A MINIMAX INEQUALITY FOR OPERATORS AND A RELATED NUMERICAL RANGE

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Introduction

Let E and F be two normed spaces over the same field K, which may be either the field of real numbers or the field of complex numbers. Denote by $L(E, F)$ the space of all bounded linear transformations of E into F, with the supremum norm. If A and B are any two elements of $L(E, F)$, the inequality

$$\sup_{|x|\leqslant 1} \inf_{\lambda \in K} |Ax+\lambda Bx| \leqslant \inf_{\lambda \in K} |A+\lambda B| = \inf_{\lambda \in K} \sup_{|x|\leqslant 1} |Ax+\lambda Bx| \qquad (*)$$

is immediate.

We prove in this paper that, provided E and F have dimension at least two, equality in the above relation is attained for every pair A, B in $L(E, F)$ if and only if both E and F are inner product spaces (if either E or F is one-dimensional, then equality holds trivially). The proof of this theorem (Theorem 3.1) is divided into two stages. In the first stage we reduce the case of arbitrary E and F to the case where both E and F are of dimension exactly two, and in the second stage we prove the theorem for this case.

To simplify statements we shall say that the pair E, F posesses the minimax property if equality holds in $(*)$ for each pair A, B in $L(E, F)$. Thus, our result is that a pair E, F has the minimax property if and only if both E and F are inner product spaces (provided both E and F have dimension strictly greater than one).

A new concept of considerable importance in this investigation is a subset $W(A, B)$ of K^2, assigned to each pair A, B in $L(E, F)$. It can be described as a joint numerical range of A and B and is defined by

$$W(A, B) = \{[\langle Ax, y\rangle, \langle Bx, y\rangle];\ x \in E,\ y \in F',\ |x|\,|y| \leqslant 1\}.$$

This set enables us to formulate conveniently the conditions for equality in (*), but it also seems to be interesting in its own right. It turns out that, given two fixed operators A and B, $W(A, B)$ is convex if and only if equality holds for all pairs of linear combinations of A and B. If E and F are inner product spaces the convexity of $W(A, B)$ is closely related to the classical theorem of Hausdorff and Toeplitz on the convexity of the numerical range of one operator.

The dual space (i.e. the space of all bounded linear functionals) of a given normed space E will be denoted by E', and the same notation will also serve for the adjoint A' of an operator A. Finally we remark what has already been apparent, namely that we are going to use the same simple bar notation for the norm in all the spaces involved, as well as for the absolute value of scalars.

The references [1], [2], and [3] are general references on the theory of vector spaces and tensor products. In particular, Proposition 1, p. 28 of [2] summarizes the needed background on tensor products.

The problem treated here is a generalization of one treated by T. Seidman. He showed in [4] that if E and F are one and the same Hilbert space, $B=I$, and A belongs to a special class of operators (including in particular the normal ones), then equality holds in (*). On the other hand, by way of the concept of a joint numerical range, this paper makes contact with the recent quite extensive literature on numerical ranges in Banach spaces. In particular, the paper by Zenger [5] contains a result related to our main result. Let G be the group of complex $n \times n$-matrices (elements of $L(C^n, C^n)$) that have one non-zero entry in each row and column, all of absolute value one. Zenger shows that if a norm on C^n is invariant under G, and if moreover for this norm the range of values of any element of $L(C^n, C^n)$ is a convex set, then the norm must be Euclidean, in this case of course a multiple of the standard l_n^2 norm. It seems reasonable to conjecture that one should be able to weaken Zenger's restrictions and still show that the norm must be Euclidean, using the methods from the last section of the present paper.

1. The joint numerical range

Let us begin with a precise formulation of our problem and the corresponding definitions. Let M be a one-dimensional *affine* subspace of $L(E, F)$ which does not pass through the origin. Consider the obvious inequality

$$\sup_{|x|\leqslant 1} \inf_{A\in M} |Ax| \leqslant \inf_{A\in M} \sup_{|x|\leqslant 1} |Ax| = \inf_{A\in M} |A|. \tag{1}$$

Definition 1.1. We shall say that the triple (E, F, M) has the minimax property if equality occurs in (1). If P is a two-dimensional *linear* subspace of $L(E, F)$ we say that

(E, F, P) has the minimax property if (E, F, M) has the minimax property for each one-dimensional affine subspace $M \subset P$, with $0 \notin M$. Finally, we say that the pair (E, F) has the minimax property if (E, F, P) has it for each two-dimensional subspace $P \subset L(E, F)$.

LEMMA 1.2. *If M is a one-dimensional affine subspace of $L(E, F)$ not passing through the origin, then*

$$\sup_{|x|\leqslant 1} \inf_{A\in M} |Ax| = \sup_{|x|\leqslant 1} \sup_{|y|\leqslant 1} \inf_{A\in M} \operatorname{Re} \langle Ax, y\rangle = \sup_{|y|\leqslant 1} \inf_{A'\in M'} |A'y|$$

where M' denotes the image in $L(F', E')$ of the set M under the transposition mapping.

Proof. By means of the two relations below

$$\inf_{A\in M} |Ax| = \sup \{ \inf_{A\in M} \operatorname{Re} \langle Ax, y\rangle; y \in F', |y| \leqslant 1\}$$

$$\inf_{A'\in M'} |A'y| = \sup \{ \inf_{A'\in M'} \operatorname{Re} \langle x, A'y\rangle; x \in E, |x| \leqslant 1\}.$$

Proof of the second relation (the proof of the first is similar): If

$$\{A'y;\ A' \in M'\} = \{y_0 + \lambda y_1;\ \lambda \in K\}$$

and $[y_0]$ is the class of y_0 in $E'/(\operatorname{span} y_1)$, then

$$\inf_{A'\in M'} |A'y| = |[y_0]| = \sup \{\operatorname{Re} \langle x, y_0\rangle; x \in E, \langle x, y_1\rangle = 0, |x| \leqslant 1\}$$

$$\leqslant \sup \{\inf_{\lambda} \operatorname{Re} \langle x, y_0 + \lambda y_1\rangle; x \in E, |x| \leqslant 1\} \leqslant \inf_{A'\in M'} |A'y|.$$

For if $N = \{x \in E; \langle x, y_1\rangle = 0\}$, then span $y_1 = N^0$, the annihilator of N in E', and the quotient space E'/N^0 is isomorphic to N' by the natural homomorphism (cf. [1], ch. IV, § 5, proposition 10).

LEMMA 1.3. *The triple (E, F, M) has the minimax property if and only if the triple (F', E', M') has it.*

Proof. An immediate consequence of Lemma 1.2.

Our main tool in this investigation is the fact that the tensor product $E \otimes F'$ equipped with the norm

$$g(t) = \inf \{\textstyle\sum |x_i| |y_i|;\ t = \sum x_i \otimes y_i\}$$

may be isometrically imbedded as a w^*-dense subspace in $L(E, F)'$. The unit ball of the normed space $(E \otimes F', g)$ is the closed convex hull of the set

$$U = \{x \otimes y; x \in E, y \in F', |x|\,|y| \leqslant 1\}$$

of simple tensors of norm at most one.

Definition 1.4. Let P be a two-dimensional subspace of $L(E, F)$. We denote by $W(P)$ the set of those elements of P' which admit a representation of the form

$$A \to \langle Ax, y \rangle, \quad A \in P$$

with $x \in E$, $y \in F'$, and $|x|\,|y| \leqslant 1$. If $\{A_0, A_1\}$ is a basis for P and if we use the dual basis for P' an affinely equivalent image of $W(P)$ in K^2 is obtained:

$$\{[\langle A_0 x, y \rangle, \langle A_1 x, y \rangle];\ |x|\,|y| \leqslant 1\}$$

which we denote by $W(A_0, A_1)$. The set $W(P)$ will be called the numerical range of P, the set $W(A_0, A_1)$ the joint numerical range of A_0 and A_1.

THEOREM 1.5. *Let P be a two-dimensional subspace of $L(E, F)$. Then (E, F, P) has the minimax property if and only if the closure of $W(P)$ is convex.*

Proof. The set of all one-dimensional affine subspaces M of P which do not pass through zero is in one-to-one correspondence with the set of all non-zero vectors $t \in P'$ by means of

$$M = \{A \in P;\ \langle A, t \rangle = 1\}.$$

Denote by $C(W(P))$ the closed convex hull in P' of the set $W(P)$ and note that both $W(P)$ and $C(W(P))$ have the property that they contain with each point z also all points of the form λz, $|\lambda| \leqslant 1$. Keeping this in mind, it is easy to see that the theorem will be proved if we prove the following two relations

$$\sup_{|x| \leqslant 1} \inf_{A \in M} |Ax| = \sup \{|\lambda|;\ \lambda t \in W(P)\} \tag{2}$$

$$\inf_{A \in M} \sup_{|x| \leqslant 1} |Ax| = \sup \{|\lambda|;\ \lambda t \in C(W(P))\}. \tag{3}$$

To prove (2) we use Lemma 1.2. If follows that

$$\sup_{|x| \leqslant 1} \inf_{A \in M} |Ax| = \sup_{x \otimes y \in U} \inf_{A \in P, \langle A, t \rangle = 1} \operatorname{Re} \langle Ax, y \rangle = \sup \{|\lambda|;\ \lambda t \in W(P)\}.$$

The equation (3) is an immediate consequence of the fact that $C(W(P))$ is the unit ball of P'. The proof is complete.

2. Reduction theorems

Having transformed the minimax property of the pair (E, F) into a statement concerning convexity of a set in K^2—a property which involves only two points of the numerical range at a time—it is to be expected that the minimax property of the pair (E, F) may be reduced to the behavior of two-dimensional subspaces and quotient spaces of E and F. This is indeed the case as the following two propositions show.

PROPOSITION 2.1. *Let E and F be two normed spaces. Suppose we are given normed spaces E_0 and F_0 and mappings $Q \in L(E, E_0)$, $V \in L(F_0, F)$ such that Q' and V are isometries. Then the mapping which assigns to each $X \in L(E_0, F_0)$ the operator $VXQ \in L(E, F)$ is an isometry of $L(E_0, F_0)$ into $L(E, F)$. Moreover, if the pair (E, F) has the minimax property then the pair (E_0, F_0) has the minimax property as well.*

Proof. It is obvious from the definition of the supremum norm of an operator that if the left factor in a product of operators is an isometry, then it can be cancelled without changing the norm. Hence

$$|VXQ| = |XQ| = |Q'X'| = |X'| = |X|,$$

proving the isometry statement. Let M be an affine subspace of $L(E_0, F_0)$, of dimension one, and let M_1 be the image of M in $L(E, F)$ by the mapping $X \to VXQ$. Using the minimax property of (E, F), and Lemma 1.2, we get

$$\begin{aligned}\sup_{|x|\leqslant 1}\ \inf_{A\in M} |Ax| &= \sup_{|y|\leqslant 1}\ \inf_{A'\in M'} |A'y| = \sup\inf |Q'A'y| \\ &= \sup\inf |AQx| = \sup\inf |VAQx| = \inf_{VAQ\in M_1} |VAQ| = \inf_{A\in M} |A|.\end{aligned}$$

Since M is arbitrary, this finishes the proof of Proposition 2.1.

PROPOSITION 2.2. *Let E, F be two normed spaces. Suppose we are given normed spaces E_0 and F_0 and mappings $H \in L(E_0, E)$, $S \in L(F, F_0)$ which are both contractions. Then the mapping R which assigns to every $X \in L(E, F)$ the operator $SXH \in L(E_0, F_0)$ is a contraction (of $L(E, F)$ into $L(E_0, F_0)$) and moreover*

$$W(R(A), R(B)) \subset W(A, B)$$

for each pair $A, B \in L(E, F)$.

Proof. The contractiveness of R follows from the submultiplicativity of the norm. The second statement is also obvious, since

$$\begin{aligned}W(R(A), R(B)) &= \{[\langle SAHx, y\rangle, \langle SBHx, y\rangle];\ x \in E_0,\ y \in F_0',\ |x|\,|y| \leqslant 1\} \\ &= \{[\langle A(Hx), S'y\rangle, \langle B(Hx), S'y\rangle];\ Hx \in E,\ S'y \in F',\ |x|\,|y| \leqslant 1\}\end{aligned}$$

and H and S' are contractions, so that $|Hx|\,|S'y| \leqslant 1$ follows from $|x|\,|y| \leqslant 1$.

3. The main theorem

THEOREM 3.1. *Let E and F be two normed spaces of dimension strictly greater than one over the same field K (which may be either the field of real numbers or the field of complex numbers). Then the pair (E, F) has the minimax property if and only if both E and F are inner product spaces.*

Proof. Suppose that E and F are inner product spaces and that A, B is a pair in $L(E, F)$. Let $[\xi_1, \eta_1]$ and $[\xi_2, \eta_2]$ be two points in $W(A, B)$, i.e. there are points $x_i \in E$, $y_i \in F'$, $i=1, 2$, such that $|x_i|\,|y_i| \leqslant 1$ and

$$\xi_i = \langle Ax_i, y_i\rangle,\ \eta_i = \langle Bx_i, y_i\rangle \quad \text{for } i=1, 2.$$

Let E_0 be the subspace of E spanned by x_1 and x_2, and $H \in L(E_0, E)$ the injection mapping; also let F_0 be the quotient space

$$F/(\text{span}\,(y_1, y_2))^0$$

with $S \in L(F, F_0)$ the canonical mapping. If R is the mapping defined in Proposition 2.2, then obviously the two points $[\xi_1, \eta_1]$ and $[\xi_2, \eta_2]$ also belong to $W(R(A), R(B))$. It follows from (2.2) that if we can prove that $W(R(A), R(B))$ is convex then the whole segment with endpoints $[\xi_1, \eta_1]$ and $[\xi_2, \eta_2]$ must belong to $W(A, B)$. Since E_0 and F_0 are inner product spaces of dimension at most two, it follows by Theorem 1.5 that in order to prove that the pair (E, F) has the minimax property it suffices to consider the case when both E and F have dimension exactly two over K—the one-dimensional cases are trivial anyway.

Moreover, by making some more trivial transformations we may assume that A and B are operators from the same two-dimensional inner product space into itself, and that the minimum of $|A+\lambda B|$ as λ ranges over K is attained for $\lambda=0$. The sup norm of the operator $T=A+\lambda B$ is given by the formula

$$|T|^2 = (\text{tr}\, T'T + ((\text{tr}\, T'T)^2 - 4|\det T|^2)^{\frac{1}{2}})/2. \tag{4}$$

This formula defines a real valued function on K, which can also be considered as the upper envelope of a family of positive quadratic functions

$$\lambda \to T \to |Tx|^2 = |Ax+\lambda Bx|^2 \tag{5}$$

as x ranges over $|x| \leqslant 1$ (cf. (*)). If, for $\lambda=0$, i.e. for $T=A$ the expression under the square root sign in (4) differs from zero, then the function $\lambda \to |A+\lambda B|^2$ is actually smooth at $\lambda=0$, hence its infimum must, by compactness, be the same as the infimum of one of the individual functions in the family (5). But that means that equality holds in (*).

If the square root expression in (4) vanishes for $\lambda=0$, then the upper envelope of the family (5) may have a corner at $\lambda=0$ and one has to proceed differently. But in that case the operator A must be an isometry and there is no loss of generality to assume $A=I$. We must then prove that there is a vector x of length one such that

$$1 \leqslant |x+\lambda Bx|^2 = 1+2\operatorname{Re}\lambda \langle Bx, x\rangle + |\lambda|^2|Bx|^2 \text{ for all } \lambda,$$

i.e. such that $\langle Bx, x\rangle = 0$. Suppose this were not so; then 0 would be an exterior point of the ordinary numerical range of B, which is a compact, convex set. Multiplying B by some number of modulus one, if necessary, we may assume that for some $\varepsilon>0$

$$\operatorname{Re}\langle Bx, x\rangle \geqslant \varepsilon \text{ whenever } |x| = 1.$$

But then, if p is a positive valued parameter

$$|I-pB|^2 \leqslant 1-2\varepsilon p + |B|^2 p^2,$$

which yields a contradiction for small p. This proves the sufficiency of the condition of Theorem 3.1.

Suppose now that the pair (E, F) has the minimax property. In order to prove that both E and F are inner product spaces it suffices according to the classical theorem of Jordan and v. Neumann to show that every two-dimensional quotient space E_0 of E and every two-dimensional subspace F_0 of F are inner product spaces. If E and F are given, denote by Q the canonical quotient mapping of E onto E_0 (then Q' is an isometry, cf. [1], loc. cit.), and by V the canonical embedding of F_0 into F. By Proposition 2.1 the pair (E_0, F_0) has the minimax property as well. Hence it suffices to prove the necessity part of Theorem 3.1 for the special case that both E and F are of dimension two over K.

Let H be a two-dimensional inner product space, with inner product denoted by (z, u) for arbitrary elements z, u in H. Let $\{x, y\}$ be an orthonormal basis for H and fix arbitrary bases in E and F so that it makes sense to talk about determinants of operators. Construct operators

$$T \in L(E, H), \quad R \in L(H, F)$$

such that

$$\begin{aligned} &|T| = |R| = 1, \\ &|\det T| = \max\{|\det A|;\ A\in L(E, H),\ |A|=1\}, \\ &|\det R| = \max\{|\det A|;\ A\in L(H, F),\ |A|=1\}. \end{aligned} \tag{6}$$

The operators T and R are unique in the following sense.

LEMMA 3.2. *If T and R are arbitrary solutions of* (6), *then the set of all solutions are given by the expressions*

$$\{UT;\ U\colon H\to H \text{ is an isometry}\}, \quad \text{and} \quad \{RU;\ U\colon H\to H \text{ is an isometry}\}.$$

Proof. We prove the statement about $\{UT\}$, the other is proved quite similarily or by duality. It is obvious that UT satisfies (6) if U is an isometry. Suppose T_1 satisfies (6), and put $U = T_1 T^{-1}$. Factor $U = PV$ where P is positive definite and V is an isometry. By hypothesis, $\det P = 1$, i.e. P has two positive eigenvalues, the product of which is one. If P is the identity then we have already proved that U is an isometry, and $T_1 = UT$; so we may assume that the opposite holds or, in other words, that neither eigenvalue of P equals one. Let T_2 in $L(E, H)$ be defined by

$$T_2 = P^{\frac{1}{2}} VT,$$

where $P^{\frac{1}{2}}$ is the unique positive definite operator whose square is P. It follows immediately that $|\det T_2| = |\det T|$, and since $T_2' T_2 = T' V' T_1$ that $|T_2| \leqslant 1$. We will show that $|T_2| < 1$, and this is a contradiction, because if $T_3 = |T_2|^{-1} T_2$, then $|\det T_3| > |\det T|$, and $|T_3| = 1$. Assume then, that $|T_2 x| = 1$ for some x in E with $|x| = 1$. Consequently

$$1 = (P^{\frac{1}{2}} VTx, P^{\frac{1}{2}} VTx) = (VTx, T_1 x)$$

and thus, because $|VTx|$, $|T_1 x| \leqslant 1$ by hypothesis,

$$VTx = T_1 x = PVTx$$

which is to say that P has an eigenvector VTx corresponding to the eigenvalue 1. This contradiction completes the proof of Lemma 3.2, and we will now go on to complete the proof of Theorem 3.1.

The space $E(F)$ is itself a Hilbert space if and only if $|T^{-1}| = 1 (|R^{-1}| = 1)$. We will prove the necessity part of Theorem 3.1 by constructing, in case E and F are not both Hilbert spaces, operators A and B in $L(E, F)$ such that strict inequality holds in (*). We may also, without loss of generality, put on the extra condition $|T^{-1}| \leqslant |R^{-1}|$, because the other case, $|R^{-1}| \leqslant |T^{-1}|$, can be proved either quite similarly or else by invoking duality, cf. Lemma 1.3. Moreover, we use the freedoms given by the isometries in Lemma 3.2 to arrange so that the basis vector x in H becomes maximal for T^{-1} and minimal for R. We thus assume

$$\begin{aligned} &1 \leqslant |T^{-1}| \leqslant |R^{-1}|; \quad 1 < |R^{-1}| \\ &|T^{-1} x| = |T^{-1}|; \quad |Rx|^{-1} = |R^{-1}|. \end{aligned} \tag{7}$$

Now we define the two elements C and D of $L(H, H)$ by

$$Cz = |R^{-1}| (z, x) x + |T^{-1}|^{-1} (z, y) y$$

$$Dz = (z, x) y$$

for all z in H. Put $A = RCT$ and $B = RDT$. Note that $CD = |T^{-1}|^{-1} D$ and that therefore

$$\det(C+\lambda D)=\det C\det(I+\lambda|T^{-1}|D)=|R^{-1}|/|T^{-1}|$$

for all $\lambda\in K$. We claim that

$$|T^{-1}|^{-1}<\min_{\lambda\in K}|A+\lambda B|\;(=|A+\lambda_0 B| \text{ for some } \lambda_0\in K).$$

Suppose not, i.e. that $|A+\lambda_0 B|\leqslant|T^{-1}|^{-1}$. Then $|R(C+\lambda_0 D)|\leqslant 1$ so that by the defining property of R

$$|\det R\det(C+\lambda_0 D)|\leqslant|\det R|,\quad \text{i.e. } |R^{-1}|/|T^{-1}|\leqslant 1.$$

Hence by (7) we have already contradiction unless $|T^{-1}|=|R^{-1}|$, and in this remaining case $C+\lambda_0 D$ would have to be an isometry (by Lemma 3.2), which is absurd for any value of λ_0 (take $z=x$).

It remains now to show that

$$\max_{|u|\leqslant 1}\min_{\lambda\in K}|Au+\lambda Bu|\leqslant|T^{-1}|^{-1}. \tag{9}$$

The minimum in (9) can be computed using Lemma 1.2. For u in E, v in F' we have

$$\min_{\lambda}\operatorname{Re}\,[|R^{-1}|(Tu,x)\langle v,Rx\rangle+|T^{-1}|^{-1}(Tu,y)\langle v,Ry\rangle+\lambda(Tu,x)\langle v,Ry\rangle]$$
$$=\begin{cases}\operatorname{Re}|R^{-1}|(Tu,x)\langle v,Rx\rangle & \text{if } \langle v,Ry\rangle=0,\\ \operatorname{Re}|T^{-1}|^{-1}(Tu,y)\langle v,Ry\rangle & \text{if } (Tu,x)=0,\\ 0 & \text{otherwise.}\end{cases}$$

Since $|T|=|R|=1$, and $|Rx|=|R^{-1}|^{-1}$, (9) follows from Lemma 1.2 provided we show that

$$|(Tu,x)|\leqslant|T^{-1}|^{-1} \text{ for all } u \text{ in } E \text{ with } |u|\leqslant 1. \tag{10}$$

To see this, let w be an element of E', with $|w|=1$, such that $\langle T^{-1}x,w\rangle=|T^{-1}x|=|T^{-1}|$. Then

$$(x,T'^{-1}w)=|T^{-1}|=|T'^{-1}|,$$

and this implies that $T'^{-1}w=|T^{-1}|x$, i.e. $w=|T^{-1}|T'x$. We have thus proved that $|T'x|=|T^{-1}|^{-1}$, which is equivalent to (10).

Together (8) and (9) show that unless both E and F are Hilbert spaces, one can construct operators A and B in $L(E,F)$ such that strict inequality holds in (*). This completes the proof of Theorem 3.1.

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