ON COHOMOLOGY OF KLEINIAN GROUPS

III. *Singular Eichler Integrals*

BY

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This paper is a direct sequel to Bers' paper [5] on Eichler integrals with singularities. We use Bers' result to give a new proof of the structure theorems of [15] and [16] for the Eichler cohomology groups of Kleinian groups Γ that represent surfaces of finite type. This new proof depends on the description of the first cohomology group of Γ with holomorphic and smooth coefficients. For the sake of completeness, we outline simple methods for obtaining the descriptions of these groups.

We also obtain a Riemann–Roch type theorem for meromorphic Eichler integrals, and, in special cases, a lower bound on the number of linearly independent holomorphic Eichler integrals.

In addition to the author's papers on cohomology of Kleinian groups [14], [15], [16], [17], the reader is referred to the work of Ahlfors [1], [2], Bers [4], [5], and Lehner [20]. The more special Fuchsian case has been treated by Eichler [8], Gunning [10], [11], Bers [3], Husseini and Knopp [12], Knopp [13], and Lehner [18], [19].

1. Cohomology

Let Γ be a (non-elementary) Kleinian group with region of discontinuity Ω. Let Δ denote an arbitrary Γ-invariant open subset of Ω. For convenience, we assume in this section $\infty \notin \Omega$.

Let r and s be two half-integers such that $r+s$ is an integer. We let the group Γ act on the right on smooth (C^∞) functions on Δ by setting for a smooth function φ on Δ $\varphi\gamma = \gamma^*_{r,s}\varphi$, where

$$(\gamma^*_{r,s}\varphi)(z) = \varphi(\gamma z)\,\gamma'(z)^r\,\overline{\gamma'(z)}^s, \quad z \in \Delta.$$

We obtain this way the Γ-module $C^\infty_{r,s}(\Delta)$, and the submodule $\mathcal{A}_r(\Delta)$ of $C^\infty_r(\Delta) = C^\infty_{r,0}(\Delta)$ consisting of holomorphic functions on Δ.

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We fix once and for all an integer $q \geqslant 2$. For us, the most important Γ-module is Π_{2q-2}, the Γ-module of polynomials of degree $\leqslant 2q-2$, viewed as a submodule of $\mathcal{A}_{1-q}(\Delta)$.

We denote the nth cohomology group of Γ with coefficients in a Γ-module $\mathcal{D}$ by $H^n(\Gamma, \mathcal{D})$. See [22, p. 115] for definitions.

A cohomology class $p \in H^1(\Gamma, \Pi_{2q-2})$ is called Δ-*parabolic* if for every (cyclic) parabolic subgroup Γ_0 of Γ that corresponds to a puncture on Δ/Γ (see, for example, [16] for details), we have

$$p \mid \Gamma_0 = 0. \tag{1.1}$$

The subspace of Δ-parabolic cohomology classes is denoted by $PH^1_\Delta(\Gamma, \Pi_{2q-2})$. If (1.1) holds for every cyclic parabolic subgroup of Γ, then we say p is (*strongly*) *parabolic*. We denote this space by $PH^1(\Gamma, \Pi_{2q-2})$.

From the general theory of cohomology of groups we need two facts. First,

$$H^0(\Gamma, \mathcal{D}) = \mathcal{D}(\Gamma) = \{p \in \mathcal{D};\ p\gamma = p \text{ all } \gamma \in \Gamma\}.$$

(Note that $\mathcal{A}_r(\Delta)(\Gamma)$ will be written as $\mathcal{A}_r(\Delta, \Gamma)$. Similarly, $C^{\infty}_{r,s}(\Delta)\,(\Gamma) = C^{\infty}_{r,s}(\Delta, \Gamma)$.) Second, if

$$0 \longrightarrow \mathcal{A} \xrightarrow{\ f\ } \mathcal{B} \xrightarrow{\ g\ } \mathcal{C} \longrightarrow 0$$

is a short exact sequence of Γ-modules and Γ-linear maps, then we have a long exact sequence

$$0 \longrightarrow \mathcal{A}(\Gamma) \xrightarrow{\ f\ } \mathcal{B}(\Gamma) \xrightarrow{\ g\ } \mathcal{C}(\Gamma) \xrightarrow{\ \delta\ } H^1(\Gamma, \mathcal{A}) \xrightarrow{\ f\ } H^1(\Gamma, \mathcal{B}) \longrightarrow \dots,$$

where δ is the familiar connecting morphism.

For the convenience of the reader, we recall the definition of the first cohomology group. The group $H^1(\Gamma, \mathcal{D})$ is the group of crossed homomorphisms (mappings $\chi\colon \Gamma \to \mathcal{D}$ such that $\chi_{\gamma_1 \circ \gamma_2} = \chi_{\gamma_1}\gamma_2 + \chi_{\gamma_2}$, for all $\gamma_1, \gamma_2 \in \Gamma$, where χ_γ is the value of the mapping χ at $\gamma \in \Gamma$) factored by the group of principal homomorphisms (mappings of the above type with $\chi_\gamma = p\gamma - p$ for some $p \in \mathcal{D}$, all $\gamma \in \Gamma$).

2. Automorphic forms, generalized Beltrami coefficients, and potentials

Let Δ be a Γ-invariant union of components of the region of discontinuity Ω of a Kleinian group Γ. Denote by λ the Poincaré metric on Δ. A meromorphic function φ on Δ is called an (*automorphic*) q-*form* if

$$\gamma_q^* \varphi = \varphi, \quad (\text{where } \gamma_q^* = \gamma_{q,0}^*) \text{ all } \gamma \in \Gamma. \tag{2.1}$$

A q-form is called *integrable* if

$$\iint_{\Delta/\Gamma} \lambda(z)^{2-q} |\varphi(z)\, dz \wedge d\bar{z}| < \infty. \tag{2.2}$$

An integrable φ has at worse simple poles in Δ. The Banach space of integrable *holomorphic* q-forms is denoted by $A_q(\Delta, \Gamma)$. A q-form φ is *bounded* if

$$\sup \{ \lambda(z)^{-q} |\varphi(z)|;\ z \in \Delta \} < \infty. \tag{2.3}$$

Every bounded form is holomorphic. The Banach space of bounded forms is denoted by $B_q(\Delta, \Gamma)$.

If φ is integrable and ψ is bounded, then we define the *Petersson scalar product* by

$$(\varphi, \psi) = \iint_{\Delta/\Gamma} \lambda(z)^{2-2q} \varphi(z) \overline{\psi(z)}\, dz \wedge d\bar{z}. \tag{2.4}$$

It is well known (the proofs are outlined in [17]) that the Petersson scalar product establishes an anti-linear topological isomorphism between $B_q(\Delta, \Gamma)$ and the dual space of $A_q(\Delta, \Gamma)$.

We describe next what it means for a q-form to be meromorphic or holomorphic at a cusp. Let φ be a meromorphic q-form on Δ. Let $\zeta \in \Lambda$, the limit set of Γ, be a cusp. We say φ is *meromorphic* at ζ if its projection to Δ/Γ can be extended to be meromorphic at the puncture determined by ζ. Choose a Möbius transformation A taking ∞ into ζ and $V_c = \{z \in \mathbf{C};\ 0 \leqslant \operatorname{Re} z < 1,\ \operatorname{Im} z > c\}$ onto a cusped region belonging to ζ. Then $\psi = A_q^* \varphi$ is a meromorphic q-form for $A^{-1} \circ \Gamma \circ A$ (defined on $A^{-1}(\Delta)$). We say φ is *holomorphic* at ζ if

$$\lim_{z \to \infty} \psi(z) = \varrho \quad (z \in V_c) \tag{2.5}$$

exists (and is finite). This definition does not depend on the choice of the Möbius transformation A. We say that φ satisfies the *cusp condition* at ζ, if ϱ defined by (2.5) is zero. In this case we also have that

$$\lim_{z \to \zeta} \varphi(z) = 0 \quad (z \in A(V_c)).$$

Every bounded form satisfies the cusp condition. So does an integrable form provided it has only finitely many poles in the corresponding cusped region.

We shall also have to study bounded *measurable* q-forms; that is, measurable functions φ on Δ that satisfy (2.1) and (2.3). The Banach space (of equivalence classes) of bounded measurable q-forms will be denoted by $L_q^{(\infty)}(\Delta, \Gamma)$. It is clear that every element $\psi \in L_q^{(\infty)}(\Delta, \Gamma)$

gives rise to a linear functional on $A_q(\Delta, \Gamma)$ via (2.4). (One defines similarly the space of bounded measurable (r, s)-forms, for all half-integers r and s with $r+s$ an integer.)

If $\mu \in L_q^{(\infty)}(\Delta, \Gamma)$, then $\nu = \lambda^{2-2q}\bar{\mu}$ is called a *generalized Beltrami coefficient*. A *potential* F for ν is a continuous function F on $\mathbf{C}$ such that

$$F(z) = O(|z|^{2q-2}),\ z \to \infty,$$

and $\bar{\partial}F = \partial F/\partial \bar{z}$, in the sense of distributions, is a measurable function with

$$\bar{\partial}F \,|\, \Delta = \nu$$

and

$$\bar{\partial}F \,|\, \mathbf{C} - \Delta = 0 \text{ a.e.}$$

It is easy to check that if F is a potential for ν, then

$$p_\gamma = \gamma_{1-q}^* F - F,\ \gamma \in \Gamma,$$

defines a Π_{2q-2}-cocycle whose cohomology class is strongly parabolic and depends only on ν.

The existence of potentials has been established by Bers [4], [5]. It is easy to see that we have defined the (anti-linear) *Bers map* (see [15])

$$\beta^* : L_q^{(\infty)}(\Delta, \Gamma) \to PH^1(\Gamma, \Pi_{2q-2}).$$

It is known ([4], [17]) that for μ_1 and $\mu_2 \in L_q^{(\infty)}(\Delta, \Gamma)$, we have $\beta^*\mu_1 = \beta^*\mu_2$ whenever $(\varphi, \mu_1) = (\varphi, \mu_2)$, all $\varphi \in A_q(\Delta, \Gamma)$. We may thus view β^* as the linear map

$$\beta^* :\ (A_q(\Delta, \Gamma))^* \to PH^1(\Gamma, \Pi_{2q-2}),$$

where, as usual, $(-)^*$ denotes the dual space of $(-)$.

3. Cohomology with holomorphic and smooth coefficients

Throughout this section Δ represents a Γ-invariant open subset of the region of discontinuity of a (non-elementary) Kleinian group Γ.

PROPOSITION 3.1. *We have*

$$H^1(\Gamma, C_{r,s}^\infty(\Delta)) = \{0\}.$$

Outline of proof. (See [15] for details.) Let p be a cocycle representing a cohomology class of $H^1(\Gamma, C_{r,s}^\infty(\Delta))$. Let η be a partition of unity for Γ on Δ; that is, a smooth function η on Δ such that

(a) $0 \leqslant \eta \leqslant 1$,

(b) for each $z \in \Delta$, there is a neigborhood U of z and a finite subset J of Γ such that $\eta | \gamma(U) = 0$ for each $\gamma \in \Gamma - J$, and

(c) $\sum_{\gamma \in \Gamma} \eta(\gamma z) = 1$, $z \in \Delta$.

Define

$$f(z) = -\sum_{\gamma \in \Gamma} \eta(\gamma z) p_\gamma(z), \quad z \in \Delta.$$

Then $f \in C^\infty_{r,s}(\Delta)$, and p is the coboundary of f.

PROSOSITION 3.2. *If Δ/Γ has no compact components, then*

$$H^1(\Gamma, \mathcal{A}_r(\Delta)) = \{0\}.$$

The proof of the above proposition is based on

LEMMA 3.3. *Let W be an open Riemann surface and $n \in \mathbf{Z}$. If μ is a smooth $(n, 1)$-differential on W, then there exists a smooth n-differential ν on W such that $\bar{\partial}\nu = \mu$.*

(If ν is an n-differential on W, and $\nu = f(z)dz^n$, in terms of some local coordinate z, then $\bar{\partial}\nu = (\partial f/\partial \bar{z})dz^n \overline{dz}$.)

Outline of proof. We first assume that W is a relatively compact subset of a Riemann surface X of finite type, and that $X - \mathrm{Cl}\, W$ ($\mathrm{Cl}\, W$ = closure of W) has non-empty interior. Furthermore, we assume that μ is defined in a neighborhood of $\mathrm{Cl}\, W$. It involves no loss of generality to assume that X has the unit disc Δ as its universal covering space, since this can always be achieved by puncturing X at three (or more) points that are not in $\mathrm{Cl}\, W$. By multiplying μ by a power of a nowhere vanishing holomorphic abelian differential on $\mathrm{Cl}\, W$, we may assume $n = -1$.

Consider the space of integrable holomorphic quadratic differentials on X. This is a finite dimensional space that may be identified with $A_2(\Delta, \Gamma)$, where Γ is the covering group of X. Let π: $\Delta \to X$ be the corresponding covering map and let $\Delta_0 = \pi^{-1}(W)$. The $(-1, 1)$-differential μ lifts to a smooth density (bounded $(-1, 1)$-form) defined on $\mathrm{Cl}\, \Delta_0$, that is also be denoted by μ. We extend μ to Δ as a bounded density (generalized Beltrami coefficient for $q = 2$) and require that

$$(\varphi, \lambda^2 \bar{\mu}) = 0, \quad \text{all } \varphi \in A_2(\Delta, \Gamma).$$

We now choose 3 distinct fixed points of hyperbolic elements of Γ and let F be a potential for μ that vanishes at these points. Then $\bar{\partial}F=\mu$, and F induces the zero Π_2-cocycle; that is $\gamma^*_{-1}F=F$, all $\gamma\in\Gamma$. Thus the projection of F to W is the required ν.

To solve the general case, it is convenient to assume that $n=0$. (The existence of a nowhere vanishing holomorphic abelian differential on W is a consequence of the generalized Weierstrass theorem (see, Florack [9] or Royden [23]).) Using the solution on compact regions, a normal exhaustion of W, and the first approximation theorem of Behnke–Stein [7] (see also Behnke–Sommer [6]), standard arguments complete the proof of the Lemma.

We now exhibit a useful exact sequence of Γ-modules:

$$0 \longrightarrow \mathcal{A}_r(\Delta) \xrightarrow{\ i\ } C_r^\infty(\Delta) \xrightarrow{\ \bar{\partial}\ } C_{r,1}^\infty(\Delta) \longrightarrow 0.$$

Exactness is a consequence of the previous lemma. The corresponding long exact cohomology sequence begins with

$$0 \longrightarrow \mathcal{A}_r(\Delta,\Gamma) \xrightarrow{\ i\ } C_r^\infty(\Delta,\Gamma) \xrightarrow{\ \bar{\partial}\ } C_{r,1}^\infty(\Delta,\Gamma) \xrightarrow{\ \delta\ } H^1(\Gamma,\mathcal{A}_r(\Delta)) \xrightarrow{\ i\ } H^1(\Gamma, C_r^\infty(\Delta)) \xrightarrow{\ \bar{\partial}\ } \dots.$$

Since $H^1(\Gamma, C_r^\infty(\Delta))=\{0\}$, we obtain

PROPOSITION 3.4. *We have*

$$H^1(\Gamma,\mathcal{A}_r(\Delta)) \cong \frac{C_{r,1}^\infty(\Delta,\Gamma)}{\bar{\partial}C_r^\infty(\Delta,\Gamma)}.$$

We are now ready to return to the

Proof of Proposition 3.2. It clearly suffices to assume that Δ/Γ is connected. If Δ does not contain any elliptic elements, then the result follows from Proposition 3.4 and Lemma 3.3. For the general case, let $\mu\in C_{r,1}^\infty(\Delta,\Gamma)$. There is a function $f\in C_r^\infty(\Delta,\Gamma)$ such that $\bar{\partial}f=\mu$ near each elliptic fixed point in Δ (see [15]). Let $\mu_1=\mu-\bar{\partial}f$, and $\tilde{\mu}_1$ its projection to Δ/Γ. Note that $\tilde{\mu}_1$ vanishes near the ramified points on Δ/Γ. Choose $\tilde{\nu}_1$ such that $\bar{\partial}\tilde{\nu}_1=\tilde{\mu}_1$. Then $\tilde{\nu}_1$ is holomorphic near the ramified points. To insure that the lift ν_1 of $\tilde{\nu}_1$ to Δ will be holomorphic at the elliptic fixed points, we have to subtract from $\tilde{\nu}_1$ a holomorphic n-differential $\tilde{\varphi}$ so that $\tilde{\nu}_1-\tilde{\varphi}$ vanishes of sufficiently high order at the ramified points. We then set $\nu=\nu_1+f$.

Remark. Let M be a Riemann surface and $\mathcal{O}$ the sheaf of germs of holomorphic functions on M. Then the first cohomology group of M with coefficients in $\mathcal{O}$, $H^1(M,\mathcal{O})$, is isomorphic to

$$\frac{\text{smooth (0,1)-differentials on } M}{\bar{\partial}\text{ (smooth functions on } M)}.$$

If we represent M as Δ/Γ where Δ is the unit disc and Γ is a fixed point free Fuchsian group (we are eliminating certain surfaces), then

$$H^1(M, O) \cong \frac{C^\infty_{0,1}(\Delta, \Gamma)}{\bar{\partial} C^\infty(\Delta, \Gamma)}.$$

(For obvious reasons C_0^∞ is abbreviated by C^∞.) We have shown that for open Riemann surfaces M, we have

$$H^1(M, O) = \{0\}.$$

4. Eichler integrals with singularities

In this section we outline the results of Bers' paper [5]. Our definitions differ slightly from Bers'.

Let Δ be a Γ-invariant union of components of a Kleinian group Γ. An *Eichler integral* (of order $1-q$ for Γ on Δ) is a function E holomorphic on Δ, except for isolated singularities, such that for every $\gamma \in \Gamma$, there is a $p_\gamma \in \Pi_{2q-2}$ such that

$$\gamma^*_{1-q} E - E = p_\gamma | \Delta',$$

where $\Delta' = \{z \in \Delta;\ E \text{ is holomorphic at } z\}$. In this case, $\gamma \mapsto p_\gamma$ is a one cocycle called the *period* (pd) of E. If E is an Eichler integral, so is $E+p$ for every $p \in \Pi_{2q-2}$. Two Eichler integrals are *identified* whenever they differ by an element of Π_{2q-2}. The cohomology class in $H^1(\Gamma, \Pi_{2q-2})$ of pd E depends only on the equivalence class of E modulo Π_{2q-2}. Henceforth, all spaces of Eichler integrals will be taken modulo Π_{2q-2}.

Note that Proposition 3.2 states that the period map is surjective when Δ/Γ has no compact components (even when we require all Eichler integrals to be holomorphic on Δ).

If E is an Eichler integral, then $\partial^{2q-1}E$ is a q-form on Δ' ($\partial = \partial/\partial z$). We shall say that E is *meromorphic, holomorphic*, or satisfies the *cusp condition* at a cusp if $\partial^{2q-1}E$ is meromorphic, holomorphic, or satisfies the cusp condition respectively. The space of Eichler integrals on Δ that are meromorphic (or holomorphic) on Δ and its cusps is denoted by $E^{\text{mer}}_{1-q}(\Delta, \Gamma)$ (or $E^{\text{hol}}_{1-q}(\Delta, \Gamma)$).

An Eichler integral E is said to be Δ-*parabolic* (or *parabolic*) if its period is. The condition of Δ-parabolicity can be expressed in terms of the Fourier series expansions for E about the cusps in Δ. The spaces of meromorphic and holomorphic parabolic Eichler integrals are denoted by $PE^{\text{mer}}_{1-q}(\Delta, \Gamma)$ and $PE^{\text{hol}}_{1-q}(\Delta, \Gamma)$.

Note that $E \in PE_{1-q}^{\mathrm{hol}}(\Delta, \Gamma)$ if and only if $E \in E_{1-q}^{\mathrm{hol}}(\Delta, \Gamma)$ and E satisfies the cusp condition at each cusp of Δ. (For Δ/Γ of finite type, holomorphic Eichler integrals have been called *quasi-bounded* integrals, and the holomorphic parabolic integrals have been called *bounded* integrals in [16].)

We set $\bar{\Delta} = \Delta \cup \{\text{cusps on } \Delta\}$. By a *distinguished* neighborhood of a point $z \in \bar{\Delta}$ we mean an open disc $U \subset \Delta$ such that

(i) $z \in U$ if $z \in \Delta$, and $z \in \mathrm{Cl}\, U$ if $z \notin \Delta$,
(ii) $\gamma(U) = U$ for $\gamma \in \Gamma_z$, the stabilizer of z, and
(iii) $\gamma(U) \cap U = \emptyset$ for $\gamma \in \Gamma - \Gamma_z$.

Let $z \in \bar{\Delta}$. By a *principal part* of an Eichler integral at z we mean a pair (U, h), where

$$U \text{ is a distinguished neighborhood of } z, \tag{4.1}$$

$$h \text{ is holomorphic in } U - \{z\}, \text{ and} \tag{4.2}$$

$$\gamma_{1-q}^{*} h = h \text{ for } \gamma \in \Gamma_z. \tag{4.3}$$

Two principal parts (U_1, h_1) and (U_2, h_2) at z are *equivalent* if $h_1 - h_2$ is (can be extended to be) holomorphic at z.

It is quite clear that if E is an Eichler integral, then the principal part of E at $z \in \Delta$ does determine a principal part of an Eichler integral. The same is true at $z \in \bar{\Delta} - \Delta$ provided that E is holomorphic in a half plane belonging to the cusp z.

Let H be a principal part of an Eichler integral at $z \in \bar{\Delta}$. We *associate* with H a *linear functional* l on the space of automorphic q-forms φ that are holomorphic (and satisfy the cusp condition at z if $z \notin \Delta$) at z. We set

$$l(\varphi) = 2\pi i \,\mathrm{Res}_z\, h\varphi, \tag{4.4}$$

where the residue of $h\varphi$ is computed on the, not necessarily connected, Riemann surface Δ/Γ.

Bers [5] has shown that l defined by (4.4) is a continuous linear functional on $A_q(\Delta, \Gamma)$ and may hence be viewed as an element of $(A_q(\Delta, \Gamma))^*$. Clearly, a finite set of principal parts defines in an obvious way a continuous linear functional on the space of integrable holomorphic q-forms.

Let there be given a system $\mathcal{H} = \{H_1, \ldots, H_r\}$ of finitely many principal parts at non-equivalent points $z_1, \ldots, z_r$ of $\bar{\Delta}$. If E is an Eichler integral on Δ that is holomorphic at all points of $\bar{\Delta}$ non-equivalent to $z_1, \ldots, z_r$, and if H_j is the principal part of E at z_j $(j = 1, \ldots, r)$ then we call $\mathcal{H}$ a *complete system* of principal parts for E.

PROPOSITION 4.1. *Given $\mathcal{H}$, a finite system of principal parts, and the corresponding linear functional l, then there exists a parabolic Eichler integral E such that $\mathcal{H}$ is a complete system of principal parts for E. Furthermore,* pd $E=\beta^* l$.

The above proposition is the main result of [5].

5. Divisors on $\overline{\Delta/\Gamma}$

Let Γ be a Kleinian group and Δ an invariant union of components of the region of discontinuity of Γ. We assume throughout this section that Δ/Γ is of finite type. We denote by $\overline{\Delta/\Gamma}$ the (not necessarily connected) Riemann surfaces obtained by the adjoining the punctures to Δ/Γ; that is, $S=\overline{\Delta/\Gamma}\cong\bar{\Delta}/\Gamma$. By a *divisor* d on $\overline{\Delta/\Gamma}$ we mean a formal sum

$$d=\sum_{x\in S} n(x)\,x,$$

where $n(x)\in\mathbf{Z}$ and $n(x)=0$ for all but finitely many $x\in S$. There is, of course, a natural partial ordering on the additive group of divisors.

If φ is a meromorphic q-form that is not identically zero on a component S_0 of S, then we define the *(reduced) divisor of* φ on S_0 by

$$(\varphi)=(\Phi)=\sum_{x\in S_0}\operatorname{ord}_x\Phi,$$

where Φ is the q-differential obtained by projecting φ to S. (For convenience we let ∞ be the divisor of the zero form, where $\infty\geqslant d$, for all divisors d on $\overline{\Delta/\Gamma}$.) If $z\in\Delta$, then it is easy to see that if $r=\operatorname{ord}_z\varphi$, $R=\operatorname{ord}_{\pi(z)}\Phi$, $\nu=\operatorname{ord}\Gamma_z$, where $\pi\colon\Delta\to\Delta/\Gamma$ is the natural projection map, then

$$R=\frac{1}{\nu}(r+q)-q. \tag{5.1}$$

If $z\in\bar{\Delta}-\Delta$, then

$$R=r-q. \tag{5.2}$$

To interpret (5.2) we must explain what is meant by the order of an meromorphic q-form at a puncture. If ∞ is the fixed point corresponding to the puncture generated by the parabolic element $z\mapsto z+1$, and if the Fourier series expansion for φ in the cusped region determined by the puncture is given by

$$\varphi(z)=\sum_{n=r}^{\infty}a_n e^{2\pi i n z}\quad\text{for }\operatorname{Im} z>0,$$

with $a_r \neq 0$, then we set $\mathrm{ord}_\infty \varphi = r$. Using (5.1) and (5.2), it is thus possible to define (φ) directly in terms of the zeros and poles of φ.

For every integer q (not necessarily $\geqslant 2$) it is convenient to introduce a *q-canonical ramification divisor* α^q by

$$\alpha^q = \sum_{x \in S} n_q(x)\, x,$$

where

$$n_q(x) = -\left[q\left(1 - \frac{1}{\nu(x)}\right)\right], \tag{5.3}$$

where $[y]$ is the greatest integer $\leqslant y$ and $1 \leqslant \nu(x) < \infty$ is the ramification number of $x \in \Delta/\Gamma$, and where

$$n_q(x) = \begin{cases} 1-q & \text{if} \quad q > 0 \\ -q & \text{if} \quad q \leqslant 0 \end{cases} \tag{5.4}$$

for $x \in S - \Delta/\Gamma$ (points of ramification number ∞). Note that (5.4) is the limiting case of (5.3) with $\nu \to \infty$.

It is easy to check (see [1]) that

$$\alpha^q + \alpha^{1-q} = 0. \tag{5.5}$$

If d is any divisor on S, then we denote by $A_q(d)$ the space of meromorphic q-forms on Δ whose reduced divisors are multiples of $\alpha^q + d$. Note that for $q \geqslant 2$, $A_q(0) = A_q(\Delta, \Gamma) = B_q(\Delta, \Gamma)$.

From now on we return to our original convention $q \geqslant 2$.

The zero set of an Eichler integral is not invariant under the group Γ. It is hence impossible to define a space consisting of Eichler integrals that are multiples of an arbitrary divisor d. However, for $d \leqslant 0$, we can introduce two spaces:

$E_{1-q}(d)$, the space of meromorphic Eichler integrals whose reduced polar divisors are multiples of $\alpha^{1-q} + d$;

$PE_{1-q}(d)$, the corresponding space of parabolic Eichler integrals.

By the (*reduced*) *polar divisor* of an Eichler integral F we mean

$$\sum_{x \in S} n(x)\, x,$$

where $n(x) = n_{1-q}(x)$ if F is holomorphic at x, and $n(x)$ is the value of R given by (5.1) or (5.2) with $-r$, the order of the pole of F at x and q replaced by $(1-q)$, if F is not holomorphic at x.

It is clear that $E_{1-q}(0) = E^{\mathrm{hol}}_{1-q}(\Delta, \Gamma)$ and $PE_{1-q}(0) = PE^{\mathrm{hol}}_{1-q}(\Delta, \Gamma)$.

6. Structure of $H^1(\Gamma, \Pi_{2q-2})$

In this section we derive some of the consequences of Proposition 4.1. We assume throughout that Γ is a Kleinian group with region of discontinuity Ω, and that Δ is an invariant union of components of Ω such that Δ/Γ is of finite type.

THEOREM 6.1. *Let $\mathcal{H}$ be a finite set of principal parts at non-equivalent points of $\bar{\Delta}$. Then $\mathcal{H}$ is a complete system of principal parts for a meromorphic $(1-q)$-form E on $\overline{\Delta/\Gamma}$ if and only if the linear functional l associated with $\mathcal{H}$ vanishes on $A_q(0)$.*

Proof. If $l(\varphi)=0$ for all $\varphi\in A_q(0)$, then there is a $p\in\Pi_{2q-2}$ and an Eichler integral E (with $\mathcal{H}$ as a complete system of principal parts) such that $E+p$ is a meromorphic $(1-q)$-form by Proposition 4.1. The converse follows since $E\varphi$ is a 1-form for all $\varphi\in A_q(0)$, whenever E is a $(1-q)$-form, and the sum of the residues of $E\varphi$ (on $\overline{\Delta/\Gamma}$) is zero.

Let $k=\dim A_q(0)$. We shall say that $k+1$ divisors $\{d_1, ..., d_{k+1}\}$ are *adopted* to $A_q(0)$ if

$$0=d_0<d_1<d_2<...<d_{k+1}, \tag{6.1}$$

and if there is a basis $\{\varphi_1, ..., \varphi_k\}$ of $A_q(0)$ with

$$(\varphi_j)\geqslant \alpha^q+d_j, \quad \text{and} \quad (\varphi_j)\not\geqslant \alpha^q+d, \tag{6.2}$$

for all divisors d with $d_j<d\leqslant d_{j+1}$, $j=1, ..., k$. Select points $x_j\in\overline{\Delta/\Gamma}$ such that

$$d_j+x_j\leqslant d_{j+1} \quad \text{for } j=0, ..., k. \tag{6.3}$$

Let E_j $(j=1, ..., k)$ be a strongly parabolic Eichler integral whose polar divisor is $\alpha^{1-q}-d_j-x_j$. Such integrals exist by Proposition 4.1. Set $-d=d_k+x_k$. Let $E^*_{1-q}(d)$ be the space of Eichler integrals spanned by $E_1, ..., E_k$ and $E_{1-q}(0)$, and $PE^*_{1-q}(d)$, the space spanned by $E_1, ..., E_k$ and $PE_{1-q}(0)$. Obviously $PE^*_{1-q}(d)\subset PE_{1-q}(d)$ and $E^*_{1-q}(d)\subset E_{1-q}(d)$.

THEOREM 6.2. *We have*

$$H^1(\Gamma, \Pi_{2q-2})\cong E^*_{1-q}(d),$$

and

$$PH^1_\Delta(\Gamma, \Pi_{2q-2})\cong PE^*_{1-q}(d).$$

Proof. If Δ/Γ has no compact components, let $\Delta_\circ=\Delta$. If $S=\Delta/\Gamma$ has compact components, select a point in each such component and let $S_\circ$ be S punctured at these points. Let $\Delta_\circ=\pi^{-1}(S_\circ)$ where π: $\Delta\to\Delta/\Gamma$ is the natural projection map. In either case $\overline{\Delta/\Gamma}=\bar{\Delta}/\Gamma$ is a finite union of compact surfaces, and $\Delta_\circ/\Gamma$ is a finite union of open surfaces.

The isomorphism of the theorem is the period map, pd. Let $E\in E^*_{1-q}(d)$ and assume pd $E=0$. We may without loss of generality assume that E is a meromorphic $(1-q)$-form. Write

$$E=\sum_{i=0}^{k} c_i E_i, \tag{6.4}$$

with $c_i \in \mathbf{C}$, $c_0=1$, $E_0 \in E_{1-q}(0)$. Let j be the largest integer $\leqslant k$ such that $c_j \neq 0$. If $j>0$, then $E\varphi_j$ is a meromorphic 1-form with a single simple pole x_j. Since the sum of the residues of a 1-form is zero, we conclude $j=0$. Thus the Eichler integral E is a holomorphic $(1-q)$-form. By Riemann–Roch or Gauss–Bonnet, $E=0$. Thus pd is a monomorphism. We must show that this linear map is surjective.

Let p be a cocyle that represents a cohomology class of $H^1(\Gamma, \Pi_{2q-2})$. By Proposition 3.2, there is an Eichler integral $\hat{E}_1$ holomorphic on $\Delta_\circ$ such that pd $\hat{E}_1=p$. There are finitely many non-equivalent points in $\bar{\Delta}-\Delta_\circ$. The Eichler integral $\hat{E}_1$ determines a principal part at each of these points. Let $\mathcal{H}_1$ be this finite system of principal parts, and l_1 the associated linear functional.

Let E_i^* be the linear functional associated to the principal part of E_i, $i=1, ..., k$. Observe

$$E_i^*(\varphi_j)=\varepsilon_i \delta_{ij} \quad \text{for } i=1, \ldots, k, \; j=i, \ldots, k,$$

with $\varepsilon_i \in \mathbf{C}-\{0\}$, and δ_{ij} the Kronecker delta function. Thus the $k \times k$ matrix

$$(E_i^*(\varphi_j)) \quad i, j=1, \ldots, k \tag{6.5}$$

is non singular, and $\{E_1^*, ..., E_k^*\}$ are linearly independent linear functionals on $A_q(0)$. In particular, we can choose constants b_j, $j=1, ..., k$, such that

$$l_1=\sum_{j=1}^{k} b_j E_j^* \quad \text{on } A_q(0).$$

Let $\mathcal{H}_2$ be the finite system of principal parts of $\hat{E}_2=\sum_{j=1}^{k} b_j E_j$, and $\mathcal{H}=\mathcal{H}_1-\mathcal{H}_2$. The linear functional associated to $\mathcal{H}$ is zero on $A_q(0)$. Thus there exists a meromorphic $(1-q)$-form $\hat{E}_3$ such that $\mathcal{H}$ is a complete system of principal parts for $\hat{E}_3$. In particular, $\hat{E}_4=\hat{E}_1-\hat{E}_2-\hat{E}_3$ is a holomorphic Eichler integral, and pd $(\hat{E}_2+\hat{E}_4)=$ pd $(\hat{E}_1-\hat{E}_3)=$ pd $(\hat{E}_1)$.

COROLLARY 2. *There exist divisors d such that*

$$\text{pd: } E_{1-q}(d) \xrightarrow{\cong} H^1(\Gamma, \Pi_{2q-2})$$

and

$$\text{pd: } PE_{1-q}(d) \xrightarrow{\cong} PH^1_\Delta(\Gamma, \Pi_{2q-2}).$$

Proof. If suffices to construct divisors $\{d_1, ..., d_{k+1}\}$ as in (6.1) that satisfy (6.2) and such that there exist points x_j $(j=0, ..., k)$ with

$$d_j + x_j = d_{j+1}. \tag{6.3)'$$

Such divisors can clearly be found, since we may work with each component of Δ/Γ separately.

COROLLARY 2. *We have canonical isomorphisms*

$$H^1(\Gamma, \Pi_{2q-2}) \cong (A_q(0))^* + E_{1-q}(0),$$

and

$$PH^1_\Delta(\Gamma, \Pi_{2q-2}) \cong (A_q(0))^* + PE_{1-q}(0).$$

The above is (essentially) the Main Theorem of [16].

The next corollary is a sort of Riemann–Roch Theorem for Eichler integrals.

COROLLARY 3. *The following is a commutative diagram with exact rows for every divisor* $d \leqslant 0$:

$$\begin{array}{ccccccccc}
0 \to A_{1-q}(d) & \xrightarrow{i} & PE_{1-q}(d) & \xrightarrow{\text{pd}} & PH^1_\Delta(\Gamma, \Pi_{2q-2}) & \xrightarrow{*} & (A_q(-d))^* \to 0 \\
\downarrow id & & \cap\downarrow & & \cap\downarrow & & \downarrow id \\
0 \to A_{1-q}(d) & \xrightarrow{i} & E_{1-q}(d) & \xrightarrow{\text{pd}} & H^1(\Gamma, \Pi_{2q-2}) & \xrightarrow{*} & (A_q(-d))^* \to 0,
\end{array}$$

where i is the inclusion map, pd *is the period map, and * is defined as follows: If $p \in H^1(\Gamma, \Pi_{2q-2})$, then $p = \text{pd}\, E$, with E a meromorphic Eichler integral. We set $*p = E^*$, where E^* is the restriction to $A_q(-d)$ of the linear functional associated to the (finite) system of principal parts of E.*

Proof. Clearly * is well defined. The inclusion map i is injective since there are no automorphic polynomial forms for Γ. Let $E \in E_{1-q}(d)$ and assume pd $E=0$. Then we can find a representative for E that gives the zero cocycle; that is, $E \in A_{1-q}(d)$. We have shown kernel pd$\subset$image i. The reverse inclusion is, of course, trivial. Next assume that $p \in \text{pd}\, E_{1-q}(d)$. Then $p = \text{pd}\, E$ with $E \in E_{1-q}(d)$. Hence $E\varphi$ is holomorphic on $\bar{\Delta}$ for all $\varphi \in A_q(-d)$. Thus $E^* | A_q(-d) = 0$; that is, $p \in$kernel *. Hence image pd$\subset$kernel *. To establish the reverse inclusion, choose $k+1$ ($k = \dim A_q(0)$) divisors $\{d_1, ..., d_{k+1}\}$ adopted to $A_q(0)$. Furthermore, we may assume that (see 6.3)

$$d_{j_0} + x_{j_0} \leqslant -d \leqslant d_{j_0+1} \tag{6.6}$$

for some j_0, $j_0 = 0, ..., k$.

Recall the strongly parabolic Eichler integrals $E_1, ..., E_k$ constructed in the proof of

Theorem 6.2. If $p \in H^1(\Gamma, \Pi_{2q-2})$, then $p = \text{pd } E$ with E given by (6.4). If $p \in$ kernel *, then (6.6) shows that $E \in E_{1-q}(d)$. It remains to show that * is surjective. But this is an easy consequence of the arguments in the proof of Theorem 6.2.

7. Holomorphic Eichler integrals

Let Δ be an invariant component of a finitely generated Kleinian group Γ. In this section we obtain lower bounds for $\dim E_{1-q}^{\text{hol}}(\Delta, \Gamma)$ and $\dim PE_{1-q}^{\text{hol}}(\Delta, \Gamma)$.

Recall that Δ/Γ is a Riemann surface of finite type. Let Δ' be the complement in Δ of the elliptic fixed points in Δ. Then Δ'/Γ is again of finite type. Thus $\Delta' \to \Delta'/\Gamma$ exhibits Δ' as a planar regular covering surface of Δ'/Γ. By Maskit's planarity theorem [21] there exists a finite set of simple, closed, mutually disjoint, orientation preserving loops

$$\{u_1, \ldots, u_k\} \tag{7.1}$$

on $S = \Delta'/\Gamma$, and there exists a set of positive integers

$$\{\alpha_1, \ldots, \alpha_k\}$$

so that every loop u on Δ' may be deformed to a loop of the form

$$\prod_{i=1}^{k} W_i U_i W_i^{-1}, \tag{7.2}$$

where U_i is some lifting of $u_i^{\alpha_i}$, and W_i some curve in Δ', $i = 1, \ldots, k$. Note that $\Delta' \to S$ is the highest regular covering of S, for which the loops $u_i^{\alpha_i}$ lift to loops. The covering group N of the covering $\Delta' \to S$ is then the smallest normal subgroup of $\pi_1(S)$, the fundamental group of S, that contains all the elements $u_i^{\alpha_i}$. (We have suppressed all reference to base points. Thus for $\pi_1(S)$ we must choose a base point $0 \in S$ and we view $u_i^{\alpha_i}$ as an element of $\pi_1(S)$ by joining $u_i^{\alpha_i}$ to 0 by a curve f_i and taking the homotopy class of $f_i \circ u_i^{\alpha_i} \circ f_i^{-1}$.)

Consider k loops $U_1^*, \ldots, U_k^*$ lying over the loop $u_1^{\alpha_1}, \ldots, u_k^{\alpha_k}$ and eliminate from the set $\{U_1^*, \ldots, U_k^*\}$ loops that bound a punctured disc in Δ'. Note that every puncture on Δ' arises from a fixed point of an elliptic element in Γ; that is, the punctures on Δ' are precisely the points in $\Delta - \Delta'$. We obtain this way $K \leqslant k$ loops. We choose K as low as possible. Let us assume that we have chosen in this manner the loops $U_1^*, \ldots, U_K^*$ that cover the loops $u_1^{\alpha_1}, \ldots, u_K^{\alpha_K}$. In this way every element in $\pi_1(\Delta)$ may be represented by a loop of the form (7.2). However, the product extends from 1 to K.

THEOREM 7.1. *We have*

$$\dim PE^{\text{hol}}_{1-q}(\Delta, \Gamma) \geqslant \dim B_q(\Delta, \Gamma) - (2q-1)K,$$

and

$$\dim E^{\text{hol}}_{1-q}(\Delta, \Gamma) \geqslant \dim B_q(\Delta, \Gamma) + n - (2q-1)K,$$

where n *is the number of punctures on* Δ/Γ.

Proof. Let $\varphi \in B_q(\Delta, \Gamma)$, and set

$$F(z) = \frac{1}{(2q-2)!}\int_{z_0}^{z} (z-\zeta)^{2q-2}\varphi(\zeta)\, d\zeta,$$

where $z_0 \in \Delta$ is fixed, $z \in \Delta$ is arbitrary, and we integrate over an arbitrary path in Δ from z_0 to z. It is clear that F is locally well defined and $\partial^{2q-1}F = \varphi$. Thus $F \in PE^{\text{hol}}_{1-q}(\Delta, \Gamma)$ if and only if for every $C \in \pi_1(\Delta)$ and every $z \in \mathbf{C}$, we have

$$\int_C (z-\zeta)^{2q-2}\varphi(\zeta)\, d\zeta = 0. \tag{7.3}$$

(Recall that $\pi_1(\Delta)$ is in general a free group on infinitely many generators.) Note that (7.3) may be rewritten as

$$\int_C \zeta^j \varphi(\zeta)\, d\zeta = 0, \quad j = 0, 1, \ldots, 2q-2. \tag{7.3$'$}$$

We have observed that C is homotopic to a curve of the type (7.2). Thus

$$\int_C (z-\zeta)^{2q-2}\varphi(\zeta)\, d\zeta = \sum_{i=1}^{K}\left(\int_{W_i} + \int_{U_i} + \int_{W_i^{-1}}\right)(z-\zeta)^{2q-2}\varphi(\zeta)\, d\zeta.$$

Since

$$\left(\int_{W_i} + \int_{W_i^{-1}}\right)(z-\zeta)^{2q-2}\varphi(\zeta)\, d\zeta = 0,$$

we conclude that

$$\int_C (z-\zeta)^{2q-2}\varphi(\zeta)\, d\zeta = \sum_{i=1}^{K}\int_{U_i} (z-\zeta)^{2q-2}\varphi(\zeta)\, d\zeta.$$

Thus in (7.3) we need consider only curves $C = U_i$, some i. Note that if

$$\int_{U_i^*} (z-\zeta)^{2q-2}\varphi(\zeta)\, d\zeta = 0, \quad i = 1, \ldots, K, \tag{7.4}$$

then

$$\int_{U_i} (z-\zeta)^{2q-2}\varphi(\zeta)\, d\zeta = 0, \quad \text{all } U_i. \tag{7.5}$$

For given a U_i, it lies over $u_i^{\alpha_i}$. So does U_i^*. Thus there is an element $\gamma \in \Gamma$ so that

$$U_i = \gamma U_i^*.$$

We compute using the identity

$$(\gamma\zeta - \gamma z)^2 = (\zeta - z)^2 \gamma'(\zeta)\gamma'(z),$$

$$\int_{\gamma(U_i^*)} (\gamma z - \zeta)^{2q-2} \varphi(\zeta)\, d\zeta = \int_{U_i^*} (\gamma z - \gamma\zeta)^{2q-2} \varphi(\gamma\zeta)\gamma'(\zeta)\, d\zeta$$

$$= \gamma'(z)^{q-1} \int_{U_i^*} (z-\zeta)^{2q-2} \varphi(\gamma\zeta)\gamma'(\zeta)^q d\zeta = \gamma'(z)^{q-1} \int_{U_i^*} (z-\zeta)^{2q-2} \varphi(\zeta)\, d\zeta.$$

Thus we have shown that (7.4) implies (7.5); and, hence, a necessary and sufficient condition for F to be well defined is (7.4). Recalling the equivalence of (7.3) and (7.3)', we see that (7.4) imposes $(2q-1)K$ conditions on the elements of $B_q(\Delta, \Gamma)$. We have verified the first inequality of the theorem. The second is a direct consequence of the first and the Riemann–Roch theorem.

8. Generalizations

Let Γ be a Kleinian group and $q \geqslant 2$ an integer or half-integer. Let χ be a character on Γ (that is, a homomorphism of Γ into the multiplicative group of complex numbers of modulus one). We may define an action of Γ on Π_{2q-2} by

$$(p\gamma)(z) = p(\gamma z)\gamma'(z)^{1-q}\chi(\gamma), \quad z \in \mathbf{C},\ p \in \Pi_{2q-2},\ \gamma \in \Gamma. \tag{8.1}$$

If q is a half-integer (and not an integer), we must, of course, assume that it is possible to select branches of $\{(\gamma')^{\frac{1}{2}};\ \gamma \in \Gamma\}$ such that for all γ_1 and $\gamma_2 \in \Gamma$ and all $z \in \mathbf{C}$, we have

$$(\gamma_1 \circ \gamma_2)'(z)^{\frac{1}{2}} = \gamma_1'(\gamma_2 z)^{\frac{1}{2}} \gamma_2'(z)^{\frac{1}{2}}. \tag{8.2}$$

We obtain in this manner a cohomology group that may be denoted by $H^1(\Gamma, \Pi_{2q-2}, \chi)$. The preceeding development generalizes to this setting. There are, of course, some obvious changes. The orders of zeros of automorphic forms at elliptic fixed points depend on the character χ. So does the concept of parabolicity of cohomology classes.

If $\chi(\gamma) \neq 1$ for every $\gamma \in \Gamma$ that generates a parabolic subgroup of Γ that corresponds to a puncture on Δ/Γ, then

$$PH^1_\Delta(\Gamma, \Pi_{2q-2}, \chi) = H^1(\Gamma, \Pi_{2q-2}, \chi).$$

By invariance of parabolicity under conjugation it suffices to assume $\gamma(z)=z+1$. Let p be a cocycle that represents a cohomology class of $H^1(\Gamma, \Pi_{2q-2}, \chi)$. We must show that there is a $v \in \Pi_{2q-2}$ such that $p_\gamma = v\gamma - v$. Consider the linear map $\delta: \Pi_{2q-2} \to \Pi_{2q-2}$, where

$$(\delta v)(z) = \chi v(z+1) - v(z),$$

with $\chi = \chi(\gamma) \neq 1$. The map δ is clearly injective, and thus surjective.

Problems. (1) For what Kleinian groups Γ is it possible to choose a selection of branches of $\{(\gamma')^{\frac{1}{2}};\ \gamma \in \Gamma\}$ that satisfy (8.2) for all $z \in \mathbf{C}$, all $\gamma_1, \gamma_2 \in \Gamma$?

(2) Describe the character group of a Kleinian group.

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