SUBGROUPS OF IA AUTOMORPHISMS OF A FREE GROUP

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1. Introduction

Generators and defining relations for the group A_n of automorphisms of a free group of rank n were derived by J. Nielsen [11]. For $n=2$, this is a fairly easy task, but for $n \geqslant 3$ it requires very difficult combinatorial arguments which have not been simplified since the appearance of Nielsen's paper. In order to obtain an easier approach to the investigation of A_n and a better understanding of its structure, it seems natural to study its subgroups.

For all n, the elements of A_n which induce the identical automorphism in the commutator quotient group F_n/F'_n form a normal subgroup K of A_n. Bachmuth [1] calls this the group of IA automorphisms of F_n. Magnus [8] showed that this subgroup is generated by the automorphisms

$$K_{ij}: a_i \to a_j a_i a_j^{-1}$$

$$a_k \to a_k, \quad k \neq i$$

and

$$K_{ijk}: a_i \to a_i a_j a_k a_j^{-1} a_k^{-1}$$

$$a_m \to a_m, \quad m \neq i$$

where $a_1, a_2, \ldots, a_n$ are a set of free generators of F_n, and where the subscripts of each of these automorphisms are distinct members of the set $\{1, 2, \ldots, n\}$. In the present paper, we will study certain interesting subgroups of K, in the case $n=3$. In this case, K has a minimal set of nine generators, as K_{ijk}^{-1} is easily seen to be K_{ikj}. Some, although not all, of our results can be obtained for $n>3$ by the same methods.

In section 3, generators and defining relations for the subgroup K_1 of those automorphisms in K which keep two generators of the free group fixed will be presented. In section 4, generators for the subgroup $\tilde{K}_3$ of those automorphisms in K which leave one generator of the free group fixed will be found. Then, in section 5, the group of those automorphisms which take every generator of F into a conjugate of itself will be studied. It will be shown that this group is just C (the subgroup generated by the double indexed generators of K), and a set of defining relations for this group will be found. In section 6, a theorem about T (the group generated by the triple indexed generators of K) being free will be stated, with some discussion of the proof. And finally, in section 7, some comments about some known relations in K will be made, leading to some conjectures about the structure of K.

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2. Notation

We will use $a=a_1$, $b=a_2$, and $c=a_3$ to denote a set of free generators of $F=F_3$.

If μ is an automorphism given by $a\mu=\alpha$, $b\mu=\beta$, $c\mu=\gamma$, then μ will frequently be denoted by μ: $(a; b; c)\rightarrow(\alpha; \beta; \gamma)$ or in some contexts just by $(\alpha; \beta; \gamma)$.

If $\mu, \nu \in A=A_3$, then $\mu\nu$ means first apply ν to $(a; b; c)$ and then apply μ to the result.

According to Nielsen [12, page 23], P, Q, O, and U can be chosen as a set of generators of A, where

$$P\colon (a; b; c)\rightarrow(b; a; c), \quad Q\colon (a; b; c)\rightarrow(a; c; b),$$

$$O\colon (a; b; c)\rightarrow(a^{-1}; b; c) \quad \text{and} \quad U\colon (a; b; c)\rightarrow(ab; b; c).$$

The subgroup of K generated by the double indexed K_{ij} will be called C (for conjugation) and its normal closure in K will be called N.

The subgroup of K generated by the triple indexed K_{ijk} will be called T.

F' will denote the commutator subgroup of F.

If M and N are elements of a group, then the notation $M\rightleftarrows N$ means that M and N commute.

And finally, gp $\langle g_1, ..., g_n\rangle$ will denote the group generated by $g_1, ..., g_n$.

3. The group K_1 of those automorphisms in K which leave two generators of F fixed

If the generators of F are a, b, and c, then by K_1 is meant the group of automorphisms which take $(a; b; c)\rightarrow(aw; b; c)$, where $w\in F'$.

Clearly K_{12}, K_{13}, and K_{123} are in the group K_1 as is any word in these generators.

THEOREM 1. *K is generated by K_{12}, K_{13}, and K_{123}.*

The proof of the theorem relies on the following lemma.

LEMMA 1. *If* μ: $(a; b; c) \to (w(a, b, c); b; c)$ *is an automorphism of the free group F and* $w(a, b, c)$ *is freely reduced, then* $w(a, b, c)$ *contains a exactly once.*

Proof. Clearly w contains a at least once since w, b, and c must be free generators for F, which implies $a \in \text{gp}\langle w, b, c\rangle$.

Suppose there is some automorphism in which $w(a, b, c)$ contains a more than once. Let α be such an automorphism in which w is of minimal length.

$$\alpha\colon (a; b; c) \to (w_\alpha(a, b, c,); b; c).$$

Clearly $w_\alpha(a, b, c)$ must begin and end in some power of a [not necessarily the same power for the beginning and end], since if it ends in b^ε then applying the automorphism $U^{-\varepsilon}$ will result in an automorphism with a w of shorter length. [Similarly if it ends in c^ε or begins in b^ε or c^ε then it could be shortened.] Therefore, $w_\alpha(a, b, c) = a^\beta v(a, b, c) a^\gamma$, where $v(a, b, c)$ does not begin or end in a. But since α is an automorphism, w_a, b, and c must be free generators of F and hence $a = u(w_\alpha, b, c)$—i.e.

$$a = w_\alpha^{p(1)} b^{q(1)} c^{r(1)} \dots w_\alpha^{p(k)} b^{q(k)} c^{r(k)},$$

where the exponents are integers, some of which may be zero. Now w_α^p must begin and end in a, since F is a free group. Therefore, there can be no cancellation between $w_\alpha^{p(i)}$ and $b^{q(j)}$ or $c^{r(j)}$ where $j = i$ or $i-1$. Therefore, again since F is free, all $q(i)$ and $r(i)$ must be zero, so $a = u(w_\alpha, b, c,) = w_\alpha^p$. But this can only happen if $p = \pm 1$ and $w_\alpha = a^{\pm 1}$. This contradicts the assumption that w_α contains a twice. Therefore, the lemma is proved.

By the lemma, any automorphism in K_1 must take $a \to u(b, c) a v(b, c)$ where $u(b, c) v(b, c) \in F'(b, c)$ since we are dealing with an IA automorphism of F.

The proof of Theorem 1 now proceeds as follows: If $k \in K_1$, k: $a \to u(b, c) a v(b, c)$ then

$$u^{-1}(K_{12}, K_{13}) k\colon a \to a v(b, c) u(b, c),$$

where by $u(K_{12}, K_{13})$ is meant the image of $u(b, c)$ in gp $\langle K_{12}, K_{13}\rangle$ under the mapping $b \to K_{12}$, $c \to K_{13}$.

Therefore, the theorem need only be proved for those automorphisms in K_1 which take $a \to a w(b, c)$ where $w(b, c) \in F'(b, c)$. Suppose in such an automorphism k,

$$w(b, c) = w_1(b, c) c^\gamma b^\beta w_2(b, c),$$

where $\beta = \pm 1$ and $\gamma = \pm 1$.

$$w_2(K_{12}, K_{13}) k\colon a \to w_2(b, c) a w_1(b, c) c^\gamma b^\beta.$$

$$\text{If } \beta=\gamma=-1, \text{ then apply } K_{123};$$
$$\text{if } \beta=\gamma=1, \text{ then apply } K_{13}^{-1}K_{123}K_{12}K_{13};$$
$$\text{if } \gamma=1, \beta=-1, \text{ then apply } K_{13}^{-1}K_{132}K_{13};$$
$$\text{and if } \gamma=-1, \beta=1, \text{ then apply } K_{12}^{-1}K_{132}K_{12}.$$

In any of these cases the result will be

$$w_2(b, c)\,aw_1(b, c)\,b^{\beta}c^{\gamma},$$

and then applying $w_2^{-1}(K_{12}, K_{13})$ we will get:

$$aw_1(b, c)\,b^{\beta}c^{\gamma}w_2(b, c).$$

Therefore, given any automorphism in K_1 of the form

$$a \to aw_1(b, c)c^{\gamma}b^{\beta}w_2(b, c),$$

multiplying this automorphism by the proper automorphism in gp $\langle K_{12}, K_{13}, K_{123}\rangle$ results in an automorphism taking

$$a \to aw_1(b, c)\,b^{\beta}c^{\gamma}w_2(b, c).$$

Continuing this process, the above automorphism can be brought into the form $a \to ab^{\varrho}c^{\sigma}$ simply by multiplying by the proper elements of gp $\langle K_{12}, K_{13}, K_{123}\rangle$. But, since the resulting automorphism is in K_1, and hence in K, $\varrho=\sigma=0$.

Therefore, any automorphism in K_1 can be changed to the identity automorphism by multiplying by an automorphism in gp $\langle K_{12}, K_{13}, K_{123}\rangle$, and so the theorem is proven.

Note, by the way, that any mapping taking $(a; b; c)$ into $(aw(b, c); b; c)$ is an automorphism for an arbitrary $w(b,c)$, since it can be generated by U and QUQ. If $w(b,c) \in F'(b,c)$, then this automorphism will be in K and hence in K_1, and hence will be generated by K_{12}, K_{13}, K_{123}.

Now that the generators for the group K_1 are known, one would like to find defining relations. In order to do this, it is useful to introduce new generators for K_1 which facilitate this process. Let

$$w_{\beta\gamma} = b^{\beta}c^{\gamma}b^{-1}c^{-1}bc^{1-\gamma}b^{-\beta}.$$

Then the $w_{\beta\gamma}$ are free generators of $F'(b, c)$. [This is a consequence of a theorem proved in reference 6.] Let $R_{\beta\gamma}$ be the automorphism of F given by $(a; b; c)$ going into $(aw_{\beta\gamma}; b; c)$. By the note above $R_{\beta\gamma}$ is clearly an automorphism in K_1. Similarly, define

$$L_{\beta\gamma}\colon (a; b; c) \to (w_{\beta\gamma}a; b; c).$$

$L_{\beta\gamma}=OR_{\beta\gamma}^{-1}O$, so $L_{\beta\gamma}$ is an automorphism, and since it is clearly in K it is in K_1. Now $K_{123}=R_{11}$, so K_1 is generated by K_{12}, K_{13}, $R_{\beta\gamma}$, and $L_{\beta\gamma}$. The relations below (3.1) are easily seen to be true.

$$\left.\begin{aligned} K_{12}^{\varepsilon}R_{\beta\gamma}K_{12}^{-\varepsilon}&=R_{\beta+\varepsilon,\gamma}\\ K_{12}^{\varepsilon}L_{\beta\gamma}K_{12}^{-\varepsilon}&=L_{\beta+\varepsilon,\gamma}\end{aligned}\right\}\quad \varepsilon=\pm 1$$

$$K_{13}R_{\beta\gamma}K_{13}^{-1}=\begin{cases} R_{11}R_{21}\dots R_{\beta 1}R_{\beta,\gamma+1}R_{\beta 1}^{-1}\dots R_{21}^{-1}R_{11}^{-1} & \text{if } \beta>0\\ R_{0,\gamma+1} & \text{if } \beta=0\\ R_{01}^{-1}R_{-1,1}^{-1}\dots R_{\beta+1,1}^{-1}R_{\beta,\gamma+1}R_{\beta+1,1}\dots R_{-1,1}R_{01} & \text{if } \beta<0\end{cases}$$

$$K_{13}L_{\beta\gamma}K_{13}^{-1}=\begin{cases} L_{11}^{-1}L_{21}^{-1}\dots L_{\beta 1}^{-1}L_{\beta,\gamma+1}L_{\beta 1}\dots L_{21}L_{11} & \text{if } \beta>0\\ L_{0,\gamma+1} & \text{if } \beta=0\\ L_{01}L_{-1,1}\dots L_{\beta+1,1}L_{\beta,\gamma+1}L_{\beta+1,1}^{-1}\dots L_{-1,1}^{-1}L_{01}^{-1} & \text{if } \beta<0\end{cases} \tag{3.1}$$

$$K_{13}^{-1}R_{\beta\gamma}K_{13}=\begin{cases} R_{00}R_{-1,0}\dots R_{\beta+1,0}R_{\beta,\gamma-1}R_{\beta+1,0}^{-1}\dots R_{-1,0}^{-1}R_{00}^{-1} & \text{if } \beta<0\\ R_{0,\gamma-1} & \text{if } \beta=0\\ R_{10}^{-1}\dots R_{\beta 0}^{-1}R_{\beta,\gamma-1}R_{\beta 0}\dots R_{10} & \text{if } \beta>0\end{cases}$$

$$K_{13}^{-1}L_{\beta\gamma}K_{13}=\begin{cases} L_{00}^{-1}L_{-1,0}^{-1}\dots L_{\beta+1,0}^{-1}L_{\beta,\gamma-1}L_{\beta+1,0}\dots L_{-1,0}L_{00} & \text{if } \beta<0\\ L_{0,\gamma-1} & \text{if } \beta=0\\ L_{10}L_{20}\dots L_{\beta 0}L_{\beta,\gamma-1}L_{\beta 0}^{-1}\dots L_{10}^{-1} & \text{if } \beta>0\end{cases}$$

Using these relations, any time $R_{\beta\gamma}^{\pm 1}$ or $L_{\beta\gamma}^{\pm 1}$ is followed, in some given word in K_1, by $K_{12}^{\pm 1}$ or $K_{13}^{\pm 1}$ it can be replaced by $K_{12}^{\pm 1}$ or $K_{13}^{\pm 1}$ followed by some word in the $R_{\beta\gamma}$, and $L_{\beta\gamma}$. Eventually the given word can be brought into the form $w_1(K_{12}, K_{13})w_2(R_{\beta\gamma}, L_{\beta\gamma})$. Also,

$$K_{12}K_{13}K_{12}^{-1}K_{13}^{-1}=R_{11}^{-1}L_{11}, \tag{3.2}$$

so, using the relations (3.1) and (3.2), any word in gp $\langle K_{12}, K_{13}, R_{\beta\gamma}, L_{\beta\gamma}\rangle$ can be changed into the form $K_{12}^{\sigma}K_{13}^{\varrho}w_3(R_{\beta\gamma}, L_{\beta\gamma})$. Such an automorphism takes $(a; b; c)$ into

$$(b^{\sigma}c^{\varrho}f_1af_2c^{-\varrho}b^{-\sigma};\, b;\, c)$$

where f_1 and f_2 are in $F'(b, c)$. This cannot be the identity unless $\varrho=\sigma=0$. Therefore, any relation in gp $\langle K_{12}, K_{13}, R_{\beta\gamma}, L_{\beta\gamma}\rangle$ can be brought by means of (3.1) and (3.2) into the form $w_3(R_{\beta\gamma}, L_{\beta\gamma})=1$. But since $w_{\beta\gamma}$ is a free set of generators of $F'(b, c)$, then clearly gp $\langle R_{\beta\gamma}\rangle$ is free as is gp $\langle L_{\beta\gamma}\rangle$. Also since multiplication on the right is completely independent of multiplication on the left,

$$R_{\beta\gamma}\rightleftarrows L_{\varrho\sigma}. \tag{3.3}$$

This gives a set of defining relations for gp $\langle R_{\beta\gamma}, L_{\beta\gamma}\rangle$.

Therefore the following presentation for K_1 is obtained:

$$K_1 = \langle K_{12}, K_{13}, R_{\beta\gamma}, L_{\beta\gamma} \,|\, (3.1), (3.2), (3.3)\rangle.$$

One would like to have a presentation for K_1 in terms of the generators K_{12}, K_{13}, and K_{123}. But

$$R_{\beta\gamma} = K_{12}^{\beta} K_{13}^{\gamma-1} K_{12}^{-1} K_{123} K_{12} K_{13}^{1-\gamma} K_{12}^{-\beta}$$

and

$$L_{\beta\gamma} = K_{12}^{\beta} K_{13}^{\gamma} K_{12}^{-1} K_{13}^{-1} K_{123} K_{12} K_{13}^{1-\gamma} K_{12}^{-\beta}.$$

Therefore, substituting these expressions in (3.1), (3.2), (3.3), one gets a set of defining relations for K_1 in terms of K_{12}, K_{13}, and K_{123}.

4. The subgroup $\tilde{K}_3$ of those automorphisms in K which leave one generator of F fixed

$\tilde{K}_3$ stands for the group of automorphisms of F which take $(a; b; c)$ into $(aw; bu; c)$ where $w, u \in F'$.

Clearly, K_{12}, K_{13}, K_{123}, K_{21}, K_{23}, and K_{213} are in this group, as is any word generated by them, and again the converse is true.

THEOREM 2. *K_{12}, K_{13}, K_{123}, K_{21}, K_{23}, and K_{213} generate $\tilde{K}_3$.*

The method of proof of this theorem is based upon the work of Magnus [8, section 6] in finding the generators of K.

The proof depends on the following lemma, due to Nielsen. (This lemma is an easy consequence of the fact that any set of free generators of the free group can be changed into (a, b, c) by elementary Nielsen transformations without increasing total length [11] (or see [9, Theorem 3.1]).

NIELSEN'S LEMMA. *Let α, β, c be free generators of F, where α and β are words in (a, b, c). Then $(\alpha; \beta; c)$ can be changed into $(a; b; c)$ by the following processes:*

1. $\alpha \to \alpha\beta^{\pm 1}$ *or* $\beta^{\pm 1}\alpha$	2. $\alpha \to \beta^{\pm 1}$	3. $\alpha \to \alpha c^{\pm 1}$ *or* $c^{\pm 1}\alpha$ *or* α
$\beta \to \beta$	$\beta \to \alpha^{\pm 1}$	$\beta \to \beta c^{\pm 1}$ *or* $c^{\pm 1}\beta$ *or* β
$c \to c$	$c \to c$	$c \to c$

without ever increasing $|\alpha| + |\beta| + |c|$, where $|\;|$ *means length in terms of* a, b, *and* c.

Nielsen's lemma gives generators for the subgroup A_3 of those automorphisms of F which leave c fixed. These can easily be shown equivalent to the following automorphisms:

μ_1: $(\alpha; \beta; c) \to (\alpha\beta; \beta; c)$, μ_2: $(\alpha; \beta; c) \to (\beta\alpha; \beta; c)$,

μ_3: $(\alpha; \beta; c) \to (\beta; \alpha; c)$, μ_4: $(\alpha; \beta; c) \to (\alpha c; \beta; c)$,

μ_5: $(\alpha; \beta; c) \to (\alpha; c\beta; c)$.

Now $K \cap \tilde{A}_3$ consists of those automorphisms of F which leave c fixed, and which induce the identity automorphism of F/F'. But this is clearly just $\tilde{K}_3$. Since K is normal in A, $\tilde{K}_3 = K \cap \tilde{A}_3$ is normal in $\tilde{A}_3$. Also, $A/K = G$, the full 3 by 3 modular group ([12, page 28] or [8, section 6]), so $\tilde{A}_3/\tilde{K}_3 = \tilde{G}_3$, some subgroup of G. Clearly a matrix will be in $\tilde{G}_3$ only if it is in G and it is of the form.

$$\begin{Vmatrix} a_{11} & a_{12} & e \\ a_{21} & a_{22} & f \\ 0 & 0 & 1 \end{Vmatrix}$$

where $a_{11}a_{22} - a_{12}a_{21} = \pm 1$, and all entries are integers.

Conversely, given a matrix of the above form, the matrix

$$\begin{Vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{Vmatrix}$$

is in the 2 by 2 modular group, and hence corresponds to an automorphism μ of the free group generated by a and b ([12], or [8, p. 168]). But then the given matrix corresponds to the automorphism $\mu_4^e \mu_5^f \mu$. Therefore, a three by three matrix is in $\tilde{G}_3$ if and only if it has integer entries with determinant ± 1 and its third row is 0 0 1.

Suppose a presentation for $\tilde{G}_3$ can be found; then $\tilde{K}_3$ is the normal subgroup in $\tilde{A}_3$ generated by the preimages of the relators in $\tilde{G}_3$.

The group of matrices of the form

$$\begin{Vmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{Vmatrix}$$

with determinant ± 1 is generated by the matrices

$$P = \begin{Vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{Vmatrix}, \quad U = \begin{Vmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix}$$

with defining relations

$$P^2 = (PUPU^{-1}PU)^2 = (PU^{-1}PU^2)^4 = 1 \tag{4.1}$$

as is easily seen from [12, page 8].

Also, the group of matrices of the form

$$\begin{Vmatrix} 1 & 0 & e \\ 0 & 1 & f \\ 0 & 0 & 1 \end{Vmatrix}$$

are generated by

$$Q = \begin{Vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix}, \quad S = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{Vmatrix}$$

with defining relations

$$QSQ^{-1}S^{-1} = 1. \tag{4.2}$$

The group of matrices

$$\begin{Vmatrix} a_{11} & a_{12} & e \\ a_{21} & a_{22} & f \\ 0 & 0 & 1 \end{Vmatrix}$$

is easily seen to be a splitting extension of the group of matrices

$$\begin{Vmatrix} 1 & 0 & e \\ 0 & 1 & f \\ 0 & 0 & 1 \end{Vmatrix} \quad \text{by the group} \quad \begin{Vmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{Vmatrix},$$

and the action is given by conjugation, resulting in the relations

$$PQP^{-1}S^{-1} = PSP^{-1}Q^{-1} = UQU^{-1}Q^{-1} = USU^{-1}S^{-1}Q^{-1} = 1. \tag{4.3}$$

Therefore, $\tilde{G}_3 = \langle P, Q, S, U \mid (4.1), (4.2), (4.3) \rangle$.

Now, as was seen above, $\tilde{A}_3$ is generated by $\mu_1, \mu_2, \mu_3, \mu_4$, and μ_5. The natural mapping of $\tilde{A}_3 \to \tilde{G}_3$ takes $\mu_1 \to U$, $\mu_2 \to U$, $\mu_3 \to P$, $\mu_4 \to Q$, and $\mu_5 \to S$. Therefore, $\tilde{K}_3$ is the normal subgroup of $\tilde{A}_3$ generated by all possible preimages of (4.1), (4.2), and (4.3) substituting μ_1 or μ_2 for U, μ_3 for P, μ_4 for Q, and μ_5 for S. To show that this is just the group generated by K_{12}, K_{13}, K_{123}, K_{21}, K_{23}, and K_{213}, it is only necessary to show that each such preimage of a defining relation is in this group, and that this group is normal in $\tilde{A}_3$.

This is easily checked (see [4] for more detail), and so Theorem 2 is proved.

As of yet, no set of defining relations for $\tilde{K}_3$ has been found, and it seems as if this may be as difficult as finding relations for all of K.

5. The subgroup C^* of those automorphisms which take each generator of F into a conjugate of itself

THEOREM 3. *The group C^* of those automorphisms which take each generator of F into a conjugate of itself is just C, the group generated by the double indexed generators of K.*

Proof. Clearly $C \subseteq C^*$. Conversely, given an automorphism in C^*, apply an inner automorphism so that the resulting automorphism takes $c \to c$. Since the group of inner automorphisms is generated by $K_{12}K_{32}$, $K_{21}K_{31}$, and $K_{13}K_{23}$, to prove the theorem it suffices to show that any automorphism of the form

$$(a;\ b;\ c) \to (TaT^{-1};\ SbS^{-1};\ c)$$

is in C. Applying K_{13}^{ε} and K_{23}^{δ} to such an automorphism, we can obtain another of the same type in which neither T nor S begins with c. Since this is an automorphism, at least half of $Ta^{\pm 1}T^{-1}$ or $Sb^{\pm 1}S^{-1}$ must be cancelled in $Ta^{\pm 1}T^{-1}Sb^{\pm 1}S^{-1}$ [9, Theorem 3.2]. But both are of odd length so more than half of one of them must be cancelled—i.e. either $a^{\pm 1}T^{-1}$ is cancelled by S or $Sb^{\pm 1}$ is cancelled by T^{-1}. In the first case, applying $K_{21}^{\pm 1}$ shortens the total length while, in the second, applying $K_{12}^{\pm 1}$ has the same effect. Continuing in this manner—applying K_{13}^{ε} or K_{23}^{δ} to cancel any c's that appear at the extremes of either of the first two components at any stage, and otherwise applying $K_{12}^{\pm 1}$ or $K_{21}^{\pm 1}$ (whichever shortens the total length) the identity automorphism must eventually be reached, since at each step the total length decreases. But this proves the theorem, since in reducing the arbitrary automorphism to the identity only double indexed automorphisms were used.

The following is a presentation for C.

THEOREM 4. $C = \langle K_{12}, K_{13}, K_{21}, K_{23}, K_{31}, K_{32} \mid K_{ij} \rightleftarrows K_{kj}, K_{ij} \rightleftarrows K_{ik}K_{jk}, i \neq j \neq k \neq i \rangle$.

This theorem is also proven by Levinger as an outgrowth of more general considerations [5, Theorem 6.1]. However, our approaches differ, so the theorem is presented here.

To prove the theorem, note first that the group I of inner automorphisms of F is generated by $I_1 = K_{21}K_{31}$, $I_2 = K_{12}K_{32}$, and $I_3 = K_{13}K_{23}$, and is a free group. Then we need the following lemma.

LEMMA 2. $C/I \cong \mathrm{gp}\,\langle K_{12}, K_{13}, K_{21} \rangle$ *and the extension splits.*

Proof of Lemma. Using the relations

$$\begin{aligned} &(1)\quad K_{ij} \rightleftarrows K_{kj} \quad i \neq j \neq k \neq i \\ &(2)\quad K_{ij}^{\varepsilon} I_k K_{ij}^{-\varepsilon} = I_k \quad \text{if } i \neq k \\ &(3)\quad K_{ij}^{\varepsilon} I_i K_{ij}^{-\varepsilon} = I_j^{\varepsilon} I_i I_j^{-\varepsilon} \end{aligned} \tag{5.1}$$

any word in C can be brought into the form

$$v_1(I)\, v_2(K_{12}, K_{13}, K_{21}).$$

I.e., first replace K_{23}^{ε}, K_{31}^{ε}, and K_{32}^{ε} respectively by $I_3^{\varepsilon}K_{13}^{-\varepsilon}$, $I_1^{\varepsilon}K_{21}^{-\varepsilon}$, and $I_2^{\varepsilon}K_{12}^{-\varepsilon}$. Then, using the above relations, bring the inner automorphisms to the left, without changing any of the K_{ij} remaining. Clearly, $v_2(K_{12}, K_{13}, K_{21})$ can be chosen as the coset representative of the above element, and so, to finish the proof of the lemma, it is enough to show that no word in gp $\langle K_{12}, K_{13}, K_{21}\rangle$ other than the identity is an inner automorphism. (This shows firstly that exactly those relations which hold in gp $\langle K_{12}, K_{13}, K_{21}\rangle$ hold in C/I, and secondly that the coset representative $v_{2,3}$ of a product must be the product $v_{2,1}v_{2,2}$ of the coset representatives, otherwise $v_{2,1}v_{2,2}v_{2,3}^{-1}$ would be in I but not a relator in gp $\langle K_{12}, K_{13}, K_{21}\rangle$).

So suppose some inner automorphism is also in gp $\langle K_{12}, K_{13}, K_{21}\rangle$. Since it is an inner automorphism, it takes $(a; b; c)$ into $(waw^{-1}; wbw^{-1}; wcw^{-1})$. But gp $\langle K_{12}, K_{13}, K_{21}\rangle \subseteq \tilde{K}_3$, and hence any element of it leaves c fixed. Therefore, w must be c^β for some β. But the inner automorphism taking $(a; b; c)$ into $(c^\beta a c^{-\beta}; c^\beta b c^{-\beta}; c)$ is just $K_{13}^\beta K_{23}^\beta$. Therefore,

$$K_{13}^\beta K_{23}^\beta = u(K_{12}, K_{13}, K_{21})$$

or

$$K_{13}^\beta K_{23}^\beta u^{-1}(K_{12}, K_{13}, K_{21}) = 1.$$

The exponent sum of K_{23} in this relation is β, but K/K' is free abelian of rank 9 [2, page 7], and therefore the exponent sum of each generator of K must be zero in any relation. Therefore, $\beta = 0$, and so the inner automorphism involved is just the identity, so the lemma is proven.

Note that the relations (5.1) just give the action of gp $\langle K_{12}, K_{13}, K_{21}\rangle$ on I, and so, since I is free, all that remains to be done to find a presentation for C is to find one for gp $\langle K_{12}, K_{13}, K_{21}\rangle$.

LEMMA 3. gp $\langle K_{12}, K_{13}, K_{21}\rangle$ *is free of rank* 3.

The lemma finishes the proof because $C/I \cong$ gp $\langle K_{12}, K_{13}, K_{21}\rangle$. I is free; gp $\langle K_{12}, K_{13}, K_{21}\rangle$ is free; the extension is a splitting extension, so a trivial factor set can be chosen; and the action of gp $\langle K_{12}, K_{13}, K_{21}\rangle$ on I is given by the relations (5.1). Therefore, $C = $ gp $\langle K_{12}, K_{13}, K_{21}, I_1, I_2, I_3 \mid (5.1)\rangle$. But

$$I_1 = K_{21}K_{31}, \quad I_2 = K_{12}K_{32}, \quad \text{and} \quad I_3 = K_{13}K_{23},$$

and the third line of (5.1) is derivable from the second

$$(K_{ij}^\varepsilon I_i K_{ij}^{-\varepsilon} = K_{ij}^\varepsilon K_{kj}^\varepsilon K_{kj}^{-\varepsilon} I_i K_{ij}^{-\varepsilon} = I_j^\varepsilon I_i K_{kj}^{-\varepsilon} K_{ij}^{-\varepsilon} = I_j^\varepsilon I_i I_j^{-\varepsilon}),$$

which is equivalent to the relations

$$K_{ij} \rightleftarrows K_{ik}K_{jk}.$$

Therefore,

$$C = \text{gp}\,\langle K_{12}, K_{13}, K_{21}, K_{23}, K_{31}, K_{32} \mid K_{ij} \rightleftarrows K_{kj},\ K_{ij} \rightleftarrows K_{ik}K_{jk},\quad i \neq j \neq k \neq i\rangle, \quad \text{q.e.d.}$$

The proof of Lemma 3 is quite messy and relies on the type of combinatorial arguments due to Nielsen. The details of the proof will not be presented here, but may be found in [4].

6. The subgroup T generated by the triple indexed generators of K

Bachmuth proved that T is a free group on the three free generators K_{123}, K_{213}, and K_{312} [1, Theorem 4]. The method of proof is to find a representation of K, and to show that the elements of the representation corresponding to K_{123}, K_{213}, and K_{312} generate a free group of rank 3. I too use this idea to prove this theorem, but I use a representation different from the one used by Bachmuth and I believe my proof is simpler, so I am including it here.

Burau [3] gave a matrix representation for the Braid group. An explanation of the method used to get this representation may be found in a paper by Magnus and Peluso [10]. This same approach may be used to find matrix representations for many groups of automorphisms. As a K-characteristic subgroup of the free group F of rank 3, we choose H, the normal closure of ab^{-1} and bc^{-1}; and we denote the groupring of F/H by R. Then R is isomorphic to the ring of polynomials with integral coefficients in $v^{\pm}$, where v is an indeterminate, and H/H' is a free R-module of rank two on which the automorphisms of K act as linear mappings. In this representation, K_{123}, K_{213}, and K_{312} correspond respectively to the matrices

$$\left\|\begin{matrix} 1 & v-v^2 \\ 0 & 1 \end{matrix}\right\|, \quad \left\|\begin{matrix} 1+v^2-v & v^2-v \\ v-v^2 & 1+v-v^2 \end{matrix}\right\|, \quad \left\|\begin{matrix} 1 & 0 \\ v^2-v & 1 \end{matrix}\right\|.$$

To show these matrices generate a free group, it is enough to find a particular value of v for which the group is free. Putting $v=4$, we get the matrices

$$\left\|\begin{matrix} 1 & -12 \\ 0 & 1 \end{matrix}\right\|, \quad \left\|\begin{matrix} 13 & 12 \\ -12 & -11 \end{matrix}\right\|, \quad \left\|\begin{matrix} 1 & 0 \\ 12 & 1 \end{matrix}\right\|.$$

Letting

$$N=\left\|\begin{matrix} 1 & 2 \\ 0 & 1 \end{matrix}\right\|, \quad M=\left\|\begin{matrix} 1 & 0 \\ 2 & 1 \end{matrix}\right\|,$$

these matrices are just N^{-6}, $(N^{-1}M)^6$, and M^6 respectively. But N and M generate a free group of rank 2 [7], and so it is clear that N^{-6}, $(N^{-1}M)^6$, and M^6 generate a free group of rank 3.

But then T is free, since it is generated by three generators and has a free quotient group of rank 3.

7. Conjectures

In conclusion, we would like to venture some as yet unproved conjectures, some supportive evidence for which may be found in [4].

1. K is not finitely related.

2. If N represents the normal closure of C in K, then $N \cap T = 1$, or, alternatively, $K/N \cong T$, which would imply that K has a free quotient group of rank 3.

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