A THEOREM ON NEVANLINNA DEFICIENCIES

BY

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We shall prove the following

THEOREM. *Let $f(z)$ be meromorphic and of finite lower order μ in the finite plane, and let $a_1, a_2, \ldots$ be its set of Nevanlinna deficient values. Then*

$$\sum_{\nu} \delta^{\frac{1}{2}}(a_\nu, f) < \infty. \tag{1}$$

This problem seems to have first been considered in 1939 by O. Teichmüller [16; p. 167] who suggested that, in addition to the classical Nevanlinna defect relation

$$\sum_{\nu} \delta(a_\nu, f) \leqslant 2,$$

certain conditions including finite order might imply

$$\sum_{\nu} \delta^{\frac{1}{3}}(a_\nu, f) < \infty. \tag{2}$$

In 1957 W. Fuchs [5] established (2) under only the assumption that $f(z)$ be of finite lower order. This work was subsequently refined by V. Petrenko [13], and I. Ostrovskii and I. Kazakova [9] who concentrated primarily on the bounds for the sum (2); an alternative proof of Fuchs's theorem was given in 1965 by A. Edrei [2; p. 85].

A major advance was made by W. Hayman [8; p. 90] who proved that if $f(z)$ has finite lower order then

$$\sum_{\nu} \delta^{\frac{1}{2}+\varepsilon}(a_\nu, f) < \infty$$

for every $\varepsilon > 0$.

Following Hayman's approach, Petrenko [14], in 1966, proved the convergence of $\sum \delta^{\frac{1}{2}}(a_\nu, f)\, (\log e/\delta(a_\nu, f))^{-1}$ and in the following year E. Bombieri and P. Ragnedda [1] proved the convergence of $\sum (\delta(a_\nu, f)\, \sigma(\delta(a_\nu, f)))^{\frac{1}{2}}$ for suitable functions $\sigma(t)$ satisfying $\int_0 \sigma(t)/t\, dt < \infty$.

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As to the best possible nature of our Theorem, Hayman has shown that a construction technique due essentially to A. Goldberg [6] will yield examples of meromorphic functions of finite order for which $\sum \delta^{\frac{1}{3}-\varepsilon}(a_\nu, f)$ diverges for every $\varepsilon > 0$. In fact, as Hayman has observed [8; p. 98], the convergence of (1) may be made arbitrarily slow in the sense that, given any convergent positive series $\sum c_n$, there exists a meromorphic function of finite order such that $\delta^{\frac{1}{3}}(a_\nu, f) > Kc_\nu$ $(\nu = 1, 2, \ldots)$ for some constant $K > 0$. The necessity of finite lower order in the Theorem can be seen by examples due to Fuchs and Hayman [8; p. 80].

I should like to express my gratitude to the late Sir Edward Collingwood for several discussions which inspired this proof.

1. Notations and conventions

Our study of $f(z)$ will be confined to annuli around the Pólya peaks $\{r_m\}$ of order μ of $T(r, f)$. For the basic properties and existence of Pólya peaks the reader is referred to [2; p. 82]. The relevant property for our investigation is that for any fixed $\sigma \geq 1$,

$$T(r, f) \leq \left(\frac{r}{r_m}\right)^{\mu} T(r_m, f)\,(1 + o(1)) \quad (m \to \infty) \tag{1.1}$$

for $r_m \leq r \leq \sigma r_m$.

We shall take $\{\alpha_m\}$ to represent a sequence of positive numbers tending to ∞ such that

$$\alpha_m = o(T(r_m, f)) \quad (m \to \infty) \tag{1.2}$$

and corresponding to $\{\alpha_m\}$ we define

$$\mathcal{E} = \mathcal{E}(\{\alpha_m\}) = \bigcup_m \{z = re^{i\theta}\colon r_m \leq r \leq 6r_m, \log |f'(re^{i\theta})| < -\alpha_m\}. \tag{1.3}$$

A particular sequence $\{\alpha_m\}$ will be specified in Lemma B.

We shall make some assumptions here for the sake of later convenience, and without loss of generality. First we assume that the intervals $[r_m, 6r_m]$ are pairwise disjoint. We also assume that the value ∞ is not included among the set $a_1, a_2, \ldots$ of deficient values and that this set has at least two elements. The latter condition implies that $\mu > 0$ [3; p. 297].

Throughout the proof of the Theorem, the letter K will denote constants which will not necessarily be the same at each occurrence, and which may depend upon some parameters. More precisely, the constants K which appear in the proofs may depend upon the lower order μ as well as the sizes of the first two deficiencies $\delta(a_1, f)$, $\delta(a_2, f)$.

2. Preliminary lemmas

In this section we shall prove five lemmas, none of which is essentially new. Lemmas A and C are quite standard estimates; Lemma B is substantially contained in Lemma 1 of [18]; and Lemma D is a variation of a theorem of H. Selberg [15; p. 311] (see also [17; p. 22]) in a form suitable for our applications. Lemma E is an inequality on harmonic measure which is closely related to known estimates [12, p. 76] but which takes into account the fact that the sets we consider need not be simply connected.

LEMMA A. *Let $f(z)$ be as in the statement of the Theorem and $\{r_m\}$ a sequence of Pólya peaks of order μ of $T(r,f)$. Then if $\sigma > 1$ is fixed*

$$\delta(a_1, f)\, T(r,f)\,(1+o(1)) \leqslant T(r,f') \leqslant 2T(r,f)\,(1+o(1)) \tag{2.1}$$

as $r \to \infty$ through the intervals $r_m \leqslant r \leqslant \sigma r_m$.

Proof. We first make some remarks regarding the exceptional set which arises in the study of the logarithmic derivative. A basic fact of Nevanlinna's theory is that

$$m\left(r, \frac{f'}{f-a}\right) = o(T(r,f)) \tag{2.2}$$

as $r \to \infty$ outside an exceptional r set E which has finite measure [8; p. 41]. It is important to note that E occurs in intervals where the characteristic grows very rapidly; in particular E does not depend on the value a [8; p. 41], and consideration of the growth lemma from which it arises [8; p. 38] shows that it may be taken to be disjoint from the intervals $[r_m, \sigma r_m]$. In other words (2.2) must hold as $r \to \infty$ through the intervals $r_m \leqslant r \leqslant \sigma r_m$.

The right-hand inequality of (2.1) follows from [8; p. 55], and the left-hand side from the above remarks and the elementary computation

$$\delta(a_1, f)\, T(r,f)\,(1+o(1)) \leqslant m\left(r, \frac{1}{f-a_1}\right) \leqslant m\left(r, \frac{1}{f'}\right) + m\left(r, \frac{f'}{f-a_1}\right)$$
$$\leqslant T(r,f')\,(1+o(1)) \quad (r\to\infty;\, r_m \leqslant r \leqslant \sigma r_m).$$

LEMMA B. *Let $f(z)$ be as in the Theorem and $\{r_m\}$ a sequence of Pólya peaks of order μ of $T(r,f)$. There exist $\mathcal{E}$ and $\{\alpha_m\}$ as in* (1.2) *and* (1.3), *and pairwise disjoint subsets $\mathcal{E}_k (k = 1, 2, \ldots)$ of $\mathcal{E}$ such that each $\mathcal{E}_k$ is the union of components of $\mathcal{E}$, and if $\mathcal{E}_k(r)$ is the set of arguments in $[0, 2\pi)$ of $\mathcal{E}_k \cap \{|z| = r\}$ we have*

$$\frac{1}{2\pi}\int_{\mathcal{E}_k(r)} \log \frac{1}{|f'(re^{i\theta})|}\, d\theta \geqslant \tfrac{1}{3}\,\delta(a_k, f)\, T(r,f') \quad (r > r_0(k);\, r_m \leqslant r \leqslant 6 r_m). \tag{2.3}$$

Proof. We use the same procedure here as in [18; p. 123].

By (1.1), (2.1), the Cartan identity [12; p. 177] and the fact $\mu>0$, we deduce that

$$\frac{1}{2\pi}\int_0^{2\pi} n\left(6r_m, \frac{1}{f'-te^{i\phi}}\right)d\phi \leqslant (2\pi\log 2)^{-1}\int_0^{2\pi} N\left(12r_m, \frac{1}{f'/t-e^{i\phi}}\right)d\phi$$
$$\leqslant (\log 2)^{-1}\left(\log^+\frac{1}{t}+T(12r_m,f')(1+o(1))\right)$$
$$\leqslant 2(\log 2)^{-1}\left(\log^+\frac{1}{t}+T(12r_m,f)(1+o(1))\right)$$
$$\leqslant 2(\log 2)^{-1}\left(\log^+\frac{1}{t}+12^{\mu}T(r_m,f)(1+o(1))\right)\quad (m\to\infty).$$

Applying the length-area inequality(1) [7; p. 18] we then have

$$\int_\gamma^\beta \frac{l^2(t)}{t}\,dt \leqslant 144\pi^2(\log 2)^{-1}r_m^2\left(\log^+\frac{1}{\gamma}+12^{\mu}T(r_m,f)(1+o(1))\right)\quad (m\to\infty)$$

where $l(t)$ denotes the total length of the level curves $|f'(z)|=t\,(\gamma<t<\beta)$ in $\{|z|<6r_m\}$. Taking $\gamma=\frac{1}{2}\exp(-\sqrt{T(r_m,f)})$ and $\beta=\exp(-\sqrt{T(r_m,f)})$ we deduce the existence of α_m such that

$$\sqrt{T(r_m,f)}\leqslant\alpha_m\leqslant\sqrt{T(r_m,f)}+\log 2 \tag{2.4}$$

and $$l^2(e^{-\alpha_m})\log 2\leqslant 144\pi^2(\log 2)^{-1}12^{\mu}r_m^2T(r_m,f)(1+o(1))\quad (m\to\infty)$$

so that $$l(e^{-\alpha_m})\leqslant(\log 2)^{-1}\pi 12^{1+\mu/2}r_m\sqrt{T(r_m,f)}(1+o(1))\quad (m\to\infty). \tag{2.5}$$

We thus fix the sequence $\{\alpha_m\}$ as above and note that with this choice, (2.4) implies (1.2). A simple geometric argument and (2.5) (cf. [18, p. 124]) yields that if z_1, z_2 are two points in the same component of $\mathcal{E}$ and $r_m\leqslant|z_1|\leqslant 6r_m$, $r_m\leqslant|z_2|\leqslant 6r_m$, there exists a continuous curve Γ joining z_1 and z_2 whose length does not exceed

$$K(\mu)\,r_m\sqrt{T(r_m,f)}\,(1+o(1))\quad (m\to\infty)$$

on which $|f'(z)|\leqslant e^{-\alpha_m}$ and hence

$$|f(z_1)-f(z_2)|=\left|\int_\Gamma f'(z)\,dz\right|\leqslant K(\mu)r_m\sqrt{T(r_m,f)}\exp(-\sqrt{T(r_m,f)})(1+o(1))=o(1)\quad (m\to\infty).$$

(1) While this inequality is stated for regular functions it is readily seen to be valid for meromorphic functions.

Now we consider the sets

$$\mathcal{F}_k = \bigcup_m \{z = re^{i\theta}: r_m \leqslant r \leqslant 6r_m, \log |f(re^{i\theta}) - a_k| < -\alpha_m\}. \tag{2.7}$$

Taking $\mathcal{F}_k(r)$ as the argument set in $[0, 2\pi)$ of $\mathcal{F}_k \cap \{|z| = r\}$, it follows from (1.2) and (2.7) that

$$m\left(r, \frac{1}{f - a_k}\right) = \frac{1}{2\pi} \int_{\mathcal{F}_k(r)} \log \frac{1}{|f(re^{i\theta}) - a_k|} \, d\theta + o(T(r_m, f))$$

as $r \to \infty$ through the values $r_m \leqslant r \leqslant 6r_m$. Since (2.2) holds as $r \to \infty$ through these intervals we have from (2.8) that

$$m\left(r, \frac{1}{f - a_k}\right) \leqslant \frac{1}{2\pi} \int_{\mathcal{F}_k(r)} \log^+ \frac{1}{|f'(re^{i\theta})|} \, d\theta + o(T(r_m, f)) \tag{2.9}$$

as $r \to \infty$ through the intervals $r_m \leqslant r \leqslant 6r_m$.

Let G_j be the set formed by the components of $\mathcal{E}$ which have a nonempty intersection with $\mathcal{F}_j (j = 1, 2, \ldots)$. From (2.6) and (2.7) if follows that for some integer m_k

$$\{|z| \geqslant r_{m_k}\} \cap G_k \cap G_j = \varnothing \quad (j < k). \tag{2.10}$$

We define

$$\mathcal{E}_k = G_k \cap \{|z| \geqslant r_{m_k}\}. \tag{2.11}$$

It then follows from (2.10) and (2.11) that the sets $\mathcal{E}_k$ are pairwise disjoint and each is the union of components of $\mathcal{E}$. Moreover, since $-\log |f'(re^{i\theta})| \leqslant \alpha_m = o(T(r_m, f))$ outside the set $\mathcal{E}$ as $r \to \infty$ through the intervals $r_m \leqslant r \leqslant 6r_m$ it follows from (2.1), (2.9), (2.11) and the definition of G_k that (2.3) must hold.

LEMMA C. *Let the hypotheses and notations of Lemma B be unchanged and let $\theta_k(r)$ denote the angular measure of the intersection of $\mathcal{E}_k$ and $\{|z| = r\}$ $(r_m \leqslant r \leqslant 6r_m)$. Then there exists $\theta_0 > 0$ such that*

$$\limsup_{\substack{r \to \infty \\ r_m \leqslant r \leqslant 6r_m}} \theta_k(r) \leqslant 2\pi - \theta_0 \quad (k = 1, 2, \ldots) \tag{2.12}$$

Proof. Since we have assumed that f has at least 2 deficient values we may define

$$\alpha_j = \liminf_{\substack{r \to \infty \\ r_m \leqslant r \leqslant 6r_m}} \theta_j(r) \quad (j = 1, 2).$$

Using a known lemma of Edrei and Fuchs [4; p. 322] we have

$$\frac{1}{2\pi} \int_{\mathcal{E}_j(r)} \log \frac{1}{|f'(re^{i\theta})|} \, d\theta \leqslant 22\, T(2r, f')\, \theta_j(r) \left[1 + \log^+ \frac{1}{\theta_j(r)}\right]$$

and thus, if $\alpha_1 = 0$ or $\alpha_2 = 0$ we would have by (1.1) and (2.1) an immediate contradiction to (2.3).

Taking $\theta_0 = \min\{\alpha_1, \alpha_2\}$ the result then follows.

LEMMA D. *Let D be a component of the set $\mathcal{E}$ of (1.3) and $g(z, z_0)$ its Green's function with pole at z_0. Let S be the intersection of D with $\{|z| = r\}$ and $\theta(r)$ its angular measure. Then if S is nonempty*

$$\int_S g(re^{i\theta}, z_0)\, d\theta \leqslant 2\pi^2 \omega(z_0, r) \tan \frac{\theta(r)}{4}$$

where, for the case $|z_0| < r$, $\omega(z_0, r)$ is the harmonic measure at z_0 of the component of $D \cap \{|z| < r\}$ containing z_0, with respect to the portion of the boundary on $\{|z| = r\}$; for $|z_0| > r$, $\omega(z_0, r)$ is the harmonic measure at z_0 of the component of $D \cap \{|z| > r\}$ containing z_0 with respect to the portion of the boundary on $\{|z| = r\}$; and for $|z_0| = r$, $\omega(z_0, r) = 1$.

Proof. The case $|z_0| = r$ is covered by Selberg's original theorem [15; p. 311]. Suppose now $|z_0| < r$. Then we define a function $\omega(z)$ on $D \cap \{|z| < r\}$ by taking $\omega(z)$ to be the harmonic measure of the component of $D \cap \{|z| < r\}$ containing z, with respect to the portion of the boundary of the component on $\{|z| = r\}$. Then, by the principle of monotoneity [12; p. 68] and symmetry of the Poisson kernel we have for $z = te^{i\alpha} \in D \cap \{|z| < r\}$

$$\omega(z) \leqslant \frac{1}{2\pi} \int_S \frac{(r^2 - t^2)\, d\psi}{r^2 - 2tr \cos(\psi - \alpha) + t^2} \leqslant \frac{1}{\pi} \int_0^{\theta(r)/2} \frac{(r^2 - t^2)\, d\chi}{r^2 - 2tr \cos \chi + t^2}.$$

Therefore,

$$\frac{1 - \omega(z)}{r - t} \geqslant \frac{1}{\pi} \int_{\theta(r)/2}^{\pi} \frac{(r + t)\, d\chi}{r^2 - 2tr \cos \chi + t^2}$$

and

$$\frac{\partial \omega}{\partial r} \geqslant \frac{1}{\pi r} \int_{\theta(r)/2}^{\pi} \frac{d\chi}{1 - \cos \chi} = \frac{1}{\pi r} \cot \frac{\theta(r)}{4} \quad \text{on } S. \tag{2.13}$$

By Green's formula we have

$$\int_S g \frac{\partial \omega}{\partial r}\, r d\theta = \int_S \frac{\partial g}{\partial r}\, r d\theta + 2\pi \omega(z_0). \tag{2.14}$$

Now, let T be the portion of the boundary of $D \cap \{|z| > r\}$ not in S. Another application of Green's formula yields

$$\int_T \frac{\partial g}{\partial n}\, ds - \int_S \frac{\partial g}{\partial r}\, r d\theta = 0$$

(n denoting the outward pointing normal) so that

$$\int_S \frac{\partial g}{\partial r}\, r d\theta = \int_T \frac{\partial g}{\partial n}\, ds < 0. \tag{2.15}$$

Then, using (2.13) and (2.15) in (2.14) we obtain the result for $|z_0| < r$.

As for the case $|z_0| > r$, we make the change of variables $\zeta = r^2/\bar{z}$ and apply the preceding proof.

LEMMA E. *Let the hypotheses and notations of Lemma D be unchanged. Then*

$$\omega(z_0, r) \leqslant \begin{cases} \exp\left(-\dfrac{1}{\pi} \displaystyle\int_{|z_0|}^{r} \dfrac{d\varrho}{\varrho \tan \dfrac{\theta(\varrho)}{4}} \right) & (|z_0| \leqslant r) \\ \exp\left(-\dfrac{1}{\pi} \displaystyle\int_{r}^{|z_0|} \dfrac{d\varrho}{\varrho \tan \dfrac{\theta(\varrho)}{4}} \right) & (|z_0| > r). \end{cases}$$

Proof. We assume first $|z_0| < r$ and let $z_1 \in D(|z_0| < |z_1| < r)$. Take $\omega(z_1, \varrho)$ to be the harmonic measure at z_1 of the component of $D \cap \{|z| < \varrho\}$ $(|z_1| < \varrho < r)$ containing z_1, with respect to the intersection of $|z| = \varrho$ and D. Then as in the proof of Lemma D we have

$$\omega(z_1, \varrho) \leqslant \frac{1}{\pi} \int_0^{\theta(\varrho)/2} \frac{(\varrho^2 - \varrho_1^2)\, d\chi}{\varrho^2 + \varrho_1^2 - 2\varrho\varrho_1 \cos \chi} = \frac{2}{\pi} \operatorname{arc\,tan} \left(\frac{\varrho + \varrho_1}{\varrho - \varrho_1} \tan \frac{\theta(\varrho)}{4} \right) \quad (\varrho_1 = |z_1|). \tag{2.16}$$

Following Carleman (cf. [12, p. 76]) we observe that for $|z_0| < \varrho_1 < \varrho$, the inequality (2.16) implies

$$\omega(z_0, \varrho) \leqslant \frac{2}{\pi} \left(\operatorname{arc\,tan} \left(\frac{\varrho + \varrho_1}{\varrho - \varrho_1} \tan \frac{\theta(\varrho)}{4} \right) \right) \omega(z_0, \varrho_1)$$

and thus

$$\omega(z_0, \varrho) - \omega(z_0, \varrho_1) \leqslant -\,\omega(z_0, \varrho_1) \frac{2}{\pi} \operatorname{arc\,tan} \left(\frac{\varrho - \varrho_1}{\varrho + \varrho_1} \left(\tan \frac{\theta(\varrho)}{4} \right)^{-1} \right). \tag{2.17}$$

Now, by the principle of monotoneity, $\omega(z_1, \varrho)$ is monotone decreasing with ϱ, and we may thus apply standard results (cf. [10; pp. 211, 212]) dealing with its derivative. In particular $d\omega(z_0, \varrho)/d\varrho$ exists almost everywhere and by (2.17) satisfies

$$\frac{dw}{d\varrho}(z_0, \varrho) \leqslant -\frac{\omega(z_0, \varrho)}{\pi\varrho \tan \dfrac{\theta(\varrho)}{4}}. \tag{2.18}$$

Integrating (2.18) from $|z_0|$ to r we obtain the desired inequality in the case $|z_0| < r$. The case $|z_0| = r$ is trivial; if $|z_0| > r$ the result follows from the above proof and the change of variable $\zeta = r^2/\bar{z}$.

3. Proof of the theorem

Let $\{r_m\}$ be a sequence of Pólya peaks of order μ of $T(r, f)$. By (2.3) of Lemma B, we may consider the quantities

$$\delta_k \equiv \liminf_{\substack{r\to\infty \\ r_m \leqslant r \leqslant 6r_m}} \frac{\int_{\mathcal{E}_k(r)} \log \frac{1}{|f'(re^{i\theta})|}\, d\theta}{T(r, f')} \tag{3.1}$$

in place of the respective deficiencies $\delta(a_k, f)$ in (1).

By consideration of the inequality [11; p. 25]

$$\frac{1}{r}\int_1^r \log^+ M(s, g)\, ds \leqslant K(\varkappa)\, T(\varkappa r, g) \quad (\varkappa > 1)$$

and the properties of Pólya peaks (1.1) together with (2.1) we may take sequences $\{r'_m\}$, $\{r''_m\}$ satisfying

$$r_m \leqslant r'_m \leqslant 2r_m \quad 5r_m \leqslant r''_m \leqslant 6r_m \tag{3.2}$$

and such that

$$\log^+ M\left(r'_m, \frac{1}{f'}\right) \leqslant KT(r_m, f') \qquad \log^+ M\left(r''_m, \frac{1}{f'}\right) \leqslant KT(r_m, f') \tag{3.3}$$

where $K = K(\mu, \delta(a_1, f))$ and $m > m_0$.

For each of the zeros $z_{j,k}$ $(j = 1, 2, \dots, p_{k,m})$ of $f'(z)$ in $\mathcal{E}_k \cap \{r'_m < |z| < r''_m\}$, let $g(z, z_{j,k})$ be the Green's function of the component of $\mathcal{E}_k \cap \{r'_m < |z| < r''_m\}$ containing $z_{j,k}$, and having its pole at $z_{j,k}$.

We now introduce functions $h_{1,k}(z)$ and $h_{2,k}(z)$. For $z \in \mathcal{E}_k \cap \{r'_m < |z| < r''_m\}$ let $h_{1,k}(z)$ be the harmonic measure of the component of $\mathcal{E}_k \cap \{r'_m < |z| < r''_m\}$ containing z, with respect to the portion of the boundary of the component on $\{|z| = r'_m\}$ in the case this component extends to the circumference $\{|z| = r'_m\}$; otherwise let $h_{1,k}(z) = 0$. If $z \notin \mathcal{E}_k \cap \{r'_m < |z| < r''_m\}$ for any m we again take $h_{1,k}(z) = 0$. The function $h_{2,k}(z)$ is defined in an analogous way with regard to the circumference $\{|z| = r''_m\}$.

Then, for $r'_m < r < r''_m$, it follows from (3.2) and (3.3) that

$$\frac{\int_{\mathcal{E}_k(r)} \log \frac{1}{|f'(re^{i\theta})|}\, d\theta}{T(r, f')} - \frac{2\pi\alpha_m}{T(r, f')} \leqslant \frac{1}{T(r_m, f')} \int_{\mathcal{E}_k(r)} \sum_{j=1}^{p_{k,m}} g(re^{i\theta}, z_{j,k})\, d\theta$$
$$+ K\int_{\mathcal{E}_k(r)} h_{1,k}(re^{i\theta})\, d\theta + K\int_{\mathcal{E}_k(r)} h_{2,k}(re^{i\theta})\, d\theta \tag{3.4}$$

where, in the integration of the Green's functions we take g to be zero outside its region of definition, and, as before, $\mathcal{E}_k(r)$ to be the argument set in $[0, 2\pi)$ of $\mathcal{E}_k \cap \{|z|=r\}$.

Let $\theta_k(r)$ be the measure of $\mathcal{E}_k(r)$. Then, from (3.1), (3.4), (1.2), and (2.1) we have

$$\frac{\delta_k(1+o(1))}{r\theta_k^2(r)} \leqslant \frac{1}{T(r_m, f')\, r\theta_k^2(r)} \int_{\mathcal{E}_k(r)} \sum_{j=1}^{p_{k,m}} g(re^{i\theta}, z_{j,k})\, d\theta + \frac{K}{r\theta_k^2(r)} \int_{\mathcal{E}_k(r)} h_{1,k}(re^{i\theta})\, d\theta + \frac{K}{r\theta_k^2(r)} \int_{\mathcal{E}_k(r)} h_{2,k}(re^{i\theta})\, d\theta \tag{3.5}$$

as $r \to \infty$, $r'_m < r < r''_m$.

We now integrate both sides of (3.5) from $3r_m$ to $4r_m$ and obtain

$$\delta_k(1+o(1)) \int_{3r_m}^{4r_m} \frac{dr}{r\theta_k^2(r)} \leqslant \frac{1}{T(r_m, f')} \sum_{j=1}^{p_{k,m}} \int_{3r_m}^{4r_m} \int_{\mathcal{E}_k(r)} \frac{g(re^{i\theta}, z_{j,k})}{r\theta_k^2(r)}\, d\theta\, dr + K \int_{3r_m}^{4r_m} \frac{1}{r\theta_k^2(r)} \int_{\mathcal{E}_k(r)} h_{1,k}(re^{i\theta})\, d\theta\, dr + K \int_{3r_m}^{4r_m} \frac{1}{r\theta_k^2(r)} \int_{\mathcal{E}_k(r)} h_{2,k}(re^{i\theta})\, d\theta\, dr. \tag{3.6}$$

We next estimate the first term on the right side of (3.6). To this end, we first observe that Lemma C implies the existence of a constant $K = K(\theta_0) > 0$ such that for $r'_m < r < r''_m$, and $m > m_0$

$$\tan \frac{\theta_k(r)}{4} \leqslant K\theta_k(r). \tag{3.7}$$

Now, let j_0 be the index corresponding to the maximum term in the sum in (3.6), and $z_k = z_{j_0,k}$. We assume that $3r_m \leqslant |z_k| \leqslant 4r_m$; the modifications needed for the other case will be obvious. Then, applying Lemma D, (3.7), and Lemma E we obtain, for $m > m_0(k)$

$$\begin{aligned}
\frac{1}{T(r_m, f')} \sum_{j=1}^{p_{k,m}} \int_{3r_m}^{4r_m} \int_{\mathcal{E}_k(r)} \frac{g(re^{i\theta}, z_{j,k})}{r\theta_k^2(r)}\, d\theta\, dr &\leqslant \frac{Kp_{k,m}}{T(r_m, f')} \int_{3r_m}^{4r_m} \frac{\omega(z_k, r)}{r\theta_k(r)}\, dr \\
&\leqslant \frac{Kp_{k,m}}{T(r_m, f')} \left(\int_{3r_m}^{|z_k|} \frac{\exp\left(-K\int_r^{|z_k|} \frac{dt}{t\theta_k(t)}\right)}{r\theta_k(r)}\, dr + \int_{|z_k|}^{4r_m} \frac{\exp\left(-K\int_{|z_k|}^{r} \frac{dt}{t\theta_k(t)}\right)}{r\theta_k(r)}\, dr \right) \\
&= \frac{Kp_{k,m}}{T(r_m, f')} \left(\int_{3r_m}^{|z_k|} \frac{d}{dr}\left\{\exp\left(-K\int_r^{|z_k|} \frac{dt}{t\theta_k(t)}\right)\right\} dr - \int_{|z_k|}^{4r_m} \frac{d}{dr}\left\{\left(\exp\left(-K\int_{|z_k|}^{r} \frac{dt}{t\theta_k(t)}\right)\right\} dr \right) \\
&\leqslant \frac{Kp_{k,m}}{T(r_m, f')}.
\end{aligned} \tag{3.8}$$

Next consider the remaining terms on the right side of (3.6). The same estimate for harmonic measure and (3.2) yield

$$\int_{3r_m}^{4r_m}\frac{1}{r\theta_k^2(r)}\int_{\mathcal{E}_k(r)}h_{1,k}(re^{i\theta})\,d\theta\,dr+\int_{3r_m}^{4r_m}\frac{1}{r\theta_k^2(r)}\int_{\mathcal{E}_k(r)}h_{2,k}(re^{i\theta})\,d\theta\,dr$$

$$\leqslant 2\pi\exp\left(-K\int_{r'_m}^{3r_m}\frac{dt}{t\theta_k(t)}\right)\int_{3r_m}^{4r_m}\frac{dr}{r\theta_k^2(r)}+2\pi\exp\left(-K\int_{4r_m}^{r''_m}\frac{dt}{t\theta_k(t)}\right)\int_{3r_m}^{4r_m}\frac{dr}{r\theta_k^2(r)}$$

$$\leqslant 2\pi\left(\exp\left(-K\int_{2r_m}^{3r_m}\frac{dt}{t\theta_k(t)}\right)+\exp\left(-K\int_{4r_m}^{5r_m}\frac{dt}{t\theta_k(t)}\right)\right)\int_{3r_m}^{4r_m}\frac{dr}{r\theta_k^2(r)} \qquad (3.9)$$

Combining (3.6), (3.9), and using the Cauchy–Schwarz inequality twice yields

$$\delta_k(1+o(1))\leqslant\left(\int_{3r_m}^{4r_m}\frac{dr}{r\theta_k^2(r)}\right)^{-1}\frac{Kp_{k,m}}{T(r_m,f')}+K\exp\left(-K\int_{2r_m}^{3r_m}\frac{dt}{t\theta_k(t)}\right)+K\exp\left(-K\int_{4r_m}^{5r_m}\frac{dt}{t\theta_k(t)}\right)$$

$$\leqslant\frac{Kp_{k,m}}{T(r_m,f')}\left(\int_{3r_m}^{4r_m}\frac{dr}{r\theta_k(r)}\right)^{-2}+K\exp\left(-K\int_{2r_m}^{3r_m}\frac{dt}{t\theta_k(t)}\right)$$

$$+K\exp\left(-K\int_{4r_m}^{5r_m}\frac{dt}{t\theta_k(t)}\right)\leqslant\frac{Kp_{k,m}}{T(r_m,f')}\left(\int_{3r_m}^{4r_m}\frac{\theta_k(r)}{r}\,dr\right)^2$$

$$+K\exp\left(-K\int_{2r_m}^{3r_m}\frac{dt}{t\theta_k(t)}\right)+K\exp\left(-K\int_{4r_m}^{5r_m}\frac{dt}{t\theta_k(t)}\right) \qquad (3.10)$$

as $m\to\infty$.

Using in (3.10) the simple estimates $(\sum c_\nu)^{\frac{1}{2}}\leqslant\sum c_\nu^{\frac{1}{2}}\quad(c_\nu\geqslant 0)$ and

$$\left(\frac{Kp_{k,m}}{T(r_m,f')}\left(\int_{3r_m}^{4r_m}\frac{\theta_k(r)}{r}\,dr\right)^2\right)^{\frac{1}{2}}\leqslant\frac{Kp_{k,m}}{T(r_m,f')}+\int_{3r_m}^{4r_m}\frac{\theta_k(r)}{r}\,dr,$$

we obtain

$$\delta_k^{\frac{1}{2}}(1+o(1))\leqslant\frac{Kp_{k,m}}{T(r_m,f')}+\int_{3r_m}^{4r_m}\frac{\theta_k(r)}{r}\,dr+K\exp\left(-K\int_{2r_m}^{3r_m}\frac{dt}{t\theta_k(t)}\right)+K\exp\left(-K\int_{4r_m}^{5r_m}\frac{dt}{t\theta_k(t)}\right)$$

$$\leqslant\frac{Kp_{k,m}}{T(r_m,f')}+\int_{3r_m}^{4r_m}\frac{\theta_k(r)}{r}\,dr+K\left(\int_{2r_m}^{3r_m}\frac{dt}{t\theta_k(t)}\right)^{-1}+K\left(\int_{4r_m}^{5r_m}\frac{dt}{t\theta_k(t)}\right)^{-1}$$

$$\leqslant\frac{Kp_{k,m}}{T(r,f')}+\int_{3r_m}^{4r_m}\frac{\theta_k(r)}{r}\,dr+K\int_{2r_m}^{3r_m}\frac{\theta_k(t)}{t}\,dt+K\int_{4r_m}^{5r_m}\frac{\theta_k(t)}{t}\,dt \qquad (3.11)$$

as $m\to\infty$.

In view of (3.11), we may for each k choose $m_0=m_0(k)$ such that for $m>m_0$

$$\delta_k^{\frac{1}{2}}\leqslant K\left(\frac{p_{k,m}}{T(r_m,f')}+\int_{2r_m}^{5r_m}\frac{\theta_k(r)}{r}\,dr\right),$$

where K depends only on the lower order μ and the size of the first two deficiencies $\delta(a_1,f)$, $\delta(a_2,f)$.

For a finite set $\delta_1, \ldots, \delta_n$ we thus have for all sufficiently large m

$$\sum_{k=1}^{n} \delta_k^{\frac{1}{2}} \leqslant K \sum_{k=1}^{n} \frac{p_{k,m}}{T(r_m, f')} + K \sum_{k=1}^{n} \int_{2r_m}^{5r_m} \frac{\theta_k(r)}{r} dr. \tag{3.12}$$

The second term on the right side of (3.12) is clearly bounded above by $2\pi K \log \frac{5}{2}$.

As for the first term on the right side, since $p_{j,m}$ is the number of zeros of $f'(z)$ in $\mathcal{E}_j \cap \{r_m' < |z| < r_m''\}$ it follows from (2.1) and that

$$\sum_{k=1}^{n} p_{k,m} \leqslant n\left(6r_m, \frac{1}{f'}\right) \leqslant \frac{1}{\log 2} N\left(12r_m, \frac{1}{f'}\right)$$
$$\leqslant \frac{1}{\log 2} T\left(12r_m, \frac{1}{f'}\right) \leqslant \frac{2 \cdot 12^{\mu}}{\delta(a_1, f) \log 2} T(r_m, f')\,(1 + o(1))$$

as $m \to \infty$.

Thus both sums on the right side of (3.12) are bounded independent of n and hence $\sum_{k=1}^{\infty} \delta_k^{\frac{1}{2}} < \infty$ from which (1) follows.

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