a continuous function on X. Hence f is the restriction of a Baire measurable function on X to the set $\partial X \in \mathfrak{F}_0$, and so the integral (6) is well defined.)

The above condition is *not* sufficient for extendability. There are simple examples of uniformly continuous functions on ∂X which satisfy (6), and for which $\int f\,d\mu \neq 0$ for some $\mu \in L^{\perp}(\overline{\partial X})$. In fact, set $X = [0,1] \cup \{i\} \cup \{-i\} \subset \mathbf{C}$, and consider the space $L \subset C_C(X)$ of all functions f such that $2f(0) = f(i) + f(-i)$. Here $\partial X = X \setminus \{0\}$, $L^{\perp}(\partial X) = (0)$, and $L^{\perp}(\overline{\partial X}) = (2\varepsilon_0 - \varepsilon_i - \varepsilon_{-i})$. Now an example of the desired type is furnished by $f \mid \partial X$ for any $f \in C_C(X) \setminus L$.

The *L-envelopes* of a continuous and bounded real valued function f defined on a subset Y of X containing ∂X, are the functions

$$\hat{f}(x) = \inf \{a(x) \mid f \leqslant a \mid Y, a \in L_r\}, \tag{7}$$

and

$$\check{f}(x) = \sup \{a(x) \mid f \geqslant a \mid Y, a \in L_r\}, \tag{8}$$

where L_r is the linear space of real parts of functions in L.

Clearly $\hat{f}$ is upper semi-continuous, $\check{f}$ is lower semi-continuous, and $\check{f} \leqslant \hat{f}$. Also $\check{f}(x) = f(x) = \hat{f}(x)$ for every $x \in \partial X$. (This is standard for $X = Y$, i.e. for $f \in C_R(X)$. The two equalities prevail for l.s.c. and u.s.c. functions on X, respectively. Hence both equalities are valid for continuous functions on a general $Y \supset \partial X$. For details cf e.g. [2]).

We shall say that a point $x \in \overline{\partial X}$ is a *singular Shilov point* for a continuous and bounded *real* valued function f on ∂X, if

$$\check{f}(x) \neq \hat{f}(x). \tag{9}$$

Similarly we shall say that a point $x \in \overline{\partial X}$ is a *singular Shilov point* for a continuous and bounded *complex* valued function f on ∂X if the inequality (9) is valid for either the real or the imaginary part of f (or both).

Points of ∂X are always *non-singular* Shilov points, by virtue of the above remarks. However, for every $x \in \overline{\partial X} \setminus X$ there exists a continuous and bounded real valued function f on ∂X (one may choose $f \in C_R(X) \mid \partial X$) such that x is a *singular* Shilov point for f. (Cf. e.g. [5], [9]).

Note that if f is a continuous and bounded function on ∂X for which all Shilov points are non-singular, then f is *uniformly* continuous. A forteriori f is $\mathfrak{F}_0$-measurable, and so the integral $\int f\,dm$ is well defined for any $m \in L^{\perp}(\partial X)$.

THEOREM. *A continuous and bounded real (complex) valued function f on the Choquet boundary ∂X of a real (complex) sup-norm space L over a compact Hausdorff space X can be extended to a function of the class L on X iff:*

(i) *There are no singular Shilov points for f*

(ii) $\int_{\partial X} f\,dm = 0$ *for all real (complex) measures* $m \in L^{\perp}(\partial X)$.

Proof. Necessity is obvious, and we shall prove the sufficiency in the case of a *complex* sup-norm space L.

Note first that the definition and basic properties of the Choquet boundary are independent of the requirement that L be uniformly closed, and that the Choquet boundaries of L and L_r coincide (cf. e.g. [9]).

Assume that f is a continuous and bounded complex valued function on ∂X satisfying (i), (ii). Let $f = f_1 + if_2$ where f_1, f_2 are real valued, and observe that $\check{f}_i$ and $\hat{f}_i$ have common, continuous restrictions $\tilde{f}_i$ to $\overline{\partial X}$ for $i = 1, 2$, by virtue of the hypothesis (i). In the sequel we shall write $\tilde{f} = \tilde{f}_1 + i\tilde{f}_2$.

Let $M(\overline{\partial X})$ be the Banach space of complex Baire measures on $\overline{\partial X}$, and define maps

$$\begin{array}{ccccc} & & M(\overline{\partial X}) & & \\ & & \varrho \downarrow & \searrow \varphi & \\ X & \xrightarrow{\ \psi\ } & L^* & \xrightarrow{\ \varphi'\ } & \mathbf{C} \end{array}$$

as follows:

$$\psi(x)(a) = a(x), \quad \text{all } a \in L, x \in X, \tag{10}$$

$$\varrho(\mu)(a) = \int a\,d\mu, \quad \text{all } a \in L, \mu \in M(\overline{\partial X}), \tag{11}$$

$$\varphi(\mu) = \int_{\overline{\partial X}} \bar{f}\, d\mu, \quad \text{all } \mu \in M(\overline{\partial X}), \tag{12}$$

and finally

$$\varphi'(q) = \int_{\partial X} f\, dm, \tag{13}$$

where $q \in L^*$, and m is any complex measure on $\mathfrak{F}_0$ which *represents* q (in the sense of (2)), and for which $|m|(X \setminus \partial X) = 0$. Such measures exist by the Choquet–Bishop–de Leeuw theorem, and the function φ' is well defined by virtue of the hypothesis (ii).

Clearly ψ, ϱ, φ are continuous with respect to the given topology of X, the w^*-topology of L^*, the vague topology of $M(\overline{\partial X})$ (i.e. the w^*-topology of $M(\overline{\partial X})$ considered as the Banach dual space of $C_C(X)$) and the customary topology of $\mathbf{C}$. The w^*-continuity of φ' is the crucial point. We shall derive it from the continuity of ϱ and φ after proving that the above diagram is commutative.

The proof that follows, is based on certain norm- and order- preserving properties of the linear functional φ' on L^*. Let $\mu \in M(\overline{\partial X})$, and consider the standard decomposition $\mu = (\mu_1^+ - \mu_1^-) + i(\mu_2^+ - \mu_2^-)$ into positive components. For $j = 1, 2$ the positive linear functionals $a \curvearrowright \int a\, d\mu_j^+$, $a \curvearrowright \int a\, d\mu_j^-$ on L_r can be represented by *positive* measures m_j', m_j'' on $\mathfrak{F}_0$, all vanishing off ∂X, (Choquet–Bishop–de Leeuw Theorem). Clearly the measure $m = (m_1' - m_1'') + i(m_2' - m_2'')$ represents $\varrho(\mu)$. Since $1 \in L_r$, we shall have $\|m_j'\| = \|\mu_j^+\|$, $\|m_j''\| = \|\mu_j^-\|$ for $j = 1, 2$. It follows that $\|m\| \leqslant 2\sqrt{2}\|\mu\|$, and so we obtain our first estimate:

$$|\varphi'(\varrho(\mu))| = \left| \int_{\partial X} f\, dm \right| \leqslant 3\|\mu\| \cdot \|f\|. \tag{14}$$

Writing $m_j = m_j' - m_j''$ for $j = 1, 2$, and separating into real and imaginary parts, we obtain the inequality

$$|\varphi'(\varrho(\mu)) - \varphi(\mu)| \leqslant \sum_{j,k=1}^{2} \left| \int_{\partial X} f_j\, dm_k - \int_{\overline{\partial X}} f_j\, d\mu_k \right|. \tag{15}$$

Assume for a moment that a_j, b_j are such elements of L_r that

$$a_j | \partial X \leqslant f_j \leqslant b_j | \partial X, \quad j = 1, 2. \tag{16}$$

For each of the four choices of $j, k = 1, 2$, we shall have

$$\int_{\overline{\partial X}} a_j\, d\mu_k^+ = \int_{\partial X} a_j\, dm_k' \leqslant \int_{\partial X} f_j\, dm_k' \leqslant \int_{\partial X} b_j\, dm_k' = \int_{\overline{\partial X}} b_j\, d\mu_k^+$$

and
$$\int_{\overline{\partial X}} a_j \, d\mu_k^+ \leqslant \int_{\overline{\partial X}} \bar{f}_j \, d\mu_k^+ \leqslant \int_{\overline{\partial X}} b_j \, d\mu_k^+.$$

Hence
$$\left| \int_{\partial X} f_j \, dm_k' - \int_{\overline{\partial X}} \bar{f}_j \, d\mu_k^+ \right| \leqslant \int_{\overline{\partial X}} (b_j - a_j) \, d\mu_k^+.$$

Similarly
$$\left| \int_{\partial X} f_j \, dm_k'' - \int_{\overline{\partial X}} \bar{f}_j \, d\mu_k^- \right| \leqslant \int_{\overline{\partial X}} (b_j - a_j) \, d\mu_k^-.$$

Combination of these inequalities gives

$$\left| \int_{\partial X} f_j \, dm_k - \int_{\overline{\partial X}} \bar{f}_j \, d\mu_k \right| \leqslant \int_{\overline{\partial X}} (b_j - a_j) \, d|\mu_k|. \tag{17}$$

By (15) and (17) we shall have

$$|\varphi'(\varrho(\mu)) - \varphi(\mu)| \leqslant \sum_{j,\,k=1}^{2} \int_{\overline{\partial X}} (b_j - a_j) \, d|\mu_k|. \tag{18}$$

Now we shall apply the general estimate (14) and the estimate (18) valid under the condition (16), to prove that the diagram is commutative.

Let $\mu \in M(\overline{\partial X})$, and let $\varepsilon > 0$ be arbitrary. For every Baire subset B of $\overline{\partial X}$ we define $\Phi(B)$ to be the (possibly empty) subset of L_r^4 consisting of all (a_1, b_1, a_2, b_2) such that:

$$a_j | \partial X \leqslant f_j \leqslant b_j | \partial X, \tag{19}$$

and
$$b_j | B - a_j | B \leqslant \varepsilon, \tag{20}$$

for $j = 1, 2$.

We claim that if C is a Baire subset of $\overline{\partial X}$ such that

$$|\mu| (\overline{\partial X} \setminus C) > 0, \tag{21}$$

then there exists another Baire subset B of $\overline{\partial X}$ such that

$$\Phi(B) \neq \varnothing, \quad |\mu|(B) > 0, \quad B \cap C = \varnothing \tag{22}$$

By regularity there is a compact subset B' of $\overline{\partial X} \setminus C$ such that $|\mu|(B') > 0$. Let μ' be the restriction of μ to B', i.e. $\mu'(A) = \mu(A \cap B')$ for every $A \in \mathcal{B}_0$. Let $z \in \operatorname{Supp}(\mu')$, and apply the definition (7), (8) of envelopes to construct elements $a_i, b_i \in L_r$ satisfying (19) together with the additional requirement:

$$\check{f}_i(z) - \frac{\varepsilon}{2} < a_i(z), \quad b_i(z) < \hat{f}_i(z) + \frac{\varepsilon}{2}, \quad \text{for } i = 1, 2. \tag{23}$$

The set

$$T = \left\{ x \in X \mid \check{f}_i(z) - \frac{\varepsilon}{2} \leqslant a_i(x), \ b_i(x) \leqslant \hat{f}_i(z) + \frac{\varepsilon}{2}; \ i = 1, 2 \right\}$$

is a closed neighbourhood of z. Hence $|\mu'|(B) > 0$, with $B = B' \cap T$. By the hypothesis (i), $\check{f}_i(z) = \hat{f}_i(z)$ for $i = 1, 2$, and so

$$b_i(x) - a_i(x) \leqslant \varepsilon, \quad \text{all } x \in T; \ i = 1, 2.$$

Hence the quadruple (a_1, b_1, a_2, b_2) satisfies the requirement (20), and so $\Phi(B) \neq \emptyset$ This gives (22), and the claim is proved.

Now we can apply an inductive argument to construct a finite or infinite sequence $\{B^n\}$ of pairwise disjoint Baire subsets of $\overline{\partial X}$ together with four sequences

$\{a_1^n\}, \{b_1^n\}, \{a_2^n\}, \{b_2^n\}$ from L_r such that for $n = 1, 2, \ldots$:

$$a_j^n \mid B^n \leqslant f_j \leqslant b_j^n \mid B^n, \quad j = 1, 2, \tag{24}$$

$$b_j^n \mid B^n - a_j^n \mid B^n \leqslant \varepsilon, \quad j = 1, 2. \tag{25}$$

and

$$|\mu|(B^n) > (1 - 2^{-n})\Lambda_n, \tag{26}$$

where $\Lambda_n = \sup\{|\mu|(B) \mid B \in \mathcal{B}_0, B \subset \overline{\partial X} \setminus (B^1 \cup \ldots \cup B^{n-1}), \ \Phi(B) \neq \emptyset\}$;

and such that the sequences break off after term number k iff

$$|\mu|(\overline{\partial X} \setminus (B^1 \cup \ldots \cup B^n)) = 0. \tag{27}$$

We claim that

$$|\mu|(\overline{\partial X} \setminus \bigcup_n B^n) = 0. \tag{28}$$

If the sequences break off, then (28) follows from (27). Otherwise $\lim_n \Lambda_n = 0$, since the relation (26) implies $\Lambda_n < 2|\mu|(B^n)$ for $n = 1, 2 \ldots$. Now assume (28) inexact, and consider a Baire subset B of $\overline{\partial X}$ which satisfies (22) with $\bigcup_n B^n$ in the place of C. By the definition of Λ_n, we shall have $|\mu|(B) \leqslant \Lambda_n$ for $n = 1, 2, \ldots$, which is a contradiction since $|\mu|(B) > 0$.

By (28) we can choose a natural number N such that

$$\sum_{n>N} |\mu|(B^n) < \varepsilon. \tag{29}$$

Consider the restricted measures $\mu_0, \mu_1, \ldots, \mu_N$ defined by

$$\mu_0(A) = \mu(A \setminus (B^1 \cup \ldots \cup B^N)) \tag{30}$$

and

$$\mu_n(A) = \mu(A \cap B^n), \quad n = 1, \ldots, N, \tag{31}$$

for all Baire subsets A of $\overline{\partial X}$.

Decomposing $\mu^n = \mu_1^n + i\mu_2^n$, and making use of (25), we obtain for each of the four choices of $j, k = 1, 2$:

$$\int_{\overline{\partial X}} (b_j^n - a_j^n)\, d|\mu_k^n| \leq \varepsilon |\mu|(B^n), \quad n = 1, \ldots, N. \tag{32}$$

Now we may apply the estimate (14) for μ_0, and by virtue of (24) we may apply the estimate (18) for μ_n, $n = 1, \ldots, N$. By (29) and (32) this gives:

$$|\varphi'(\varrho(\mu)) - \varphi(\mu)| \leq \sum_{n=0}^{N} |\varphi'(\varrho(\mu_n)) - \varphi(\mu_n)|$$

$$\leq 4\|\mu_0\| \cdot \|f\| + 4\varepsilon \sum_{n=1}^{N} |\mu|(B^n)$$

$$\leq 4(\|f\| + \|\mu\|)\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this gives the equality

$$\varphi'(\varrho(\mu)) = \varphi(\mu), \tag{33}$$

completing the proof that the diagram is commutative.

Let $M_1(\overline{\partial X})$ and L_1^* be the closed unit balls of $M(\overline{\partial X})$ and L^* respectively, and observe that ϱ maps $M_1(\overline{\partial X})$ *onto* L_1^*. (This does not require the full strength of the Choquet Theorem. It is merely an integral form of the Krein–Milman Theorem, cf. e.g. [9]).

Consider an arbitrary closed subset F of $\mathbf{C}$. By the commutativity of the diagram we shall have:

$$L_1^* \cap (\varphi')^{-1}(F) = \varrho(M_1(\overline{\partial X}) \cap \varphi^{-1}(F)). \tag{34}$$

By the vague continuity of $\varphi, \varphi^{-1}(F)$ is vaguely closed, and so $M_1(\overline{\partial X}) \cap \varphi^{-1}(F)$ is vaguely compact. By the continuity of ϱ, the set $L_1^* \cap (\varphi')^{-1}(F)$ is w^*-compact and hence also w^*-closed It follows that the restriction of φ' to the closed unit ball is w^*-continuous.

By the theorem of Banach–Dieudonné (or Krein–Šmuljan, cf. e.g. [6, p. 429]), φ' is itself w^*-continuous, and so there is an element $\tilde{\tilde{f}} \in L$ such that

$$\varphi'(q) = q(\tilde{\tilde{f}}), \quad \text{all } q \in L^*. \tag{35}$$

If $x \in X$ is arbitrary, then by (35) and by the definitions (10) and (13)

$$\tilde{\tilde{f}}(x) = \psi(x)(\tilde{\tilde{f}}) = \varphi'(\psi(x)) = \int_{\partial X} f\,dm, \tag{36}$$

where m is any measure on $\mathfrak{F}_0$ such that $|m|(X \setminus \partial X) = 0$ and such that m represents $\psi(x)$ in the sense of (2); or what is equivalent, if m represents x in the sense of (1).

If $x \in \partial X$, then we may choose the measure m of (36) to be the positive unit mass concentrated at x. Hence

$$\tilde{\tilde{f}}(x) = f(x), \quad \text{all } x \in \overline{\partial X}. \tag{37}$$

Thus $\tilde{\tilde{f}}$ is a function which belongs to the class L and extends f. The proof is complete.

We shall apply the theorem to the case where L is the real sup-norm space of continuous affine functions on a compact convex subset X of a locally convex Hausdorff space E. Here ∂X is the same as the *extreme boundary* $\partial_e X$, and for a continuous and bounded real valued function f on ∂X, $\bar{f}$ is the *u.s.c. concave upper envelope* of f and $\check{f}$ is the *l.s.c. convex lower envelope* of f. Also we note that $L^\perp(\partial X)$ is (canonically isomorphic to) the space $\mathfrak{N}(\partial_e X)$ of generalized *affine dependences on the extreme boundary of* X, whereas $L^\perp(X)$ is the space $\mathfrak{N}(X)$ of *generalized affine dependences on* X. (cf. [1]).

We are now able to establish the theorem of [2] without any metrizability condition.

Corollary 1. *Let X_i be a compact convex set in a locally convex Hausdorff space E_i for $i = 1, 2$. A continuous map φ of the extreme boundary $\partial_e X_1$ into X_2 can be extended to a homomorphism (continuous affine map) of X_1 into X_2 if and only if the following two requirements are satisfied:*

(i)′ *If* $\nu \in \mathfrak{N}(\partial_e X_1)$, *then the transformed measure* $\varphi\nu$ *belongs to* $\mathfrak{N}(X_2)$.

(ii)′ *The restrictions of* $\widehat{f\circ\varphi}$ *and* $\widecheck{f\circ\varphi}$ *to* $\overline{\partial_e X_1}$ *are continuous for every* $f \in E_2^*$.

Proof. Again the necessity is obvious. To prove sufficiency we assume (i)′, (ii)′ and choose an arbitrary $f \in E_2^*$. The conditions (i), (ii) of the theorem are valid with $f\circ\varphi$ in the place of f by virtue of the conditions (i)′, (ii)″ above. Hence there exists a continuous affine function $\widetilde{f\circ\varphi}$ on X_1 which extends $f\circ\varphi$.

Let $x \in X$ be arbitrary, and let m be any probability measure on $\mathcal{F}_0$ such that $m(\partial X) = 1$ and such that m represents x. (Geometrically, x is the barycenter of m). The continuous mapping φ: $\partial_e X_1 \to X_2$ has a weak m-integral z in the compact convex set K_2. (Geometrically, z is the barycenter of the transformed measure φm.) By the fact that $\widetilde{f\circ\varphi}$ is continuous and affine, and by the definition of weak integrals:

$$\widetilde{f\circ\varphi}(x) = \int_{\partial X} (f\circ\varphi)\, dm = f(z).$$

It follows that z is independent of the choice of m. Hence we may write $z = \tilde{\varphi}(x)$, obtaining:

$$\widetilde{f\circ\varphi}(x) = f(\tilde{\varphi}(x)). \tag{38}$$

Clearly $\tilde{\varphi}$ is affine, and for every $x \in \partial_e X_1$:

$$f(\varphi(x)) = \widetilde{f\circ\varphi}(x) = f(\tilde{\varphi}(x))$$

Since f was arbitrary in E_2^*, this implies $\varphi(x) = \tilde{\varphi}(x)$ for $x \in \partial_e X_1$. Hence $\tilde{\varphi}$ is an affine extension of φ to the whole set X_1.

Finally we observe that (38) gives continuity of $\tilde{\varphi}$ in the given topology of X_1 and in the weak topology of E_2. By the compactness of X_2, the latter topology coincides with the given topology of X_2, and the proof is complete.

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