MEROMORPHIC FUNCTIONS WITH MAXIMAL DEFICIENCY SUM AND A CONJECTURE OF F. NEVANLINNA

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I. Introduction

Let $f(z)$ be a meromorphic function and $\delta(\tau, f)$ be the deficiency, in the sense of Nevanlinna, of the value τ. The order λ and lower order μ of $f(z)$ are defined by the usual relations

$$\lambda = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r} \qquad \mu = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log r},$$

and the total deficiency $\Delta(f)$ by

$$\Delta(f) = \sum_{\tau} \delta(\tau, f),$$

where the summation is to be extended to all values τ, finite or infinite, such that

$$\delta(\tau, f) > 0. \tag{1.1}$$

The number of deficient values, that is the number of distinct values of τ for which (1.1) holds, will be denoted by $\nu(f)$.

In addition to the familiar notations of Nevanlinna's theory, we shall find it convenient to define, for a measurable subset J of $[0, 2\pi)$ and a meromorphic function $g(z)$, the symbol

$$m(r, g; J) = \frac{1}{2\pi} \int_J \log^+ |g(re^{i\theta})| \, d\theta.$$

The present investigation centers around the classical second fundamental theorem of Nevanlinna's theory which asserts that the total deficiency of any meromorphic function $f(z)$ satisfies the inequality

$$\Delta(f) \leq 2.$$

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THEOREM 1. *Let $f(z)$ be a meromorphic function of lower order $\mu < \infty$ and assume*

$$\Delta(f) = 2. \tag{1.2}$$

Then $\nu(f) \leqslant 2\mu$.

It is perhaps worth noting that in the important special case of entire functions, the sharp bound on $\nu(f)$ for functions satisfying (1.2) is given by $\nu(f) \leqslant \mu + 1$. This result was obtained by Edrei and Fuchs [8] in extending work of Pfluger [13], and under the condition $\lambda < \infty$. They also showed [7; p. 299] that if $\lambda < \infty$, $f(z)$ entire, and $\Delta(f) = 2$, then $f(z)$ must be of positive integral order and $\mu = \lambda$. In both these results the assumption $\lambda < \infty$ could be removed by using the methods developed in [3].

In the general case of meromorphic functions it was conjectured by F. Nevanlinna [12] that the only finite values of λ for which (1.2) is attainable are those of the form $n/2$ ($n = 2, 3, \ldots$), with $\nu(f) \leqslant 2\lambda$, and each of the deficiencies of the form $\delta(\tau_i, f) = k_i/\lambda$ where k_i is an integer.

Our Theorem 2, which follows readily from Theorem 1, shows that this conjecture is valid at least for $0 \leqslant \lambda < 3/2$.

THEOREM 2. *Let $f(z)$ be a meromorphic function of lower order $\mu < 3/2$ and order $\lambda (\leqslant \infty)$. If*

$$\Delta(f) = 2$$

then $\mu = \lambda = 1$ and $\nu(f) = 2$.

Theorem 2 extends an earlier theorem of Edrei [5; p. 55] and, in fact, the approach we use in this paper is that suggested by Edrei in [5; pp. 54–56].

In concluding this introduction, I should like to thank Professor P. Church for several helpful suggestions. I should also like to extend my sincere gratitude to Professor A. Edrei, since many of the ideas contained herein were developed while I was a student under his guidance.

II. Discussion of methods

The point of departure of our investigation is the following elementary

LEMMA A. *Let $f(z)$ be a meromorphic function and assume*

$$\Delta(f) = 2. \tag{2.1}$$

Then

$$\lim_{\substack{r \to \infty \\ r \notin E}} \frac{N\left(r, \dfrac{1}{f'}\right)}{T(r, f')} = 0, \tag{2.2}$$

and
$$\lim_{\substack{r\to\infty\\ r\notin\mathcal{E}}}\frac{T(r,f')}{T(r,f)}=2-\delta(\infty,f), \tag{2.3}$$

where $\mathcal{E}$ is a set of finite measure,

Proof. Let $\{\tau_j\}$ be a sequence of distinct finite complex numbers containing all the finite deficient values of $f(z)$. Given $\varepsilon>0$, we choose q sufficiently large so that

$$\sum_{j=1}^{q}\delta(\tau_j,f)+\delta(\infty,f)>2-\varepsilon. \tag{2.4}$$

Now, basic estimates in Nevanlinna's theory [11; pp. 33, 34, 56] yield

$$T(r,f')\leqslant T(r,f)+N(r,f)+S(r) \tag{2.5}$$

and
$$\sum_{j=1}^{q} m\left(r,\frac{1}{f-\tau_j}\right)\leqslant m\left(r,\sum_{j=1}^{q}\frac{1}{f-\tau_j}\right)+O(1)$$
$$\leqslant m\left(r,\frac{1}{f'}\right)+S(r)=T(r,f')-N\left(r,\frac{1}{f'}\right)+S(r), \tag{2.6}$$

where $S(r)=o(T(r,f))$ as $r\to\infty$ outside a possible exceptional set $\mathcal{E}$ of finite r measure, which does not depend on $\tau_1,\ldots,\tau_q$ or q [11; p. 41].

Now (2.5) implies
$$\limsup_{\substack{r\to\infty\\ r\notin\mathcal{E}}}\frac{T(r,f')}{T(r,f)}\leqslant 2-\delta(\infty,f), \tag{2.7}$$

and it follows from (2.4) and (2.6) that $T(r,f)=O(m(r,1/f'))$ as $r\to\infty$, $r\notin\mathcal{E}$. Therefore, we have by (2.6) and (2.7)

$$\frac{N\left(r,\frac{1}{f'}\right)}{T(r,f')}+\frac{T(r,f)}{T(r,f')}\sum_{j=1}^{q}\frac{m\left(r,\frac{1}{f-\tau_j}\right)}{T(r,f)}\leqslant 1+o(1)\quad (r\to\infty, r\notin\mathcal{E}), \tag{2.8}$$

and hence
$$\frac{N\left(r,\frac{1}{f'}\right)}{T(r,f')}+\frac{\sum_{j=1}^{q}\delta(\tau_j,f)}{2-\delta(\infty,f)}\leqslant 1+o(1)\quad (r\to\infty, r\notin\mathcal{E}). \tag{2.9}$$

Thus, from (2.4), (2.8) and (2.9) we deduce

$$\limsup_{\substack{r\to\infty\\ r\notin\mathcal{E}}}\frac{N\left(r,\frac{1}{f'}\right)}{T(r,f')}\leqslant\frac{\varepsilon}{2-\delta(\infty,f)}$$

and
$$\limsup_{\substack{r\to\infty \\ r\notin \mathcal{E}}} \frac{T(r,f)}{T(r,f')} \leqslant \frac{1}{2-\delta(\infty,f)-\varepsilon}.$$

Since $\varepsilon > 0$ was arbitrary, we have in view of (2.7) completed the proof of the lemma.

The exceptional set $\mathcal{E}$ introduced in Lemma A could have been deleted if we were to make the additional assumption $\lambda < \infty$. In this special case, our Lemma A is contained in Lemma A of [5]. Since the introduction of an exceptional set does not seriously hinder our computations we make no such assumption. It is however quite possible that (2.1) precludes its existence.

We shall study the behavior of the function $f'(z)$ in annuli around suitably chosen circumferences $\{|z| = r_m\}$. To make this precise we introduce the following

DEFINITION OF PÓLYA PEAKS. *Let $G(x)$ be a positive, nondecreasing continuous function defined for $t > t_0$. A sequence $\{r_m\}$ is said to be a sequence of Pólya peaks of order ϱ $(0 \leqslant \varrho < \infty)$ of $G(x)$ if it is possible to find sequences $\{r'_m\}$ and $\{r''_m\}$ such that*

$$r'_m \to \infty, \quad \frac{r_m}{r'_m} \to \infty, \quad \frac{r''_m}{r_m} \to \infty \quad \textit{as} \quad m \to \infty,$$

and
$$\frac{G(t)}{G(r_m)} \leqslant \left(\frac{t}{r_m}\right)^{\varrho} (1+o(1)) \quad (m\to\infty;\, r'_m \leqslant t \leqslant r''_m).$$

In its present form, the notion of Pólya peaks was introduced and used by Edrei [4], [5], who also proved [4] the following

EXISTENCE THEOREM FOR PÓLYA PEAKS. *Let $G(x)$ have the properties stated in the above definition. Let $\mu (< \infty)$ be the lower order and $\lambda (\leqslant \infty)$ be the order of $G(x)$. Then there exists a sequence of Pólya peaks of order ϱ for each finite ϱ such that $\mu \leqslant \varrho \leqslant \lambda$.*

Now, given any function $f(z)$ as in Theorem 1 having $\tau_1, ..., \tau_n$ $(n \geqslant 2)$ amongst its deficient values, it follows from the elements of Nevanlinna's theory that for any complex numbers $a (\neq 0)$ and b, the function

$$g(z) = \frac{a}{f(z) - b}$$

also satisfies the conditions of Theorem 1. Choosing, as is always possible, a value of b such that

$$N\left(r, \frac{a}{f(z)-b}\right) \sim T(r, f) \quad (\sim T(r, g)) \qquad (r \to \infty)$$

we would then have

$$\delta\left(\frac{a}{\tau_k - b}, g\right) = \delta(\tau_k, f) \quad (\tau_k \neq \infty),$$

$$\delta(0, g) = \delta(\infty, f),$$

$$\delta(\infty, g) = 0.$$

Thus in Theorem 1 we could always make, without loss of generality, the assumptions

$$\delta(\infty, f) = 0, \tag{2.10}$$

and for a finite collection $\tau_1, \ldots, \tau_n$ of $n \geqslant 2$ deficient values

$$\min_{\substack{i \neq j \\ i, j = 1, \ldots, n}} |\tau_i - \tau_j| > 3. \tag{2.11}$$

Furthermore, since it is known [7; p. 294] that if $\mu = 0$ then $\nu(f) \leqslant 1$, we may assume in Theorem 1 that

$$\mu > 0. \tag{2.12}$$

We are now in a position to outline the steps which lead to the proof of Theorem 1.

In Section III we deduce from (2.2) that if $f(z)$ has total deficiency 2, then $1/f'(z)$ closely approximates functions which are regular in annuli around the Pólya peaks of $T(r, f')$ (Lemma 2). We also prove in this section an important lemma which gives us bounds on the lengths of certain level curves in an annulus in which a meromorphic function is regular (Lemma 1). In Section IV we show that the sets where $f(z)$ is close to its deficient values give rise to disjoint component sets in large annuli where $f'(z)$ is very small (Lemma 3). This is done by using Lemma 1 to construct short paths in the sets where $f'(z)$ is small, and then integrating $f'(z)$ over these paths. This shows that the difference between the values of $f(z)$ at the endpoints of the path is small so that, in view of (2.11) $f(z)$ cannot be close to different deficient values at each endpoint.

In Sections V and VI we apply a localized form of the methods developed by Carleman in his proof of the Ahlfors–Carleman–Denjoy Theorem. By these means we limit the number of sets where simultaneously $f(z)$ is close to a deficient value and $1/f'(z)$ is large in much the same way Carleman obtained the bound on the number of asymptotic paths of an entire function. Thus, the number of deficient values does not exceed 2μ.

III. Preliminary lemmas

In order to apply Lemma A we shall find it useful to study the behavior of meromorphic functions $F(z)$ satisfying the condition

$$\lim_{\substack{r\to\infty \\ r\notin \mathcal{E}}} \frac{N(r,F)}{T(r,F)} = 0, \tag{3.1}$$

where $\mathcal{E}$ is a set of finite measure.

To facilitate this study we introduce the following notations.

For $\sigma > 1$, we define

$$P(z;\sigma,t) = cz^l \prod_{t/8\sigma<|b_\nu|<8\sigma t} \left(1 - \frac{z}{b_\nu}\right), \tag{3.2}$$

where $\{b_\nu\}$ is the sequence of poles of $F(z)$ (counting multiplicity) and c and l are chosen such that the function

$$F(z;\sigma,t) = P(z;\sigma,t)\,F(z) \tag{3.3}$$

satisfies

$$F(0;\sigma,t) = 1. \tag{3.4}$$

The following lemma will play a central role in our applications of the functions $F(z;\sigma,t)$.

LEMMA 1. *Let $G(z)$ be a nonrational function which is meromorphic in the disk*

$$|z| \leqslant R' \quad (0 < R' < \infty)$$

and holomorphic in the closure of the annulus

$$\mathcal{A} = \{z: \varrho < |z| < R\} \quad (1 \leqslant \varrho < R < R'). \tag{3.5}$$

Assume also that

$$G(0) = 1, \tag{3.6}$$

and that

$$\begin{cases} \mathfrak{m} = \sup_{|z|=\varrho} |G(z)| \\ \mathfrak{M} = \sup_{|z|=R} |G(z)| \end{cases} \tag{3.7}$$

satisfy the inequality

$$\mathfrak{m} < \mathfrak{M}. \tag{3.8}$$

Let the interval

$$I = \{t: \alpha < t < \beta\} \quad (\mathfrak{m} < \alpha < \beta < \mathfrak{M}). \tag{3.9}$$

be given. Then the following assertions hold:

(i) *The open set*

$$U(\gamma) = \mathcal{A} \cap \{z: |G(z)| > \gamma\} \quad (\gamma \in I) \tag{3.10}$$

has a finite number of components. If $K(\gamma)$ *is any component of* $U(\gamma)$, *then* $\overline{K(\gamma)}$ *intersects the circumference* $|z| = R$. *Furthermore, the boundary of each component of the complement (with respect to the plane* $|z| < \infty$) *of* $\overline{K(\gamma)}$ *is a Jordan curve.*

(ii) *There exists a subset* $I^* \subseteq I$ *such that*

$$\text{meas } I^* \geq \frac{\beta - \alpha}{2} \tag{3.11}$$

and such that if $\gamma \in I^*$, *the total length* $l(\gamma)$ *of the level curves*

$$|G(z)| = \gamma$$

which lie in $\mathcal{A}$ *does not exceed the bound*

$$L = 2\pi R \sqrt{\frac{\beta T(R', G)}{(\beta - \alpha) \log R'/R}}. \tag{3.12}$$

(iii) *If* γ *is an element of* I^* *as in* (ii), *and* $K(\gamma)$ *is a component of* $U(\gamma)$ *as in* (i), *then any two points* $re^{i\phi}$, $re^{i\psi}$ *both residing in* $K(\gamma)$ *may be joined by a continuous curve of length not exceeding* $4\pi R + L$ *on which* $|G(z)| \geq \gamma$.

Proof. We first prove (i). The conditions

$$\mathfrak{m} < \alpha < \beta < \mathfrak{M}$$

and the definition of $K(\gamma)$ imply that $\overline{K(\gamma)}$ does not intersect the circle $|z| = \varrho$. Hence, it follows from the maximum modulus principle that $\overline{K(\gamma)}$ intersects the circumference $|z| = R$.

We next show that there exist only finitely many points on any circle $|z| = t$ $(0 < t \leq R)$ for which $|G(z)| = \gamma$. In fact, if there were infinitely many such points, the function

$$\phi(z) = G(z)\, \overline{G\left(\frac{t^2}{\bar{z}}\right)}$$

which is meromorphic in some annulus containing the circle $|z| = t$ would have

$$\phi(te^{i\theta}) = G(te^{i\theta})\, \overline{G(te^{i\theta})} = |G(te^{i\theta})|^2 = \gamma^2$$

for infinitely many values of θ $(0 \leq \theta < 2\pi)$.

Therefore, $$\phi(z) \equiv \gamma^2$$

and by analytic continuation, $G(z)$ could be extended to the whole plane yielding a function meromorphic in the plane, and satisfying for all values of z,

$$G(z) = \frac{\gamma^2}{G\left(\frac{t^2}{\bar{z}}\right)}.$$

However, (3.6) then implies that $G(z)$ is regular at infinity and is therefore rational, a contradiction.

Now, for each component $K(\gamma)$ of $U(\gamma)$ as defined in (3.10), $\overline{K(\gamma)}$ intersects $|z| = R$. Since only finitely many points of the level set $|G(z)| = \gamma$ lie on $|z| = R$, it follows that there can be only finitely many such components.

To complete the proof of (i) we notice first that $\overline{K(\gamma)}$ is a continuum. We next prove that it is locally connected.

Assume that $\overline{K(\gamma)}$ is not locally connected. Then there is a subcontinuum H of infinitely many points, at each of which $\overline{K(\gamma)}$ is not locally connected [14; (12.3), p. 19].

Since
$$S = \{z\colon z \in \mathcal{A},\ G'(z) = 0\} \cup \{z\colon |z| = R,\ |G(z)| = \gamma\} \tag{3.13}$$

has only finitely many elements, we may assume that

$$H \cap S = \emptyset. \tag{3.14}$$

Now $K(\gamma)$ is open in the plane and thus

$$H \cap K(\gamma) = \emptyset. \tag{3.15}$$

Therefore, letting $\mathcal{B}$ denote the boundary of $K(\gamma)$ we have by (3.15)

$$H \subseteq \mathcal{B}.$$

The points in $(\mathcal{A} \cap \mathcal{B}) - S$ may be easily handled by observing that since $G(z)$ is a local homeomorphism at each point of $\mathcal{A} - S$, the set $\{z\colon z \in \mathcal{A},\ |G(z)| = \gamma\} - S$ is locally an arc at each point. Since

$$\mathcal{A} \cap \mathcal{B} \subseteq \mathcal{A} \cap \{z\colon |G(z)| = \gamma\}$$

an elementary argument shows that if $z \in (\mathcal{A} \cap \mathcal{B}) - S$, then z lies on an arc of the level set $\{z\colon |G(z)| = \gamma\}$ which separates arbitrarily small neighborhoods of z into two components; one component contained in $\overline{K(\gamma)}$ and the other disjoint from $\overline{K(\gamma)}$. Hence,

$$H \cap ((\mathcal{A} \cap \mathcal{B}) - S) = \emptyset \tag{3.16}$$

and, in view of (3.13), (3.14), (3.15), and (3.16), it now remains only to show $H \cap ((\mathcal{B} \cap \{z\colon |z| = R\}) - S) = \emptyset$.

In fact, if $z_0 \in (\mathcal{B} \cap \{z: |z| = R\}) - S$, then (3.13) implies $|G(z_0)| > \gamma$, and hence there exists an arbitrarily small disk $\mathcal{D}$ centered at z_0 on which $|G(z)| > \gamma$ and

$$\mathcal{D} \cap \{z: |z| \leqslant R\} = \mathcal{D} \cap \overline{K(\gamma)}$$

which is a connected set. This proves the local connectedness at z_0. Since we have now shown the contradiction $H = \emptyset$, we conclude that $\overline{K(\gamma)}$ is locally connected.

Since $\overline{K(\gamma)}$ is the closure of an open set in the plane it has no cut points and we therefore deduce [14; (9.3), p. 79], [14; (2.5), p. 107] that the boundary of each component of the complement of $\overline{K(\gamma)}$ is a Jordan curve. This completes the proof of (i).

Consider next the equation

$$G(z) = te^{i\phi} \quad (t > 0,\ 0 \leqslant \phi < 2\pi) \tag{3.17}$$

and let $\mathcal{N}(te^{i\phi})$ be the number of solutions of the equation (3.17) which lie in $\mathcal{A} \cap \{z: \alpha < |G(z)| < \beta\}$.

With the usual notations of Nevanlinna's theory, we have by (3.5)

$$\mathcal{N}(te^{i\phi}) \leqslant n\left(R, \frac{1}{G(z) - te^{i\phi}}\right) \leqslant \left(\log \frac{R'}{R}\right)^{-1} N\left(R', \frac{1}{G(z) - te^{i\phi}}\right)$$

$$= \left(\log \frac{R'}{R}\right)^{-1} N\left(R', \frac{1}{\dfrac{G(z)}{t} - e^{i\phi}}\right). \tag{3.18}$$

Hence, by (3.5), (3.6), (3.18), and the Cartan identity [11, p. 8]

$$\frac{1}{2\pi}\int_0^{2\pi} \mathcal{N}(te^{i\phi})\, d\phi \leqslant \left(\log \frac{R'}{R}\right)^{-1}\left(T\left(R', \frac{G(z)}{t}\right) - \log^+ \frac{1}{t}\right) \leqslant \left(\log \frac{R'}{R}\right)^{-1} T(R', G). \tag{3.19}$$

Now, (3.19) and a classical lemma due to Ahlfors [10; p. 18] lead to

$$\int_\alpha^\beta \frac{l^2(t)}{t}\, dt \leqslant 2\pi^2 R^2 \left(\log \frac{R'}{R}\right)^{-1} T(R', G) = K_0, \tag{3.20}$$

where $l(t)$ denotes the total length of the level curves $|G(z)| = t$ $(\alpha < t < \beta)$ which lie in $\mathcal{A}$.

Let J denote the subset of I for which

$$\frac{l^2(t)}{t} \geqslant \frac{2K_0}{\beta - \alpha} \quad (t \in J). \tag{3.21}$$

Then (3.20) implies
$$K_0 \geqslant \int_J \frac{l^2(t)}{t}\, dt \geqslant \frac{2K_0}{\beta - \alpha} \operatorname{meas} J$$

and hence,
$$\text{meas } J \leqslant \frac{\beta-\alpha}{2} \tag{3.22}$$

Taking
$$I^* = I - J \tag{3.23}$$

we see by (3.22) and (3.9) that (3.11) is satisfied.

Also, for any $\gamma \in I^*$, (3.21) and (3.23) imply

$$\frac{l^2(\gamma)}{\gamma} < \frac{2K_0}{\beta-\alpha}$$

and hence
$$l^2(\gamma) < \frac{2K_0\beta}{\beta-\alpha}.$$

This completes the proof of assertion (ii).

We are now in a position to construct the curve Γ between $re^{i\phi}$ and $re^{i\psi}$ of (iii) and complete the proof of Lemma 1.

We assume arbitrarily that

$$0 \leqslant \phi < \psi < 2\pi.$$

Now, each point $re^{i\phi}$, $re^{i\psi}$ is contained in an arc of $|z| = r$ on which $|G(z)| > \gamma$. If these arcs are one and the same, a suitable Γ would be an arc of $|z| = r$ having the two points as end points. Otherwise, there exist, as we have shown, finitely many arguments $\theta_1, \ldots, \theta_n$ ($\phi < \theta_1 < \theta_2 < \ldots < \theta_n < \psi$) for which

$$|G(re^{i\theta_j})| = \gamma \quad (j = 1, \ldots, n).$$

Then, the desired Γ begins with the arc from $re^{i\phi}$ to $re^{i\theta_1}$ on $|z| = r$.

In case the arc $\{z = re^{i\theta}: \theta_1 < \theta < \theta_2\}$ belongs to $K(\gamma)$, Γ continues along $|z| = r$ from $re^{i\theta_1}$ to $re^{i\theta_2}$. Otherwise $re^{i\theta_1}$ belongs to the boundary of a component W of the complement of $\overline{K(\gamma)}$. A routine argument shows that there must exist among the points $re^{i\theta_2}, re^{i\theta_3}, \ldots, re^{i\theta_n}$ at least one other element of the boundary of W. Let θ_k be the maximum of the sequence $\{\theta_j\}_{j=1}^n$ such that $re^{i\theta_k}$ is a boundary point of W. Then, in this case, the segment of Γ between $re^{i\theta_1}$ and $re^{i\theta_k}$ is a simple arc of the boundary of W whose existence follows from (i).

Thus in the former case, we have so far constructed Γ from $re^{i\phi}$ to $re^{i\theta_2}$; in the latter case Γ extends from $re^{i\phi}$ to $re^{i\theta_k}$ ($\theta_2 \leqslant \theta_k \leqslant \theta_n$). This construction process may be continued in an obvious way until Γ reaches the point $re^{i\theta_n}$. A traversal of the arc of $|z| = r$ from $re^{i\theta_n}$ to $re^{i\psi}$ then terminates Γ.

From its construction $|G(z)| \geqslant \gamma$ on Γ, and Γ is a continuous curve consisting only of points on the circumferences $|z| = r$, $|z| = R$, and points on the level set $|G(z)| = \gamma$. It is clear that Γ can intersect itself only at points of the level set $|G(z)| = \gamma$ which are boundary points of distinct components of the complement of $\overline{K(\gamma)}$. As is readily seen, this can occur only for those finitely many points at which $G'(z) = 0$. The bound on the length of Γ given in (iii) thus follows from (ii).

We shall now prove an elementary lemma which shows that the removal of the poles of $F(z)$ in certain annuli about the Pólya peaks does not destroy the essential behavior of $F(z)$ around the peaks. This will enable us to pass from $F(z)$ to regularized functions for which a local form of Carleman's method is applicable.

LEMMA 2. *Let $F(z)$ be a meromorphic function of lower order μ $(0 < \mu < \infty)$ satisfying (3.1), and $\{r_m\}$ a sequence of Pólya peaks of order μ of $T(r, F)$ with corresponding sequences $\{r'_m\}$, and $\{r''_m\}$.*

Then there exists a sequence $\{\sigma_m\}$ satisfying

$$\sigma_m \to +\infty \quad as \quad m \to \infty, \tag{3.24}$$

$$\sigma_m r_m \leq \frac{r''_m}{8}, \quad \sigma_m^{-1} r_m \geq 8 r'_m, \tag{3.25}$$

$$\delta_m = \frac{\log\left[\pi r_m (4\sigma_m + (2\sigma_m)^{1+\mu/2}\, 3\, T(r_m, F))\right]}{T(r_m, F)} \to 0 \quad as \quad m \to \infty, \tag{3.26}$$

and such that, if $P(z; \sigma_m, r_m)$ and $F(z; \sigma_m, r_m)$ are as defined in (3.2), (3.3), *and* (3.4), *we have uniformly in the intervals*

$$I_m = \left[\frac{r_m}{2\sigma_m},\ 2\sigma_m r_m\right] \tag{3.27}$$

the estimates

$$\gamma_m = \sup_{|z| \in I_m} \frac{\log^+ |P(z; \sigma_m, r_m)|}{T(r_m, F)} \to 0 \quad as \quad m \to \infty, \tag{3.28}$$

$$m(r, F(z; \sigma_m, r_m)) = m(r, F) + o(T(r_m, F)) \quad (r \in I_m;\ m \to \infty), \tag{3.29}$$

$$\log^+ M(r, F(z; \sigma_m, r_m)) \geq m(r, F) + o(T(r_m, F)) \quad (r \in I_m;\ m \to \infty), \tag{3.30}$$

$$\log^+ M(r, F(z; \sigma_m, r_m)) \leq 3m(2r, F) + o(T(r_m, F)) \quad (r \in I_m;\ m \to \infty). \tag{3.31}$$

Proof. Let $\sigma > 2$ be arbitrarily fixed, and define

$$\varepsilon_{1,m} = \frac{8\sigma r_m}{r''_m}, \quad \varepsilon_{2,m} = \frac{8\sigma r'_m}{r_m}, \tag{3.32}$$

and

$$\varepsilon_{3,m} = \frac{\log\left[\pi r_m (4\sigma + (2\sigma)^{1+\mu/2}\, 3\, T(r_m, F))\right]}{T(r_m, F)}. \tag{3.33}$$

It follows from the definition of Pólya peaks and the fact that F is not rational that

$$\varepsilon_{j,m}\to 0 \quad \text{as } m\to\infty \quad (j=1,2,3). \tag{3.34}$$

In order to deal with the exceptional set in the following estimates we observe that since $\mathcal{E}$ has finite measure there exists a sequence $\{\tilde{r}_m\}$ such that

$$16\sigma r_m \leqslant \tilde{r}_m \leqslant 17\sigma r_m, \quad r_m \notin \mathcal{E}. \tag{3.35}$$

Now if for each m we restrict the quantity r by

$$\frac{r_m}{4\sigma}\leqslant r\leqslant 4\sigma r_m \tag{3.36}$$

it follows from (3.1), (3.2), (3.35), and (3.36)

$$\begin{aligned}\log^+|P(re^{i\theta};\sigma,r_m)| &\leqslant \log^+\prod_{r_m/8\sigma<|b_\nu|<8\sigma r_m}\left|1-\frac{re^{i\theta}}{b_\nu}\right|+\log^+|cr^l|\\ &\leqslant \log^+(1+32\sigma^2)^{n(8\sigma r_m,F)}+O(\log r_m)\\ &\leqslant \frac{N(16\sigma r_m,F)}{\log 2}\log(1+32\sigma^2)+O(\log r_m)\\ &\leqslant o(1)\frac{(17)^\mu\sigma^\mu\log(1+32\sigma^2)}{\log 2}T(r_m,F) \quad (m\to\infty).\end{aligned}$$

Hence,

$$\varepsilon_{4,m}=\sup_{r_m/4\sigma\leqslant|z|\leqslant 4\sigma r_m}\frac{\log^+|P(z;\sigma,r_m)|}{T(r_m,F)}\to 0 \quad \text{as} \quad m\to\infty. \tag{3.37}$$

Now, if r is again restricted by (3.36) for each m, we deduce from (3.1), (3.2), (3.37), and Jensen's formula

$$\begin{aligned}m(r,F)&\leqslant m(r,F(z;\sigma,r_m))+m\left(r,\frac{1}{P(z;\sigma,r_m)}\right)\\ &=m(r,F(z;\sigma,r_m))+m(r,P(z;\sigma,r_m))-N\left(r,\frac{1}{P(z;\sigma,r_m)}\right)+N(r,P(z;\sigma,r_m))\\ &\quad -\log|c|\leqslant m(r,F(z;\sigma,r_m))+\log^+M(r,P(z;\sigma,r_m))+O(\log r_m)\\ &=m(r,F(z;\sigma,r_m))+o(T(r_m,F)) \quad (m\to\infty).\end{aligned} \tag{3.38}$$

On the other hand,

$$\begin{aligned}m(r,F(z;\sigma,r_m))&\leqslant m(r,F)+m(r,P(z;\sigma,r_m))\leqslant m(r,F)+\log^+M(r,P(z;\sigma,r_m))\\ &=m(r,F)+o(T(r_m,F)) \quad (m\to\infty),\end{aligned} \tag{3.39}$$

and by comparing (3.38) and (3.39) we have

$$\varepsilon_{5,m} = \sup_{r_m/4\sigma \leqslant r \leqslant 4\sigma r_m} \frac{|m(r, F(z;\sigma, r_m)) - m(r, F)|}{T(r_m, F)} \to 0 \quad \text{as} \quad m \to \infty. \tag{3.40}$$

It follows immediately from (3.40) that

$$\log^+ M(r, F(z;\sigma, r_m)) \geqslant m(r, F) + \varepsilon_{6,m} T(r_m, F) \tag{3.41}$$

where

$$\varepsilon_{6,m} \to 0 \quad \text{as} \quad m \to \infty \quad \left(\frac{r_m}{4\sigma} \leqslant r \leqslant 4\sigma r_m\right). \tag{3.42}$$

For our next estimate, we apply the Poisson–Jensen formula with $r_m/2\sigma \leqslant r \leqslant 2\sigma r_m$ and obtain by (3.1) and (3.35)

$$\log^+ M(r, F(z;\sigma, r_m)) \leqslant \sup_\theta \left\{\frac{1}{2\pi}\int_0^{2\pi} \log^+ |F(2re^{i\phi};\sigma, r_m)| \frac{(2r)^2 - r^2}{(2r)^2 + r^2 - 4r^2\cos(\theta - \phi)}\, d\phi\right.$$

$$\left. + \sum_{|b_\nu| \leqslant r_m/8\sigma} \log\left|\frac{(2r)^2 - \bar{b}_\nu re^{i\theta}}{2r(re^{i\theta} - b_\nu)}\right|\right\} \leqslant 3\,m(2r, F(z;\sigma, r_m)) + \sum_{|b_\nu| \leqslant r_m/8\sigma} \log \frac{5r}{2(r - |b_\nu|)}$$

$$\leqslant 3\,m(2r, F(z;\sigma, r_m)) + n\left(\frac{r_m}{8\sigma}, F\right)\log\frac{10}{3} \leqslant 3\,m(2r, F(z;\sigma, r_m)) + \frac{\log\frac{10}{3}}{\log 2} N(\tilde{r}_m, F)$$

$$\leqslant 3\,m(2r, F(z;\sigma, r_m)) + \frac{\log\frac{10}{3}}{\log 2}(17\sigma)^\mu o(1)\, T(r_m, F)$$

$$= 3\,m(2r, F(z;\sigma, r_m)) + o(T(r_m, F)) \quad \left(m \to \infty;\ \frac{r_m}{2\sigma} \leqslant r \leqslant 2\sigma r_m\right).$$

Therefore, by (3.40) we have

$$\log^+ M(r, F(z;\sigma, r_m)) \leqslant 3m(2r, F) + \varepsilon_{7,m} T(r_m, F), \tag{3.43}$$

where

$$\varepsilon_{7,m} \to 0 \quad \text{as} \quad m \to \infty \quad \left(\frac{r_m}{2\sigma} \leqslant r \leqslant 2\sigma r_m\right). \tag{3.44}$$

Now, by (3.34), (3.37), (3.40), (3.42), and (3.44) we may choose, for each integer $l > 1$ an index m_l such that for $\sigma = l$, we have

$$\max_{1 \leqslant j \leqslant 7} |\varepsilon_{j,m}| < \frac{1}{l} \quad (m > m_l). \tag{3.45}$$

We thus choose

$$\sigma_m = l \quad \text{for } m_l < m \leqslant m_{l+1} \quad (l = 1, 2, \ldots).$$

With this choice of $\{\sigma_m\}$, (3.32), (3.33), (3.37), (3.40), (3.41), (3.43) and (3.45) yield (3.24)–(3.31) and the lemma is proved.

IV. Components corresponding to deficient values

In this section we shall show that the deficient values of functions $f(z)$ satisfying the conditions of Theorem 1 give rise to open sets in which the methods of Carleman are applicable.

LEMMA 3. *Let $f(z)$ be as in Theorem 1 having $\tau_1, \dots, \tau_n$ $(n \geqslant 2)$ among its deficient values and assume that* (2.10), (2.11), *and* (2.12) *hold. Set*

$$F(z) = \frac{1}{f'(z)}.$$

Then there exists a sequence $\{r_m\}$ of Pólya peaks of order μ of $T(r, F)$ with corresponding sequences $\{r_m'\}$, $\{r_m''\}$, and a sequence $\{\sigma_m\}$ satisfying (3.24)–(3.26) *such that the functions*

$$F_m(z) = F(z; \sigma_m, r_m), \quad P_m(z) = P(z; \sigma_m, r_m).$$

satisfy (3.28)–(3.31).

Furthermore, there exists a sequence $\{\eta_m\}$, a constant $\mathcal{K}\,(0 < \mathcal{K} < 1)$, and disjoint open subsets $\mathcal{D}_{1,m}, \mathcal{D}_{2,m}, \dots, \mathcal{D}_{n,m}$ of

$$U_m = \{z: |F_m(z)| > e^{\eta_m T(r_m, F)}\} \cap \mathcal{A}_m \quad \left(\mathcal{A}_m = \left\{z: \frac{r_m}{\sigma_m} < |z| < \sigma_m r_m\right\}\right), \tag{4.1}$$

each of which is the union of components of U_m, such that

$$\eta_m \to 0 \quad \text{as } m \to \infty, \tag{4.2}$$

and

$$m(r_m, F_m; E_{j,m}) > \mathcal{K} T(r_m, F) \quad (j = 1, \dots, n;\ m > m_0), \tag{4.3}$$

where

$$E_{j,m} = \{\theta: 0 \leqslant \theta < 2\pi,\ r_m e^{i\theta} \in \mathcal{D}_{j,m}\} \quad (j = 1, \dots, n). \tag{4.4}$$

The sets $\mathcal{D}_{j,m}$ $(1 \leqslant j \leqslant n)$ have finitely many components, each of which extends to the outer circumference of $\mathcal{A}_m$ and is bounded away from the inner circumference.

Proof. Let $\{r_m\}$ be a sequence of Pólya peaks of order μ of $T(r, F)$ with associated sequences $\{r_m'\}$ and $\{r_m''\}$, and let $\mathcal{E}$ denote an exceptional set arising in the classical estimate of the logarithmic derivative [11; pp. 32, 41]

$$\lim_{\substack{r \to \infty \\ r \notin \mathcal{E}}} \frac{m\left(r, \dfrac{f'}{f - \tau_j}\right)}{T(r, f)} = 0. \tag{4.5}$$

It is easily shown that, by a slight shifting of the values of the sequence $\{r_m\}$, we may assume

$$r_m \notin \mathcal{E} \quad (m > m_0). \tag{4.6}$$

In fact, since $\mathcal{E}$ has finite measure, for sufficiently large m we have in each interval $[r_m, r_m+1]$ a value $\tilde{r}_m \notin \mathcal{E}$ for which the pertinent inequalities for Pólya peaks hold:

$$\frac{T(r,F)}{T(\tilde{r}_m,F)} = \frac{T(r,F)}{T(r_m,F)}\frac{T(r_m,F)}{T(\tilde{r}_m,F)} \leqslant \left(\frac{r}{r_m}\right)^{\mu}(1+o(1))$$

$$\leqslant \left(\frac{r}{\tilde{r}_m}\right)^{\mu}\left(1+\frac{1}{r_m}\right)^{\mu}(1+o(1)) = \left(\frac{r}{\tilde{r}_m}\right)^{\mu}(1+o(1)) \quad (r_m' \leqslant r \leqslant r_m''; m \to \infty).$$

Thus we can and do fix a sequence $\{r_m\}$ of Pólya peaks of order μ of $T(r, F)$ with corresponding sequences $\{r_m'\}$ and $\{r_m''\}$ satisfying the auxiliary condition (4.6).

Now, Lemma A implies the applicability of Lemma 2 to $F(z)$ and the existence of the sequence $\{\sigma_m\}$ satisfying (3.24)–(3.26) is evident. We therefore concentrate on the construction of an appropriate sequence $\{\eta_m\}$.

Let δ_m, γ_m be as in (3.26) and (3.28). We apply Lemma 1 for each m to the functions F_m in $\mathcal{A}_m$ where by (3.25), (3.30), (3.31), and the definition of Pólya peaks we have

$$\mathfrak{m}_m = \sup_{|z|=\sigma_m^{-1}r_m} \log^+|F_m(z)| \leqslant 3\,m(2\,\sigma_m^{-1}r_m, F) + o(T(r_m, F))$$

$$\leqslant 3\,\frac{2^{\mu}}{\sigma_m^{\mu}}\,T(r_m,F) + o(T(r_m,F)) \quad (m\to\infty), \tag{4.7}$$

$$\mathfrak{M}_m = \sup_{|z|=\sigma_m r_m} \log^+|F_m(z)| \geqslant m(\sigma_m r_m, F) + o(T(r_m,F)) \quad (m\to\infty). \tag{4.8}$$

It then follows from (3.2), (3.3), (3.4), (4.7), (4.8), and (3.24) that all of the assumptions of Lemma 1 are for $m > m_0$ satisfied by the functions F_m with $\varrho = \sigma_m^{-1}r_m$, $R = \sigma_m r_m$, and $R' = 2\sigma_m r_m$.

Thus taking

$$\nu_m = \max\left\{12\,\frac{2^{\mu}}{\sigma_m^{\mu}},\ 2\,\delta_m + 2\,\gamma_m\right\} \tag{4.9}$$

we have by (3.24), (3.26), (3.28), (4.7), and (4.8) that

$$\nu_m \to 0 \quad \text{as} \quad m\to\infty \tag{4.10}$$

and

$$\mathfrak{m}_m < \frac{\nu_m}{2}\,T(r_m,F) < \nu_m T(r_m,F) < \mathfrak{M}_m \quad (m > m_0). \tag{4.11}$$

Hence there exists a subset

$$I_m^* \subseteq \left[\frac{\nu_m}{2}\,T(r_m,F),\ \nu_m T(r_m,F)\right] \tag{4.12}$$

such that
$$\text{meas } I_m^* \geqslant \frac{\nu_m}{4} T(r_m, F) \tag{4.13}$$

and (i), (ii), (iii) of Lemma 1 are applicable to F_m for all elements of I_m^*.

For each m we choose η_m so that
$$\eta_m T(r_m, F) \in I_m^*. \tag{4.14}$$

With the sequence $\{\eta_m\}$ thus fixed we proceed to show the existence of the open sets $\mathcal{D}_{1,m}, \mathcal{D}_{2,m}, \ldots, \mathcal{D}_{n,m}$.

To this end we observe first that there exist positive constants K_j and e_j such that if

$$J_j(r_m) = \left\{\theta:\ 0 \leqslant \theta < 2\pi,\ \frac{1}{|f(r_m e^{i\theta}) - \tau_j|} > e^{3K_j T(r_m, f)}\right\} \tag{4.15}$$

we have
$$\text{meas } J_j(r_m) \geqslant 2e_j \quad (j = 1, \ldots, n;\ m > m_0). \tag{4.16}$$

In fact, if this were not the case, there would exist sequences $\varepsilon_m \to 0$, $\lambda_m \to 0$ such that for some j and infinitely many m,

$$\text{meas } \left\{\theta: 0 \leqslant \theta < 2\pi,\ \frac{1}{|f(r_m e^{i\theta}) - \tau_j|} > e^{\varepsilon_m T(r_m, f)}\right\} < \lambda_m$$

and, hence by taking a sequence $\{\varrho_m\}$ such that $2r_m \leqslant \varrho_m \leqslant 3r_m$ and $\varrho_m \notin \mathcal{E}(m > m_0)$ we could apply (2.3) and a lemma of Edrei and Fuchs [9; p. 322] to obtain

$$m\left(r_m, \frac{1}{f - \tau_i}\right) \leqslant \varepsilon_m T(r_m, f) + 22\, T(\varrho_m, f)\, \lambda_m \left[1 + \log^+ \frac{1}{\lambda_m}\right]$$
$$\leqslant T(r_m, f)\left(\varepsilon_m + 22 \cdot 3^{\mu} \lambda_m \left[1 + \log^+ \frac{1}{\lambda_m}\right]\right) = o(T(r_m, f)) \quad (m \to \infty)$$

contradicting the fact that τ_j is a deficient value of $f(z)$.

Next, we note that by (2.3), (3.28), (4.5), and (4.6),

$$m\left(r_m, \frac{1}{F_m(f - \tau_j)}\right) = m\left(r_m, \frac{P_m f'}{f - \tau_j}\right) \leqslant m(r_m, P_m) + m\left(r_m, \frac{f'}{f - \tau_j}\right)$$
$$= o(T(r_m, f)) \quad (j = 1, \ldots, n;\ m \to \infty),$$

and hence

$$\text{meas } \left\{\theta: 0 \leqslant \theta < 2\pi,\ \left|\frac{1}{F_m(r_m e^{i\theta})\,(f(r_m e^{i\theta}) - \tau_j)}\right| > e^{K_j T(r_m, f)}\right\} \to 0 \quad (j = 1, \ldots, n;\ m \to \infty). \tag{4.17}$$

Therefore, in view of (4.15), (4.16), (4.17) and (2.3), there exists for each j $(1 \leqslant j \leqslant n)$ a set

$$H_j(r_m) \subseteq J_j(r_m) \tag{4.18}$$

such that

$$\text{meas}\, H_j(r_m) > e_j \quad (m > m_0) \tag{4.19}$$

and

$$|F_m(r_m e^{i\theta})| > e^{2K_j T(r_m, f)} > e^{K_j T(r_m, F)} \quad (\theta \in H_j(r_m);\, m > m_0). \tag{4.20}$$

We now define $\mathcal{D}_{j,m}$ $(1 \leqslant j \leqslant n)$ to be the union over all components of U_m containing points

$$r_m e^{i\theta},\ \theta \in H_j(r_m).$$

With $\mathcal{D}_{j,m}$ thus defined, (i) of Lemma 1 implies that $\mathcal{D}_{j,m}$ has finitely many components, each of which extend to the outer circumference of $\mathcal{A}_m$. Furthermore, the conditions (4.7), (4.11), (4.12), and (4.14) imply that each set $\mathcal{D}_{j,m}$ is bounded away from the inner circumference. Also, taking

$$\mathcal{K} = \min_{1 \leqslant j \leqslant n} \frac{K_j e_j}{2\pi}$$

we have in view of (4.19) and (4.20) proved (4.3).

It remains only to show that the sets $\mathcal{D}_{j,m}$ are pairwise disjoint. For suppose this is not the case. It then follows from their construction that one of the sets $\mathcal{D}_{j,m}$ has a component containing elements $r_m e^{i\varphi} \in H_i(r_m)$ and $r_m e^{i\varphi'} \in H_k(r_m)$ with $i \neq k$. By (iii) of Lemma 1 and (4.12) we may join these two points by a continuous path Γ_m of total length not exceeding

$$l_m = 2\pi\sigma_m r_m \sqrt{\frac{2T(2\sigma_m r_m, F_m)}{\log 2}} + 4\pi\sigma_m r_m$$

on which

$$|F_m(z)| \geqslant e^{\eta_m T(r_m, F)} \quad (m > m_0).$$

By (3.25), (3.29), and (4.1) we have

$$\log l_m < \delta_m T(r_m, F) \quad (m > m_0).$$

Hence, from (4.9), (4.12) and (4.14) we deduce

$$|f(r_m e^{i\varphi}) - f(r_m e^{i\varphi'})| = \left|\int_{\Gamma_m} f'(z)\,dz\right| \leqslant \int_{\Gamma_m} \left|\frac{1}{F_m(z)}\right| |P_m(z)|\,|dz|$$
$$\leqslant e^{(-\eta_m + \gamma_m + \delta_m) T(r_m, F)} \leqslant e^{-(\gamma_m + \delta_m) T(r_m, F)} < 1 \quad (m > m_0). \tag{4.21}$$

On the other hand, (2.11), (4.15), and (4.18) imply

$$|f(r_m e^{i\varphi}) - f(r_m e^{i\varphi'})| = |f(r_m e^{i\varphi}) - \tau_i - f(r_m e^{i\varphi'}) + \tau_k + \tau_i - \tau_k| \geqslant |\tau_i - \tau_k|$$
$$- |f(r_m e^{i\varphi}) - \tau_i| - |f(r_m e^{i\varphi'}) - \tau_k| \geqslant 2 \quad (m > m_0)$$

which contradicts (4.21) and establishes the lemma.

V. Carleman's method

We now proceed to the proof of the elegant theorem of Carleman.

LEMMA 4. (Carleman's Differential Inequality). *Let $H(z)$ be meromorphic in $|z|<e^{t_3}$, nonrational, and regular in the annulus*

$$e^{t_1} \leqslant |z| \leqslant e^{t_3} \quad (-\infty < t_1 < t_2 < t_3 \leqslant \infty),$$

and $K>0$ a constant.

Let $\mathcal{D}_1, \ldots, \mathcal{D}_n$ $(n \geqslant 1)$ be disjoint open subsets of

$$E = \{z: e^{t_1} < |z| < e^{t_2}, \ |H(z)| > 1\} \tag{5.1}$$

each of which is the union of component sets of E. Assume that each $\mathcal{D}_j$ has a nonempty intersection with the circle $|z| = e^{s_0}$ $(t_1 < s_0 < t_2)$ and each is bounded away from $|z| = e^{t_1}$. If $n=1$ we make the additional assumption that $\mathcal{D}_1$ does not contain completely any of the circumferences $|z| = e^t$ $(s_0 \leqslant t < t_2)$.

If $\Lambda_j(t, \theta)$ and $\phi_j(t)$ are defined by

$$\Lambda_j(t, \theta) = 1 \ \text{ if } e^{t+i\theta} \in \mathcal{D}_j,$$

$$\Lambda_j(t, \theta) = 0 \ \text{ if } e^{t+i\theta} \notin \mathcal{D}_j,$$

$$\phi_j(t) = K \int_0^{2\pi} \Lambda_j(t, \theta) (\log |H(e^{t+i\theta})|)^2 \, d\theta, \tag{5.2}$$

and $l_j(t)$ represents the angular measure of the largest arc of $|z| = e^t$ contained in $\mathcal{D}_j$, then $\phi_j(t)$ is a twice differentiable function of t in the interval

$$t_1 < t < t_2$$

and for $s_0 \leqslant t < t_2$ satisfies

$$\phi_j(t), \ \phi_j'(t), \ \phi_j''(t) > 0 \tag{5.3}$$

and

$$\frac{2\phi_j''(t)}{\phi_j(t)} - \left(\frac{\phi_j'(t)}{\phi_j(t)}\right)^2 \geqslant \left(\frac{2\pi}{l_j(t)}\right) \quad (j = 1, \ldots, n). \tag{5.4}$$

Proof. We fix t_0 in the interval (t_1, t_2) and let

$$e^{t_0 + i\omega_1}, e^{t_0 + i\omega_2}, \ldots, e^{t_0 + i\omega_m} \quad (m \geqslant 0) \tag{5.5}$$

be all of the zeros of $H(z)$ on the circumference $|z| = e^{t_0}$. Let $\delta_1 > 0$ be chosen sufficiently small so that the annulus

$$e^{t_0 - \delta_1} \leqslant |z| \leqslant e^{t_0 + \delta_1} \quad (t_1 < t_0 - \delta_1 < t_0 + \delta_1 < t_2)$$

contains no zeros of $H(z)$ other than those in (5.5).

Next, we select $\delta_2 (0 < \delta_2 < \delta_1)$ in such a way that if $e^{t_0-\delta_2} \leqslant |z| \leqslant e^{t_0+\delta_2}$ and $|\arg z - \omega_j| \leqslant \delta_2$, then $|H(z)| < \frac{1}{2}$. Letting C denote the set of arguments θ in

$$\bigcup_{j=1}^{m} \{\theta : |\theta - \omega_j| < \delta_2\}$$

taking their determination in $[0, 2\pi)$, we have for each j

$$\Lambda_j(t, \theta) = 0 \quad (|t - t_0| \leqslant \delta_2;\ \theta \in C). \tag{5.6}$$

If we now let $W(t, \theta)$ represent either of the functions

$$(\log |H(e^{t+i\theta})|)^2 \quad \text{or} \quad (\log |H(e^{t+i\theta})|)\, \frac{\partial \log |H(e^{t+i\theta})|}{\partial t}$$

we have by continuity

$$|W(t, \theta) - W(t', \theta)| \leqslant M |t - t'| \quad (t, t' \in [t_0 - \delta_2, t_0 + \delta_2],\ \theta \in [\theta, 2\pi) - C) \tag{5.7}$$

for some constant $M > 0$.

Recall that in the proof of (i) of Lemma 1 it was shown that under the present assumptions there can be at most a finite number of points

$$e^{t_0 + i\theta_1}, e^{t_0 + i\theta_2}, \ldots, e^{t_0 + i\theta_p} \quad (p \geqslant 0)$$

on the circle $|z| = e^{t_0}$ for which $|H(z)| = 1$. If $p = 0$ take G to be the null set. Otherwise with p so defined, M as in (5.7), and any given $\varepsilon > 0$, define G as the set of all arguments θ such that

$$\bigcup_{j=1}^{p} \left\{\theta : |\theta - \theta_j| < \frac{\varepsilon}{2Mp}\right\},$$

taking their determination in $[0, 2\pi)$.

With C and G thus defined, there exists a number $\eta > 0$ such that if

$$S = [0, 2\pi) - \{C \cup G\} \tag{5.8}$$

then

$$|\log |H(e^{t_0 + i\theta})|| > \eta \quad (\theta \in S). \tag{5.9}$$

Choosing δ_3 such that

$$0 < \delta_3 < \delta_2 \tag{5.10}$$

and

$$|\log |H(e^{t_0 + i\theta})| - \log |H(e^{t + i\theta})|| < \frac{\eta}{2} \quad (|t - t_0| \leqslant \delta_3;\ \theta \in S) \tag{5.11}$$

we distinguish two cases.

If $\theta \in S$ and $\log |H(e^{t_0 + i\theta})| > \eta$, then (5.11) implies $\log |H(e^{t + i\theta})| > \eta/2$ for all points on the line segment joining $e^{t_0 + i\theta}$ and $e^{t + i\theta}$ $(|t - t_0| \leqslant \delta_3)$. Since each $\mathcal{D}_j$ is the union of components of E defined in (5.1), it follows in this case that $\Lambda_j(t_0, \theta) - \Lambda_j(t, \theta) = 0$ for $|t-t_0| < \delta_3$. If, on the other hand, for $\theta \in S$ we have $\log|H(e^{t_0+i\theta})| < -\eta$ then (5.11) implies $\log|H(e^{t+i\theta})| < -\eta/2$ if $|t-t_0| \leq \delta_3$. Then for each j, $\Lambda_j(t_0, \theta) = \Lambda_j(t, \theta) = 0$ and we thus have shown in any case

$$\Lambda_j(t_0, \theta) - \Lambda_j(t, \theta) = 0 \quad (|t-t_0| \leq \delta_3;\, \theta \in S). \tag{5.12}$$

For $0 < |t-t_0| \leq \delta_3$, we write

$$\frac{1}{t_0-t}\left(\int_0^{2\pi} \Lambda_j(t_0, \theta)\, W(t_0, \theta)\, d\theta - \int_0^{2\pi} \Lambda_j(t, \theta)\, W(t, \theta)\, d\theta\right)$$
$$= \int_0^{2\pi} \Lambda_j(t_0, \theta)\, \frac{(W(t_0, \theta) - W(t, \theta))\, d\theta}{t_0 - t} + \int_0^{2\pi} (\Lambda_j(t_0, \theta) - \Lambda_j(t, \theta))\, \frac{W(t, \theta)}{t_0-t}\, d\theta. \tag{5.13}$$

Recalling the definitions of the sets C and G, and using (5.6), (5.8), (5.10), and (5.12), we have

$$\int_0^{2\pi} (\Lambda_j(t_0, \theta) - \Lambda_j(t, \theta))\, \frac{W(t, \theta)}{t_0-t}\, d\theta$$
$$= \int_{G-C} (\Lambda_j(t_0, \theta) - \Lambda_j(t, \theta))\, \frac{W(t, \theta)}{t_0-t}\, d\theta \quad (0 < |t-t_0| \leq \delta_3). \tag{5.14}$$

Consider in (5.14) the situation when $\Lambda_j(t_0, \theta) - \Lambda_j(t, \theta) \neq 0$, $\theta \in G - C$, and

$$0 < t_0 - t \leq \delta_3. \tag{5.15}$$

Since each $\mathcal{D}_j$ is the union of component sets of E in (5.1) there must exist t' $(t \leq t' \leq t_0)$ such that $W(t', \theta) = 0$ and hence (5.7) implies

$$|W(t, \theta)| \leq M|t-t'| \leq M|t-t_0|;$$

the same inequality holds if (5.15) were replaced by $0 < t - t_0 \leq \delta_3$.

Finally, since meas $\{G - C\} < \varepsilon/M$ it follows from (5.14) that for $0 < |t-t_0| < \delta_3$, the second term on the right-hand side of (5.13) has magnitude not exceeding ε. Since $\varepsilon > 0$ could be arbitrarily small, putting

$$u = u(t, \theta) = \log|H(e^{t+i\theta})|,$$

we thus have, by a readily verified passage to the limit in (5.13),

$$\phi_j'(t) = 2K \int_0^{2\pi} \Lambda_j(t, \theta)\, u\, \frac{\partial u}{\partial t}\, d\theta \quad (t_1 < t < t_2), \tag{5.16}$$

$$\phi_j''(t) = 2K\int_0^{2\pi} \Lambda_j(t,\theta)\left(\left(\frac{\partial u}{\partial t}\right)^2 + u\frac{\partial^2 u}{\partial t^2}\right) d\theta \quad (t_1 < t < t_0). \tag{5.17}$$

Following Carleman we shall now estimate (5.17) on the interval $s_0 \leqslant t < t_0$. We note that the sets $\mathcal{D}_j$ extend from $|z| = e^{s_0}$ to $|z| = e^{t_0}$ so that $\phi_j(t) > 0$ on this interval. Further, we obtain from Laplace's equation and an integration by parts in (5.17), the equation

$$\phi_j''(t) = 2K\int_0^{2\pi} \Lambda_j(t,\theta)\left(\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial u}{\partial \theta}\right)^2\right) d\theta \geqslant 0 \quad (t_1 < t < t_2). \tag{5.18}$$

Since $\phi_j(s_0) > 0$, and the sets $\mathcal{D}_j$ are bounded away from the circle $|z| = e^{t_1}$, it is clear from (5.18) that $\phi_j'(t) > 0$ for $s_0 \leqslant t < t_2$.

Now, by the Cauchy–Schwarz inequality and (5.16) we have

$$K\int_0^{2\pi} \Lambda_j(t,\theta)\left(\frac{\partial u}{\partial t}\right)^2 d\theta \geqslant \frac{1}{4}\frac{(\phi_j'(t))^2}{\phi_j(t)} > 0 \quad (s_0 \leqslant t < t_2), \tag{5.19}$$

and by Wirtinger's inequality

$$K\int_0^{2\pi} \Lambda_j(t,\theta)\left(\frac{\partial u}{\partial \theta}\right)^2 d\theta \geqslant \left(\frac{\pi}{l_j(t)}\right)^2 \phi_j(t) > 0 \quad (s_0 \leqslant t > t_2). \tag{5.20}$$

On combining (5.18), (5.19), and (5.20) we obtain both (5.4) and the fact that $\phi_j''(t) > 0$ for $s_0 \leqslant t < t_2$. This concludes the proof of Lemma 4.

In order to apply the differential inequality of Carleman, we must return to the pertinent estimates of $F_m(z)$. We shall choose the constant K in (5.2) appropriately to "normalize" the functions ϕ_j.

LEMMA 5. *Let the assumptions and notations of Lemma 3 be unchanged. Set* $t = \log r$, $t_m = \log r_m$, *and define*

$$\phi_{j,m}(t) = \frac{1}{2\pi T^2(e^{t_m}, F)}\int_{E_{j,m}(t)} \left(\log\left|\frac{F_m(e^{t+i\theta})}{e^{\eta_m T(e^{t_m},F)}}\right|\right)^2 d\theta, \tag{5.21}$$

where
$$E_{j,m}(t) = \{\theta : 0 \leqslant \theta < 2\pi,\ e^{t+i\theta} \in \mathcal{D}_{j,m}\} \quad (j = 1, \dots, n). \tag{5.22}$$

Then, for each $j = 1, \dots, n$,

$$\phi_{j,m}(t) \leqslant 4^\mu 9\left(\frac{e^t}{e^{t_m}}\right)^{2\mu} + o(1) \quad \left(\frac{e^{t_m}}{\sigma_m} \leqslant e^t \leqslant \sigma_m e^{t_m};\ m \to \infty\right), \tag{5.23}$$

$$\phi_{j,m}(t_m) \geqslant \frac{\mathcal{K}^2}{2} \quad (m > m_0), \tag{5.24}$$

and
$$\phi'_{j,m}(t_m) \geq \frac{\mu \mathcal{K}^2}{2 \log \left(\frac{40 \cdot 4^{\mu}}{\mathcal{K}^2}\right)} \quad (m > m_0). \tag{5.25}$$

Proof. It follows from (3.25), (3.31), (4.2), and the definition of Pólya peaks that

$$\phi_{j,m}(t) \leq \frac{1}{T^2(e^{t_m}, F)} (\log^+ M(e^t, F_m) + \eta_m T(e^{t_m}, F))^2 \leq \frac{9}{T^2(e^{t_m}, F)} (T(2e^t, F) + o(T(e^{t_m}, F))^2$$
$$\leq 4^{\mu}\, 9 \left(\frac{e^t}{e^{t_m}}\right)^{2\mu} + o(1) \quad \left(\frac{e^{t_m}}{\sigma_m} \leq e^t \leq \sigma_m e^{t_m}; m \to \infty\right).$$

We obtain the bound (5.24) by applying (3.29), (4.2), (4.3), and the Cauchy–Schwarz inequality. Thus,

$$\phi_{j,m}(t_m) = \frac{1}{2\pi T^2(e^{t_m}, F)} \left(\int_{E_{j,m}(t_m)} (\log |F_m(e^{t_m + i\theta})|)^2 d\theta \right.$$
$$\left. + (\eta_m T(e^{t_m}, F))^2 \int_{E_{j,m}(t_m)} d\theta - 2\eta_m T(e^{t_m}, F) \int_{E_{j,m}(t_m)} \log |F_m(e^{t_m + i\theta})| \, d\theta \right)$$
$$\geq \frac{1}{2\pi T^2(e^{t_m}, F)} \left(\int_{E_{j,m}(t_m)} (\log |F_m(e^{t_m + i\theta})|)^2 d\theta - 4\pi\eta_m T(e^{t_m}, F)\, m(e^{t_m}, F_m) \right)$$
$$\geq \frac{1}{2\pi T^2(e^{t_m}, F)} (2\pi m^2(e^{t_m}, F; E_{j,m}(t_m)) - o(T^2(e^{t_m}, F))) \geq \mathcal{K}^2 + o(1) \quad (m \to \infty).$$

In order to prowe (5.25) we notice first that the conditions of Lemma 3 imply Lemma 4 is applicable to the functions $\phi_{j,m}(t)$ which are thus differentiable in the respective intervals $|t - t_m| < \log \sigma_m$. Now setting

$$t = t_m - \frac{1}{2\mu} \log \left(\frac{40 \cdot 4^{\mu}}{\mathcal{K}^2}\right)$$

then $t < t_m$, and by (5.23) and (5.24) we have

$$\frac{\phi_{j,m}(t_m) - \phi_{j,m}(t)}{t_m - t} \geq \frac{\mu \mathcal{K}^2}{2 \log \left(\frac{40 \cdot 4^{\mu}}{\mathcal{K}^2}\right)}. \tag{5.26}$$

Applying the mean value theorem to (5.26) and then using (5.3) we obtain (5.25) and the proof is complete.

VI. Completion of the proof of Theorem 1

Let the assumptions and notations of Lemma 3 and Lemma 5 be unchanged. Continuing with Carleman, we set

$$\psi_{j,m}(t)=\log \phi_{j,m}(t)$$

and obtain from (5.4)

$$(\psi'_{j,m}(t))^2+2\psi''_{j,m}(t)\geqslant\left(\frac{2\pi}{l_{j,m}(t)}\right)^2 \quad (t_m\leqslant t\leqslant t_m+\log\sigma_m),$$

where $l_{j,m}(t)$ is the angular measure of the largest arc of $|z|=e^t$ in $\mathcal{D}_{j,m}$.

Since (5.25) shows that $\psi'_{j,m}(t)\neq 0$ in $[t_m, t_m+\log\sigma_m]$, we have there

$$\frac{2\pi}{l_{j,m}(t)}\leqslant\sqrt{(\psi'_{j,m}(t))^2+2\psi''_{j,m}(t)}\leqslant\psi'_{j,m}(t)+\frac{\psi''_{j,m}(t)}{\psi'_{j,m}(t)}. \tag{6.1}$$

Summing in (6.1) over all $j=1,\dots,n$ we may write

$$\frac{1}{n}\sum_{j=1}^{n}\int_{t_m}^{t_m+\log\sigma_m}\int_{t_m}^{s}\frac{2\pi}{l_{j,m}(t)}\,dt\,ds\leqslant\frac{1}{n}\sum_{j=1}^{n}\int_{t_m}^{t_m+\log\sigma_m}\int_{t_m}^{s}\left(\psi'_{j,m}(t)+\frac{\psi''_{j,m}(t)}{\psi'_{j,m}(t)}\right)dt\,ds. \tag{6.2}$$

Turning first to the left-hand side of (6.2), we have

$$\frac{1}{n}\sum_{j=1}^{n}\frac{2\pi}{l_{j,m}(t)}\geqslant\frac{2\pi n}{\sum_{j=1}^{n}l_{j,m}(t)}\geqslant n,$$

and hence

$$\frac{1}{n}\sum_{j=1}^{n}\int_{t_m}^{t_m+\log\sigma_m}\int_{t_m}^{s}\frac{2\pi}{l_{j,m}(t)}\,dt\,ds\geqslant n\int_{t_m}^{t_m+\log\sigma_m}\int_{t_m}^{s}dt\,ds=\frac{n}{2}(\log\sigma_m)^2. \tag{6.3}$$

Turning next to the right-hand side of (6.2) we have by (5.23), (5.24), and (5.25)

$$\begin{aligned}\int_{t_m}^{t_m+\log\sigma_m}\int_{t_m}^{s}\left(\psi'_{j,m}(t)+\frac{\psi''_{j,m}(t)}{\psi'_{j,m}(t)}\right)dt\,ds&=\int_{t_m}^{t_m+\log\sigma_m}\psi_{j,m}(s)\,ds-\psi_{j,m}(t_m)(\log\sigma_m)\\&+\int_{t_m}^{t_m+\log\sigma_m}\log\psi'_{j,m}(s)\,ds-\log\psi'_{j,m}(t_m)(\log\sigma_m)\\&\leqslant\int_{t_m}^{t_m+\log\sigma_m}\log\left(4^{\mu}\cdot 9\left(\frac{e^s}{e^{t_m}}\right)^{2\mu}\right)ds-\psi_{j,m}(t_m)(\log\sigma_m)\\&+\int_{t_m}^{t_m+\log\sigma_m}\psi'_{j,m}(s)\,ds-\log\psi'_{j,m}(t_m)(\log\sigma_m)\\&\leqslant 2\mu\int_{t_m}^{t_m+\log\sigma_m}(s-t_m)\,ds+K_1\log\sigma_m=\mu(\log\sigma_m)^2+K_1\log\sigma_m\end{aligned} \tag{6.4}$$

for some constant K_1 and $m>m_0$.

If we apply (6.3) and (6.4) to (6.2) we obtain

$$\frac{n}{2}(\log \sigma_m)^2 \leqslant \mu(\log \sigma_m)^2 + K_1 \log \sigma_m \quad (m > m_0). \tag{6.5}$$

As $m \to \infty$ we have $\sigma_m \to \infty$ and (6.5) thus yields the bound $n \leqslant 2\mu$ as desired.

VII. Proof of Theorem 2

It follows from Theorem 1 that the conditions $\Delta(f) = 2$ and $\mu < 3/2$ imply $\nu(f) = 2$ and $\mu \geqslant 1$. Consequently $\delta(\tau_1, f) = \delta(\tau_2, f) = 1$ for some $\tau_1 \neq \tau_2$. By performing, if necessary, a suitable homographic transformation of $f(z)$ we may take

$$\tau_1 = 0, \quad \tau_2 = \infty.$$

Considering then the quantity

$$\mathcal{K}(f) = \limsup_{r\to\infty} \frac{N(r,f) + N\left(r, \frac{1}{f}\right)}{T(r,f)},$$

we have

$$\mathcal{K}(f) \leqslant \limsup_{r\to\infty} \frac{N(r,f)}{T(r,f)} + \limsup_{r\to\infty} \frac{N\left(r, \frac{1}{f}\right)}{T(r,f)} = 1 - \delta(\infty, f) + 1 - \delta(0, f) = 0. \tag{7.1}$$

Now, if we compare (7.1) with a result due to Edrei [3; p. 5] we see that

$$\mathcal{K}(f) \geqslant \frac{|\sin \pi\tau|}{A\tau + \frac{1}{2}|\sin \pi\tau|} \quad (0 < A < 12)$$

for all finite τ satisfying $\mu \leqslant \tau \leqslant \lambda$, and thus conclude that $\mu = \lambda = 1$. This completes the proof of Theorem 2.

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