ON THE TORAL STRUCTURE OF LIE p-ALGEBRAS

BY

DAVID J. WINTER

University of Michigan, Ann Arbor, Mich., U.S.A. (1)

Introduction

The purpose of this paper is to describe some of the structure of a Lie p-algebra in terms of its tori. General properties of tori are developed in section 2, where maximal tori and Cartan subalgebras are related, preservation of properties of tori under base field extension and under p-homomorphisms is studied and the invariance of the dimension of maximal tori in a solvable Lie p-algebra is proved. In section 3, exponentials are introduced for the purpose of studying the distribution of tori in a Lie p-algebra. The significance of this section is that it shows how a class of exponentials, sufficiently rich at characteristic 0 to express the conjugacy of Cartan subalgebras, can be effectively introduced at characteristic p. In section 4, it is shown that the maximal tori of a solvable Lie p-algebra over an algebraically closed field are conjugate.

The general prerequisites for the paper are contained in [4], [6]. It is well to mention here results of N. Jacobson [4], [5] and G. Seligman [6], [7] on tori, which are important for this paper, as well results of D. Barnes [1] and R. Block [2] on exponentials in Lie algebras $\mathfrak{L}$ satisfying certain conditions on the degree of nilpotency of ad $\mathfrak{L}^{\infty}$.

I would like to take this opportunity to thank George Seligman for several important observations on the original material, and James Humphreys for remarks leading to a simpler account of the material on exponentials.

1. Preliminaries

We are concerned with only finite dimensional Lie algebras and vector spaces over fields.

1.1 *Base field extension.* If $\mathfrak{L}$ is a Lie algebra or vector space over F, then the extension

---

(1) Most of this work was done at the University of Bonn in 1967-68 while the author was a National Science Foundation Postdoctoral Research Fellow.

$\mathcal{L}\otimes_F K$ of $\mathcal{L}$ to an extension field K of F is denoted $\mathcal{L}_K$, and $\mathcal{L}$ is canonically identified with $\mathcal{L}\otimes 1$.

1.2 *Fitting–Zassenhaus decomposition.* Let $\mathcal{L}$ be a nilpotent Lie algebra over F, V a module for $\mathcal{L}$. Then V has a decomposition $V = V_0(\mathcal{L}) + V_*(\mathcal{L})$ (direct) as described in [4; pp. 33–43]. The subspaces $V_0(\mathcal{L})$, $V_*(\mathcal{L})$ are $\mathcal{L}$-stable, and $V_0(\mathcal{L})$ is characterized as the maximal $\mathcal{L}$-stable subspace of V on which the elements of $\mathcal{L}$ are all nilpotent. The decomposition commutes with base extension. For α a function from $\mathcal{L}$ to F, let $V_\alpha(\mathcal{L}) = \{v\in V \mid v(N-\alpha(N))^{\dim V}=0 \text{ for } N\in\mathcal{L}\}$. This is called a *weight space* (and, in the context of $V=\mathcal{L}$ where the associated representation is ad, a *root space*) of $\mathcal{L}$ in V. Now, $V_0(\mathcal{L}) = V_o(\mathcal{L})$ where o is the zero function on $\mathcal{L}$; and, if F is algebraically closed, $V_*(\mathcal{L}) = \sum_{\alpha\neq o} V_\alpha(\mathcal{L})$.

1.3 *Lie p-algebras.* A Lie p-algebra is a restricted Lie algebra over a field of characteristic $p>0$ [Cf. 4, p. 187]. This is a Lie algebra $\mathcal{L}$ together with a "pth power mapping" $x\mapsto x^p$ from $\mathcal{L}$ to $\mathcal{L}$ such that $(cx)^p = c^p x^p$ $(c\in F,\ x\in\mathcal{L})$, $(\operatorname{ad} x)^p = \operatorname{ad} x^p$ $(x\in\mathcal{L})$, and a third condition. If the center of $\mathcal{L}$ is $\{0\}$, the third condition follows from the first and second. If $\mathcal{L}$ is abelian, the third condition is that $(x+y)^p = x^p + y^p$ $(x, y\in\mathcal{L})$.

A p-subalgebra (respectively p-ideal) of a Lie p-algebra $\mathcal{L}$ is a subalgebra (respectively deal) of the underlying Lie algebra of $\mathcal{L}$ which is closed under the pth power mapping. A p-homomorphism of Lie p-algebras is a Lie algebra homomorphism which commutes with the pth power mappings. A linear Lie p-algebra is a linear Lie algebra stable under the usual associative pth power mapping. (A linear p-algebra together with the associative pth power mapping is a Lie p-algebra). A p-representation of a Lie p-algebra $\mathcal{L}$ is a p-homomorphism from $\mathcal{L}$ into a linear Lie p-algebra.

If $\mathcal{L}$ is a Lie p-algebra over F, then $\mathcal{L}_K$ has the unique structure of a Lie p-algebra over K for any extension field K of F, by [4, p. 192].

1.4. *Notations.* Let $\mathcal{L}$ be a Lie algebra. Then $\mathcal{C}(\mathcal{L})$ denotes the center of $\mathcal{L}$ and Nilp $\mathcal{L}$ denotes the maximal nilpotent ideal of $\mathcal{L}$. The intersection $\bigcap_{i=0}^{\infty}\mathcal{L}^i$ is denoted $\mathcal{L}^\infty$. For subsets $\mathcal{A}$, $\mathcal{B}$ of $\mathcal{L}$, $\mathcal{C}_{\mathcal{A}}(\mathcal{B}) = \{a\in\mathcal{A} \mid a \text{ centralizes } \mathcal{B}\}$ and $\mathcal{N}_{\mathcal{A}}(\mathcal{B}) = \{a\in\mathcal{A} \mid \mathcal{B} \operatorname{ad} a \subset \mathcal{B}\}$.

2. Tori

We develop here some preliminary material on the tori of a Lie p-algebra $\mathcal{L}$. Some of this material is known [Cf. 4, 5, 6, 7] or is straightforward, so that details are often omitted. The chief new results are the following, the first of which is proved for perfect fields in [6].

THEOREM 2.14. *$\mathcal{H}$ is a Cartan subalgebra of $\mathcal{L}$ if and only if $\mathcal{H}$ is the centralizer of a maximal torus of $\mathcal{L}$.*

THEOREM 2.15. *If $\mathcal{J}$ is a maximal torus of $\mathcal{L}$, then $\mathcal{J}_E$ is a maximal torus of $\mathcal{L}_E$ for any extension field E of the ground field of $\mathcal{L}$.*

Throughout this section, $\mathcal{L}$ is a Lie p-algebra. The ground field of $\mathcal{L}$ is denoted F, its algebraic closure K.

Definition 2.1. An element x of $\mathcal{L}$ is *nilpotent* if $x^{p^e}=0$ for some positive integer e. A subalgebra of $\mathcal{L}$ is *nil* if it consists of nilpotent elements of $\mathcal{L}$.

Definition 2.2. $\mathcal{J}$ is a *torus* over F (or, $\mathcal{J}$ is a *toral* Lie p-algebra over F) if $\mathcal{J}$ is an abelian Lie p-algebra over F and $\mathcal{J}_K$ contains no non-zero nilpotent elements. A *torus of* $\mathcal{L}$ is a p-subalgebra of $\mathcal{L}$ which, as a Lie p-algebra, is a torus.

The following (characteristic) property of tori is proved in [4; pp. 192–3].

THEOREM 2.3. *Let $\mathcal{J}$ be a torus over F. Then $\delta(\mathcal{J})$ is diagonalizable over K for any p-representation δ of $\mathcal{J}$ over F.*

Definition 2.4. $\mathcal{L}^{p^e}$ is the span of $\{x^{p^e} \mid x\in\mathcal{L}\}$.

PROPOSITION 2.5. *Let $\mathcal{L}$ be abelian. Then, of the following conditions,* (1), (2), (3) *and* (4) *are equivalent. If F is perfect, all are equivalent.*

(1) *$\mathcal{L}$ is a torus;*

(2) *if $e_1, \ldots, e_n$ is a basis for $\mathcal{L}$, then $e_1^p, \ldots, e_n^p$ is a basis for $\mathcal{L}$;*

(3) *each $x\in\mathcal{L}$ is contained in the p-subalgebra $\langle x^p\rangle$ generated by x^p;*

(4) $\mathcal{L}=\mathcal{L}^p$;

(5) $\mathcal{L}=\{x^p \mid x\in L\}$;

(6) *$\mathcal{L}$ contains no non-zero nilpotent elements.*

Proof. We first show that (1), (2), (3) and (4) are equivalent. Suppose first that $\mathcal{L}$ is a torus and that $e_1, \ldots, e_n$ is a base for $\mathcal{L}$. We claim that $e_1^p, \ldots, e_n^p$ is a basis for $\mathcal{L}$. Thus, suppose that $\sum c_i e_i^p=0$ where $c_i\in F$ for $1\leqslant i\leqslant n$. Choose $d_i\in K$ such that $d_i^p=c_i$ for $1\leqslant i\leqslant n$. Then $0=\sum d_i^p e_i^p=(\sum d_i e_i)^p$ and, since $\mathcal{L}_K$ contains no non-zero nilpotent element, $\sum d_i e_i=0$. It follows that the d_i, hence the c_i, are all zero. Thus $e_1^p, \ldots, e_n^p$ is a basis for $\mathcal{L}$, and (1) implies (2).

We next show that (2) implies (3). Thus, assume (2), let x be a non-zero element of $\mathcal{L}$ and let x^{p^j} be the first p^eth power of x which is a linear combination of the preceding ones. Then $x, \ldots, x^{p^{j-1}}$ is linearly independent. Thus, $x^p, \ldots, x^{p^j}$ is linearly independent, so that the coefficient of x in the expression for x^{p^j} in terms of $x, .., x^{p^{j-1}}$ is non-zero. Thus $x\in\langle x^p\rangle$.

Obviously (3) implies (4).

Assume next that (4) holds for $\mathfrak{L}$. Then (4) holds for $\mathfrak{L}_K$ and the mapping $y \mapsto Y^p$ must therefore map a basis for $\mathfrak{L}_K$ onto a basis for $\mathfrak{L}_K$. In particular, $y \mapsto y^p$ is injective and $\mathfrak{L}_K$ contains no nilpotent elements. Thus, $\mathfrak{L}$ is a torus and (4) implies (1). This shows that (1)–(4) are equivalent.

Finally, assume that F is perfect. Then (5) and (6) are equivalent, since $y \mapsto y^p$ is injective if and only if surjective in this case. Also, (5) implies (4) and (1) implies (6). Thus (1) – (6) are equivalent.

Two straightforward consequences of this proposition are:

COROLLARY 2.6. *Let E be an extension field of F. Then $\mathfrak{L}_E$ is a torus if and only if $\mathfrak{L}$ is a torus.*

COROLLARY 2.7. *Let $\mathfrak{T}$ be a torus over F, δ a p-homomorphism from $\mathfrak{T}$ into $\mathfrak{L}$. Then $\delta(\mathfrak{T})$ is a torus of $\mathfrak{L}$.*

PROPOSITION 2.8. *If $\mathfrak{T}$ is a torus of* Nilp $\mathfrak{L}$, *then $\mathfrak{T}$ is central in $\mathfrak{L}$.*

Proof. Since $\mathfrak{T}$ is contained in the nilpotent ideal Nilp $\mathfrak{L}$ and $\mathfrak{L}$ ad $\mathfrak{T} \subset$ Nilp $\mathfrak{L}$, $\mathfrak{L}(\text{ad } \mathfrak{T})^n = 0$ for some n and $\mathfrak{L} \subset \mathfrak{L}_0$ (and $\mathfrak{T}$). But ad $\mathfrak{T}$ is diagonalizable over K, by 2.3, so that $\mathfrak{L}_0(\text{ad } \mathfrak{T}) = C_{\mathfrak{L}}(\mathfrak{T})$ and $\mathfrak{T}$ centralizes $\mathfrak{L}$.

Definition 2.9. An element of $\mathfrak{L}$ is *semi-simple* if it is contained in some torus of $\mathfrak{L}$.

By 2.5, an element x of $\mathfrak{L}$ is semi-simple if and only if x is contained in the p-subalgebra $\langle x^p \rangle$ generated by x^p. This condition is taken as the definition of semi-simplicity in [6].

PROPOSITION 2.10. *Let $\mathfrak{S}$ be a commutative set of semisimple elements of $\mathfrak{L}$. Then $\mathfrak{S}$ is contained in a torus of $\mathfrak{L}$.*

Proof. Let $s_1, \dots, s_m$ be a maximal F-free subset of $\mathfrak{S}$. Let $\mathfrak{S}_i$ be the p-subalgebra generated by s_i $(1 \leqslant i \leqslant m)$. Now $\mathfrak{T} = \sum \mathfrak{S}_i$ is abelian, since each $\mathfrak{S}_i$ is spanned by p^eth powers of s_i. Each $\mathfrak{S}_i$ is a torus, since $\mathfrak{S}_i$ is contained in any torus containing s_i. Thus, $\mathfrak{T}^p = \sum \mathfrak{S}_i^p = \sum \mathfrak{S}_i = \mathfrak{T}$. Thus, $\mathfrak{T}$ is a torus containing $\mathfrak{S}$, by 2.5.

PROPOSITION 2.11. *For each $x \in \mathfrak{L}$, x^{p^e} is semi-simple for some $e \geqslant 0$. If $\mathfrak{L}$ is abelian, $\mathfrak{L}^{p^e}$ is a torus for some $e \geqslant 0$.*

Proof. The p-subalgebra generated by an element x of $\mathfrak{L}$ is abelian, so the first assertion follows from the second. For the second, let $\mathfrak{L}$ be abelian. Then $\mathfrak{L}^{p^{e+1}} = (\mathfrak{L}^{p^e})^p$ and $\mathfrak{L}^p \supset \mathfrak{L}^{p^2} \supset \dots \supset \mathfrak{L}^{p^e}$ for $e \geqslant 1$, so that $\mathfrak{L}^{p^e} = (\mathfrak{L}^{p^e})^p$ and $\mathfrak{L}^{p^e}$ is a torus for e sufficiently large, by 2.5.

COROLLARY 2.12. *Let $\mathfrak{L}$ be nilpotent. Then $\mathfrak{L}$ has a unique maximal torus $\mathfrak{T}$. Moreover, $\mathfrak{T}$ is central and consists of the semi-simple elements of $\mathfrak{L}$. If $\mathfrak{V}$ is a p-module for $\mathfrak{L}$, then* $\mathfrak{V}_0(\mathfrak{T}) = \mathfrak{V}_0(\mathfrak{L})$.

Proof. Let $\mathfrak{T}$ be a maximal torus of $\mathfrak{L}$. Since $\mathfrak{L}$ is nilpotent, $\mathfrak{T}$ is central in $\mathfrak{L}$, by 2.8. If x is a semisimple element of $\mathfrak{L}$, it follows from 2.10 that $x \in \mathfrak{T}$. Thus, $\mathfrak{T}$ is the set of semi-simple elements of $\mathfrak{L}$.

Now let V be p-module for $\mathfrak{L}$. For each $x \in \mathfrak{L}$, $x^{p^e} \in \mathfrak{T}$ for some e, by 2.11, so that $V_0(x) \supset V_0(\mathfrak{T})$ for $x \in \mathfrak{L}$. Thus, $V_0(\mathfrak{L}) \supset V_0(\mathfrak{T})$. The other inclusion is trivial.

COROLLARY 2.13. *Let $\mathfrak{T}$ be a p-ideal of $\mathfrak{L}$ and let $\mathfrak{T}$ and $\mathfrak{L}/\mathfrak{T}$ be tori. Then $\mathfrak{L}$ is a torus.*

Proof. Suppose not and let $\mathfrak{S}$ be a maximal abelian subalgebra of $\mathfrak{L}$. Since $\mathfrak{T}$ is central, by 2.8, $\mathfrak{T}$ is a proper subalgebra of $\mathfrak{S}$. Since $\mathfrak{S}$ is an abelian p-subalgebra and $\mathfrak{S}/\mathfrak{T}$ a torus, $\mathfrak{S}$ is a torus, by a simple application of 2.5. Since $\mathfrak{L}$ is nilpotent ($\mathfrak{T}$ is central, as noted above), $\mathfrak{S}$ is central, by 2.8. This, together with the maximal commutativity of $\mathfrak{S}$ implies that $\mathfrak{S} = \mathfrak{L}$. Thus, $\mathfrak{L}$ is a torus.

THEOREM 2.14. *$\mathfrak{H}$ is a Cartan subalgebra of $\mathfrak{L}$ if and only if $\mathfrak{H}$ is the centralizer of a maximal torus of $\mathfrak{L}$.*

Proof. Let $\mathfrak{T}$ be a maximal torus of $\mathfrak{L}$ and $\mathfrak{H} = C_{\mathfrak{L}}(\mathfrak{T})$. Then $\mathfrak{H}/\mathfrak{T}$ contains no non-zero torus, by 2.13. Thus, $\mathfrak{H}/\mathfrak{T}$ contains no non-zero semi-simple elements. For $x \in \mathfrak{H}/\mathfrak{T}$ we can choose e such that x^{p^e} is semi-simple—that is, such that $x^{p^e} = 0$. That is, ad x is nilpotent. Thus, $\mathfrak{H}/\mathfrak{T}$ is nilpotent. Since $\mathfrak{T}$ is central in $\mathfrak{H}$, $\mathfrak{H}$ is nilpotent. To prove that $\mathfrak{H}$ is a Cartan subalgebra of $\mathfrak{L}$, it remains to show that $\mathfrak{N}_{\mathfrak{L}}(\mathfrak{H}) \subset \mathfrak{H}$. But $\mathfrak{N}_{\mathfrak{L}}(\mathfrak{H})\,\mathrm{ad}\,\mathfrak{T} \subset \mathfrak{H} = C_{\mathfrak{L}}$ (ad $\mathfrak{T}$), so that $\mathfrak{N}_{\mathfrak{L}}(\mathfrak{H}) \subset \mathfrak{L}_0$ (ad $\mathfrak{T}$) $= C_{\mathfrak{L}}$ (ad $\mathfrak{T}$) $= \mathfrak{H}$.

Suppose, conversely that $\mathfrak{H}$ is a Cartan subalgebra of $\mathfrak{L}$. It is a p-subalgebra of $\mathfrak{L}$, as the normalizer of itself. Let $\mathfrak{T}$ be the maximal torus of $\mathfrak{H}$. Then ad $\mathfrak{T}$ is the maximal torus of ad $\mathfrak{H}$. (E.g., ad $\mathfrak{H}$/ad $\mathfrak{T}$ is nil because $\mathfrak{H}/\mathfrak{T}$ is nil). Thus, $\mathfrak{H} = \mathfrak{L}_0$ (ad $\mathfrak{H}$) $= \mathfrak{L}_0$ (ad $\mathfrak{T}$) $=$ $C_{\mathfrak{L}}(\mathfrak{T})$, by 2.12.

THEOREM 2.15. *Let E be an extension field of F, $\mathfrak{T}$ a maximal torus of $\mathfrak{L}$. Then $\mathfrak{T}_E$ is a maximal torus of $\mathfrak{L}_E$.*

Proof. We know that $\mathfrak{T}_E$ is a torus of $\mathfrak{L}_E$. We now reduce the proof to the nilpotent, then the abelian, case.

Let $\mathfrak{H}$ be the centralizer of $\mathfrak{T}$. Since $\mathfrak{H}$ is a Cartan subalgebra of $\mathfrak{L}$, by 2.14, $\mathfrak{H}_E$ is a Cartan subalgebra of $\mathfrak{L}_E$. Thus $\mathfrak{H}_E$ contains a unique maximal torus of $\mathfrak{L}_E$, by 2.14. We may therefore assume, without loss of generality, that $\mathfrak{L}$ is nilpotent.

The maximal torus of $\mathcal{L}$ (respectively $\mathcal{L}_E$) is contained in the center of $\mathcal{L}$ (respectively $\mathcal{L}_E$), by 2.12. But, as is well known and easily verified, the center of $\mathcal{L}_E$ is $\mathcal{C}_E$ where $\mathcal{C}$ is the center of $\mathcal{L}$. Thus, we may assume that $\mathcal{L}$ is abelian.

Now choose e such that $\mathcal{L}^{p^e} = \mathcal{T}$, by 2.5 and 2.11. Since $\mathcal{L}$ is abelian, $\mathcal{L}_E^{p^e} = \mathcal{T}_E$. It follows that $\mathcal{T}_E$ is the maximal torus of $\mathcal{L}_E$.

The following theorem is similar to a result in [1]. We include a proof, since the theorem is needed for the sequel.

THEOREM 2.16. *Let* $0 \to \mathcal{L}' \xrightarrow{\alpha} \mathcal{L} \xrightarrow{\beta} \bar{\mathcal{L}} \to 0$ *be an exact sequence of Lie p-algebras, the* α, β *being p-homomorphisms. If* $\mathcal{T}$ *is a maximal torus of* $\mathcal{L}$, *then* $\beta(\mathcal{T})$ *is a maximal torus of* $\bar{\mathcal{L}}$. *If* $\bar{\mathcal{T}}$ *is a maximal torus of* $\bar{\mathcal{L}}$ *and* $\mathcal{T}$ *is a maximal torus of* $\beta^{-1}(\bar{\mathcal{T}})$, *then* $\mathcal{T}$ *is a maximal torus of* $\mathcal{L}$.

Proof. Suppose first that $\bar{\mathcal{T}}$ is a maximal torus of $\bar{\mathcal{L}}$ and that $\mathcal{T}$ is a maximal torus of $\beta^{-1}(\bar{\mathcal{T}}) = \mathcal{B}$. Let $\mathcal{H} = \mathcal{B}_0$ (ad $\mathcal{T}$), a Cartan subalgebra of $\mathcal{B}$ by 2.14. Since $\mathcal{B} = \mathcal{H} + \mathcal{B}_*$ (ad $\mathcal{T}$), $[\mathcal{B}, \mathcal{B}] \supset [\mathcal{T}, \mathcal{B}_*$ (ad $\mathcal{T})] \supset \mathcal{B}_*$ (ad $\mathcal{T}$). Thus, $\mathcal{B} = \mathcal{H} + [\mathcal{B}, \mathcal{B}]$. But $\beta(\mathcal{B})$ is abelian, so that $\beta([\mathcal{B}, \mathcal{B}]) = 0$ and $\beta(\mathcal{H}) = \beta(\mathcal{B}) = \bar{\mathcal{T}}$. Now $\mathcal{H}/\mathcal{T}$ is nil, so that $\beta(\mathcal{H})/\beta(\mathcal{T}) = \bar{\mathcal{T}}/\beta(\mathcal{T})$ is nil as well as toral. Thus, $\bar{\mathcal{T}} = \beta(\mathcal{T})$. It follows that $\mathcal{T}$ is a maximal torus of $\mathcal{L}$. For let $\mathcal{S}$ be a maximal torus of $\mathcal{L}$ containing $\mathcal{T}$. Then $\beta(\mathcal{S})$ is a torus containing $\bar{\mathcal{T}}$, so that $\beta(\mathcal{S}) = \bar{\mathcal{T}}$ and $\mathcal{S} \subset \beta^{-1}(\bar{\mathcal{T}}) = \mathcal{B}$. Thus, $\mathcal{S} = \mathcal{T}$ and $\mathcal{T}$ is a maximal torus of $\mathcal{L}$.

Next, let $\mathcal{T}$ be any maximal torus of $\mathcal{L}$. Let $\bar{\mathcal{T}}$ be a maximal torus of $\bar{\mathcal{L}}$ containing $\beta(\mathcal{T})$. Then $\mathcal{T}$ is a maximal torus of $\beta^{-1}(\bar{\mathcal{T}})$, whence $\beta(\mathcal{T}) = \bar{\mathcal{T}}$ as above. Thus, $\beta(\mathcal{T})$ is a maximal torus of $\bar{\mathcal{L}}$.

PROPOSITION 2.17. *Let* $\mathcal{L}$ *be solvable. Then the dimension of a maximal torus of* $\mathcal{L}$ *is a constant.*

Proof. The proof is by induction on the dimension of $\mathcal{L}$, and is trivial if $\mathcal{L}$ is abelian or of dimension one. Next, suppose that $\mathcal{L}$ is not abelian and let $\mathcal{A}$ be a minimal non-zero p-ideal of $\mathcal{L}$. Since $\mathcal{L}$ is solvable, $\mathcal{A}$ is abelian. Moreover, $\mathcal{A}$ is a torus or $\mathcal{A}$ is a nil ideal, as we see by considering the series $\mathcal{A}^{p^e}$ in the light of 2.5. Let $\bar{\mathcal{L}} = \mathcal{L}/\mathcal{A}$ and let β: $\mathcal{L} \to \bar{\mathcal{L}}$ be the canonical p-homomorphism. Let $\mathcal{T}_1$, $\mathcal{T}_2$ be maximal tori of $\mathcal{L}$. Then the dimensions of $\beta(\mathcal{T}_1)$, $\beta(\mathcal{T}_2)$ are equal, by induction, since the $\beta(\mathcal{T}_i)$ are maximal tori of $\bar{\mathcal{L}}$ by 2.16. It follows that the dimensions of $\mathcal{T}_1$, $\mathcal{T}_2$ are equal. For if $\mathcal{A}$ is nil, $\mathcal{T}_i \cap \mathcal{A} = \{0\}$ for $i = 1, 2$; and if $\mathcal{A}$ is toral, it is central and $\mathcal{T}_i \supset \mathcal{A}$ for $i = 1, 2$.

3. Exponentials

In this section, we introduce exponential operators on sets of tori and Cartan subalgebras, and develop basic properties of these operators. In the next section, conjugacy results in terms of these operators are established.

Our considerations are motivated by the importance of the automorphisms exp ad x where x is an element of a root space $\mathcal{L}_\alpha$ (ad $\mathcal{H}$) of a Lie algebra $\mathcal{L}$ of characteristic 0 relative to a Cartan subalgebra $\mathcal{H}$ of $\mathcal{L}$. Examples of difficulties to be circumvented in the present context of ground fields of characteristic p are that exp ad x is not defined until the meaning of $1/n!$ ($n \geqslant p$) is decided upon, and that exp ad x is not always an automorphism even when $(\mathrm{ad}\, x)^p = 0$.

Throughout the remainder of the section, $\mathcal{L}$ is a Lie p-algebra, F is its underlying ground field and K is the algebraic closure of F.

It is convenient to introduce the following rough substitute E^x (not to be confused with the e^x introduced later) for the exponential mapping.

Definition 3.1.
$$E^x = \sum_0^{p-1} \frac{(\mathrm{ad}\, x)^n}{n!}.$$

We begin by considering a maximal torus $\mathcal{T}$ and an element x of a root space $\mathcal{L}_\alpha$ (ad $\mathcal{T}$). The Lie p-algebra $\mathcal{B}$ generated by $\mathcal{T}$ and x is solvable and has the form $\mathcal{B} = \mathcal{T} + \mathcal{A}$ (not necessarily a direct sum of subspaces) where $\mathcal{A}$ is an abelian p-ideal of $\mathcal{B}$ and $x \in \mathcal{A}$. Now $E^x|_{\mathcal{B}} = 1 + \mathrm{ad}_{\mathcal{B}} x$ is an automorphism of $\mathcal{B}$. Thus, $\mathcal{T}E^x$ is an abelian subalgebra, but need not be a torus or p-subalgebra since $E^x|_{\mathcal{B}}$ need not be a p-automorphism. For these reasons, the following conventions are adopted.

Definition 3.2. Let $\mathcal{T}$ be a torus of $\mathcal{L}$. An element x (respectively subset $\mathcal{S}$) of $\mathcal{L}$ is *defined* at $\mathcal{T}$ if there is a p-subalgebra of $\mathcal{L}$ of the form $\mathcal{B} = \mathcal{T} + \mathcal{A}$ (not necessarily a direct sum of subspaces) where $\mathcal{A}$ is an abelian p-ideal of $\mathcal{B}$ and $x \in \mathcal{A}$ (respectively $\mathcal{S} \subset \mathcal{A}$).

Definition 3.3. Let $\mathcal{T}$ be a torus of $\mathcal{L}$, x an element of $\mathcal{L}$ defined at $\mathcal{T}$. Then $\mathcal{T}e^x$ is the maximal torus of the (abelian) p-subalgebra generated by $\mathcal{T}E^x$.

If x lies in a root space of a torus $\mathcal{T}$ of $\mathcal{L}$, then x is defined at $\mathcal{T}$, by our earlier remarks, so that $\mathcal{T}e^x$ is defined. It is this case with which we are primarily concerned. However, the additional generality which is built in costs little and is convenient later on.

THEOREM 3.4. *Let $\mathcal{T}$ be a maximal torus of $\mathcal{L}$. Then:*

(1) *if x is defined at $\mathcal{T}$, the dimensions of $\mathcal{T}$ and $\mathcal{T}e^x$ are equal;*

(2) *if $\{x_1, \ldots, x_n\}$ is defined at $\mathcal{T}$, $\mathcal{T}e^{x_1} \ldots e^{x_n} = \mathcal{T}e^{x_1+\cdots+x_n}$.*

Proof. Let $\mathcal{B} = \mathcal{T} + \mathcal{A}$ be a p-subalgebra of $\mathcal{L}$ where $\mathcal{A}$ is as in 3.2. Everything takes place in $\mathcal{B}$. Note that $E^a|_{\mathcal{B}}$ is an automorphism of $\mathcal{B}$ for $a \in \mathcal{A}$.

For (1), let $x \in \mathcal{A}$. Since $\mathcal{B}$ is solvable, it suffices, by 2.17, to show that $\mathcal{T}e^x$ is a maximal torus of $\mathcal{B}$. For this, let $\sigma = E^x|_{\mathcal{B}}$. We have the commutative diagram

$$\begin{tikzcd} \mathcal{B} \arrow[r, "\sigma"] \arrow[d, "\mathrm{ad}_{\mathcal{B}}"'] & \mathcal{B} \arrow[d, "\mathrm{ad}_{\mathcal{B}}"] \\ \mathrm{Ad}_{\mathcal{B}}\mathcal{B} \arrow[r, "\mathrm{Ad}\,\sigma"'] & \mathrm{Ad}_{\mathcal{B}}\mathcal{B} \end{tikzcd}$$

where $\mathrm{Ad}\,\sigma$ is the mapping $b \mapsto \sigma^{-1}b\sigma$ ($b \in \mathcal{B}$). Now $\mathrm{ad}_{\mathcal{B}}$ is a p-homomorphism and $\mathrm{Ad}\,\sigma$ is a p-automorphism (although σ need not be a p-automorphism). And $\mathrm{ad}\,\mathcal{T}$ is a maximal torus of $\mathrm{ad}_{\mathcal{B}}\,\mathcal{B}$, by 2.16. Thus, $(\mathrm{ad}\,\mathcal{T})\,\mathrm{Ad}\,\sigma = \mathrm{ad}\,(\mathcal{T}\sigma) = \mathrm{ad}\,\mathcal{T}E^x$ is a maximal torus of $\mathrm{ad}_{\mathcal{B}}\,\mathcal{B}$. It therefore suffices to show that $\mathcal{T}e^x$ is a maximal torus of $\mathrm{ad}^{-1}(\mathcal{T}E^x) = \mathcal{T}E^x + \mathcal{C}(\mathcal{C} = \mathcal{C}(\mathcal{B}))$, by 2.16. Let $\mathcal{S}$ be the maximal torus of $\mathcal{C}$. Since $\mathcal{T}$ is maximal, $\mathcal{S} \subset \mathcal{T}$. Thus, $\mathcal{S} \subset \mathcal{T}e^x$. Choose e such that $(\mathcal{T}E^x)^{p^e} \subset \mathcal{T}e^x$ and $\mathcal{C}^{p^e} \subset \mathcal{S}$, by 2.11. Then $(\mathcal{T}E^x + \mathcal{C})^{p^e} \subset \mathcal{T}e^x$. It follows that $\mathcal{T}e^x$ is the maximal torus of $\mathcal{T}E^x + \mathcal{C}$, by 2.5 and 2.11.

We now prove (2). For this, we assume that $x, y \in \mathcal{A}$ and follow the above notation. As above, $\mathcal{T}e^x$ is the maximal torus of $\mathcal{T}e^x + \mathcal{C}$. Similarly, $(\mathcal{T}e^x)\,e^y$ is the maximal torus of $(\mathcal{T}e^x)\,E^y + \mathcal{C} = (\mathcal{T}e^x + \mathcal{C})\,E^y \subset (\mathcal{T}E^x + \mathcal{C})\,E^y = \mathcal{T}E^{x+y} + \mathcal{C}$. Finally, $\mathcal{T}e^{x+y}$ is the maximal torus of $\mathcal{T}E^{x+y} + \mathcal{C}$, so that $\mathcal{T}e^x e^y \subset \mathcal{T}e^{x+y}$. Since their dimensions are equal, $\mathcal{T}e^x e^y = \mathcal{T}e^{x+y}$. Now working in $\mathcal{B}$ so that relevant tori are maximal, (2) follows easily by induction.

PROPOSITION 3.5. *Let* $0 \to \mathcal{L}' \xrightarrow{\alpha} \mathcal{L} \xrightarrow{\beta} \bar{\mathcal{L}} \to 0$ *be an exact sequence of Lie p-algebras, where α and β are p-homomorphisms. Let $\mathcal{T}$ be a maximal torus of $\mathcal{L}$, x an element of $\mathcal{L}$ defined at $\mathcal{T}$. Then $\beta(x)$ is defined at $\beta(\mathcal{T})$ and $\beta(\mathcal{T}e^x) = \beta(\mathcal{T})\,e^{\beta(x)}$.*

Proof. $\beta(\mathcal{T})$ is a maximal torus of $\bar{\mathcal{L}}$, by 2.16, and $\beta(x)$ is obviously defined at $\beta(T)$. Since the root vector components x_α of x lie in any $\mathcal{A}$ of the kind specified in 3.2, the set $\{x_\alpha\}$ is defined at $\mathcal{T}$ and $\mathcal{T}e^x = \mathcal{T}\prod e^{x_\alpha}$, by 3.4.

It follows that we may assume without loss of generality that $x = x_\alpha \in \mathcal{L}_\alpha\,(\mathrm{ad}\,T)$ for some $\alpha \neq 0$. Let $\mathcal{T}_0$ be the kernel of α and choose $t \in \mathcal{T}$ such that $[t, x] = x$. Now $\mathcal{T} = \mathcal{T}_0 \oplus Ft$ and $\mathcal{T}e^x = (\mathcal{T}(1 + \mathrm{ad}\,x))^{p^e} = \mathcal{T}_0 \oplus F(t+x)^{p^e}$ for some e. Choose such an e and observe that $\beta(\mathcal{T}e^x) = \beta(\mathcal{T}_0) + F(\beta(t) + \beta(x))^{p^e}$, a torus. But the same reasoning as above shows that the latter algebra is $\beta(\mathcal{T})\,e^{\beta(x)}$. That is, $\beta(\mathcal{T}e^x) = \beta(\mathcal{T}_0) + F(\beta(t) + \beta(x))^{p^e} = \beta(\mathcal{T})\,e^{\beta(x)}$.

LEMMA 3.6. *Let $\mathcal{T}$ be a torus of $\mathcal{L}$, x an element of a root space $\mathcal{L}_\alpha$ $(\mathrm{ad}\,\mathcal{T})$ such that $(\mathrm{ad}\,x)^p = 0$. Then $\mathrm{ad}\,(\mathcal{T}E^x)$ is diagonalizable over K and $\mathcal{C}_{\mathcal{L}}(\mathcal{T}E^x) = \mathcal{C}_{\mathcal{L}}(\mathcal{T})\,E^x$.*

Proof. By base field extension considerations, we can assume without loss of generality that $\mathrm{ad}\,\mathcal{T}$ is diagonalizable. Now $\mathcal{L} = \sum \mathcal{L}_\beta\,(\mathrm{ad}\,\mathcal{T})$. Take $y \in \mathcal{L}_\beta\,(\mathrm{ad}\,\mathcal{T})$. Since

$y\,(\mathrm{ad}\,x)^n \in L_{\beta+n\alpha}\,(\mathrm{ad}\,\mathcal{T})$ for all n, we have the following for $t\in\mathcal{T}$: $(yE^x)\ \mathrm{ad}\,(tE^x)=(yE^x)$ $(\mathrm{ad}\,t\ (I+\mathrm{ad}\,x))=\sum_0^{p-1}(y\,(\mathrm{ad}\,x)^n/n!)\ (\mathrm{ad}\,t-\alpha\,(\mathrm{ad}\,t)\ \mathrm{ad}\,x)=\sum_0^{p-1}((\beta\ (\mathrm{ad}\,t)+n\alpha\ (\mathrm{ad}\,t))$ $(y(\mathrm{ad}\,x)^n/n!)-\alpha\,(\mathrm{ad}\,t)\,(y\,(\mathrm{ad}\,x)^{n+1}/n!))=\beta\,(\mathrm{ad}\,t)\,yE^x+\sum\alpha\,(\mathrm{ad}\,t)\,((ny\,(\mathrm{ad}\,x)^n/n!)-(y(\mathrm{ad}\,x)^{n+1}/n!)$ $=\beta\,(\mathrm{ad}\,t)\ yE^x-\alpha\,(\mathrm{ad}\,t)\ (y(\mathrm{ad}\,x)^p/(p-1)!)=\beta\,(\mathrm{ad}\,t)\ yE^x$. Thus, for $Z\in\mathfrak{L}_\beta\ (\mathrm{ad}\,\mathcal{T})\,E^x$, $Z\,\mathrm{ad}$ $(tE^x)=\beta\ (\mathrm{ad}\,t)\ Z$. Since E^x is non-singular, $\mathfrak{L}=\sum\mathfrak{L}_\beta\,(\mathrm{ad}\ T)\,E^x$. It follows that $\mathrm{ad}\,(\mathcal{T}E^x)$ is diagonalizable and that $\mathfrak{L}_\beta\,(\mathrm{ad}\,\mathcal{T})\,E^x=\mathfrak{L}_{\beta'}\,(\mathrm{ad}\,\mathcal{T}E^x)$ where $\beta'(\mathrm{ad}\,tE^x)=\beta\ (\mathrm{ad}\,t)\ (t\in\mathcal{T})$. Taking $\beta=0$, we have $C_{\mathfrak{L}}(\mathcal{T})\,E^x=C_{\mathfrak{L}}(\mathcal{T}E^x)$.

THEOREM 3.7. *Let $\mathcal{T}$ be a maximal torus of $\mathfrak{L}$, x an element of $\mathfrak{L}$ defined at $\mathcal{T}$. Then $\mathcal{T}$ is a Cartan subalgebra of $\mathfrak{L}$ if and only if $\mathcal{T}e^x$ is a Cartan subalgebra of $\mathfrak{L}$.*

Proof. It suffices to show that for $\mathcal{T}$ a Cartan subalgebra of $\mathfrak{L}$, $\mathcal{T}e^x$ is a Cartan subalgebra of $\mathfrak{L}$. For then, if $\mathcal{T}e^x$ is a Cartan subalgebra of $\mathfrak{L}$, $\mathcal{T}=(\mathcal{T}e^x)\,e^{-x}$ is a Cartan subalgebra of $\mathfrak{L}$.

Thus, assume that $\mathcal{T}$ is a Cartan subalgebra of $\mathfrak{L}$. We show that $\mathcal{T}e^x$ is a Cartan subalgebra of $\mathfrak{L}$, that is, that $\mathcal{T}e^x=C_{\mathfrak{L}}(\mathcal{T}e^x)$. As in the proof of 3.5, the set $\{x_\alpha\}$ of root space components of x is defined at $\mathcal{T}$. Since $\mathcal{T}e^x=\mathcal{T}\prod e^{x_\alpha}$, by 3.4, we may therefore assume that $x\in\mathfrak{L}_\alpha\ (\mathrm{ad}\,\mathcal{T})$ for some $\alpha\neq 0$. Let $\mathcal{T}_0$ be the kernel of α. Now $C_{\mathfrak{L}}(\mathcal{T}_0)$ contains $\mathcal{T}$, x, $\mathcal{T}e^x$ and $C(\mathcal{T}e^x)$. Since what we are to show is that $\mathcal{T}e^x=C(\mathcal{T}e^x)$, we may work in $C_{\mathfrak{L}}(\mathcal{T}_0)$. That is, we may assume with no loss in generality that $\mathcal{T}_0$ is central. But $x^p\in\mathcal{T}_0$. For $0=\mathcal{T}\,(\mathrm{ad}\,x)^p=\mathcal{T}\ \mathrm{ad}\ x^p$, so that $x^p\in C_{\mathfrak{L}}(\mathcal{T})=\mathcal{T}$; and $0=[x,x^p]=\alpha(x^p)\,x$, so that $x^p\in\mathcal{T}_0$. Thus, $(\mathrm{ad}\,x)^p=0$ and $C_{\mathfrak{L}}(\mathcal{T}E^x)=C_{\mathfrak{L}}(\mathcal{T})\,E^x=\mathcal{T}e^x$, by 3.6. Thus, it remains to show that $C_{\mathfrak{L}}(\mathcal{T}e^x)=C_{\mathfrak{L}}(\mathcal{T}E^x)$. But this is true since $\mathrm{ad}\ \mathcal{T}E^x$ is diagonalizable, by 3.6, and since $\mathfrak{L}_0\ (\mathrm{ad}\ \mathcal{T}e^x)=\mathfrak{L}_0\ (\mathrm{ad}\ \mathcal{T}E^x)$, by 2.12.

We now define exponential operators on Cartan subalgebras as follows. The preceding theorem ensures that, in doing this, no ambiguities are introduced.

Definition 3.8. Let $\mathcal{H}$ be a Cartan subalgebra of $\mathfrak{L}$ with maximal torus $\mathcal{T}$. Then an element x (respectively subset $\mathcal{S}$) of $\mathfrak{L}$ is *defined* at $\mathcal{H}$ if x (respectively $\mathcal{S}$) is defined at $\mathcal{T}$. If x (respectively $\mathcal{S}$) is defined at $\mathcal{H}$, then $\mathcal{H}e^x=C_{\mathfrak{L}}(\mathcal{T}e^x)$.

It is not known whether $\mathcal{T}e^x$ is a maximal torus of $\mathfrak{L}$ for every maximal torus $\mathcal{T}$ of $\mathfrak{L}$. For practical purposes, however, this difficulty is circumvented as follows, since the above becomes true if "maximal torus" is replaced by "torus of maximal dimension".

Definition 3.9. The *rank* of a nilpotent Lie p-algebra is the dimension of its maximal torus.

PROPOSITION 3.10. *Let $\mathcal{H}$ be a Cartan subalgebra of maximal rank. Then for x defined at $\mathcal{H}$, $\mathcal{H}e^x$ is a Cartan subalgebra of maximal rank.*

Proof. If $\mathcal{T}$ is a torus of maximal dimension and if x is defined at $\mathcal{T}$, then $\mathcal{T}e^x$ is a torus of maximal dimension, by 3.4.

In the following application of this material, we establish the existence of large numbers of Cartan subalgebras of $\mathcal{L}$.

COROLLARY 3.11. *Let $\mathcal{L}$ have a torus $\mathcal{T}$ of maximal dimension such that* ad $\mathcal{T}$ *is diagonalizable over $\mathfrak{F}$. Then $\mathcal{L}$ is spanned by the union of Cartan subalgebras of $\mathcal{L}$ of maximal rank.*

Proof. $\mathcal{L}=\sum \mathcal{L}_\alpha$ (ad $\mathcal{T}$). Let $\mathcal{H}$ be the Cartan subalgebra $C_{\mathcal{L}}(\mathcal{T})$. Then $\mathcal{H}$ is of maximal rank and the elements x of root spaces $\mathcal{L}_\alpha$ (ad $\mathcal{T}$) are defined at $\mathcal{H}$. For such x, $\mathcal{H}e^x$ is a Cartan subalgebra of $\mathcal{L}$ of maximal rank, and we claim that $x\in\mathcal{H}+\mathcal{H}e^x$. We may assume that $\alpha\neq 0$, since $x\in\mathcal{H}$ for $\alpha=0$. Now choose $t\in\mathcal{T}$ such that $\alpha\,(\text{ad}\,t)=-1$. Then $\mathcal{H}e^x$ contains $t(I+\text{ad}\,x)=t+x$, and $x\in\mathcal{H}+\mathcal{H}e^x$ since $t\in\mathcal{H}$. Thus, the span of the Cartan subalgebras of $\mathcal{L}$ of maximal rank contains the $\mathcal{L}_\alpha$ (ad $\mathcal{T}$), hence contains $\mathcal{L}$.

The following observation is needed for an application in the next section.

PROPOSITION 3.12. *Let $\mathcal{H}$ be a Cartan subalgebra of $\mathcal{L}$ of maximal rank, x an element of a root space $\mathcal{L}_\alpha$* (ad $\mathcal{T}$) *such that* $(\text{ad}\,x)^p=0$. *Then $\mathcal{H}e^x=\mathcal{H}E^x$.*

Proof. Let $\mathcal{T}$ be the maximal torus of $\mathcal{H}$. Then $\mathcal{H}E^x=C_{\mathcal{L}}(\mathcal{T})\,E^x=C_{\mathcal{L}}(\mathcal{T}E^x)=C_{\mathcal{L}}(\mathcal{T}e^x)=\mathcal{H}e^x$, by 3.6 and 2.12.

4. Conjugacy

We now apply the material of section 3 on exponentials to the problem of conjugacy of tori and Cartan subalgebras of a Lie p-algebra $\mathcal{L}$. The precise meaning of conjugacy is taken to be the following.

Definition 4.1. Two tori $\mathcal{T}$, $\mathcal{T}'$ (respectively Cartan subalgebras $\mathcal{H}$, $\mathcal{H}'$) are *conjugate in $\mathcal{L}$ under S* ($S\subset\mathcal{L}$) if there exist $x_1, \ldots, x_n\in S$ and tori $\mathcal{T}_i$ (respectively Cartan subalgebras $\mathcal{H}_i$) for $1\leqslant i\leqslant n$ such that x_i lies in some root space of $\mathcal{T}_i$ (respectively $\mathcal{H}_i$) in $\mathcal{L}$ for $1\leqslant i\leqslant n$ and $\mathcal{T}=\mathcal{T}_1$, $\mathcal{T}_ie^{x_i}=\mathcal{T}_{i+1}$ for $1\leqslant i\leqslant n-1$ and $\mathcal{T}_ne^{x_n}=\mathcal{T}'$ (respectively $\mathcal{H}=\mathcal{H}_1$, $\mathcal{H}_ie^{x_i}=\mathcal{H}_{i+1}$ for $1\leqslant i\leqslant n-1$ and $\mathcal{H}_ne^{x_n}=\mathcal{H}'$.

We prove, among other results, that any two maximal tori of a solvable Lie p-algebra $\mathcal{L}$ over an algebraically closed field are conjugate under $\mathcal{L}^\infty$. The question as to whether this generalizes to an arbitrary Lie p-algebra $\mathcal{L}$ over an algebraically closed field reduces to the question as to whether any two maximal solvable subalgebras of $\mathcal{L}$ containing maximal tori of $\mathcal{L}$ contain a common maximal torus of $\mathcal{L}$. The latter question is the analog of Bruhat's lemma for algebraic groups, and is not within the scope of this paper.

Throughout the section, $\mathcal{L}$ is a Lie p-algebra over an algebraically closed field F.

LEMMA 4.2. *Let $\mathcal{T}$ be a diagonalizable linear Lie algebra over F, V the underlying vector space of $\mathcal{T}$. Suppose that $V_0(\mathcal{T})=\{0\}$. Let f be a function from $\mathcal{T}$ into V such that $f(s)t=f(t)s$ for s, $t\in\mathcal{T}$. Then there exists $v\in V$ such that $f(t)=vt$ for $t\in\mathcal{T}$.*

Proof. Choose $s\in\mathcal{T}$ such that $V_0(s)=\{0\}$. This is possible since F is infinite and $\mathcal{T}$ is a diagonalizable subalgebra of Hom V. Let $v=f(s)s^{-1}$. Then for $t\in\mathcal{T}$, $vt=(f(s)s^{-1})t=(f(s)t)s^{-1}=(f(t)s)s^{-1}=f(t)$.

THEOREM 4.3. *Suppose that $\mathcal{L}=\mathcal{M}+\mathcal{U}$ (not necessarily a direct sum of subspaces) where $\mathcal{M}$ is a p-subalgebra of $\mathcal{L}$ and $\mathcal{U}$ is a nil p-ideal of $\mathcal{L}$. Let $\mathcal{T}$ be a torus of $\mathcal{L}$ of maximal dimension. Then some conjugate $\mathcal{T}'$ of $\mathcal{T}$ under $\mathcal{U}$ is contained in $\mathcal{M}$.*

Proof. Suppose first that $\mathcal{U}$ is abelian and that $\mathcal{M}\cap\mathcal{U}=\{0\}$. Then $\mathcal{L}=\mathcal{M}+\mathcal{U}$ (direct sum of subspaces). Let $\mathcal{U}_0=\mathcal{U}_0$ (ad $\mathcal{T}$) and $\mathcal{U}_*=\mathcal{U}_*$ (ad $\mathcal{T}$). Then $\mathcal{L}=(\mathcal{M}+\mathcal{U}_0)+\mathcal{U}_*$ (direct sum of subspaces). For $s\in\mathcal{T}$, let $f(s)$ be the corresponding projection of s on $\mathcal{U}$, $f_*(s)$ the corresponding projection on $\mathcal{U}_*$:

$s=m+f(s)$ with $m\in\mathcal{M}$, $f(s)\in\mathcal{U}$;

$f(s)=u_0+f_*(s)$ with $u_0\in\mathcal{U}_0$ and $f_*(s)\in\mathcal{U}_*$.

Then for s, $t\in\mathcal{T}$, $0=[s,t]=[m+f(s),\ n+f(t)]=[m,n]+[m,f(t)]+[f(s),n]$ where $s=m+f(s)$, $t=n+f(t)$, and $[m,n]=0$ and $[f(t),s]=[f(t),m]=[f(s),n]=[f(s),t]$. Thus, $[f(t),s]=[f(s),t]$ and, consequently, $[f_*(t),s]=[f_*(s),t]$ for s, $t\in\mathcal{T}$. Thus, there exists $x\in\mathcal{U}_*$ such that $f_*(t)=[x,t]$ for all $t\in\mathcal{T}$, by 4.2. We can now show that ad $\mathcal{T}E^x\subset$ ad $\mathcal{M}$, which then is used to show that some conjugate of $\mathcal{T}$, namely $\mathcal{T}e^x$, is contained in $\mathcal{M}$. Thus, let $s\in\mathcal{T}$. Then, $s=m+u_0+f_*(s)$ ($m\in\mathcal{M}$, $u_0\in\mathcal{U}_0$) and $sE^x=s(I+\operatorname{ad}x)=s-f_*(s)=m+u_0$. Since $\mathcal{U}$ is abelian, E^x is an automorphism of $\mathcal{L}$. Moreover, Ad E^x (the mapping ad $y\mapsto E^{-x}(\operatorname{ad}y)E^x=\operatorname{ad}(yE^x)$) is a p-automorphism of ad $\mathcal{L}$. Now $[m,u_0]=0$, since $m=s-u_0-f_*(s)$, so that $[\operatorname{ad}m,\operatorname{ad}u_0]=0$. Choosing e such that $u_0^{p^e}=0$, we have $\operatorname{ad}(s^{p^e}E^x)=((\operatorname{ad}s)^{p^e})\operatorname{Ad}E^x=((\operatorname{ad}s)\operatorname{Ad}E^x)^{p^e}=\operatorname{ad}(sE^x)^{p^e}=\operatorname{ad}(m+u_0)^{p^e}=\operatorname{ad}m^{p^e}\in\operatorname{ad}\mathcal{M}$. Since $\mathcal{T}=\mathcal{T}^{p^e}$ for all e, it follows readily that ad $\mathcal{T}E^x\subset$ ad $\mathcal{M}$.

We next note that $\mathcal{T}e^x$ is in fact conjugate to $\mathcal{T}$ in the sense of 4.1. This is true since, for $x=\sum_1^n x_i$ with $x_i\in\mathcal{L}_{\alpha_i}$ (ad $\mathcal{T}$), $\{x_i\}$ is defined at $\mathcal{T}$ and the tori $\mathcal{T}_j=\mathcal{T}e^{\sum_1^j x_i}$ have the property that x_j is a root vector with respect to $\mathcal{T}_j$ (the root spaces of all the $\mathcal{T}_j$ coincide since $\mathcal{U}$ is abelian). Now $\mathcal{T}$ and $\mathcal{T}e^x$ are conjugate under $\mathcal{U}$, via the $\mathcal{T}_j$, x_j.

It remains only to show that $\mathcal{T}e^x\subset\mathcal{M}$. But, since Ad E^x is a p-automorphism and since the following diagram is commutative, ad $\mathcal{T}e^x=$ ad $\mathcal{T}E^x$ as in the proof of 3.4:

$$\begin{tikzcd} L \arrow[r, "E^x"] \arrow[d, "\mathrm{ad}"'] & L \arrow[d, "\mathrm{ad}"] \\ \mathrm{ad}\, L \arrow[r, "\mathrm{Ad}\, E^x"'] & \mathrm{ad}\, L \end{tikzcd}$$

Thus, ad $\mathcal{T}e^x \subset$ ad $\mathcal{M}$, by the preceding paragraph. Thus $\mathcal{T}e^x \subset \mathcal{M} + C(\mathcal{L}) = \mathcal{M} + C_u(\mathcal{L})$. Let $s \in \mathcal{T}e^x$. Then $s = m + u$ ($m \in M$, $u \in C_u(\mathcal{L})$) and $s^{p^e} = m^{p^e} \in \mathcal{M}$ where e is chosen such that $u^{p^e} = 0$. Choosing e such that $U^{p^e} = \{0\}$, we have $\mathcal{T}e^x = (\mathcal{T}e^x)^{p^e} \subset \mathcal{M}$, as asserted.

We now prove the theorem in general by induction on dim $\mathcal{L}$. The proof for dim $\mathcal{L} = 1$ is trivial. If $\mathcal{U} = \{0\}$, there is nothing to prove. Thus, suppose that $\mathcal{U} \neq \{0\}$. Then $C_u(\mathcal{U}) \neq 0$ (since $\mathcal{U}$ is nilpotent). Let $\mathcal{V} = C_u(\mathcal{U})$ if $\mathcal{M} \cap C_u(\mathcal{U}) = \{0\}$ and $\mathcal{V} = \mathcal{M} \cap C_u(\mathcal{U})$ if $\mathcal{M} \cap C_u(U) \neq \{0\}$. Then $\mathcal{V}$ is an abelian nil p-ideal of $\mathcal{L}$ of positive dimension, and either $\mathcal{V} \subset \mathcal{M}$ or $\mathcal{M} \cap \mathcal{V} = \{0\}$. Let $\overline{\mathcal{L}} = \mathcal{L}/\mathcal{V}$ and let $0 \to \mathcal{V} \to \mathcal{L} \xrightarrow{\beta} \overline{\mathcal{L}} \to 0$ be the associated exact sequence of Lie p-algebras. Let $\overline{\mathcal{M}} = \beta(\mathcal{M})$, $\overline{\mathcal{U}} = \beta(\overline{\mathcal{U}})$, $\overline{\mathcal{T}} = \beta(\mathcal{T})$. Then $\overline{\mathcal{L}} = \overline{\mathcal{M}} + \overline{\mathcal{U}}$ and $\overline{\mathcal{T}}$ is a torus of $\overline{\mathcal{L}}$ of maximal dimension, by 2.16, since $\mathcal{V}$ is nil. (Cf. proof to 2.17). By the induction hypothesis, there exist $\bar{x}_1, \ldots, \bar{x}_n$ in $\overline{\mathcal{U}}$ and tori $\overline{\mathcal{T}}_i$ of $\overline{\mathcal{L}}$ such that $\overline{\mathcal{T}} = \overline{\mathcal{T}}_1$, $\overline{\mathcal{T}}_i e_i^{\bar{x}_i} = \overline{\mathcal{T}}_{i+1}$ ($1 \leqslant i \leqslant n-1$) and $\overline{\mathcal{T}}_n e^{\bar{x}_n} \subset \overline{\mathcal{M}}$. Choose maximal tori $\mathcal{T}_i$ of $\mathcal{L}$ such that $\beta(\mathcal{T}_i) = \overline{\mathcal{T}}_i$, by 2.16 ($1 \leqslant i \leqslant n$). By conservation of root-space properties under passage to quotients, we can find x_i in $\mathcal{L}$ such that x_i is a root vector of $\mathcal{T}_i$ and $\beta(x_i) = \bar{x}_i$ ($1 \leqslant i \leqslant n$). Now $\beta(\mathcal{T}_i e^{x_i}) = \beta(\mathcal{T}_i) e^{\beta(x_i)} = \overline{\mathcal{T}}_i e^{x_i} = \overline{\mathcal{T}}_{i+1} = \beta(\mathcal{T}_{i+1})$ ($1 \leqslant i \leqslant n-1$). Similarly, $\beta(\mathcal{T}_n e^{x_n}) = \overline{\mathcal{T}}_n e^{\bar{x}_n} \subset \overline{\mathcal{M}}$. Thus, $\mathcal{T} + \mathcal{V} = \mathcal{T}_1 + \mathcal{V}$, $\mathcal{T}_i e^{x_i} + \mathcal{V} = \mathcal{T}_{i+1} + \mathcal{V}$ ($1 \leqslant i \leqslant n-1$) and $\mathcal{T}_n e^{x_n} + \mathcal{V} \subset \mathcal{M} + \mathcal{V}$. By the preceding paragraph, $\mathcal{T}$ is conjugate to $\mathcal{T}_1$ under $\mathcal{V}$, $\mathcal{T}_i e^{x_i}$ is conjugate to $\mathcal{T}_{i+1}$ under $\mathcal{V}$ ($1 \leqslant i \leqslant n-1$) and, in the case that $\mathcal{M} \cap \mathcal{V} = \{0\}$, some conjugate of $\mathcal{T}_n e^{x_n}$ under $\mathcal{V}$ is contained in $\mathcal{M}$. In the remaining case $\mathcal{M} \cap \mathcal{V} \neq \{0\}$, we have $\mathcal{V} \subset \mathcal{M}$, whence $\mathcal{T}_n e^{x_n} \subset \mathcal{M}$. Thus, some conjugate of $\mathcal{T}$ under $\mathcal{U}$ is contained in $\mathcal{M}$ in each case.

THEOREM 4.4. *Let $\mathcal{L}$ be a solvable Lie p-algebra. Then any two maximal tori $\mathcal{T}$, $\mathcal{T}'$ are conjugate under $\mathcal{L}^\infty$.*

Proof. Recall that the maximal tori of $\mathcal{L}$ have constant dimension, by 2.17.

We prove that two given maximal tori $\mathcal{T}$, $\mathcal{T}'$ are conjugate, by induction on dim $\mathcal{L}$. The assertion is trivial if dim $\mathcal{L} = 1$. Now assume that dim $\mathcal{L} > 1$ and that the assertion is true for lower dimensions. Let $\mathcal{A}$ be a minimal non-zero abelian p-ideal of $\mathcal{L}$. Then $\mathcal{A}$ is a nil-p-ideal or a torus, as we see from 2.5 upon considering the series $\mathcal{A}^{p^e}$. Let

$0 \to \mathcal{A} \to \mathcal{L} \xrightarrow{\beta} \mathcal{L}/\mathcal{A} \to 0$ be the associated exact sequence and let $\beta(\mathcal{L}) = \overline{\mathcal{L}}$, $\beta(\mathcal{T}) = \overline{\mathcal{T}}$, $\beta(\mathcal{T}') = \overline{\mathcal{T}}'$. Then $\overline{\mathcal{T}}$, $\overline{\mathcal{T}}'$ are maximal tori of $\overline{\mathcal{L}}$, and there exist, by induction, $\bar{x}_1, ..., \bar{x}_n$ in $\overline{\mathcal{L}}$ and tori $\overline{\mathcal{T}}_1, ..., \overline{\mathcal{T}}_n$ of $\overline{\mathcal{L}}$ with $\bar{x}_i$ a root vector of $\overline{\mathcal{T}}_i$ $(1 \leqslant i \leqslant n)$ such that $\overline{\mathcal{T}} = \overline{\mathcal{T}}_1$, $\mathcal{T}_i e^{\bar{x}_i} = \overline{\mathcal{T}}_{i+1}$ $(1 \leqslant i \leqslant n-1)$ and $\overline{\mathcal{T}}_n e^{x_n} = \overline{\mathcal{T}}'$. Again we can choose x_i in $\mathcal{L}$ and maximal tori $\mathcal{T}_i$ of $\mathcal{L}$ such that x_i is a root vector for $\mathcal{T}_i$, $\beta(x_i) = \bar{x}_i$ and $\beta(\mathcal{T}_i) = \overline{\mathcal{T}}_i$ $(1 \leqslant i \leqslant n)$. (Compare with the proof of 4.3). Then $\beta(\mathcal{T}) = \beta(\mathcal{T}_1)$, $\beta(\mathcal{T}_i e^{x_i}) = \overline{\mathcal{T}}_i e^{x_i} = \overline{\mathcal{T}}_{i+1} = \beta(\mathcal{T}_{i+1})$ $(1 \leqslant i \leqslant n-1)$ and $\beta(\mathcal{T}_n e^{x_n}) = \overline{\mathcal{T}}_n e^{x_n} = \overline{\mathcal{T}}' = \beta(\mathcal{T}')$. Thus, $\mathcal{T} + \mathcal{A} = \mathcal{T}_1 + A$, $\mathcal{T}_i e^{x_i} + \mathcal{A} = \mathcal{T}_{i+1} + \mathcal{A}$ for $1 \leqslant i \leqslant n-1$ and $\mathcal{T}_n e^{x_n} + \mathcal{A} = \mathcal{T}' + \mathcal{A}$. If $\mathcal{A}$ is nil. $\mathcal{T}$ and $\mathcal{T}'$ are conjugate under $\mathcal{L}$ by 4.3. Otherwise $\mathcal{A}$ is a toral ideal, hence is central by 2.8, and therefore is contained in the maximal tori $\mathcal{T}$, $\mathcal{T}'$ and $\mathcal{T}_i$ $(1 \leqslant i \leqslant n)$. Thus, $\mathcal{T}$ and $\mathcal{T}'$ are conjugate under $\mathcal{L}$ in both cases. But conjugacy under $\mathcal{L}$ implies conjugacy under $\mathcal{L}^\infty$, for if $\mathcal{S}$ is a torus and $x \in \mathcal{L}_\alpha(\operatorname{ad} \mathcal{S})$ with $\alpha \neq 0$, then $[Fx, S] = Fx$ so that $x \in \mathcal{L}^\infty$.

We have as corollary the following improvement of results of [1], [2].

COROLLARY 4.5. *Let $\mathcal{L}$ be a solvable Lie p-algebra such that* $(\operatorname{ad} x)^p = 0$ *for* $x \in \mathcal{L}^\infty$. *Then any two Cartan subalgebras $\mathcal{H}$, $\mathcal{H}'$ of $\mathcal{L}$ have the same dimension and are conjugate in the sense that there exist $x_i \in \mathcal{L}^\infty$ $(1 \leqslant i \leqslant n)$ such that $\mathcal{H}' = \mathcal{H}E^{x_1} \ldots E^{x_n}$.*

Proof. Use 3.12 and 4.4.

COROLLARY 4.6. *If $\mathcal{L}$ is solvable and some Cartan subalgebra of $\mathcal{L}$ is a torus, then every Cartan subalgebra of $\mathcal{L}$ is a torus.*

Proof. This follows from 4.4 and 3.7.

References

[1]. BARNES, D. W., On Cartan subalgebras of Lie algebras. *Math. Z.*, 101 (1967), 350–355.
[2]. BLOCK, R. W., On the extensions of Lie algebras. To appear.
[3]. CHEVALLEY, C., (Séminaire C. Chevalley). *Classification des Groupes de Lie Algébriques.* Tome I. Ecole Normale Supérieure, Paris, 1955.
[4]. JACOBSON, N., *Lie Algebras.* Interscience, New York, 1962.
[5]. —— Restricted Lie algebras of characteristic p. *Trans. Amer. Math. Soc.*, 50 (1941), 15–25.
[6]. SELIGMAN, G., *Modular Lie Algebras.* Springer, New York, 1967.
[7]. —— Some results on Lie p-algebras. *Bull. Amer. Math. Soc.*, 73 (1967), 528–30.
[8]. WINTER, D. J., On groups of automorphisms of Lie algebras. *J. of Algebra*, 8 (1968), 131–142.
[9]. —— Solvable and nilpotent subalgebras of Lie algebras. *Bull. Amer. Math. Soc.*, 4 (1969).

Received February 27, 1969