REMOVABLE SINGULARITIES OF SOLUTIONS OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS

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1. Introduction

Suppose $P(x, D)$ is a linear partial differential operator on an open set Ω contained in $\mathbf{R}^n$ and that A is a closed subset of Ω. Given a class $\mathfrak{F}(\Omega)$ of distributions on Ω, the set A is said to be *removable* for $\mathfrak{F}(\Omega)$ if each $f \in \mathfrak{F}(\Omega)$, which satisfies $P(x, D)f = 0$ in $\Omega - A$, also satisfies $P(x, D)f = 0$ in Ω. The problem considered in this paper is the following. Given a class $\mathfrak{F}(\Omega)$ of distributions on Ω, what restriction on the size of A will ensure that A is removable for $\mathfrak{F}(\Omega)$. We obtain results for $L^p_{\mathrm{loc}}(\Omega)$ $(p \leqslant \infty)$, $C(\Omega)$, and $\mathrm{Lip}_\delta(\Omega)$.

The first result of this kind was the Riemann removable singularity theorem: if a function f is holomorphic in the punctured unit disk and $f(z) = o(|z|^{-1})$ as z approaches zero, then f is holomorphic in the whole disk. Bochner [1] generalized Riemann's result by considering the class $\mathfrak{F}(\Omega)$ of functions f on Ω such that $f(x) = o(d(x, A)^{-q})$ uniformly for x in compact subsets of Ω, and giving a condition on the size of A which insures that A is removable for $\mathfrak{F}(\Omega)$ (Theorem 2.5 below). Bochner's theorem is remarkable in that the condition on the size of A only depends on the order of the operator $P(x, D)$. The theorem applies, therefore, to systems of differential operators, such as exterior differentiation in $\mathbf{R}^n$ and $\bar{\partial}$ (the Cauchy–Riemann operator) in $\mathbf{C}^n$. The same can be said for the other results in this paper. The proof of Bochner's theorem provided the motivation for our results.

It is interesting to note that a very general result (Corollary 2.4) for $L^p_{\mathrm{loc}}(\Omega)$ (due to Littman [7]) is an easy corollary of Bochner's work. Here the condition on the singular set A is expressed in terms of Minkowski content.

In section 4 the case of $L^p_{\mathrm{loc}}(\Omega)$ is studied again, and results in section 2 are improved by replacing Minkowski content with Hausdorff measure. In addition, the cases $C(\Omega)$

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and $\mathrm{Lip}_\delta(\Omega)$ are considered. The facts about Hausdorff measure needed in section 4 are developed in section 3. There are two lemmas here which may have independent interest.

A problem of a slightly different kind is considered in section 5. Here the singular set A is a hyper-surface in Ω. Instead of restricting the growth of the function f near A, a better result (Theorem 5.2) is obtained by requiring the jump across A (in a weak sense) of f, and some normal derivatives of f, to be zero. As an illustrative example we obtain a new proof of a strong form of the classical Schwarz reflection principle.

In section 6 we are concerned with the case where A is a d-dimensional smooth submanifold of Ω. First (Theorem 6.1) we examine a generalization of the question of removable singularities: given $f \in L^p_{\mathrm{loc}}(\Omega)$ which satisfies $Pf=0$ in $\Omega - A$, what restrictions does this place on the distribution Pf supported on A? Theorem 6.1 (a) provides a new proof of part (a) of Theorem 4.1 (for A smooth).

Bochner's Theorem 2.5 is not sharp for the Laplacian in $\mathbf{R}^2$ and $A=\{0\}$, or more generally for elliptic operators whose order is the same as the codimension of A. By utilizing the theory of pseudo-differential operators we obtain (in section 6) sharp results for these cases.

Throughout the paper Ω will denote an open subset of n-dimensional euclidean space $\mathbf{R}^n$ and A will denote a relatively closed subset of Ω. The linear differential operator $P(x, D)=\Sigma\, a_\alpha(x)\, D^\alpha$ will be assumed to have matrix coefficients $a_\alpha \in C^\infty(\Omega)$. The reader will note that this is unnecessarily restrictive for most of the results. Here $\alpha=(\alpha_1, \ldots, \alpha_n)$ is a multi-index, $|\alpha|=\alpha_1+\ldots+\alpha_n$, and $D^\alpha = D_1^{\alpha_1} \ldots D_n^{\alpha_n}$ where $D_j=(1/i)\partial/\partial x_j$. The formal adjoint of $P(x, D)$ is the operator defined by ${}^tP(x, D)\,\varphi=\Sigma_{|\alpha|\leqslant m}(-1)^{|\alpha|}D^\alpha(a_\alpha\varphi)$.

We will let $d(x, B)$ denote the euclidean distance from the point x to the set $B\subset\mathbf{R}^n$. Then $B_\varepsilon=\{x\in\mathbf{R}^n\colon d(x, B)<\varepsilon\}$ is the ε-neighborhood of B. Let χ_B denote the characteristic function of the set B and $\lambda(B)$ the Lebesgue measure of B. For $1\leqslant p\leqslant\infty$, p' is defined by $(1/p)+(1/p')=1$. For $f\in\mathcal{D}'(\Omega)$, supp f will denote the support of f. The pairing between $\mathcal{D}'(\Omega)$ and $C_0^\infty(\Omega)$ will be denoted by $(f, \varphi)=f(\varphi)$.

2. Two theorems of Bochner

Bochner's basic theorem is (see [1]):

THEOREM 2.1. *Suppose $f\in L^1_{\mathrm{loc}}(\Omega)$ satisfies $P(x, D)f=0$ in $\Omega-A$. If*

$$\liminf_{\varepsilon\to 0+}\varepsilon^{-m}\|\chi_{K_\varepsilon}f\|_1=0$$

for each compact set $K\subset A$ then $P(x, D)f=0$ in Ω.

The proof depends upon the following lemma, also due to Bochner [1].

LEMMA 2.2. *Suppose $K \subset \mathbf{R}^n$ is compact. Then for every $\varepsilon > 0$, there is a $\varphi_\varepsilon \in C_0^\infty(\mathbf{R}^n)$ with $\varphi_\varepsilon \equiv 1$ in a neighborhood of K and supp $\varphi_\varepsilon \subset K_\varepsilon$ such that $|D^\alpha \varphi_\varepsilon(x)| \leqslant C_\alpha \varepsilon^{-|\alpha|}$ for all x where C_α is independent of ε.*

Proof. Let
$$\varphi_\varepsilon(x) = \varepsilon^{-n} \int \chi_{K_{\varepsilon/2}}(y)\, \psi\left(\frac{x-y}{\varepsilon}\right) dy,$$

where $\psi \in C_0^\infty$ has its support contained in $\{x: |x| \leqslant \frac{1}{3}\}$ and satisfies $\int \psi(x)\,dx = 1$. Then $\varphi_\varepsilon(x) = 1$ if $x \in K_{\varepsilon/6}$, supp $\varphi_\varepsilon \subset K_{\varepsilon/2} + \{x: |x| \leqslant \varepsilon/3\} \subset K_\varepsilon$ and $D^\alpha \varphi_\varepsilon(x) = \varepsilon^{-n-|\alpha|} \int \chi_{K_{\varepsilon/2}}(y)$ $D^\alpha \psi((x-y)/\varepsilon)\,dy$. Hence $|D^\alpha \varphi_\varepsilon(x)| \leqslant C_\alpha \varepsilon^{-|\alpha|}$ with $C_\alpha = \|D^\alpha \psi\|_1$.

Proof of Theorem 2.1. Suppose $\varphi \in C_0^\infty(\Omega)$ and let $K = (\operatorname{supp} \varphi) \cap A$. Since supp $P(x, D) f \subset A$, we have with the φ_ε of the lemma, $(P(x, D) f, \varphi) = (P(x, D) f, \varphi_\varepsilon \varphi) = (f, {}^tP(x, D)(\varphi_\varepsilon \varphi))$. By the above lemma $\|{}^tP(x, D)(\varphi_\varepsilon \varphi)\|_\infty \leqslant C\varepsilon^{-m}$, and hence $|(P(x, D) f, \varphi)| \leqslant C\varepsilon^{-m} \|\chi_{K_\varepsilon} f\|_1$ for all $\varepsilon > 0$, which implies $(P(x, D) f, \varphi) = 0$.

Before proceeding to Bochner's second theorem we define three set functions. Let d be a non-negative real number. The d-dimensional *lower Minkowski content* of a bounded set A is defined by
$$M_d(A) = \liminf_{\varepsilon \to 0+} \varepsilon^{d-n} \lambda(A_\varepsilon).$$

The *upper Minkowski content* of A, $M^d(A)$, is defined similarly using lim sup. For each $\varepsilon > 0$, let $\Lambda_d^\varepsilon(A) = \inf \{\sum_{i=1}^\infty r_i^d\}$, where the infimum is over all coverings of A by countable collections of balls $\{S_i\}$, where each ball S_i has radius $r_i \leqslant \varepsilon$. The d-dimensional *Hausdorff measure* of A, denoted $\Lambda_d(A)$, is $\lim_{\varepsilon \to 0} \Lambda_d^\varepsilon(A)$. Hausdorff measure is a regular metric outer measure and hence $\Lambda_d(A) = 0$ if and only if $\Lambda_d(K) = 0$ for all compact subsets K of A. In general, $c_d \Lambda_d(A) \leqslant M_d(A) \leqslant M^d(A)$, where c_d is a constant depending only on d. There are examples ([6] and [4]) to show that the reverse inequalities are not true in general. However, $c_d \Lambda_d$, M_d, and M^d all agree with d-dimensional Lebesgue measure on compact subsets of a d-dimensional smooth submanifold of $\mathbf{R}^n$.

It is interesting to note that some very general results are easy corollaries of Bochner's theorem. Since by Hölder's inequality $\|\chi_{K_\varepsilon} f\|_1 \leqslant \lambda(K_\varepsilon)^{1/p'} \|\chi_{K_\varepsilon} f\|_p$, Theorem 2.1 gives the following corollary.

COROLLARY 2.3. *Let $1 \leqslant p \leqslant \infty$. Suppose $f \in L^p_{\text{loc}}(\Omega)$ and $P(x, D) f = 0$ in $\Omega - A$. If $\lim_{\varepsilon \to 0+} \inf\, [\varepsilon^{d-n} \lambda(K_\varepsilon)]^{1/p'} \|\chi_{K_\varepsilon} f\|_p = 0$, (where $d = n - mp'$) for each compact set $K \subset A$, then $P(x, D) f = 0$ in Ω.*

As an immediate consequence of Corollary 2.3, we have the following result of Littman [7] (Littman used a different set function, which however is comparable to lower Minkowski content).

COROLLARY 2.4. (a) ($p<\infty$). *Suppose $M_{n-mp'}(K)<\infty$ for all compact sets $K\subset A$. Then each $f\in L^p_{loc}(\Omega)$ which satisfies $P(x, D)f=0$ in $\Omega-A$ also satisfies $P(x, D)f=0$ in Ω.*

(b) ($p=\infty$). *Suppose $M_{n-m}(K)=0$ for all compact sets $K\subset A$. Then each $f\in L^\infty_{loc}(\Omega)$ which satisfies $P(x, D)f=0$ in $\Omega-A$ also satisfies $P(x, D)f=0$ in Ω.*

Bochner's second theorem generalizes the classical Riemann removable singularity theorem.

THEOREM 2.5. *Suppose $M^{n-m-q}(K)<\infty$ for all compact sets $K\subset A$. If $f\in L^1_{loc}\Omega)$ satisfies $f(x)=o(d(x, A)^{-q})$ uniformly for x in compact subsets of Ω and $P(x, D)f=0$ in $\Omega-A$, then $P(x, D)f=0$ in Ω.*

Proof. If $q\leqslant 0$, the theorem is a trivial consequence of Theorem 2.1, so suppose $q>0$. Let $K\subset A$ be compact. The hypothesis implies that

$$\|\chi_{K_\varepsilon} d(x, A)^{-q}\|_1 \leqslant C\varepsilon^m. \tag{2.1}$$

Let $K_j=\{x\in K_\varepsilon \,|\, d(x,A)<\varepsilon 2^{-j}\}$. Then $\int_{K_\varepsilon} d(x,A)^{-q}dx \leqslant \sum_{j=0}^\infty \int_{K_j-K_{j+1}} (\varepsilon/2^{j+1})^{-q}dx \leqslant \sum_{j=0}^\infty (\varepsilon/2^{j+1})^{-q}\lambda(K_j)$. By hypothesis, there is a constant c such that $\lambda(K_j)\leqslant c(\varepsilon 2^{-j})^{m+q}$ for all j. Therefore the above sum is less than $(c\sum_{j=0}^\infty (\frac{1}{2})^{-q+mj})\,\varepsilon^m$ which proves (2.1).

For each $\varepsilon>0$ there is a constant c_ε such that $|\chi_{K_\varepsilon}(x)f(x)|\leqslant c_\varepsilon \chi_{K\varepsilon} d(x, A)^{-q}$ with $c_\varepsilon\to 0$ as $\varepsilon\to 0$. Therefore $\|\chi_{K_\varepsilon} f\|_1\leqslant Cc_\varepsilon\varepsilon^m$ by (2.1), which implies that $P(x, D)f=0$ in Ω by Theorem 2.1.

It is possible that upper Minkowski content could be replaced in Theorem 2.5 by lower Minkowski content or Hausdorff measure. However, "o" cannot be replaced by "O", at least for A a d-dimensional linear subspace of $\mathbf{R}^n$. First, assume that $d=0$. If $n>2$, consider the fundamental solution $c_n|x|^{2-n}$ of the Laplacian. If $n=2$, consider the fundamental solution $E(x, t)$ of the wave equation. ($E(x, t)$ is the characteristic function of the positive light cone $\{(x, t)\colon t^2-x^2>0 \text{ and } t>0\}$.) If $n=1$, consider the fundamental solution $\frac{1}{2}|x|$ of $d^2/(dx^2)$. Examples for arbitrary integral d can be obtained by tensoring the above examples with the identity on $\mathbf{R}^d$ (i.e., consider the function defined above on $\mathbf{R}^{n-d}$ as a function on $\mathbf{R}^n$ independent of the last d variables).

If $n-d\geqslant 3$, then Corollary 2.3 is sharp for the Laplacian. Consider the function $\int_{\mathbf{R}^d}(|x'|^2+|x''-y''|^2)^{(2-n)/2}dy''$. However, if $n-d=2$, Theorem 2.5 is not sharp for the Laplacian. Sharp results for this case are included in the results of section 6.

3. Fundamental lemmas

This section contains two lemmas which extend Lemma 2.2.

Before proceeding with Lemma 3.1 we take a closer look at Hausdorff measure. For each integer k, there is a space filling collection of closed cubes of length 2^{-k}, the vertices of which have coordinates of the form $p2^{-k}$ where p is an integer. Such a cube will be called a dyadic cube of length 2^{-k}. Two dyadic cubes will be called *disjoint* if their intersection has no interior. Let $A\subset \mathbf{R}^n$, and for each $\varepsilon>0$ define $L_d^\varepsilon(A)=\inf \sum_{i=1}^\infty s_i^d$ where the infimum is over all coverings of A by countable collections of dyadic cubes $\{Q_i\}$ with length $s_i\leqslant\varepsilon$. Define $L_d(A)=\lim_{\varepsilon\to 0}L_d^\varepsilon(A)$. Note that the collections $\{Q_i\}$ may as well be taken to be disjoint, since if the intersection of two dyadic cubes has non-empty interior, one is contained in the other.

Each dyadic cube of length s is contained in a ball of radius $(\sqrt{n}/2)s$. Similarly, each ball of radius r can be covered by 3^n dyadic cubes of length 2^{-k}, where $2^{-k-1}<r\leqslant 2^{-k}$. These facts can be used to show that there are constants c_d and C_d such that $c_dL_d(A)\leqslant\Lambda_d(A)\leqslant C_dL_d(A)$ for any set $A\subset\mathbf{R}^n$. Thus L_d is comparable with Hausdorff measure. L_d is much easier to work with as the following lemma shows.

For a cube Q of length s, we let $\frac{3}{2}Q$ denote the cube with the same center and length $3s/2$.

LEMMA 3.1. *Let $\{Q_i \mid 1\leqslant i\leqslant N\}$ be a finite disjoint collection of dyadic cubes of length s_i. For each i, there is a function $\varphi_i\in C_0^\infty(\mathbf{R}^n)$ with supp $\varphi_i\subset\frac{3}{2}Q_i$ such that $\sum_{i=1}^N\varphi_i(x)=1$ for all $x\in\bigcup_{i=1}^N Q_i$. Furthermore, for each multi-index α, there is a constant C_α, depending only on α, for which $|D^\alpha\varphi_i(x)|\leqslant C_\alpha s_i^{-|\alpha|}$ for all x and $1\leqslant i\leqslant N$.*

Proof. In the proof we will use $C_{\beta^1,\dots,\beta^r}$ to indicate a constant depending only on the one or several multi-indices used as subscripts. It need not be the same constant in each application.

Assume $s_1\geqslant s_2\geqslant\dots\geqslant s_N$. Choose $\psi\in C_0^\infty(\mathbf{R}^n)$ such that $\psi(x)\equiv 1$ if $|x_i|\leqslant 1$ for $1\leqslant i\leqslant n$ and $\psi(x)=0$ if $|x_i|\geqslant 3/2$ for some i. Let $\psi_k(x)=\psi(2(x-x_k)/s_k)$, where x_k is the center of the cube Q_k. Define for $1\leqslant k\leqslant N$, $\varphi_1=\psi_1$, $\varphi_{k+1}=\psi_{k+1}\prod_{j=1}^k(1-\psi_j)$. Then $\varphi_k\in C_0^\infty(\mathbf{R}^n)$ and supp $\varphi_k\subset$ supp $\psi_k\subset\frac{3}{2}Q_k$. An easy inductive proof shows that $\sum_{j=1}^k\varphi_j=1-\prod_{j=1}^k(1-\psi_j)$ for $k=1, 2, \dots, N$, and hence $\sum_{j=1}^N\varphi_j(x)=1$ if $x\in\bigcup_{j=1}^N Q_j$.

It remains to prove the estimate on the derivatives of φ_j. Let $\theta_k=\sum_{j=1}^k\varphi_j=1-\prod_{j=1}^k(1-\psi_j)$. Since $s_k\geqslant s_{k+1}$, it suffices to prove the estimate for θ_k. For integers $\nu_1, \dots, \nu_r$ where $1\leqslant\nu_i\leqslant k$, define

$$g_{\nu_1,\ldots,\nu_r}=\begin{cases}0 & \text{if } \nu_i=\nu_j \text{ for some } i\neq j\\ \prod\limits_{\substack{i\neq\nu_1,\ldots,\nu_r\\ i\leqslant k}}(1-\psi_i) & \text{if all } \nu_i \text{ are distinct.}\end{cases}$$

Then there are constants $C_{\beta^1,\ldots,\beta^r}$ depending only on the multi-index subscripts, such that

$$D^\alpha\theta_k=\sum C_{\beta^1,\ldots,\beta^r}\Big(\sum_{\nu_1,\ldots,\nu_r=1}^{k} g_{\nu_1,\ldots,\nu_r}(D^{\beta^1}\psi_{\nu_1})(D^{\beta^2}\psi_{\nu_2}),\ldots,(D^{\beta^r}\psi_{\nu_r})\Big),$$

where the sum is over all sets of multi-indices $\{\beta^1,\ldots,\beta^r\}$ for which $|\beta^i|\geqslant 1$ and $\beta^1+\ldots+\beta^r=\alpha$. Therefore

$$|D^\alpha\theta_k(x)|\leqslant\sum C_{\beta^1,\ldots,\beta^r}\Big(\sum_{\nu_1=1}^{k}|D^{\beta^1}\psi_{\nu_1}(x)|\Big)\ldots\Big(\sum_{\nu_r=1}^{k}|D^{\beta^r}\psi_{\nu_r}(x)|\Big).$$

Consider a typical sum: $\sum_{\nu=1}^{k}|D^\beta\psi_\nu(x)|$. Note that $D^\beta\psi_\nu(x)=0$ unless $x\in\frac{3}{2}Q_\nu$. Furthermore, if $x\in\frac{3}{2}Q_\nu$, $|D^\beta\psi_\nu(x)|\leqslant C_\beta s_\nu^{-|\beta|}$. Therefore $\sum_{\nu=1}^{k}|D^\beta\psi_\nu(x)|\leqslant C_\beta\sum s_\nu^{-|\beta|}$ where the last sum is over those cubes Q_ν, with length $s_\nu\geqslant s_k$, for which $x\in\frac{3}{2}Q_\nu$. It is easily seen that for each non-negative integer p there are at most 2^n dyadic cubes Q_ν of length $s_\nu=s_k2^p$, for which $x\in\frac{3}{2}Q_\nu$. Hence

$$\sum_{\nu=1}^{k}|D^\beta\psi_\nu(x)|\leqslant 2^nC_\beta\sum_{p=0}^{\infty}(s_k2^p)^{-|\beta|}\leqslant C_\beta s_k^{-|\beta|}.$$

Therefore we have

$$|D^\alpha\theta_k(x)|\leqslant\sum C_{\beta^1,\ldots,\beta^r}(C_{\beta^1}s_k^{-|\beta^1|}),\ldots,(C_{\beta^r}s_k^{-|\beta^r|})\leqslant C_\alpha s_k^{-|\alpha|}. \tag{3.1}$$

LEMMA 3.2. *Suppose $K\subset\mathbf{R}^n$ is compact. Given $d=n-mp'$ and $\varepsilon>0$, there is a $\varphi_\varepsilon\in C_0^\infty(\mathbf{R}^n)$ with $\varphi_\varepsilon\equiv 1$ in a neighborhood of K and supp $\varphi_\varepsilon\subset K_\varepsilon$, such that for $|\alpha|\leqslant m$, $\|D^\alpha\varphi_\varepsilon\|_{p'}\leqslant C_\alpha\varepsilon^{m-|\alpha|}(\Lambda_{n-mp'}(K)+\varepsilon)^{1/p'}$, where C_α is independent of ε.*

Proof. For each $\varepsilon>0$, choose a covering of K by a finite collection $\{Q_k\}$ of dyadic cubes of length $s_k\leqslant\varepsilon$, with $\bigcup\frac{3}{2}Q_k\subset K_\varepsilon$, and $\sum s_k^{n-mp'}\leqslant L_{n-mp'}(K)+\varepsilon$. We may assume $s_1\geqslant s_2\geqslant\ldots\geqslant s_N$. Let $\{\varphi_k\}$ be the partition of unity for $\{Q_k\}$ constructed in Lemma 3.1 and define $\varphi_\varepsilon=\Sigma\varphi_k$. Then supp $\varphi_\varepsilon\subset\bigcup\frac{3}{2}Q_k\subset K_\varepsilon$ and $\varphi(x)\equiv 1$ on $\bigcup Q_k$.

For $k=1,2,\ldots,N$ let $T_k=\frac{3}{2}Q_k-\bigcup_{j>k}\frac{3}{2}Q_j$. Then $\{T_k\}$ is a disjoint collection of sets with $\bigcup T_k=\bigcup\frac{3}{2}Q_k$, $T_k\subset\frac{3}{2}Q_k$ for all k, and $\varphi_j(x)=0$ if $j>k$ and $x\in T_k$. Therefore if $x\in T_k$, $\varphi_\varepsilon(x)=\sum_{j=1}^{k}\varphi_j(x)=\theta_k(x)$. By (3.1) $|D^\alpha\varphi_\varepsilon(x)|\leqslant C_\alpha s_k^{-|\alpha|}$ for all $x\in T_k$, where C_α is a constant not depending on ε or k. Hence

$$\|D^\alpha\varphi_\varepsilon\|_{p'}^{p'}=\sum\int_{T_k}|D^\alpha\varphi_\varepsilon(x)|^{p'}dx\leqslant C_\alpha\sum s_k^{-|\alpha|p'+n}$$
$$\leqslant C_\alpha\varepsilon^{(m-|\alpha|)p'}\sum s_k^{n-mp'}\leqslant C_\alpha\varepsilon^{(m-|\alpha|)p'}(L_{n-mp'}(K)+\varepsilon).$$

4. Results for $L^p_{\text{loc}}(\Omega)$, $C(\Omega)$ and $Lip_\delta(\Omega)$

Our first task in this section is to improve Corollary 2.4 by replacing lower Minkowski content with Hausdorff measure.

THEOREM 4.1. (a) $(p<\infty)$. *Suppose* $\Lambda_{n-mp'}(K)<\infty$ *for each compact set* $K\subset A$. *Then each* $f\in L^p_{\text{loc}}(\Omega)$ *which satisfies* $P(x,D)f=0$ in $\Omega-A$, *also satisfies* $P(x,D)f=0$ *in* Ω.

(*b*) $(p=\infty)$. *Suppose* $\Lambda_{n-m}(A)=0$. *Then each* $f\in L^\infty_{\text{loc}}(\Omega)$ *which satisfies* $P(x,D)f=0$ *in* $\Omega-A$ *also satisfies* $P(x,D)f=0$ *in* Ω.

Carleson [3] obtained part (a) for the Laplacian in $\mathbf{R}^n$ and A compact. Later Serrin [9] extended Carleson's work to elliptic operators of second order with Hölder continuous coefficients.

Proof of Theorem 4.1. Suppose $\varphi\in C_0^\infty(\Omega)$ and let $K=A\cap\operatorname{supp}\varphi$. Since $\operatorname{supp}(P(x,D)f)\subset A$, we have with the φ_ε of Lemma 3.2, $(Pf,\varphi)=(Pf,\varphi_\varepsilon\varphi)=(f,{}^tP(\varphi_\varepsilon\varphi))$. By Hölder's inequality and Lemma 3.2,

$$|(Pf,\varphi)|\leqslant\|\chi_{K_\varepsilon}f\|_p\,\|{}^tP(\varphi_\varepsilon\varphi)\|_{p'}\leqslant C\,\|\chi_{K_\varepsilon}f\|_p(\Lambda_{n-mp'}(K)+\varepsilon)^{1/p'}. \tag{4.1}$$

Parts (a) and (b) follow immediately.

Remark. If $\inf\{\|\psi\|_{p',m}:\psi\in C_0^\infty(\Omega),\ \psi\equiv 1$ in a neighborhood of $K\}=0$ then K is removable for $L^p_{\text{loc}}(\Omega)$ $(p\leqslant\infty)$. For $p<\infty$, this result is due to Littman [7]. It follows immediately from (4.1), with χ_{K_ε} replaced by 1, and φ_ε replaced by a suitably chosen ψ.

For fixed $d=n-mp'$, Theorem 4.1(a) says that A is removable for L^p_{loc} if $p\geqslant(n-d)/(n-d-m)$. At least for linear subspaces A of dimension d, the allowable range of p cannot be improved. First assume $d=0$. For $n=2$, an example is provided by $1/\pi z$, the fundamental solution of the Cauchy–Riemann equations, which belongs to L^p_{loc} for $p<2$. If $n\geqslant 3$, then $c_n|x|^{2-n}$, the fundamental solution of the Laplacian, belongs to L^p_{loc} for $p<n/(n-2)$. To get examples for more general d, it is only necessary to tensor this example with the identity on $\mathbf{R}^d$. If $(n-d)\geqslant 3$, the allowable range of p is sharp for the Laplacian since the function $\int_{\mathbf{R}^d}(|x'|^2+|x''-y''|^2)^{(2-n)/2}dy''$ belongs to L^p_{loc} for $p<(n-d)/(n-d-m)$.

In Theorem 4.1(b), it is not possible in general to replace the condition $\Lambda_{n-m}(A)=0$ by $\Lambda_{n-m}(A)<\infty$. Examples illustrating this are the fundamental solution of d/dx in $\mathbf{R}$ or the wave equation in $\mathbf{R}^2$, and the function $\sqrt{z(1-z)}$ for the Cauchy–Riemann operator in $\mathbf{C}$. These examples also illuminate Theorem 4.2(a) below.

Next we prove a result for $C(\Omega)$ and give a condition which implies Pf is a measure.

THEOREM 4.2. *Suppose* $\Lambda_{n-m}(K)<\infty$ *for each compact set* $K\subset A$.

(a) *If* $f \in L^{\infty}_{\text{loc}}(\Omega)$ *satisfies* $P(x, D)f = 0$ *in* $\Omega - A$, *then* $P(x, D)f$ *is a measure supported in* A.

(b) *If* $f \in C(\Omega)$ *satisfies* $P(x, D)f = 0$ *in* $\Omega - A$, *then* f *also satisfies* $P(x, D)f = 0$ *in* Ω.

Proof. Let $K \subset A$ be compact and suppose $\varphi \in C_0^\infty(\Omega)$ with $(\operatorname{supp} \varphi) \cap A \subset K$. Then with the φ_ε of Lemma 3.2, we have

$$(Pf, \varphi) = (Pf, \varphi_\varepsilon \varphi) = (f, {}^tP(\varphi_\varepsilon \varphi)) = (f\, {}^tP\varphi_\varepsilon, \varphi) + \sum_{|\beta| < m} (f, \psi_\beta D^\beta \varphi_\varepsilon),$$

with $\psi_\beta \in C_0^\infty(\Omega)$ depending on φ. By Lemma 3.2 we have

$$|(f, \psi_\beta D^\beta \varphi_\varepsilon)| \leqslant C\|f\|_\infty \|D^\beta \varphi_\varepsilon\| \leqslant C\varepsilon^{m-|\beta|}(\Lambda_{n-m}(K) + \varepsilon)^{1/p'}.$$

Therefore Pf is the weak limit in $\mathcal{D}'(\Omega)$ of the net of functions $\{f\,{}^tP\varphi_\varepsilon\}$.

By Lemma 3.2, $\|{}^tP\varphi_\varepsilon\|_1 \leqslant C(\Lambda_{n-m}(K) + \varepsilon)$. Since $\Lambda_{n-m}(K) < \infty$, this proves that $\|{}^tP\varphi_\varepsilon\|_1 \leqslant C$ independent of ε. If f is bounded, the net $\{f\,{}^tP\varphi_\varepsilon\}$ is bounded in $L^1(\Omega)$, and hence its weak limit in $\mathcal{D}'(\Omega)$ must be a measure. This proves (a).

Since the net $\{{}^tP\varphi_\varepsilon\}$ is bounded in $L^1(\Omega)$ it has a subsequence which converges weakly in $C(\Omega)'$. The limit of this subsequence must be zero, since the net $\{\varphi_\varepsilon\}$, and therefore the net $\{{}^tP\varphi_\varepsilon\}$ converges to zero in $\mathcal{D}'(\Omega)$. Now suppose $f \in C(\Omega)$. Then multiplication by f is continuous in $C(\Omega)'$. Hence a subsequence of the net $\{f\,{}^tP\varphi_\varepsilon\}$ converges weakly to zero in $C(\Omega)'$. Since the net itself converges weakly to Pf in $\mathcal{D}'(\Omega)$, $Pf = 0$.

For k a negative integer we make the following definitions: $f \in L^p_{k,\text{loc}}(\Omega)$ if for each set $\omega \subset\subset \Omega$ there are functions $g_\alpha \in L^p(\omega)$ such that $f = \sum_{|\alpha| \leqslant -k} D^\alpha g_\alpha$ in ω; $f \in C_1(\Omega)$ if for each set $\omega \subset\subset \Omega$ there are functions $g_\alpha \in C(\omega)$ such that $f = \sum_{|\alpha| \leqslant -k} D^\alpha g_\alpha$ in ω (this definition of $L^p_{k,\text{loc}}$ is standard for $1 < p < \infty$).

The following is an easy extension of Theorem 4.1 and Theorem 4.2. Let $k < m$ be an integer.

THEOREM 4.3. (a) $(p < \infty)$ *Suppose* $\Lambda_{n-(m-k)p'}(K) < \infty$ *for each compact subset* $K \subset A$. *Then each* $f \in L^p_{k,\text{loc}}(\Omega)$ *which satisfies* $P(x, D)f = 0$ *in* $\Omega - A$, *also satisfies* $P(x, D)f = 0$ *in* Ω.

(b) $(p = \infty)$ *Suppose* $\Lambda_{n-m+k}(A) = 0$. *Then each* $f \in L^\infty_{k,\text{loc}}(\Omega)$ *which satisfies* $P(x, D)f = 0$ *in* $\Omega - A$, *also satisfies* $P(x, D)f = 0$ *in* Ω.

(c) *Suppose* $\Lambda_{n-m+k}(K) < \infty$ *for each compact subset* $K \subset A$. *If* $f \in L^\infty_{k,\text{loc}}(\Omega)$ *satisfies* $P(x, D)f = 0$ *in* $\Omega - A$, *then* $P(x, D)f$ *is a measure supported in* A.

(d) *Suppose* $\Lambda_{n-m+k}(K) < \infty$ *for each compact subset* $K \subset A$. *Then each* $f \in C^k(\Omega)$ *which satisfies* $P(x, D)f = 0$ *in* $\Omega - A$, *also satisfies* $P(x, D)f = 0$ *in* Ω.

Proof. First assume that k is a positive integer. For suitable partial differential operators Q_α of order $\leqslant m-k$ we have $P(x, D)=\sum_{|\alpha|\leqslant k} Q_\alpha(x, D) D^\alpha$. Then $(Pf, \varphi)=(Pf, \varphi\varphi_\varepsilon)=\sum_{|\alpha|\leqslant k}(D^\alpha f, {}^tQ_\alpha(\varphi\varphi_\varepsilon))$. Now the proof proceeds analogously to the proofs of Theorems 4.1 and 4.2.

If k is a negative integer, then

$$(Pf, \varphi)=(Pf, \varphi\varphi_\varepsilon)=\sum_{|\alpha|\leqslant -k}(PD^\alpha g_\alpha, \varphi\varphi_\varepsilon)$$

and the rest of the proof is analogous to the proofs of Theorems 4.1 and 4.2.

Denote by $\text{Lip}_\delta(\Omega)$, $0<\delta<1$, the space of all functions f defined in Ω which satisfy a Hölder condition of order δ uniformly on compact subset of Ω, i.e., for each compact set $K\subset\Omega$, there is a constant C such that $|f(x)-f(y)|\leqslant C|x-y|^\delta$ for all x and y belonging to K.

THEOREM 4.4. *Suppose $\Lambda_{n-m+\delta}(A)=0$. Then each $f\in \text{Lip}_\delta(\Omega)$ which satisfies $P(x, D)f=0$ in $\Omega-A$, also satisfies $P(x, D)f=0$ in Ω.*

Remark. Carleson ([2] and [3]) has proved this result if $P(x, D)$ is the Laplacian and A is compact. In this case Carleson has also shown that the condition $\Lambda_{n-m+\delta}(A)=0$ is necessary.

Proof. Let $\varphi\in C_0^\infty(\Omega)$ and let $K=A\cap(\text{supp}\,\varphi)$. Let $\{Q_i\}$ be a finite disjoint collection of dyadic cubes which covers K. Let x_k be the center of Q_k and s_k the length. We assume $s_k\leqslant 1$. Let $\{\varphi_k\}$ be the partition of unity for $\{Q_k\}$ constructed in Lemma 3.1. Then $(Pf, \varphi)=\sum(Pf, \varphi_k\varphi)=\sum_k(f, {}^tP(\varphi_k\varphi))$. For each k we have $(f, {}^tP(\varphi_k\varphi))=\sum_{|\alpha|\leqslant m}(f, D^\alpha(a_\alpha\varphi_k\varphi))$. If $|\alpha|<m$, we have $|(f, D^\alpha(a_\alpha\varphi_k\varphi))|\leqslant\int_{3/2\,Q_k}|f(x)D^\alpha(a_\alpha\varphi_k\varphi)(x)|\,dx\leqslant C_\alpha s_k^{n-|\alpha|}\leqslant C_\alpha s_k^{n-m+\delta}$ where for the last inequality we use the assumption $s_k\leqslant 1$. For $|\alpha|=m$ we note that $\int D^\alpha(a_\alpha\varphi_k\varphi)\,dx=0$, so $|(f, D^\alpha(a_\alpha\varphi_k\varphi))|=|\int_{3/2Q_k}(f(x)-f(x_k))D^\alpha(a_\alpha\varphi_k\varphi)(x)\,dx|\leqslant C_\alpha s_k^{n-m+\delta}$ since $f\in\text{Lip}_\delta(\Omega)$. Thus for each k, $|(f, {}^tP(\varphi_k\varphi))|\leqslant Cs_k^{n-m+\delta}$ so $|(Pf, \varphi)|\leqslant C\sum s_k^{n-m+\delta}$. Since this is true for all coverings $\{Q_k\}$, and $\Lambda_{n-m+\delta}(K)=0$, we must have $(Pf, \varphi)=0$.

Remark. It should be pointed out that the proof is valid if $\delta=1$, and hence provides an alternate proof of Theorem 4.3 for $L^\infty_{1,\text{loc}}(\Omega)$.

Let $C^{k+\delta}(\Omega)$ (k an integer and $0<\delta<1$) denote the class of functions $f\in C^k(\Omega)$ such that $D^\alpha f\in\text{Lip}_\delta(\Omega)$ for $|\alpha|=k$. Theorem 4.4 has the following extension for $k<m$.

THEOREM 4.5. *Suppose $\Lambda_{n-m+k+\delta}(A)=0$. Then each $f\in C^{k+\delta}(\Omega)$, which satisfies $P(x, D)f=0$ in $\Omega-A$, also satisfies $P(x, D)f=0$ in Ω.*

Examples. 1. Theorem 4.1(b) yields as an easy corollary the following. Suppose Ω is a connected open subset of $\mathbf{R}^n$ and A is a closed subset of Ω. If $\Lambda_{n-1}(A)=0$, then $\Omega-A$ is connected. To see this, suppose $\Omega - A$ is not connected. Then there is a non-constant function f which is constant on components of $\Omega - A$. Since $df = \sum (\partial f/\partial x_i)\, dx_i = 0$ in $\Omega - A$, Theorem 4.1 (b) implies that $df = 0$ in Ω and hence f is a constant function on Ω.

2. Let Ω be a bounded open set in $\mathbf{R}^n$. If $\Lambda_{n-1}(\partial\Omega) < \infty$, then by Theorem 4.2 (a), $d\chi_\Omega$ is a measure. Similarly, if Ω is a bounded open subset of $\mathbf{C}^n$, with $\Lambda_{2n-1}(\partial\Omega) < \infty$, and f is a holomorphic function in a neighborhood of Ω, then $\bar\partial(f\chi_\Omega) = \Sigma f(\partial\chi_\Omega/\partial\bar z_i)\, d\bar z_i$ is a measure.

3. Let $A \subset \Omega \subset \mathbf{C}^n$ and suppose $\Lambda_{2n-1}(A) = 0$. Then by Theorem 4.1 (b) every locally bounded function, which is holomorphic in $\Omega - A$, is holomorphic in Ω.

If A is a sub-variety of Ω then $\Lambda_{2n-2}(K) < \infty$ for each compact subset K of A [4]. Hence by Theorem 4.1 (a), every function which is locally square integrable in Ω and holomorphic in Ω minus a proper sub-variety is holomorphic in Ω.

These statements provide two different improvements of the well-known result that a bounded function which is holomorphic outside a variety, extends to a holomorphic function across the variety.

4. The results of this section apply to such otherwise badly behaved operators as the Hans–Lewy example (i.e., the induced Cauchy–Riemann operator on $S^3 \subset \mathbf{C}^2$).

5. Removable singularities on hypersurfaces

In this section we provide two generalizations of the classical result that a continuous function on an open set Ω in the complex plane which is holomorphic in Ω off the real axis, is holomorphic in Ω. Notice that this statement follows from part (b) of Theorem 4.2. In fact, Theorem 4.3 part (d) provides us with our first generalization.

THEOREM 5.1. *Suppose* $\Lambda_{n-1}(K) < \infty$ *for each compact subset* $K \subset A$. *Then each* $f \in C^{m-1}(\Omega)$ *which satisfies* $P(x, D)f = 0$ *in* $\Omega - A$, *satisfies* $P(x, D)f = 0$ *in* Ω.

In Theorem 5.1, the set A is not required to have any smoothness, whereas the function f is assumed to be smooth across A. In the second generalization, A is an $n-1$ dimensional C^∞ submanifold of an open set in $\mathbf{R}^n$, which by a change of coordinates we can assume to be (locally) the hyperplane $\{x : x_n = 0\}$. Let $\mathbf{R}^{n-1}$ denote this hyperplane, let Ω denote an open set in $\mathbf{R}^n$, and let A denote $\mathbf{R}^{n-1} \cap \Omega$ as well as $\{x' : (x', 0) \in \Omega\}$. For convenience we make the assumption that $A \times [-a, a] \subset \Omega$ for some $a > 0$. Let $x' = (x_1, \ldots, x_{n-1})$ and $x = (x', x_n)$. As before, let $P(x, D) = \Sigma\, a_\alpha D^\alpha$ with each $a_\alpha \in C^\infty(\Omega)$. The *normal order* of $P(x, D)$ with respect to A is the largest α_n for which $a_\alpha \not\equiv 0$. Let χ_t^+ and χ_t^- denote the characteristic functions of the sets $\{x \in \Omega: x_n > t\}$ and $\{x \in \Omega: x_n < t\}$ respectively. Let δ_t denote the distribution defined by $\delta_t(\psi) = \int \psi(x', t)\,dx'$ for all $\psi \in C_0^\infty(\mathbf{R}^n)$.

THEOREM 5.2. *Suppose $P(x, D)$ has normal order $m \geqslant 1$ with respect to A. If $f \in C^\infty(\Omega - A)$ satisfies $P(x, D)f = 0$ in $\Omega - A$, and if, for $k = 0, \ldots, m-1$, both the limits $\lim_{\varepsilon\to 0^+} D_n^k f(x', \varepsilon)$ and $\lim_{\varepsilon\to 0^-} D_n^k f(x', \varepsilon)$ exist in $\mathcal{D}'(A)$ (weakly), and are equal, then $F = \lim_{\varepsilon\to 0^+} (\chi_\varepsilon^+ + \chi_\varepsilon^-) f$ exists in $\mathcal{D}'(\Omega)$ (strongly), and $P(x, D)F = 0$ in Ω.*

Proof. Since $P(x, D)$ is of normal order m with respect to A, there are functions $a_{k,\beta} \in C^\infty(\Omega)$, depending only on the coefficients of $P(x, D)$ such that for any $g \in C^\infty(\Omega)$, $\varphi \in C_0^\infty(\Omega)$ and $t \in \mathbf{R}$

$$(P(x, D)(\chi_t^+ g), \varphi) = (\chi_t^+ P(x, D) g, \varphi) + \sum_{k=1}^{m-1} \int_{\mathbf{R}^{n-1}} D_n^k g(x', t) \sum_\beta a_{k,\beta}(x', t) D^\beta \varphi(x', t)\,dx'. \quad (5.1)$$

This is Green's formula for $P(x, D)$. Let $\varphi_k = \sum_\beta a_{k,\beta} D^\beta \varphi$. Then Green's formula can be rewritten as

$$(P(x, D)(\chi_t^+ g), \varphi) = \sum_{k=0}^{m-1} [D_n^k g \cdot \delta_t](\varphi_k) + (\chi_t^+ P(x, D) g, \varphi). \quad (5.2)$$

Since $(P(x, D)(\chi_t^- g), \varphi) = (P(x, D) g, \varphi) - (P(x, D)(\chi_t^+ g), \varphi)$, (5.2) implies

$$(P(x, D)(\chi_t^- g)\, \varphi) = -\sum_{k=0}^{m-1} [D_n^k g \cdot \delta_t](\varphi_k) + (\chi_t^- P(x, D) g, \varphi). \quad (5.3)$$

Since $P(x, D) f = 0$ in $\Omega - A$, (5.2) for $t = \varepsilon$ and (5.3) for $t = -\varepsilon$ imply

$$(P(x, D)[\chi_\varepsilon^+ f + \chi_{-\varepsilon}^- f], \varphi) = \sum_{k=1}^{m-1} [(D_n^k f) \cdot \delta_\varepsilon - (D_n^k f) \cdot \delta_{-\varepsilon}](\varphi_k). \quad (5.4)$$

In the following lemma we will prove that $\lim_{\varepsilon\to 0^+} (\chi_\varepsilon^+ f + \chi_{-\varepsilon}^- f)$ exists in $\mathcal{D}'(\Omega)$, and that $\lim_{\varepsilon\to 0^+} [(D_n^k f) \cdot \delta_\varepsilon - (D_n^k f) \cdot \delta_{-\varepsilon}] = 0$ in $\mathcal{D}'(\Omega)$ for $k = 0, \ldots, m-1$. This will, of course, complete the proof of the theorem.

LEMMA 5.3. *Suppose $g \in C^\infty(\Omega^+)$ where $\Omega^+ = \{x \in \Omega: x_n > 0\}$ and $v \in \mathcal{D}'(A)$. The following conditions are equivalent.*

(a) $\lim_{\varepsilon\to 0^+} g(x, \varepsilon) = v$ *weakly in* $\mathcal{D}'(A)$.

(b) $\lim_{\varepsilon\to 0^+} g(x, \varepsilon) = v$ *strongly in* $\mathcal{D}'(A)$.

(c) $\lim_{\varepsilon\to 0^+} g \cdot \delta_\varepsilon = v \otimes \delta$ *strongly in* $\mathcal{D}'(\Omega)$.

In addition, the above conditions imply

(d) *The net $\{\chi_\varepsilon^+ g\}$ converges in the strong topology on $\mathcal{D}'(\Omega)$.*

Proof. Obviously, (c) implies (a). For the sake of completeness we include the standard proof that (a) implies (b). Condition (a) says that the map of [0, 1] into $\mathcal{D}'(A)$ defined by $\varepsilon \to g(x', \varepsilon)$ is continuous with value v at $\varepsilon = 0$ (where $\mathcal{D}'(A)$ has the weak topology). The image of a compact set under a continuous map is compact. Therefore, $\{g(x', \varepsilon): 0 < \varepsilon \leqslant a\}$ is relatively weakly compact, and hence weakly bounded, in $\mathcal{D}'(A)$. Since $C_0^\infty(A)$ is barrelled, the Banach–Steinhaus Theorem is applicable. It says a weakly convergent, weakly bounded net in $\mathcal{D}'(A)$ is uniformly convergent on precompact sets in $C_0^\infty(A)$. By Ascoli's Theorem each bounded set in $C_0^\infty(A)$ is precompact. Therefore, a weakly convergent, weakly bounded net in $\mathcal{D}'(A)$ is uniformly convergent on bounded sets in $C_0^\infty(A)$; that is, strongly convergent.

Next we prove (b) implies (c). Suppose B is a bounded set in $C_0^\infty(A \times (-a, a))$. We must show $\int g(x', \varepsilon)\psi(x', \varepsilon)\,dx'$ converges, uniformly for $\psi \in B$, to $v \otimes \delta(\psi)$. Since $\int g(x', \varepsilon)\,\psi(x', 0)\,dx'$ converges, uniformly for $\psi \in B$, to $v(\psi(x', 0)) = v \otimes \delta(\psi)$, it is sufficient to prove that $\int g(x', \varepsilon)$ $[\psi(x', \varepsilon) - \psi(x', 0)]\,dx'$ converges, uniformly for $\psi \in B$, to zero. The set $\{g(x', \varepsilon): 0 < \varepsilon \leqslant a\}$ is weakly bounded and hence equicontinuous since $C_0^\infty(A)$ is barrelled. Therefore, for each compact set $K \subset A$ there exists an integer N and a constant C such that

$$\left| \int g(x', \varepsilon)\, \varphi(x')\, dx' \right| \leqslant C \sum_{|\alpha| \leqslant N} \sup |D^\alpha \varphi|, \tag{5.5}$$

for all $\varphi \in C_0^\infty(A)$ with supp $\varphi \subset K$ and for all $0 < \varepsilon \leqslant a$. Now, there exists a compact set $K \subset A$ such that supp $\psi \subset K \times (-a, a)$ for all $\psi \in B$. Therefore, $|\int g(x', \varepsilon)(\psi(x', \varepsilon) - \psi(x', 0))\,dx'| \leqslant C \sum_{|\alpha| \leqslant N} \sup |D^\alpha(\psi(x', \varepsilon) - \psi(x', 0))|$, for all $\psi \in B$ and $0 < \varepsilon \leqslant a$. The right-hand side converges, uniformly for $\psi \in B$, to zero.

To prove (d) we show that $\{\chi_\varepsilon^+ g\}$ is a Cauchy net in $\mathcal{D}'(A \times (-a, a))$. Suppose B is a bounded set in $C_0^\infty(A \times (-a, a))$. Then there exists a compact set $K \subset A$ and constants C_α such that for all $\psi \in B$ and $-a < \varepsilon < a$, supp $\psi(x', \varepsilon) \subset K$ and sup $|D^\alpha \psi(x', \varepsilon)| \leqslant C_\alpha$. Therefore, by (5.5) there is a constant C such that $|\int g(x', \varepsilon)\psi(x', \varepsilon)\,dx'| \leqslant C$ for all $\psi \in B$ and $0 < \varepsilon < a$. Therefore, if $\varepsilon_1 > \varepsilon_2 > 0$, then

$$|(\chi_{\varepsilon_2}^+ g - \chi_{\varepsilon_1}^+ g)(\psi)| = \left| \int_{\varepsilon_2}^{\varepsilon_1} g(x', x_n)\, \psi(x', x_n)\, dx'\, dx_n \right| \leqslant C(\varepsilon_1 - \varepsilon_2)$$

for all $\psi \in B$.

Remark. A strong form of the Schwarz reflection principle can be stated as follows. Suppose $f \in C^\infty(\Omega^+)$ satisfies $\Delta f = 0$ in Ω^+ and $\lim_{\varepsilon \to 0^+} f(x', \varepsilon) = 0$ weakly in $\mathcal{D}'(A)$. Let $g(x', x_n) = -f(x', -x_n)$. Then $\lim_{\varepsilon \to 0^+} (\chi_\varepsilon^+ f + \chi_{-\varepsilon}^- g) = F$ satisfies $\Delta F = 0$ in Ω.

This can be deduced from Theorem 5.2 as follows. By Lemma 5.3, $\lim_{\varepsilon \to 0^+} (\chi_\varepsilon^+ f + \chi_\varepsilon^- g)$ exists in $\mathcal{D}'(\Omega)$. Obviously $\Delta F = 0$ in $\Omega - A$. Since $D_n f(x', \varepsilon) = D_n g(x', -\varepsilon)$, it remains to show that $\lim_{\varepsilon \to 0^+} D_n f(x', \varepsilon)$ exists in $\mathcal{D}'(A)$. Now $D_n^2(\chi_\varepsilon^+ f) = \chi_\varepsilon^+ D_n^2 f + (D_n f)\delta_\varepsilon + D_n(f\delta_\varepsilon)$.

The nets $\chi_\varepsilon^+ f$ and $f\delta_\varepsilon$ have limits in $\mathcal{D}'(\Omega)$. Also, since $\Delta f=0$ in Ω^+, $\chi_\varepsilon^+ D_n^2 f = -\sum_{i=1}^{n-1} D_i^2(\chi_\varepsilon^+ f)$ has a limit. Therefore $(D_n f)\delta_\varepsilon$ converges in $\mathcal{D}'(\Omega)$.

Remark. In the special case of the Cauchy–Riemann operator $\partial/\partial\bar z = \frac{1}{2}(D_1+iD_2)$, Theorem 5.2 can be improved by requiring that $\lim_{\varepsilon\to 0^+} f(x+i\varepsilon)-f(x-i\varepsilon)=0$ (weakly in $\mathcal{D}'(A)$), instead of requiring that the individual limits of $f(x+i\varepsilon)$ and $f(x-i\varepsilon)$ exist and are equal. Suppose f is holomorphic on $\Omega - A$ and $\lim_{\varepsilon\to 0^+} f(x_1+i\varepsilon)-f(x_1-i\varepsilon)=0$ in $\mathcal{D}'(A)$. Let $g(x_1, x_2)=f(x_1+ix_2)-f(x_1-ix_2)$. Then by the above version of the Schwarz reflection principle g extends to a harmonic function G on Ω. Therefore $\partial G/\partial z$ is holomorphic in Ω. Also $\partial G/\partial z = \partial f/\partial z$ in $\Omega - A$. Given an open disk D contained in Ω, pick H holomorphic in D with $\partial H/\partial z = \partial G/\partial z$ in D. Thus, in $D-A$, both $\partial/\partial\bar z(H-f)=0$ and $\partial/\partial z(H-f)=0$. It follows that there exists a function F holomorphic in D with $F=f$ in $D-A$.

6. Special results for linear subspaces

The previous results (except Theorem 4.2 (b)) all address directly the problem of when singularities are removable. The next theorem examines the more general question. Suppose the singularities are contained in a certain set A. What restrictions does this place on the distribution Pf?

As before Ω will denote an open set in $\mathbf{R}^n$. We will use the decomposition $\mathbf{R}^n = \mathbf{R}^d \times \mathbf{R}^{n-d}$ with $z=(x, y)\in\mathbf{R}^n$, $x\in\mathbf{R}^d$, and $y\in\mathbf{R}^{n-d}$. We will let A^d denote $\Omega\cap(\mathbf{R}^d\times\{0\})$ as well as $\{x\in\mathbf{R}^d: (x,0)\in\Omega\}$. For an n multi-index α we will write $\alpha=(\beta,\gamma)$ where β and γ are d and $(n-d)$ multi-indices respectively. As before $P(z, D)$ will denote a differential operator $\sum a_\alpha(z) D^\alpha$ with $a_\alpha(z)\in C^\infty(\Omega)$. The *normal order* m of $P(z, D)$ with respect to A^d is the largest $|\gamma|$ such that $a_{(\beta,\gamma)}(z)\not\equiv 0$ for some β.

Let δ denote the Dirac measure in $\mathbf{R}^{n-d}$. If $u\in\mathcal{D}'(\Omega)$ with supp $u\subset A^d$, then there exist unique distributions $u_\gamma\in\mathcal{D}'(A^d)$, with $\{\operatorname{supp} u_\gamma\}$ locally finite, such that $u=\sum u_\gamma\otimes D^\gamma\delta$ (see Schwartz [8]). Consequently, if $f\in\mathcal{D}'(\Omega)$, then $Pf=0$ in $\Omega - A^d$ if and only if $Pf = \sum u_\gamma\otimes D^\gamma\delta$ with $u_\gamma\in\mathcal{D}'(A^d)$ and $\{\operatorname{supp} u_\gamma\}$ locally finite.

THEOREM 6.1. (a) *Suppose $f\in L^p_{k,\mathrm{loc}}(\Omega)$ $(p<\infty)$ and $P(z,D)f=0$ in $\Omega - A^d$. Then $P(z, D)f\in\mathcal{D}'(\Omega)$ has a finite decomposition $\sum_{|\gamma|\leqslant N} u_\gamma\otimes D^\gamma\delta$ with $N<(m-k)-(n-d)/p'$.*

(b) *Suppose $f\in L^1_{\mathrm{loc}}(\Omega)$ and $f(z)=o(d(z,A)^{-q})$ uniformly for z in compact subsets of Ω. If $P(z,D)f=0$ in $\Omega-A$ then $P(z,D)f\in\mathcal{D}'(\Omega)$ has a finite decomposition as above with $N<m-(n-d)+q$.*

Remark. If $f\in L^\infty_{k,\mathrm{loc}}(\Omega)$ then $f\in L^p_{k,\mathrm{loc}}(\Omega)$ for all $p<\infty$ which by part (a) implies $N\leqslant(m-k)-(n-d)$. On the other hand, part (b) is not a consequence of part (a).

Proof. Pick $\psi \in C_0^\infty(\mathbf{R}^{n-d})$ with supp $\psi \subset \{y: |y| \leq 1\}$ and $\psi \equiv 1$ near zero. Let $\psi_\varepsilon(y) = \varepsilon^{|\gamma_0|}(y/\varepsilon)^{\gamma_0}\psi(y/\varepsilon)$. Then $D^\gamma \psi_\varepsilon(0) = 0$ unless $\gamma = \gamma_0$, and $D^{\gamma_0}\psi_\varepsilon(0) = \gamma_0!$. Suppose K is a compact subset of A^d and $\varphi \in C_0^\infty(A^d)$ with supp $\varphi \subset K$. Let $u = Pf$. Then $u(\varphi \otimes \psi_\varepsilon) = \gamma_0!\, u_{\gamma_0}(\varphi)$ independent of $\varepsilon > 0$. On the other hand, $u(\varphi \otimes \psi_\varepsilon) = (Pf, \varphi \otimes \psi_\varepsilon) = (f, {}^tP(\varphi \otimes \psi_\varepsilon))$. Now $D^\alpha(\varphi \otimes \psi_\varepsilon) = D^\beta \varphi \otimes D^\gamma \psi_\varepsilon$, and $\|D^\gamma \psi_\varepsilon\|_\infty \leq C_\gamma\, \varepsilon^{|\gamma_0| - |\gamma|}$. Therefore, $|u_{\gamma_0}(\varphi)| \leq C\varepsilon^{|\gamma_0| - m}\|\chi_{K_\varepsilon} f\|_1$, where C is a constant depending on φ but not on $\varepsilon > 0$.

We will give the proof of part (a) for $k = 0$. See section 4 for the definition of $L^p_{k,\mathrm{loc}}(\Omega)$. The proof for k an integer follows similarly (see the proof of Theorem 4.4(a) and (b)). By Hölder's inequality $\|\chi_{K_\varepsilon} f\|_1 \leq \lambda(K_\varepsilon)^{1/p'}\|\chi_{K_\varepsilon} f\|_p \leq C\varepsilon^{((n-d)/p')}\|\chi_{K_\varepsilon} f\|_p$. Therefore $|u_{\gamma_0}(\varphi)| \leq C\varepsilon^r \|\chi_\varepsilon f\|_p$ with $\chi = |\gamma_0| - (m - (n-d)/p')$. The right hand side has limit zero as ε approaches zero unless $r < 0$. Hence $u_{\gamma_0} = 0$ unless $|\gamma_0| < m - (n-d)/p'$.

To prove part (b) notice that (2.1) implies that $\|\chi_{K_\varepsilon} f\|_1 \leq C_\varepsilon \varepsilon^{n-d-q}$ with $\lim_{\varepsilon \to 0} C_\varepsilon = 0$. Hence $|u_{\gamma_0}(\varphi)| \leq CC_\varepsilon \varepsilon^r$ where $r = |\gamma_0| - (m - (n-d) + q)$. This implies $u_{\gamma_0} = 0$ unless $r < 0$.

Remark. The *tangential order* m_t of $P(x, D)$ with respect to A^d is the largest $|\beta|$ such that $a_{\beta,\gamma}(z) \not\equiv 0$ for some γ. A careful look at the proof of Theorem 6.1 shows that we have $|u_{\gamma_0}(\varphi)| \leq C\|\varphi\|_{p',m_t}\|\psi_\varepsilon\|_{p',m}\|\chi_{K_\varepsilon} f\|_p$. Hence $u_\gamma \in L^p_{-m_t,\mathrm{loc}}$.

Theorem 2.5 is not always the best possible result for elliptic operators. For example, if f is harmonic in $\mathbf{R}^2 - \{0\}$ and satisfies $f(x) = o(\log 1/|x|)$, then f is harmonic in $\mathbf{R}^2$. Our next goal is to use Theorem 6.1 to generalize this result. First, though, we need a result about pseudo-differential operators.

Let Q be a pseudo-differential operator of class $L^{-m}(\Omega)$ where $m = n - d$ (this is $L^{-m}_{1,0}(\Omega)$ in the notation of Hörmander [5]). Then $Q = Q' + Q''$ where $Q'\varphi(z) = (2\pi)^{-n}\int e^{iz\cdot\zeta}\, q(z,\zeta)\hat{\varphi}(\zeta)\,d\zeta$ and $Q''\varphi(z) = \int_\Omega E(z,w)\varphi(w)\,dw$ for all $\varphi \in C_0^\infty(\Omega)$. Here $E \in C^\infty(\Omega \times \Omega)$ and $q \in S^{-m}(\Omega)$. That is, $q \in C^\infty(\Omega \times \mathbf{R}^n)$, and for every compact subset $K \subset \Omega$ and all multi-indices α and β there is a constant $C_{\alpha,\beta,K}$ such that

$$|D_z^\alpha D_\zeta^\beta q(z,\zeta)| \leq C_{\alpha,\beta,K}(1 + |\zeta|)^{-m-|\beta|} \quad \text{for all } z \in K,\ \zeta \in \mathbf{R}^n.$$

In Theorem 6.2 we assume that $q = q_0 + q_1$ where $q_1 \in S^{-m-1}(\Omega)$ and $q_0 \in S^{-m}(\Omega)$ has the property that $q_0(z, \tau\zeta) = \tau^{-m} q_0(z,\zeta)$ for $\tau \geq 1$ and $|\zeta| \geq 1$. For such an operator Q, fixed $y \in \mathbf{R}^m$ with $|y| = 1$, and $\lambda > 0$, we define operators R_λ: $\mathcal{E}'(A^d) \to C^\infty(A^d)$ by $R_\lambda u(x) = Q(u \otimes \delta)(x, \lambda y)$ for $u \in \mathcal{E}'(A^d)$. This makes sense since $Q(u \otimes \delta)$ is infinitely differentiable on the set $\{(x,y) \in \Omega \mid y \neq 0\}$. Let $k(x) = (2\pi)^{-m}\int_{|\eta|=1} q_0(x, 0, 0, \eta)\,d\sigma_\eta$.

THEOREM 6.2. *Let $u \in \mathcal{E}'(A^d)$. Then* $(\log(1/\lambda))^{-1} R_\lambda u$ *converges in $\mathcal{D}'(A^d)$ to ku.*

Proof. Let $R'_\lambda u(x) = Q'(u \otimes \delta)(x, \lambda y)$ and $R''_\lambda u(x) = Q''(u \otimes \delta)(x, \lambda y)$. Assume $u \in C_0^\infty(A^d)$. Clearly $R''_\lambda u(x)$ is bounded uniformly for x in compact sets and $\lambda \leqslant 1$. Hence we need only consider R'_λ.

Let $\psi \in C_0^\infty(\mathbf{R}^m)$ satisfy $\int \psi(y)\,dy = 1$ and define $\psi_\varepsilon(y) = \varepsilon^{-m}\psi(y/\varepsilon)$. Then $\hat{\psi}_\varepsilon(\eta) = \hat{\psi}(\varepsilon\eta)$ and ψ_ε converges to δ in $\mathcal{D}'(\mathbf{R}^m)$ as $\varepsilon \to 0$.

We have
$$Q'(u \otimes \psi)(x, y) = (2\pi)^{-d} \int e^{ix\cdot\xi} r_\psi(x, y, \xi)\, \hat{u}(\xi)\, d\xi,$$

where
$$r_\psi(x, y, \xi) = (2\pi)^{-m} \int e^{iy\cdot\eta} q(x, y, \xi, \eta)\, \hat{\psi}(\eta)\, d\eta.$$

Hence for $\gamma_0 \neq 0$,

$$\begin{aligned} y^{\gamma_0} r_\psi(x, y, \xi) &= (2\pi)^{-m} \int D_\eta^{\gamma_0}(e^{iy\cdot\eta})\, q(x, y, \xi, \eta)\, \hat{\psi}(\eta)\, d\eta \\ &= (-1)^{|\gamma_0|} \int e^{iy\cdot\eta} \sum_{\gamma \leqslant \gamma_0} \binom{\gamma_0}{\gamma} D_\eta^{\gamma_0 - \gamma} q(x, y, \xi, \eta)\, D_\eta^{\gamma} \hat{\psi}(\eta)\, d\eta. \end{aligned}$$

Substitute ψ_ε for ψ and let $\varepsilon \to 0$. For $\gamma = 0$ the expression on the right converges to

$$(-1)^{|\gamma_0|} \int e^{iy\cdot\eta} D_\eta^{\gamma_0} q(x, y, \xi, \eta)\, d\eta.$$

For $0 \neq \gamma < \gamma_0$, the term on the right converges to zero since $D_\eta^{\gamma_0 - \gamma} q \in S^{-m-|\gamma_0 - \gamma|}(\Omega)$ and $D^\gamma \hat{\psi}_\varepsilon(\eta) = \varepsilon^{|\gamma|}(D^\gamma \hat{\psi})(\varepsilon\eta)$. For $\gamma = \gamma_0$, the expression is bounded by

$$\varepsilon^{|\gamma_0|} \int |q(x, y, \xi, \eta)\, D^{\gamma_0} \hat{\psi}(\varepsilon\eta)|\, d\eta. \tag{6.1}$$

First note that $\varepsilon^{|\gamma_0|} \int_{|\eta| \leqslant 1} |q D^{\gamma_0} \hat{\psi}|\, d\eta$ converges to zero. Then note that

$$\begin{aligned} \varepsilon^{|\gamma_0|} \int_{|\eta| \geqslant 1/\varepsilon} |q(x, y, \xi, \eta)\, D^{\gamma_0} \hat{\psi}(\varepsilon\eta)|\, d\eta &= \varepsilon^{|\gamma_0| - m} \int_{|\eta| \geqslant 1} |q(x, y, \xi, \eta/\varepsilon)\, D^{\gamma_0} \hat{\psi}(\eta)|\, d\eta \\ &\leqslant C \varepsilon^{|\gamma_0|} \int_{|\eta| \geqslant 1} (\varepsilon(1 + |\xi|) + |\eta|)^{-m} |D^{\gamma_0} \hat{\psi}(\eta)|\, d\eta, \end{aligned}$$

which also converges to zero with ε. The remaining contribution to (6.1) is bounded by

$$C \varepsilon^{|\gamma_0|} \int_{1 \leqslant |\eta| \leqslant 1/\varepsilon} |q(x, y, \xi, \eta)|\, d\eta \leqslant C \varepsilon^{|\gamma_0|} \int_1^{1/\varepsilon} r^{m-1}(1 + |\xi| + r)^{-m}\, dr \leqslant C \varepsilon^{|\gamma_0|} \log 1/\varepsilon,$$

which again converges to zero with ε. This proves that $y^{\gamma_0} r_{\psi_\varepsilon}(x, y, \xi)$ converges uniformly for (x, y) in compact sets to $(-1)^{|\gamma_0|} \int e^{iy\cdot\eta} D_\eta^{\gamma_0} q(x, y, \xi, \eta)\, d\eta$ as ε approaches zero. Hence

$$R'_\lambda u(x) = (2\pi)^{-d} \int e^{ix\cdot\xi} r_\lambda(x,\xi)\, \hat{u}(\xi)\, d\xi,$$

where
$$r_\lambda(x,\xi) = \lambda^{-2}(2\pi)^{-m} \int e^{i\lambda y\cdot\eta} \Delta_\eta q(x, \lambda y, \xi, \eta)\, d\eta. \tag{6.2}$$

The following lemma completes the proof of the theorem.

LEMMA 6.3. *$r_\lambda \in S^0(A^d)$ for each $\lambda > 0$ and $(\log 1/\lambda)^{-1} r_\lambda$ converges in $S^0(A^d)$ to k as λ approaches zero.*

Proof. That $r_\lambda \in S^0(A^d)$ is clear. We will show that $(\log (1/\lambda))^{-1} r_\lambda(x,\xi)$ converges to $k(x)$ uniformly for x in compact sets. The lemma then follows by differentiating under the integral sign in (6.2) and iterating the argument.

The contribution of q_1 to (6.2) can be integrated by parts to obtain $(2\pi)^{-m}\int e^{i\lambda y\cdot\eta} q_1(x, \lambda y, \xi, \eta)\, d\eta$. This expression is bounded uniformly for x in compact sets and $\lambda \leqslant 1$. Hence its contribution to the lim $(\log 1/\lambda)^{-1} r_\lambda(x,\xi)$ is zero.

Next consider

$$\lambda^{-2}(2\pi)^{-m} \int_{|\eta|\geqslant 1/\lambda} e^{i\lambda y\cdot\eta} \Delta_\eta q_0(y, \lambda y, \xi, \eta)\, d\eta = (2\pi)^{-m} \int_{|\eta|\geqslant 1} e^{iy\cdot\eta} \Delta_\eta q_0(x, \lambda y, \lambda\xi, \eta)\, d\eta.$$

Here we make a change of variables and use the homogeneity of q_0. Again this quantity is uniformly bounded for x in compact sets and $\lambda \leqslant 1$. Therefore the only contribution to the limit of $(\log (1/\lambda))^{-1} r_\lambda$ comes from

$$\lambda^{-2}(2\pi)^{-m} \int_{|\eta|\leqslant 1/\lambda} e^{i\lambda y\cdot\eta} \Delta_\eta q_0(x, \lambda y, \xi, \eta)\, d\eta.$$

Now apply Green's formula to this expression. The boundary terms are bounded uniformly for x in compact sets and $\lambda \leqslant 1$. In addition notice that $\int_{|\eta|\leqslant 1} e^{i\lambda y\cdot\eta} q_0(x, \lambda y, \xi, \eta)\, d\eta$ is bounded uniformly for x in compact sets and $\lambda \leqslant 1$. Hence the interesting part of r_λ is

$$(2\pi)^{-m} \int_{1\leqslant|\eta|\leqslant 1/\lambda} e^{i\lambda y\cdot\eta} q_0(x, \lambda y, \xi, \eta)\, d\eta = (2\pi)^{-m} \int_1^{1/\lambda} \frac{dr}{r} \int_{|\eta|=1} e^{ir\lambda y\cdot\eta} q_0(x, \lambda y, \xi/r, \eta)\, d\sigma_\eta.$$

By the mean value theorem we have a constant C such that

$$|q_0(x, \lambda y, \xi/r, \eta) - q_0(x, \lambda y, 0, \eta)| \leqslant C\left(1 + \frac{|\xi|}{r}\right)^{-m-1} \frac{|\xi|}{r}$$

for x in compact sets, $\lambda \leqslant 1$ and $|\eta| = 1$. Since $\int_1^{1/\lambda} |\xi| r^{-2} (1 + |\xi|/r)^{-m-1} dr$ is bounded independently of ξ, we need only consider

$$(2\pi)^{-m}\int_1^{1/\lambda}\frac{dr}{r}\int_{|\eta|=1}e^{ir\lambda y\cdot\eta}q_0(x,\lambda y,0,\eta)\,d\sigma_\eta.$$

This equals

$$(2\pi)^{-m}\log(1/\lambda)\int_{|\eta|=1}q_0(x,\lambda y,0,\eta)\,d\sigma_\eta$$
$$+(2\pi)^{-m}\int_1^{1/\lambda}\frac{e^{ir\lambda y\cdot\eta}-1}{r}\,dr\int_{|\eta|=1}q_0(x,\lambda y,0,\eta)\,d\sigma_\eta.$$

The second integral on the right is bounded uniformly for x in compact sets and $\lambda\leqslant 1$ (make the change of variables $\varrho=\lambda r$ and note that the integrand is continuous at zero). Hence $(\log(1/\lambda))^{-1}r_\lambda$ converges uniformly for x in compact sets to k.

For a differential operator $P(z, D)=\sum_{|\alpha|\leqslant m}a_\alpha(z)D^\alpha$ in Ω, let $P_m(z, D)=\sum_{|\alpha|=m}a_\alpha(z)D^\alpha$ denote its principal part. Again let $z=(x, y)$ and $\zeta=(\xi,\eta)$.

THEOREM 6.4. *Let $P(z, D)$ be an elliptic differential operator in Ω, with the property that*

$$\int_{|\eta|=1}P_m(x,0,0,\eta)^{-1}d\sigma_\eta \tag{6.3}$$

never vanishes. Suppose f satisfies $P(z, D)f=0$ in $\Omega-A^d$ and $f(x, y)=o(\log 1/|y|)$ as $y\to 0$ uniformly in compact sets. Then $P(z, D)f=0$ in Ω.

Proof. The hypothesis implies that $f\in L^p_{\text{loc}}(\Omega)$ for all $p<\infty$. By Theorem 6.1 $P(z, D)f=u\otimes\delta$ for some $u\in\mathcal{D}'(A^d)$. Let $Q\in L^{-m}(\Omega)$ be a parametrix for P. If $\psi\in C_0^\infty(A^d)$, we have $Q((\psi u)\otimes\delta)=(\psi\otimes 1)f+g$ where $g\in C^\infty(\Omega)$. Thus $Q((\psi u)\otimes\delta)(x, y)=o(\log 1/|y|)$ as $y\to 0$ uniformly on compact sets. On the other hand, by Theorem 6.2

$$\lim_{\lambda\to 0}(\log 1/\lambda)^{-1}\int Q((\psi u)\otimes\delta)(x,\lambda y)\varphi(x)\,dx=(k\psi u)(\varphi),$$

where $k(x)=(2\pi)^{-m}\int_{|\eta|=1}P_m(x, 0, 0, \eta)^{-1}d\sigma_\eta$. Hence $ku=0$. Since k is never zero, we have $u=0$.

Remark. If $P_m(x, 0, 0, \eta)$ is real (6.3) is clearly never zero. Hence the theorem applies to all elliptic operators with real principal part.

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