MARCEL RIESZ IN MEMORIAM

BY

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Marcel Riesz was born in Györ, Hungary, November 16, 1886 and died in Lund, Sweden, on September 4, 1969. He studied in Budapest, Göttingen and Paris. In 1911 Mittag-Leffler invited him to come to Sweden where he taught at Stockholms Högskola. In 1926 he was appointed professor of mathematics at the university of Lund. After retiring from this position in 1952 he went to the United States where he was visiting research professor at the universities of Maryland and Chicago and other places. He returned to Lund in 1962. He was a member of the Swedish Academy of Sciences, the Physiographical Society in Lund, Videnskabernes Selskab in Copenhagen, had honorary degrees from the universities of Copenhagen and Lund and was an honorary member of the Swedish mathematical society.

Marcel Riesz was the youngest member of a generation of brilliant Hungarian mathematicians that included among others Leopold Fejér, Riesz's elder brother Frederick Riesz and Alfred Haar. His first paper ([1], 1906) is an exposition in Hungarian of a subject of current interest at the time, namely various summation methods for Taylor series of analytic functions. One of these methods, due to Mittag-Leffler, sums the series in a star-shaped region bounded by singular points, the Mittag-Leffler star. Common interest in

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these matters was the beginning of the association with Mittag-Leffler. They seem to have met for the first time in Stockholm in 1908.

In the period from 1908 to 1916, Riesz did very successful work in the summability theory of power series, trigonometric series and Dirichlet series. The starting point was Fejér's discovery that the Fourier series of a function f is summable by arithmetic means to $f(x)$ at every point of continuity. Lebesgue had shown that the same holds at all Lebesgue points. The current scales of summation methods were those of Hölder and Cesàro, recently shown to be equivalent. The Cesàro mean $s_n^{(\alpha)}$ of order $\alpha > -1$ of a sequence $\{s_n\}_0^\infty$, $s_0 = 0$, or a series with partial sums s_n is by definition the quotient of the coefficients of the nth terms in the formal power series for $(1-x)^{-\alpha}\sum_0^\infty s_n x^n$ and $(1-x)^{-\alpha-1}$. The Cesàro limit is $\lim s_n^{(\alpha)}$ for $n \to \infty$. When $\alpha = 1$, this procedure sums the series by arithmetic means and the result of Lebesgue applies. One of the classical results in Fourier series, namely that Fejér's (and Lebesgue's) theorem holds for summation of order $\alpha > 0$ is essentially due to Riesz. He announced it in 1909 [7] and published his proof much later [25]. In the meantime there were proofs by Chapman in 1910–11 and by Hardy in 1913. Riesz's main contribution, however, was in another direction. He observed that Cesàro's summation is equivalent to another one which has the advantage of being applicable also to functions $s(x)$ of locally bounded variation for $x \geq 0$ such that $s(0) = s(0+) = 0$. Its means are

$$s_\alpha(x) = x^{-\alpha}\int_0^x (x-t)^\alpha ds(t) \tag{1}$$

and the limit is defined accordingly. In case of a series we put $s(x) = s_n$ when $n \leq x < n+1$. The right-hand side of (1) appears also in the classical Riemann–Liouville integral of a function f,

$$I^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} f(t)\,dt. \tag{2}$$

The semi-group properties of the operator $\alpha \to I^\alpha$, namely that $I^\alpha I^\beta = I^{\alpha+\beta}$ and $I^0 = $ identity has immediate application to the means (1). The corresponding summation method applies to generalized Dirichlet series

$$f(z) = \int_0^\infty e^{-zt} ds(t),$$

where z is a complex number. The means are

$$x^{-\alpha}\int_0^x e^{-zt}(x-t)^\alpha ds(t)$$

and the α-sum of $f(z)$ is obtained by letting $x\to\infty$. These integrals were introduced by Riesz in 1909 [7] under the name of typical means. They had an instant success and led to a Cambridge Tract on Dirichlet series 1915 written jointly by Hardy and Riesz and reprinted in 1952.

Some ten years later, Riesz had established himself as an expert in the theory of trigonometric series and together with E. Hilb wrote the article Neuere Untersuchungen über trigonometrische Reihen in Encyklopädie der Mathematischen Wissenschaften. It would carry us too far to penetrate in detail Riesz's contributions to this intricate and much studied subject, but at least three more things stand out. His interpolation formula [14] for trigonometric polynomials has a permanent place in interpolation theory and provides simple proofs of, e.g., the well-known inequalities of Bernstein and Markoff with the best constants. One recurrent theme in Riesz's work is a theorem by Fatou that says that the power-series of an analytic function converges at every point of the boundary of the circle of convergence where the function is regular and the coefficients of the series tend to zero. Riesz gave a beautiful proof [17] of this result that also showed that the convergence is locally uniform on every arc of regularity. This proof is available in Landau's little booklet *Darstellung und Begründung einiger neuerer Ergebnisse der Funktionentheorie.* Riesz also extended Fatou's theorem to Dirichlet series and to summability situations. His best known result in Fourier series, however, lies in another direction. Every trigonometric series with real coefficients can be written as

$$u(e^{i\theta})=\sum_0^\infty \operatorname{Re} c_n e^{in\theta}. \tag{3}$$

Formally, this is the real part of the power series

$$f(z)=\sum_0^\infty c_n z^n, \quad z=re^{i}$$

for $r=1$. Its imaginary part

$$v(e^{i\theta})=\sum_0^\infty \operatorname{Im} c_n e^{in\theta}$$

is called the conjugate series to (3). By Parseval's formula

$$\int_0^{2\pi} u(e^{i\theta})^2 d\theta=\pi\sum|c_n|^2=\int_0^{2\pi} v(e^{i\theta})^2 d\theta, \tag{4}$$

provided we normalize so that $c_0=0$. Hence $u\to v$ is an isometry $L^2\to L^2$ where L^2 refers to the interval $0,2\pi$. The connection between conjugate series was an early subject of study. In particular, as early as 1912, W. H. and G. C. Young had considered the question whether the map $u\to v$ might be, if not an isometry, then at least a homeomorphism $L^p\to L^p$,

$1 < p < \infty$, but they had some indirect evidence for a negative answer. Here was an outstanding unsolved problem and when Riesz wrote to Hardy in 1923 that he had found a proof (see [33]), Hardy wrote back "some months ago you said 'j'ai démontré que 2 sér. trig. conjuguées sont toujours en même temps les séries de F de fonctions de la classe $L^p(p>1)\ldots$'. *I want the proof.* Both I and my pupil Titchmarsh have tried in vain to prove it ...". And in the next letter "Very many thanks—you supply all that is essential, I have sent on your letter to Titchmarsh. Most elegant and beautiful. Of course p. 2 is the real point. It is amazing that none of us should have seen it before (even for $p=4$!)". What Hardy is refering to is Riesz's use of Cauchy's formula which gives $\int_0^{2\pi} f(e^{i\theta})^p d\theta = 0$ when p is an integer. Taking the real part and putting $p=2$ proves (4) while, if $p=4$,

$$\int_0^{2\pi} (u^4 - 6\,u^2v^2 + v^4)\,d\theta = 0.$$

This identity and a moment's reflection show that $\int_0^{2\pi} v^p d\theta \leqslant c_p \int_0^{2\pi} u^p d\theta$ when $p=4$ and, more generally, when p is an even integer. The remaining arguments of the proof were more technical but they eventually lead Riesz to a basic result in analysis, his convexity theorem [32]. It combines elementary form with great power and runs as follows: the logarithm of the maximum $M_{\alpha\beta}$ of the absolute value of a bilinear form

$$A = \sum_{j=1}^{m} \sum_{k=1}^{n} a_{jk} x_j y_k$$

under the conditions

$$\sum_{j=1}^{m} |x_j|^{1/\alpha} \leqslant 1, \quad \sum_{k=1}^{n} |y_k|^{1/\beta} \leqslant 1$$

is a convex function of α, β in the triangle

$$\alpha \leqslant 1, \; \beta \leqslant 1, \; \alpha + \beta \geqslant 1.$$

A little later, Riesz's student Olof Thorin proved convexity when $\alpha \geqslant 0$, $\beta \geqslant 0$ provided the variables x and y are allowed to be complex. The convexity has many applications, one of them to Riesz's theorem about conjugate functions. Let c_p be the best constant in the inequality $\|v\|_p \leqslant c_p \|u\|_p$ where $\|f\|_p = (\int_0^{2\pi} |f(e^{i\theta})|^p d\theta)^{1/p}$. Then, by Hölder's inequality, c_p is the supremum of the absolute value of the bilinear form $A(u, w) = \int_0^{2\pi} u(e^{i\theta}) w(e^{i\theta}) d\theta$ when $\|u\|_p \leqslant 1$ and $\|w\|_q \leqslant 1$ where $p^{-1} + q^{-1} = 1$. Further, $c_p = c_q$. Hence, since $c_2 = 1$ and, by the argument given above, $c_p < \infty$ when p is an even integer, it follows that $c_p < \infty$ when $1 < p < \infty$. Riesz's convexity theorem has been the model for similar results, in recent times also for the theory of interpolation spaces.

Before Riesz moved to Lund he also worked with the moment problem ([24]), in particular the Hamburger moment problem: find a real measure $d\mu \geqslant 0$ on the real line with given moments

$$c_n = \int_{-\infty}^{+\infty} x^n \, d\mu(x).$$

Hans Hamburger's almost complete analysis of this problem in 1920 attracted the attention of many mathematicians, among them Riesz and Torsten Carleman. Here Carleman, who could fit the problem into his theory of singular integral equations, was perhaps more successful. Riesz was prepared for the subject through his friend Erik Stridsberg who took a lively interest in Stieltjes's classical paper from 1894 about the moment problem on the half-axis $x \geqslant 0$.

The meat of the moment problem is the analysis of the set of solutions and the problem of uniqueness. Existence of a solution μ is a relatively simple matter. It is obviously necessary that, for every real polynomial P,

$$P(x) = \sum a_n x^n \geqslant 0 \Rightarrow \sum a_n c_n \geqslant 0 \tag{5}$$

and it is easy to see that this is equivalent to the condition that all the quadratic forms $\sum_0^n \sum_0^n c_{j+k} y_j y_k$ are $\geqslant 0$. To prove that (5) is also sufficient one extends the non-negative linear functional $\varphi(P) = \Sigma a_n c_n$ from polynomials to the space of real continuous functions of at most polynomial growth. This procedure is now familiar to every student of analysis who knows the Hahn–Banach theorem, but it is interesting to note that Riesz used it in 1918 (see [24], the third note p. 2) in a lecture to the Stockholm Mathematical Society, well before Banach (1923, 1929) and Hahn (1926).

Riesz's work after he moved to Lund marks a break with the past. He acquired new interests, starting work in potential theory and wave propagation including Dirac's equation of the electron and relativity theory. He also took a continuing interest in elementary number theory. His most important contributions are in potential theory and wave propagation. In both cases he invented new multi-dimensional analogues of the Riemann-Liouville integral (2). The one used in potential theory is the fractional potential of order α of a mass distribution $f dx$,

$$I^\alpha f(x) = H_m(\alpha)^{-1} \int |x-y|^{\alpha-m} f(y) \, dy. \tag{6}$$

Here $|x-y|$ is the euclidean distance in R^m, $0 < \alpha < m$, and

$$H_m(\alpha) = \pi^{m/2} \Gamma\left(\frac{\alpha}{2}\right) \Big/ \Gamma\left(\frac{m-\alpha}{2}\right)$$

is chosen so that $I^\alpha I^\beta = I^{\alpha+\beta}$ when $\alpha>0$, $\beta>0$, $\alpha+\beta<m$. By a passage to the limit, I^0 is the identity. In particular, if

$$U^\alpha \mu(x) = H_m(\alpha)^{-1}\int |x-y|^{\alpha-m}\, d\mu(x)$$

is the α-potential of a mass distribution μ one has

$$\int U^\alpha \mu(x) U^\beta \nu(x)\, dx = H_m(\alpha+\beta)^{-1}\int |x-y|^{\alpha+\beta-m}\, d\mu(x)\, d\nu(x)$$

when $\alpha>0, \beta>0, \alpha+\beta<m$. Hence

$$\int U^\alpha \mu(x)\, d\mu(x) = H_m(\alpha)\int (U^{\alpha/2}\mu(x))^2\, dx \tag{6}$$

is non-negative. When $\alpha=2<m$, this represents the self-energy of the mass distribution μ with respect to the Newtonian potential. In this case the equilibrium distribution μ_F on a compact set F had been characterized by Gauss as having minimal energy in the class of mass distributions μ supported by F and having a given total mass. After his student Otto Frostman had put this principle on a strict basis, Riesz noted that the formula (6) indicates that the equilibrium distribution with respect to an α-potential could be defined by this principle of Gauss. In the case $0<\alpha\leqslant 2$, which is the condition for the kernel $|x-y|^{\alpha-m}$ to be subharmonic, this program met with complete success in Frostman's well-known thesis. The equilibrium distribution μ_F exists, it is unique, its potential on F is essentially constant, there is a notion of α-capacity and there are generalizations of the balayage process and Green's function. After Frostman, Riesz himself wrote on the subject [42]. Viewed in perspective, these contributions opened the way to the present-day variety of generalizations and refinements of classical potential theory.

At some points in his work on α-potentials, Riesz uses the fact that if f is smooth enough, (6) is a regular analytic function of α when $c<\operatorname{Re}\alpha<m$ with a given c and that $\Delta I^\alpha = I^{\alpha-2}$ where $\Delta=(\partial/\partial x_1)^2+\dots+(\partial/\partial x_m)^2$ is Laplace's operator. Analyticity properties became very important in the integral

$$I^\alpha f(x) = H_m(\alpha)^{-1}\int_{x-C} r(x-y)^{\alpha-m} f(y)\, dy \tag{7}$$

associated with the wave operator

$$\Delta = (\partial/\partial x_1)^2 - \dots - (\partial/\partial x_m)^2.$$

Here $r^2(x)=x_1^2-\dots-x_m^2$ is the square of the Lorentz distance and $C: r^2(x)\geqslant 0$, $x_1>0$ is the forward light-cone so that $x - C$ is the retrograde light-cone with its vertex at x. Finally,

$$H_m(\alpha) = \pi^{(m-2)/2}\, 2^{\alpha-1}\, \Gamma\left(\frac{\alpha}{2}\right) \Gamma\left(\frac{\alpha+2-m}{2}\right). \tag{8}$$

Clearly, (7) is an analytic function of α when the integral converges, i.e. when Re $\alpha > m-2$. The factor H_m is chosen so that

$$\Delta I^\alpha = I^{\alpha-2},\ I^\alpha I^\beta = I^{\alpha+\beta} \tag{9}$$

for large α and β. The main point is now that, if f is smooth enough, the right side of (7) is analytic in the halfplane Re $\alpha > -c$ with any given c, that (9) holds under corresponding circumstances and that $I^0 = 1$. The real explanation for this is perhaps the following. Let f have compact support and let

$$F(\xi + i\eta) = \int e^{-x(\xi+i\eta)} f(x)\, dx$$

be the Fourier–Laplace transform of f. Then

$$I^\alpha f(x) = (2\pi)^{-m} \int e^{x(\xi+i\eta)}\, r(\xi+i\eta)^{-\alpha} F(\xi+i\eta)\, d\eta,$$

where the integral is independent of the choice of $\xi \in C$ and the right side is indeed entire analytic in α when f is infinitely differentiable. Although Riesz was conscious of this, he always preferred direct proofs avoiding the use of the Fourier–Laplace transform. In his last published paper [59] he gives a variant of an earlier direct proof that I^0 is the identity. This point of view is connected with his intense interest in special relativity and questions of invariance under the Lorentz group.

It is probable that Riesz conceived the integral (7) during a series of seminars in the beginning of the thirties conducted by his colleague Nils Zeilon. The subject was Hadamard's theory of finite parts of integrals as exposed in his well-known book on the Cauchy problem. Hadamard gives explicit formulas for the solution of Cauchy's problem with data on a space-like surface for the wave equation with variable coefficients. In 1938 Riesz was ready with a beautiful variant of these formulas based on an operation I^α analogous to (7) and with similar properties relative to the wave operator with variable coefficients. He published an account in a note added to a lecture [43] given a year earlier where he treats the case of constant coefficients. In the lecture he remarked among other things that the fact that $1/H_m(2) = 0$ when $m > 2$ is even reflects Huygens' principle and he finished by giving an invariant formula for the solution of Cauchy's problem when $m = 4$ and analyzing it geometrically. In the fifties, during his stay in the United States, Riesz devoted much thought to this formula and extended it to characteristic surfaces in an intriguing paper [55].

The short paper [43] contains the main body of Riesz's work on the wave equation and it still makes impressive reading. The detailed account came in 1949 in a massive paper [47] in the Acta Mathematica. Considering that Riesz was a natural perfectionist and, in his later years, a slow worker, this paper is the result of an enormous effort on his part. It was made possible only through collaboration with his friend K. G. Hagstroem who graciously offered his hospitality and his services as a secretary during many summers in the forties.

Riesz's work on the wave equation did not contribute to the existence theory of Cauchy's problem and one of its main goals, the proper interpretation of divergent integrals, has since been attained in a much larger framework through Laurent Schwartz's theory of distributions. But the method of analytical continuation which makes sense also in this theory, is a lasting contribution.

Around 1930, the teaching of quantum mechanics and relativity theory became a necessity at every self-respecting university, but at the time not many physicists had the mathematical training necessary for these subjects. For many years Riesz lectured periodically on tensors and matrices to a motley crew of physicists and mathematicians. He also started research of his own and published his first paper in relativistic quantum theory in 1946. His main interest was the Dirac equation and the Clifford algebra. In [53] he introduced what amounts to a Clifford structure on a Lorentz manifold. Riesz also applied his work on the wave equation to the classical relativistic theory of the electron [49, 50]. The popular lecture [44] on the models of non-euclidean geometry, a masterpiece in its kind, is a byproduct of his interest in the geometry of relativity.

Riesz wrote clearly and well and paid much attention to form. His favourite language was French and his style, steeped in the classical tradition, sometimes borders on the precious. Mathematical research always involves competition for fame and a place in the hierarchy, but he made it seem a gentleman's game. He was of course no stranger to ambition and had to assert himself both in Sweden and in the cosmopolitan world he came from. He admired his illustrious brother Frederick and they had cordial relations. They wrote one paper [22] together, but otherwise there is a clear distance in content between their work, perhaps a result of a conscious effort on Marcel's part. Seen together they had much physical resemblance but very different temperaments, Frederick calm with great poise and Marcel quick and restless in comparison. Marcel Riesz knew an astonishing number of mathematicians and over the years made and kept many friends among them.

Mittag-Leffler had made Stockholms Högskola a center of mathematical research. It had a peak of activity before and around the turn of the century and its other mathematicians, Bendixson, von Koch and Fredholm were famous names. Some ten years later, however, their scientific activity was on the wane for various reasons. Riesz filled a mathematical vacuum. He learnt Swedish quickly and he was very active in the local mathematical society where he soon became the dominating figure. He was lively, accessible, an enthusiastic teacher and a good lecturer with a thorough knowledge of his field. His charming expository lecture [15] from 1913, written in Swedish, has a distinct personal touch reflecting these qualities. In 1923, Riesz lost a competition for a chair in Lund to Carleman. Shortly afterwards von Koch died and Bendixson, Fredholm and Phragmén made a move to appoint Riesz as von Koch's successor. The move failed and the call went to Carleman. Shortly afterwards, Riesz got a position in Lund. At least in the beginning, he must have felt his stay in Lund as an exile. He had been very successful in Stockholm where, among others, Harald Cramér and Einar Hille had been his personal students.

Lund did not have much of a mathematical tradition but Riesz's arrival meant a change of atmosphere. He was now an international star, active with his own research and he also had the time and incentive to broaden his interests. Frostman's thesis was a success and there were others after him. Lars Hörmander was one of Riesz's last personal students in Sweden. Riesz's work on fractional potentials was the origin of the contributions from Lund to the theory of partial differential operators.

I met Marcel Riesz in 1937, my first year at the university. He then had a small circle of graduate students. Each one got personal attention. Riesz loved to talk about mathematics and he appreciated having listeners. He could go on for hours and when he was in good form, his grip on the listener never slackened. Riesz lived alone and these personal lectures took place sometimes in his home, sometimes in his favourite café and sometimes over the telphone. The subjects were mostly those that occupied him at the time, fractional potentials, relativity theory and the Clifford algebra. He was not inclined to give away thesis subjects, but if somebody had a promising idea he was a very stimulating partner and spared no effort. He worked constantly, often at late hours and periodically with great intensity. These habits did not change much with advancing age and eventually took their toll. After about ten years in the United States he had a breakdown that forced him to return to Sweden and begin a more sedate life.

Although Riesz's interest in administrative matters was strictly limited to those concerning his own subject, he had great influence in the faculty merely by his good sense, his wit and his charm. He was a shrewd judge of personalities and human affairs and had an uncanny knack for the right word at the right moment. His memory was extraordinary and he was a fascinating teller of stories. Unfortunately he never wrote his memoirs. He had a long and active life and bore the burden of his last illness with great courage.

Bibliography

The works of Marcel Riesz in chronological order.

[1]. Representation of the analytical continuation of a given powerseries I, II (Hungarian). *Math. és Phys. Lapok*, XVI (1906), 1–25, 96–108.

[2]. *Summierbare trigonometrische Reihen und Potenzreihen* (Hungarian). Thesis, Budapest, 1908.

[3]. The summability of a power series on the boundary of the circle of convergence (Hungarian). *Hung. Acad. Sc. Math. és Természettud. értisitö* (1908).

[4]. Sur les séries trigonometriques. *C. R. Paris*, 145 (1907).

[5]. Sur les séries de Dirichlet. *Ibid.*, 148 (1908).

[6]. Sur la sommation des séries de Dirichlet. *Ibid.*, 149 (1909).

[7]. Sur les séries de Dirichlet et les séries entières. *Ibid.*, 149 (1909).

[8]. Sur un problème d'Abel. *Rend. Circ. Mat. Pal.*, 30 (1910), 339–348.

[9]. Une méthode de sommation équivalente à la méthode des moyennes arithmétiques. *C. R. Paris*, 152 (1911).

[10]. Über einen Satz des Herrn Fatou. *J. f. Math.*, 140 (1911), 89–99.

[11]. Über summierbare trigonometrische Reihen. *Math. Ann.* 71 (1911) 54–75.

[12]. Sur la représentation analytique des fonctions définies par des séries de Dirichlet. *Acta Math.*, 35 (1911), 253–270.

[13]. Formule d'interpolation pour la dérivée d'un polynome trigonométrique. *C. R. Paris*, 158 (1914).

[14]. Eine trigonometrische Interpolationsformel und einige Ungleichungen für Polynome. *Jahresber. Deutsch. Math. Ver.*, 23 (1914), 253–270.

[15]. A survey of the theory of trigonometric series (Swedish). *3 Skand. Mat. Kongr.* 1913. Printed 1915, 107–127.

[16]. with G. H. Hardy. *The General Theory of Dirichlet's Series.* Camb. Tracts in Math., 18 (1915), 1–78. Reprinted 1952.

[17]. Neuer Beweis des Fatouschen Satzes. *Gött. Nachr.*, (1916). 4 pp.

[18]. Sätze über Potenzreihen. *Ark. Mat. Fys. Astr.*, 11, no. 12 (1916), 1–16.

[19]. Sur l'hypothèse de Riemann. *Acta Math.*, 40 (1916), 185–190.

[20]. Über einen Satz des Herrn Serge Bernstein. *Acta Math.*, 40 (1916), 337–347.

[21]. Ein Konvergenzsatz für Dirichletsche Reihen. *Acta Math.*, 40 (1916), 349–361.

[22]. with F. Riesz. Über die Randwerte einer analytischen Funktion. *4. congr. des math. scand.* Stockholm 1916 (1920), 27–44.

[23]. Sur le principe de Phragmén–Lindelöf. *Proc. Camb. Phil. Soc.*, 20 (1920), 205–207, 21 (1921), 6.

[24]. Sur le problème des moments. Première Note. *Ark. Mat. Fys. Astr.*, 16, no. 12 (1921), 1–23. Deuxième Note, *ibid.*, 16, no. 19 (1922) 1–21. Troisième Note, *ibid.*, 17, no. 16 (1923), 1–52.

[25]. Sur la sommation des séries de Fourier. *Acta Szeged sect. math.*, 1 (1923), 104–113.

[26]. Sur un théorème de la moyenne et ses applications. *Ibid.*, 1 (1923), 114–126.

[27]. Sur le problème des moments et le théorème de Parseval correspondent. *Ibid.*, 1 (1923), 209–225.

[28]. Sur l'équivalence de certaines méthodes de sommation. *Proc. London Math. Soc.*, (2) 22 (1924), 51–64.

[29]. Sur les fonctions conjuguées et les séries de Fourier. *C. R. Paris*, 178 (1924).

[30]. Über die Summierbarkeit durch typische Mittel. *Acta Szeged sect. math.*, 2 (1924), 51–64.

[31]. with E. Hilb. Neuere Untersuchungen über trigonometrische Reihen. *Enc. d. Math. Wiss.*, II C 10 (1924), 1189–1228.

[32]. Sur les maxima des formes bilinéaires et sur les fonctionelles linéaires. *Acta Math.*, 49 (1927), 465–497.

[33]. Sur les fonctions conjuguées. *Math. Zeitschrift*, 27 (1927), 218–244.

[34]. Sur certaines inégalités dans la théorie des fonctions avec quelques remarques sur les géométries non-euclidiennes. *Proc. Roy. Physiogr. Soc. Lund*, 1 (1931), 1–21.

[35]. Sur les ensembles compacts de fonctions sommables. *Acta Szeged sect. math.*, 6, 2–3 (1933), 136–142.

[36]. Zum Eindeutigkeitssatz der fastperiodischen Funktionen. *Comm. sém. math. Lund*, 1 (1933), 1–9.

[37]. Eine Bemerkung über den Eindeutigkeitssatz der Theorie der fastperiodischen Funktionen. *Mat. Tidskr. B*, 1 (1934), 11–13.

[38]. Intégrale de Riemann–Liouville et solution invariantive du problème de Cauchy pour l'équation des ondes. *C. R. Congres Int. des math. Oslo* (1937).

[39]. Modules réciproques. *Ibid.*

[40]. Potentiels de divers ordres et leurs fonctions de Green. *Ibid.*

[41]. Volumes mixtes et facteurs invariants dans la théorie des modules. *Ibid.*

[42]. Intégrales de Riemann–Liouville et potentiels. *Acta Szeged sect. math.*, 9 (1937–38), 1–42. Rectification ..., *ibid.*, 116–118.

[43]. L'intégrale de Riemann-Liouville et le problème de Cauchy pour l'équation des ondes. *Soc. math. de France. Conf. à la réunion int. des math.* Paris, Juillet, 1937, 1–18.

[44]. An intuitive picture of non-euclidean geometry. Geometric excursions into relativity theory (Swedish). *Acta Physiogr. Soc. Lund.*, NF 2, vol. 38, no. 9 (1943), 1–14.

[45]. Sur certaines notions fondamentales en théorie quantique relativiste. *10. congres de math. scand.* Copenhagen, 1946.

[46]. Eléments de probabilité en théorie quantique relativiste. *Försäkr. mat. studier till. Filip Lundberg.* Stockholm, 1946.

[47]. L'intégrale de Riemann–Liouville et le problème de Cauchy. *Acta Math.*, 81 (1949), 1–223.

[48]. Remarque sur les fonctions holomorphes. *Acta Szeged. sect. math.*, 12 (1950), 53–56.

[49]. Sur le potentiel de Liénard–Wiechert attaché à une ligne d'univers. *C. R. Paris*, 234 (1952), 2159–2161.

[50]. Sur le potentiel retardé attaché à un courant continu. *Ibid.*, 2260–2261.

[51]. Court exposé des propriétés principales de la mesure de Lebesgue. *Ann. soc. pol. math.*, 25 (1952), 298–308.

[52]. Sur le lemme de Zolotareff et sur la loi de réciprocité des restes quadratiques. *Math. Scand.*, 1 (1952), 156–169.

[53]. L'équation de Dirac en relativité générale. *12. congr. des math. scand.* Lund, 1954, 241–259.

[54]. with A. E. LIVINGSTONE. A short proof of a classical theorem in the theory of Fourier integrals. *Amer. Math. Monthly*, 62 (1955), 434–437.

[55]. Problems related to characteristic surfaces. *Proc. conf. diff. eq. ded. to A. Weinstein.* Univ. of Maryland (1956), 57–71.

[56]. *A special characteristic surface—a new relativistic model for a particle*? Fund. Res. in Appl. Math. Dept. Math. Univ. of Maryland. Interim technical report 25 (1957).

[57]. *Clifford numbers and spiners, ch. 1–4.* Lecture notes, Inst. Fluid Dyn. Appl. Math. Univ. of Maryland, 1957–58.

[58]. A geometric solution of the wave equation in space-time of even dimension. *Comm. Pure Appl. Math.*, 13 (1960), 329–351.

[59]. The analytical continuation of the Riemann–Liouville integral in the hyperbolic case. *Canad. J. Math.*, 13 (1961), 37–48.