# THE RESIDUAL LIMIT SETS OF KLEINIAN GROUPS

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The study of Kleinian groups, having venerable roots in the work of Poincaré and Klein, has experienced a resurgence in interest in the period following the discovery of the finiteness theorem by L. Ahlfors. This interest has centered about the geometric and function-theoretic properties of the sets of discontinuity of finitely generated Kleinian groups. Using these recent results and various properties of plane continua, we will discuss the structure of the limit sets of finitely generated Kleinian groups. It is well known that the limit sets of (non-elementary) Kleinian groups are perfect, nowhere dense and have positive capacity. For $G$ a Kleinian group, we denote by $\Lambda_0(G)$, the *residual limit set of* $G$, which we define to be the set of those limit points of $G$ not lying in the boundary of any component of the set of discontinuity of $G$. It was classically stated, though incorrect, that $\Lambda_0(G)$ is always void, e.g. in Fricke–Klein ([6] p. 136) we find the following assertion— *Wir sehen, dass die beiden Fixpunkte einer hyperbolischen oder loxodromischen Substitution stets auf einer und derselben Grenzcurve, d. h. auf einem und demselben geschlossenen Zuge der Berandung des Netzes liegen.* The error recurs explicitly in Lehner ([8], p. 108). A counterexample was given by the author in [1]. Indeed, it is shown there that $\Lambda_0(G)$ may have positive areal measure when $G$ is an infinitely generated group.

We are concerned here with the properties of the residual limit set. It is shown that if $\Lambda_0(G)$ is not void, it has a perfect subset. This is related to the zero-measure problem of Ahlfors ([3]). When $G$ is finitely generated, a necessary and sufficient condition for $\lambda$ to lie in $\Lambda_0(G)$ is the existence of a nested sequence of Jordan curves, each lying in the limit set, converging to $\lambda$. As a corollary we find that hyperbolic or loxodromic fixpoints lie in $\Lambda_0(G)$ if and only if the two fixed points of the transformation are separated by a Jordan curve which lies in the limit set. Perhaps most interesting for finitely generated groups is the

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relationship between the triviality of $\Lambda_0(G)$ and $G$ being a function group, i.e. having an invariant component of the set of discontinuity. $\Lambda_0(G)$ is void if and only if $G$ is a function group or has a subgroup of index 2 which is quasi-Fuchsian. Two examples are given of groups exhibiting specific properties of $\Lambda_0(G)$, one for $G$ finitely generated and one infinitely generated.

There is a concept related to that of residual limit set, namely the relative residual limit set. The latter is defined to be

$$\Lambda_0(G, \Omega_i) = (\mathrm{Bd}\, \Omega_i(G)) - \bigcup_j \mathrm{Bd}\, \Omega_{ij}(G)$$

where $\Omega_i$ is a component of $G$ and the $\Omega_{ij}$ are a complete list of the components of $\mathrm{Ext}\, \Omega_i$. It follows from recent, but not yet published, work of Maskit that, for $G$ finitely generated, $\Lambda_0(G, \Omega_i)$ consists of the limit sets of finitely generated totally degenerate groups and some discrete set of points. We thus obtain a complete classification of the limit sets of finitely generated Kleinian groups. The limit set is a union of quasi-circles ($=\mathrm{Bd}\, \Omega_{ij}$), limit sets of degenerate groups, the residual limit set and a discrete set.

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## § 1. Preliminaries

Kleinian groups are discontinuous groups of conformal motions of $\hat{\mathbf{C}}$, the Riemann sphere, and can be identified with certain of the discrete subgroups of $\mathbf{PSL}(2, \mathbf{C})$. A formal definition follows.

Let $\Gamma$ denote the group of all Möbius transformations and $G$ be a subgroup of $\Gamma$. Both $\Gamma$ and $G$ act as topological transformation groups on the space $\hat{\mathbf{C}}$. We define the *orbit* of a point $z \in \hat{\mathbf{C}}$ under the action of $G$ to be the set

$$O_z = \{z' \mid z' = \gamma(z) \text{ for some } \gamma \in G\}$$

$O_z$ as a subset of $\hat{\mathbf{C}}$ has a derived set, $O_z'$. The *limit set* of $G$, $\Lambda(G)$, is defined to be

$$\Lambda(G) = \bigcup_{z \in \hat{\mathbf{C}}} O_z'$$

We may then define the *ordinary set* (or *set of discontinuity*) of $G$ to be

$$\Omega(G) = \hat{\mathbf{C}} - \Lambda(G).$$

The subgroup $G$ of $\Gamma$ is called *discontinuous* if $\Omega(G)$ is not void. If $G$ is discontinuous and $\Lambda(G)$ contains more than two points, $G$ is called Kleinian. This last definition is not standard. According to some authors, any discontinuous subgroup of $\Gamma$ is Kleinian. Those we call Kleinian are then called non-elementary Kleinian. We assume unless otherwise stated that $\infty \in \Omega(G)$. This assumption has no effect on the topological properties under discussion.

The limit set of a Kleinian group is perfect and nowhere dense. The ordinary set $\Omega(G)$ is open and has one, two or a countably infinite number of components, $\Omega_i$. It is easily shown (Lehner [8] p. 105), that Bd $\Omega_i \subset \Lambda(G)$. The statement in the reference incorrectly asserts that $\Lambda(G) = \cup \operatorname{Bd} \Omega_i$. We define the *residual* limit set $\Lambda_0(G)$ to be

$$\Lambda_0(G) = \Lambda(G) - \cup \operatorname{Bd} \Omega_i.$$

We shall need the following classical results.

THEOREM 1.1 (Lehner [8] p. 105). *If $G$ is a Kleinian group, $S \subset \hat{\mathbf{C}}$ a closed set containing at least two points and invariant under $G$ then $S \supset \Lambda(G)$.*

THEOREM 1.2 (Lehner [8] p. 73). *A convergent sequence of Möbius transformations converges either to a constant or to a Möbius transformation. In the latter case, the convergence is uniform on compact sets avoiding the pole of the limit transformation.*

THEOREM 1.3 (Lehner [8] p. 103). *If $G$ is a Kleinian group and $\lambda \in \Lambda(G)$, then $\lambda \in O'_z$ (i) for each $z \in \Omega(G)$ and (ii) each $z \in \Lambda(G)$ with at most one exception. (The exception only occurs for elementary groups, but we will not need this fact.)*

The modern results that we will need are more geometric in nature.

THEOREM 1.4 (Ahlfors [3], with complements in Bers [5] and Greenberg [7]). *If $G$ is a finitely generated Kleinian group, then $\Omega(G)/G$ is a finite union of Riemann surfaces of finite type.*

If $G$ is a Kleinian group and $A \subset \hat{\mathbf{C}}$, we denote by $G_A$ the stability subgroup of $A$. If $A$ is a component $\Omega_i$ of $\Omega(G)$, we will frequently write $G_i$ instead of $G_{\Omega_i}$.

THEOREM 1.5 (Ahlfors [4]). *If $G$ is a finitely generated Kleinian group, then $G_i = G_{\Omega_i}$ is also a finitely generated Kleinian group and $\Omega_i$ is a component (of the ordinary set) of $G_i$.*

If $\Omega_i$ is a component of $G$, we define a complementary component of $\Omega_i$ to be a component of $\Omega(G_i) - \Omega_i$.

THEOREM 1.6 (Accola [2]). *If $\Omega_i$ is a component of a Kleinian group $G$, then every complementary component of $\Omega_i$ is simply connected.*

A *quasi-circle* is the image of a circle under a quasi-conformal homeomorphism of the sphere. A *quasi-Fuchsian group* is a group which leaves a Jordan curve invariant. We will call a group *q-Fuchsian* if it is a finitely generated quasi-Fuchsian group, whose limit set is exactly the invariant Jordan curve, i.e. is quasi-Fuchsian of the first kind. The limit set of a $q$-Fuchsian group will be called a *q-circle*.

THEOREM 1.7. (Maskit [10]). *A finitely generated Kleinian group with two invariant components is q-Fuchsian.*

If $G$ has one or two components then $\Lambda_0(G)=\emptyset$. Thus the generic group considered here has infinitely many components. Let $\Omega_i$ be a component of $G$, a finitely generated Kleinian group, and $G_i$ be its stability group. If $\Omega_{ij}$ is a complementary component of $\Omega_i$, then the subgroup $G_{ij}$ of $G_i$ which stabilizes $\Omega_{ij}$ is $q$-Fuchsian, by Theorems 1.5 and 1.7. In which case we define the $q$-circle $C_{ij}=\Lambda(G_{ij})$ to be a *separator* of $G$. We denote by $S=S(G)$, the *set of separators* of $G$ with the obvious definition.

PROPOSITION 1.1. *If $G$ is a finitely generated Kleinian group whose ordinary set is not connected then*

$$\Lambda(G)=\overline{\bigcup_{C\in S(G)} C}$$

*where $S(G)$ is the set of separators of $G$.*

*Proof.* Since $\Omega(G)$ is not connected there exists at least one component possessing a complementary component. The boundary of this complementary component is a $q$-circle lying in $S(G)$, and therefore $S(G)$ is not void. $S(G)$ is invariant under the action of $G$, hence by Theorem 1.1 the right hand side contains the left. The reverse containment follows easily since each point lying in an element of $S(G)$ must lie in $\Lambda(G)$ and $\Lambda(G)$ is closed.

If $\lambda\in\Lambda_0(G)$, then $\lambda\notin C$ for any $C\in S(G)$ since the $q$-circles $C$ were chosen to lie in the boundaries of the components, $\Omega_i$. Thus, by Theorem 1.2, for each $\lambda\in\Lambda_0(G)$ there is a sequence $(C_i)$ of $q$-circles in $S(G)$ and points $z_i\in C_i$, such that the $z_i$ converge to $\lambda$. We define $L_1(G)$ to be the set of $\lambda\in\Lambda_0(G)$ having such a sequence $(C_i)$ with the additional property that $\lambda\in\operatorname{Int} C_i$ for each $i$. We then let $L_2(G)=\Lambda_0(G)-L_1(G)$.

PROPOSITION 1.2. *If $G$ is a finitely generated Kleinian group, then there is at most a finite number of q-circles in $S(G)$ which are pairwise inequivalent under the action of $G$.*

*Proof.* There is at most a finite number of pairwise inequivalent components $\Omega_1, \dots \Omega_j$ by Theorem 1.4. By applying Theorem 1.5 and again Theorem 1.4, each of these has at most a finite number of pairwise inequivalent complementary components. Each of the $q$-circles in $S(G)$ is the common boundary of one of the $\Omega_i$ and of one of the complementary components of $\Omega_i$. Only finitely many of the complementary components, hence only finitely many of the $q$-circles in $S(G)$, are pairwise inequivalent.

The following results from plane topology are extracted directly from Whyburn [11]. The page numbers given refer to that book.

THEOREM 1.8 (Zoretti, p. 109). *If $K_1$ is a component of $K_2$ which is a compact subset of the plane, then there exists a Jordan curve $J$ enclosing $K_1$, such that $J \cap K = \emptyset$ and each point of $J$ is at distance less than $\varepsilon$ from $K_1$.*

An *irreducible continuum $M$* with property $P$ is a continuum satisfying property $P$ such that no proper subcontinuum of $M$ satisfies property $P$.

THEOREM 1.9 (p. 17). *If $K$ is any closed subset of a continuum $M$, then $M$ contains an irreducible subcontinuum containing $K$.*

If $a$ and $b$ are points, in a set $M$, then by an $\varepsilon$-chain in $M$ between $a$ and $b$ we mean a finite set of points in $M$

$$a = x_1, x_2, x_3, \ldots, x_n = b$$

such that the distance between $x_i$ and $x_{i+1}$ is less than $\varepsilon$.

A set $M$ is *well chained* if for every $\varepsilon > 0$ there is an $\varepsilon$-chain between any two points of $M$.

THEOREM 1.11 (p. 15). *A compact set is connected if and only if it is well chained.*

A plane continuum $M$ will be called an *E-continuum* if for every $\varepsilon > 0$, $M$ contains at most a finite number of components having diameter greater than $\varepsilon$.

THEOREM 1.12 (p. 113). *A plane continuum $M$ is locally connected if and only if it is an E-continuum and every component of $M$ has a locally connected boundary.*

The following result is essential to the discussion of $L_2(G)$.

THEOREM 1.13 (p. 114). *If $p \in M$, a locally connected plane continuum, but $p$ does not lie on the boundary of any component of $M$, then for any $\varepsilon > 0$, $M$ contains a Jordan curve of diameter less than $\varepsilon$ enclosing $p$.*

A metric space $X$ has *property S* if for each $\varepsilon > 0$ $M$ can be written as a finite union of connected sets of diameter less than $\varepsilon$.

THEOREM 1.14 (p. 20). *If $X$ has property $S$, it is locally connected.*

THEOREM 1.15 (p. 107). *If $N$ is a locally connected continuum in the plane and $a$ and* *b are points lying in different complementary components of $N$, then there exists a Jordan curve $J \subset N$ separating a and b.*

A trivial modification of the proof given in the reference, allows us to assume $J$ is part of the boundary of the unbounded complementary component of $N$.

## § 2. Description of the residual limit set

In this section we will prove three technical theorems on the residual limit sets of finitely generated Kleinian groups which we apply in the following two sections. The first states that a residual limit point $\lambda$ is either contained in a nested sequenc of $q$-circles or lies in the same component of $\Lambda(G)$ as does a $q$-circle. The second gives some separation properties of separators. The third theorem gives some properties of a subgroup of $G$ related to a fixed $\lambda \in L_2(G)$.

If $A$ is a connected subset of the limit set, we denote by $\chi(A) = \chi_G(A)$ the component of $\Lambda(G)$ which contains $A$. Throughout this section, $G$ denotes a finitely generated Kleinian group.

THEOREM 2.1. *If $G$ is a finitely generated Kleinian group then*

$$\Lambda_0(G) \subset L_1(G) \cup [\bigcup_{C \in S(G)} \chi(C)].$$

*Proof.* Without loss of generality we may assume that $\Lambda(G)$ is a compact subset of the plane. If $\lambda \in \Lambda_0(G)$, then, by Theorem 1.8, for $\varepsilon > 0$ there exists a Jordan curve $J_\varepsilon$ such that:

(i) $\chi(\lambda) \subset \operatorname{Int} J$

(ii) $J \cap \Lambda(G) = \varnothing$

(iii) $d(z, \chi(\lambda)) < \varepsilon$ for all $z$ on the curve $J$.

Setting $\varepsilon = 1/n$ we get a sequence $(J_n)$ of Jordan curves converging to $\chi(\lambda)$. Since each curve does not meet the limit set, $J_n \subset \Omega(G)$ and the connectedness of $J_n$ imply the existence of a component $\Omega_n$ for which $J_n \subset \Omega_n$. We examine the following cases:

*Case 1:* If infinitely many of the $J_n$ lie in distinct components, then $\lambda$ lies in the complementary component $\Omega_{nk}$ of $\Omega_n$. Thus the $q$-circles $C_{nk'}$ of which there are infinitely many, all contain $\lambda$ in their interiors. This is precisely the condition that $\lambda$ lie in $L_1(G)$.

*Case 2:* If only finitely many $J_n$ lie in distinct components, then, by passing to a subsequence, again denoted $J_n$, each $J_n$ is contained in the same component $\Omega_0$. Since

$\lambda \notin \mathrm{Bd}\,\Omega_0$, $\lambda$ lies in a complementary component $\Omega_{01}$ of $\Omega_0$. If $C$ is the corresponding $q$-circle, then $J_n$ converges to $\chi(C)$. But $J_n$ also converges to $\chi(\lambda)$. Thus $\chi(\lambda)=\chi(C)$. Since $C$ is a separator for $G$, the theorem is proved.

LEMMA 1.2. *If* $\infty \in \Omega(G)$, the diameters of the separators in $S(G)$ *form a null sequence.*

*Proof.* Let $\{C_n\}$ be a list of the separators in $S(G)$. By Proposition 1.2, they fall into finitely many equivalence classes, where $C_i \sim C_j$ if there is a $\gamma \in G$ such that $\gamma(C_i)=C_j$. Clearly, it suffices to show that the diameters of the separators in an equivalence class form a null sequence. If this is not true, then there exists a subsequence, again denoted $\{C_n\}$, of equivalent separators whose diameters are strictly bigger than some number $d>0$. We assume $C_n=\gamma_n(C_0)$, $\gamma_n \in G$. By conjugation and passing to another subsequence, we may assume that $\gamma_n(\mathrm{Ext}\,C_0) \subset \mathrm{Int}\,C_0$. It follows, from Montel's theorem, that $\{\gamma_n\}$ is a normal family and has a convergent subsequence $\{\gamma_{n_i}\}$. By Theorem 1.2 and discreteness of $G$, $\{\gamma_{n_i}\}$ converges uniformly to a constant on $C_0 \cup \mathrm{Ext}\,C_0$, in particular on $C_0$. So diam $(C_{n_i}) \to 0$, which contradicts the assumption.

LEMMA 2.2. *If* $\infty \in \Omega(G)$ *and* $C_1$, $C_2 \in S(G)$, *then either* $C_2 \subset \overline{\mathrm{Ext}\,C_1}$ *or* $C_2 \subset \overline{\mathrm{Int}\,C_1}$. *(Separators don't cross each other.)*

*Proof.* By definition, $C_1$ and $C_2$ lie in the boundaries of components, say $\Omega_1$ and $\Omega_2$ respectively, of $G$. If $C_1$ and $C_2$ are distinct and $C_2$ intersects both $\overline{\mathrm{Ext}\,C_1}$ and $\overline{\mathrm{Int}\,C_1}$, then there exist points $\lambda_1 \in \mathrm{Bd}\,\Omega_2 \cap \mathrm{Ext}\,C_1$ and $\lambda_2 \in \mathrm{Bd}\,\Omega_2 \cap \mathrm{Int}\,C_1$. Since $\Omega_2$ is connected, $\Omega_2 \subset \Omega(G)$ and $C_1 \subset \Lambda(G)$, this is clearly impossible and the lemma is established.

The above lemma contains the separation property which is peculiar to finitely generated groups and allows us to prove the remaining results of this section. The following notion is then meaningful in the context of separators. We define a set of Jordan curves $\mathfrak{C}$ in $\hat{\mathbf{C}}$ to be *spherical nest* if for every three curves $C_1$, $C_2$, $C_3$ in $\mathfrak{C}$ one of the three, say $C_1$, has the property that $C_2$ and $C_3$ lie in distinct closed regions complementary to $C_1$. $\mathfrak{C}$ is said to be *a spherical nest rel* $(z_1, z_2)$ if for each $C \in \mathfrak{C}$ $z_1$ and $z_2$ lie in distinct open regions complementary to $C$. If $\mathfrak{C}$ is a spherical nest rel $(z_1, z_2)$ then $C \in \mathfrak{C}$ is *maximal* if the component $D$, of $\hat{\mathbf{C}}-C$ containing $z_2$, intersects no element of $\mathfrak{C}$.

We now proceed to consider subcontinua of $\Lambda(G)$ and their stability groups. If $\lambda \in L_2(G)$, then, by Proposition 1.1, we can choose a sequence of separators having $\lambda$ as an accumulation point. The next theorem shows that this sequence may be chosen to consist of maximal separators and discusses their properties.

THEOREM 2.2. *If $\lambda \in L_2(G)$ and both $z$ and $\infty$ are in $\Omega(G)$ then:*

(i) *the family of separators which separate $z$ and $\lambda$ form a finite spherical nest rel $(z, \lambda)$, denoted $\mathfrak{C}(z)$.*

(ii) *$\mathfrak{C}(z)$ contains a maximal element $C(z)$ (which is, of course, a separator).*

(iii) *$C(z) \subset \chi(\lambda)$.*

(iv) *if $z' \in \Omega(G)$ and lie in the same region complementary to $C(z)$ as does $z$, then $C(z) = C(z')$.*

(v) *$\bigcup \{C(z)\colon z \in \Omega(G)\}$ accumulates at $\lambda$.*

*Proof.* Lemma 2.2 yields that the family of separators separating $z$ and $\lambda$ is a spherical nest rel $(z, \lambda)$, denoted $\mathfrak{C}(z)$. Since connectedness is a conjugation invariant, the proof of case 2 of Theorem 2.1 tells us that $\mathfrak{C}(z)$ contains a separator in $\chi(\lambda)$ (send $z$ to $\infty$). In particular, $\mathfrak{C}(z)$ is not void. If $\mathfrak{C}(z)$ contained infinitely many separators, it would contain a subsequence of separators, each of whose interior would contain $\lambda$. Thus $\lambda \in L_1(G)$. Thus $\mathfrak{C}(z)$ is a finite spherical nest rel $(z, \lambda)$ and therefore contains, by Lemma 2.2, a maximal element $C(z)$. We denote the component of $\hat{\mathbf{C}} - C(z)$ containing $z$ by $B(z)$. $\{C : C \in \mathfrak{C}(z)\} \subset \overline{B(z)}$. But some $C_0 \in \mathfrak{C}(z)$ lies in $\chi(\lambda)$. Plane separation then demands that $C(z)$ lie in $\chi(\lambda)$, and we have proved (i), (ii) and (iii).

To prove (iv) we note that $C(z) \in \mathfrak{C}(z')$ for each $z' \in B(z)$. If $C(z) \neq C(z')$, then $C(z) \subset \overline{B(z')}$ and $B(z) \subset B(z')$. Thus $C(z') \in \mathfrak{C}(z)$ and we contradict the maximality of $C(z)$.

Since $\lambda$ does not lie on any separator we can find a sequence $z_n \to \lambda$, $z_n \in \Omega(G)$ for which $C(z_n)$ are distinct. Since diam $C(z_n)$ is a null sequence, $\bigcup C(z_n)$ accumulates at $\lambda$.

Since $C_n = C(z_n) \in \{C(z)\colon z \in \Omega(G)\}$ we have proved (v) and the proof of the theorem is complete.

For $\lambda \in L_2(G)$, we define $\mathfrak{M}(\lambda)$ to be the set of *maximal separators for* $\lambda$. Formally:

$$\mathfrak{M}(\lambda) = \{C \in S(G) : C = C(z) \text{ for some } z \in \Omega(G) \text{ and fixed } \lambda\}.$$

(The dependence of $C(z)$ on $\lambda$ has been suppressed in the previous discussion.) The *web of* $\lambda$, $\Phi(\lambda)$ is the closure of the set of maximal separators for $\lambda$. The *web subgroup of* $\lambda$, $H(\lambda)$, is the subgroup of $G$ which stabilizes $\Phi(\lambda)$. In the previous theorem we showed $\Phi(\lambda)$ contains $\lambda$ and infinitely many separators, each lying in $\chi(\lambda)$. It follows that $\Phi(\lambda) \subset \chi(\lambda)$ since $\chi(\lambda)$ is a continuum. We cannot prove directly that the web of $\lambda$ is connected. To do so we first note that there is an irreducible subcontinuum $\hat{\Phi}(\lambda)$ of $\chi(\lambda)$ which contains $\Phi(\lambda)$. This follows from Theorem 1.9 since $\chi(\lambda)$ is a continuum and $\Phi(\lambda)$ is closed. In the following two lemmas show $\hat{\Phi}(\lambda)$ cannot contain any points other than those in $\Phi(\lambda)$.

LEMMA 2.3. *Let $\{C_n\}$ be a list of the maximal separators in $\mathfrak{M}(\lambda)$ and $D_n$ the component of $\hat{\mathbf{C}} - C_n$ which contains $\lambda$. Then $\Phi(\lambda) = \bigcap \overline{D_n}$.*

*Proof.* Assume $\lambda=\infty$. Then $D_n=\text{Ext } C_n$. The $C_n$ have disjoint interiors by Lemma 2.2. Thus $\overline{D_n}\supset C_m$ for each $m\neq n$, hence for all $m$, since $C_n=\text{Bd } D_n$. Thus $\Phi(\lambda)\subset\overline{D_n}$ for each $n$. If $z\in(\bigcap\overline{D_n})-\Phi(\lambda)$ then $z$ lies at a uniformly positive distance from $\overline{\text{Int } C_n}$ for each $n$. We may assume $x\in\Omega(G)$ since $\Omega(G)$ is dense in $\hat{\mathbf{C}}$. But then there is a maximal separator $C_K\in\mathcal{M}(\lambda)$ for which $z\in\hat{\mathbf{C}}-(\overline{D_K})$. This is a contradiction.

LEMMA 2.4. $\hat{\Phi}(\lambda)=\Phi(\lambda)$.

*Proof.* In the notation of the previous lemma if $z\in\hat{\Phi}(\lambda)-\Phi(\lambda)$, there exists $D_n$ such that $z\in\hat{\mathbf{C}}-\overline{D_n}$. Since $\hat{\Phi}(\lambda)$ is well-chained (Theorem 1.11) so is $\hat{\Phi}(\lambda)\cap(C_n\cup\text{Ext } C_n)=\hat{\Phi}(\lambda)\cap\overline{D_n}$. Thus $\hat{\Phi}(\lambda)\cap\overline{D_n}$ is a proper subcontinuum of $\hat{\Phi}(\lambda)$ containing $\Phi(\lambda)$ which yields a contradiction.

LEMMA 2.5. *At most countably many $\lambda\in L_2(G)$ have distinct webs, $\Phi(\lambda)$.*

*Proof.* If there are uncountably many distinct $\Phi(\lambda)$ for $\lambda\in L_2(G)$, then there is a $C_0\in S(G)$ such that $C_0\in\mathcal{M}(\lambda)\cap\mathcal{M}(\lambda')$ where $\Phi(\lambda')\neq\Phi(\lambda)$. If $\mathcal{M}(\lambda)=\mathcal{M}(\lambda')$ then $\lambda$ and $\lambda'$ would have identical webs, so we may assume that there is a separator $C_{ij}\in\mathcal{M}(\lambda)-\mathcal{M}(\lambda')$, $C_{ij}=(\text{Bd }\Omega_i)\cap(\text{Bd }\Omega_{ij})$ where $\Omega_i$ and $\Omega_{ij}$ have the usual meaning.

*Case 1.* If $C_{ij}$ separates $\lambda$ and $\lambda'$. Then $\lambda$ and $\lambda'$ lie in distinct complementary components of $\Omega_i$. Then for $C_0$ to be maximal with respect to $\lambda$ (resp. $\lambda'$) it is necessary that it lie in the same component of $\hat{\mathbf{C}}-C_{ij}$ as $\lambda$ (resp. $\lambda'$). But this is impossible.

*Case 2.* If $C_{ij}$ does not separate $\lambda$ and $\lambda'$. Assume $\lambda$ and $\lambda'$ lie in Ext $C_{ij}$. Since $C_{ij}\notin\mathcal{M}(\lambda')$. There is an element of $\mathcal{M}(\lambda')$, say $C_1$, which separates $\lambda'$ and Int $C_{ij}$. $C_1$ does not separate Int $C_{ij}$ and $\lambda$ since $C_{ij}\in\mathcal{M}(\lambda)$. Then use the argument of case 1 on $C_1$, and obtain a contradiction. Thus we have shown that if $\mathcal{M}(\lambda)\cap\mathcal{M}(\lambda')\neq\emptyset$, we have $\Phi(\lambda)=\Phi(\lambda')$.

The next lemma shows that web subgroups are non-trivial.

LEMMA 2.6. *If $C_{ij}\in\mathcal{M}(\lambda)$, then $G_{ij}\subset H(\lambda)$.*

*Proof.* We wish to show that if $C_{ij}\in\mathcal{M}(\lambda)$ and $\gamma\in G$, $\gamma(C_{ij})=C_{ij}$, $\gamma(\text{Int } C_{ij})=\text{Int } C_{ij}$, then $\gamma(\Phi(\lambda))=\Phi(\lambda)$. We assume $\infty\in\Omega_i$. Again since $\Phi(\lambda)$ was defined to be the closure of those separators in $\mathcal{M}(\lambda)$, it suffices to show $\mathcal{M}(\lambda)$ is invariant under $\gamma$.

If not, there is a separator $C_{Kl}\in\mathcal{M}(\lambda)$ for which $C_{K'l'}=\gamma(C_{Kl})\notin\mathcal{M}(\lambda)$. We first note that $\gamma$ is a homeomorphism of Int $C_{ij}$ and $\Phi(\lambda)\subset C_{ij}\cup$ Int $C_{ij}$. (See figure 1). Since $C_{Kl}$ separates $\Omega_K$ from the component of $(\text{Ext }\Omega_K)\cap\text{Int } C_{ij}$ which contains $\lambda$, the same must be true of $C_{K'l'}$ with respect to $(\text{Ext }\Omega_{K'})\cap\text{Int } C_{ij}$. Thus $\lambda\in\Omega_{K'l'}=\gamma(\Omega_{Kl})$. If $C_{K'j'}\in\mathcal{M}(\lambda)$ then there exists a $C_0\notin\mathcal{M}(\lambda)$ so that, for each $z\in\Omega_{K'}$ $C_0=C(z)$ (in the notation of Theorem 2.2.)

Since $C_{ij}$, $C_0$, $C_{K'l'}$ is a spherical nest, so is $C_{ij}$, $\gamma^{-1}(C_0)$, $C_{Kl}$. $\lambda$ must lie in one of the

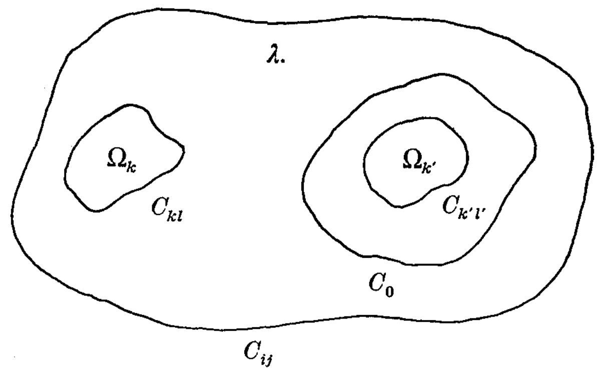


Fig. 1.

open sets Int $C_{ij} \cap$ Ext $\gamma^{-1}(C_0)$ or Int $\gamma^{-1}(C_0) \cap$ Ext $C_{Kl}$. In the first case we contradict the maximality of $C_{Kl}$ with respect to $z \in \Omega_K$ and in the second, we contradict the maximality of $C_{ij}$ with respect to $\infty$.

We have proved most of the following theorem.

THEOREM 2.3. *If $G$ is a finitely generated Kleinian group and $\lambda \in L_2(G)$, with web $\Phi(\lambda)$ and web subgroup $H = H(\lambda)$, then:*

(i) *$H$ is Kleinian*

(ii) $\Lambda(H) = \Phi(\lambda)$

(iii) *$\Lambda(H)$ is connected*

(iv) *each component of $\Omega(H)$ is simply connected and bounded by a q-circle in $\mathfrak{M}(\lambda)$*

(v) *the subgroup of $H$ fixing a component $\Omega_i(H)$ of $\Omega(H)$ is finitely generated and q-Fuchsian*

(vi) $\lambda \in \Lambda(H)$

(vii) *only countably many distinct $\lambda \in L_2(G)$ yield distinct groups $H$*

(viii) *only finitely many $\Omega_i(H)$ are inequivalent under the action of $H$.*

*Proof.*

(i) By Lemma 2.6. $H$ contains the subgroup $G_{ij}$ corresponding to each $C_{ij} \subset \Phi(\lambda)$. Thus $\Lambda(H) \supset \Lambda(G_{ij}) = C_{ij}$ and $H$ is not elementary. Since $H \subset G$ it is discontinuous and hence Kleinian.

(ii) Since $\Phi(\lambda)$ contains more than 2 points, is closed and invariant under H, by Theorem 1.1, we have $\Lambda(H) \subset \Phi(\lambda)$. $\Lambda(H)$ is closed and by part (i) contains a set of $q$-circles dense in $\Phi(\lambda)$, hence $\Lambda(H) \supset \Phi(\lambda)$.

(iii) follows from part (ii) and Lemma 2.4.

(iv) Since $\Lambda(H)$ is connected, each component of $\Omega(H)$ is simply connected. If $\Omega_i(H)$ is a component of $\Omega(H)$ we must show that $\operatorname{Bd}\Omega_i(H)$ is a $q$-circle contained in $\Phi(\lambda)$. We may assume $\infty \in \Omega_i(H) \cap \Omega_j(G)$. Since $\Omega_j(G) \cap \Omega_i(H) \neq \emptyset$, we have $\Omega_j(G) \subset \Omega_i(H)$. Further, by the construction carried out in Lemma 2.3, there is a $q$-circle $\hat{C} \subset \mathcal{M}(\lambda)$ which separated $\Omega_j(G)$ from $\lambda$. Clearly $\Omega_i(H)$ is contained in $\operatorname{Ext}\hat{C}$, and $\Phi(\lambda) \subset \hat{C} \cup \operatorname{Int}\hat{C}$. It follows that $\operatorname{Bd}\Omega_i(H) = \hat{C}$.

(v) If $H_i$ is the subgroup of $H$ leaving $\Omega_i(H)$ invariant and $\operatorname{Bd}\Omega_i(H) = \hat{C}$ as in part (iv), then $H_i$ leaves $\hat{C}$ fixed, $\hat{C} \in S(G)$, implies $\hat{C}$ is some $C_{kl}$ corresponding to a subgroup $G_{kl}$ of $G$. In Lemma 2.6 we showed that $G_{kl} \subset H$. Since $G_{kl}$ is finitely generated and $q$-Fuchsian, it suffices to show $G_{kl} = H_i$. We note that $H_i$ contains no elements $\gamma: \operatorname{Ext} C \to \operatorname{Int} C$ since $\Omega_i(H)$ is simply-connected and connected and $\Omega(H) - \Omega_i(H)$ has infinitely many components. If such a $\gamma$ did exist then $\Omega_i(H)$ and $\Omega(H) - \Omega_i(H)$ would have to be homeomorphic which is impossible. Thus if $\gamma \in H_i$, $\gamma$ leaves $\Omega_k$ invariant, i.e. $\gamma \in G_k$ and also $\gamma$ leaves $\Omega_{kl}$ invariant hence lies in $G_{kl}$. We have shown that $H_i = G_{kl}$.

(vi) follows from (ii) and Theorem 2.1 (v).

(vii) follows from (ii) and Lemma 2.5.

(viii) We first prove the following lemma:

LEMMA 2.7. *If at most finitely many $q$-circles $C_{ij} = \operatorname{Bd}\Omega_k(H)$ are pairwise inequivalent under $H$, then at most finitely many $\Omega_i(H)$ are inequivalent under the action of $H$.*

*Proof*. We show that if $\gamma \in H$, $\gamma(C_{ij}) = C_{i'j'}$, $C_{ij} = \operatorname{Bd}\Omega_k(H)$ and $C_{i'j'} = \operatorname{Bd}\Omega_{k'}(H)$, then $\gamma(\Omega_k(H)) = \Omega_{k'}(H)$. If not, then $\gamma$ maps $\Omega_k(H)$ into $\Omega(H) \cap \operatorname{Ext}\Omega_{k'}(H)$ hence onto that set. But then we would have $\Omega_k(H)$, which is connected and simply connected, homeomorphic to $\Omega(H) \cap \operatorname{Ext}\Omega_{k'}(H)$, which is not even connected. This is clearly contradictory and the lemma is established.

By Proposition 1.2, we know that there are at most finitely many $q$-circles $C_{ij} \subset \Phi(\lambda)$ which are not equivalent under $G$. If $C_1$ and $C_2$ are $q$-circles lying in $\Phi(\lambda)$, and $\gamma \in G$ for which $\gamma(C_1) = C_2$ then it suffices to show $\gamma \in H$, since if so any equivalence under $G$ of $q$-circles in $\Phi(\lambda)$ is also an equivalence under $H$.

Since $\Phi(\lambda) = \Lambda(H)$ was defined to be the closure of a set of $q$-circles, $\mathcal{M}(\lambda)$, it further suffices to show that if $C_{ij} \in \mathcal{M}(\lambda)$ then $\gamma(C_{ij}) = C_{i'j'} \in \mathcal{M}(\lambda)$. The proof of part (viii) has been reduced to the following lemma.

LEMMA 2.8. *For fixed $\lambda \in L_2(G)$, if $C_{kl'} C_{k'l'}$, $C_{ij} \in \mathcal{M}(\lambda)$ and $\gamma \in G$, $\gamma(C_{k'l'}) = C_{k'l'}$, then $C_{i'j'} = \gamma(C_{ij}) \in \mathcal{M}(\lambda)$.*

*Proof*. (See Figure 2.) Assume $\lambda \in \operatorname{Ext} C_\alpha$, $\alpha = (kl)$, $(k'l')$, $(ij)$. If $C_{i'j'}$ is not maximal with respect to $\lambda$, then there exists $C_0 \in \mathcal{M}(\lambda)$ for which $\lambda$ and $\operatorname{Int} C_{i'j'}$ lie in different components

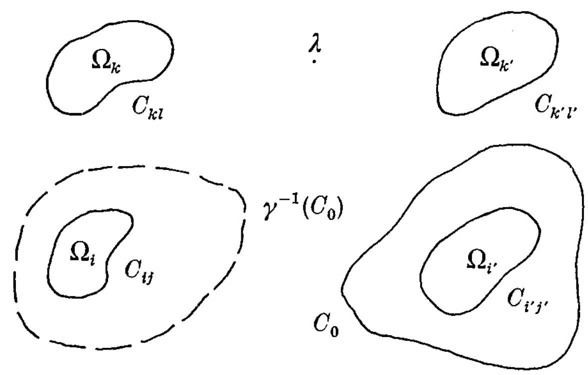


Fig. 2.

complementary to $C_0$. But then $C_0$ separates Int $C_{i'j'}$ from Int $C_{k'l'}$ (unless $C_0 = C_{k'l'}$ we consider that case below).

If $C_0 \neq C_{k'l'}$ $\gamma^{-1}(C_0)$ separates Int $C_{kl}$ from $C_{ij}$ and we obtain the same contradiction as in Lemma 2.6. If $C_0 = C_{k'l'}$, $\gamma$: Ext $C_{kl} \to$ Int $C_{k'l'}$. But $\gamma: \Omega_k \to \Omega_{k'}$, and $\Omega_k \subset$ Int $C_{kl}$ and $\Omega_{k'} \subset$ Int $C_{k'l'}$, so this is impossible.

The proofs of the lemma and theorem are now complete.

## § 3. The structure of the residual limit set

We are now in a position to describe some of the structure of the Residual Limit Set. The first theorem of this section deals with the non-triviality of $\Lambda_0(G)$ whenever it is non-void. The theorem is applicable to finitely and infinitely generated groups and in asserting not that $\Lambda_0(G)$ has a perfect subset, we show that measure theoretic questions about $\Lambda_0(G)$ are non-trivial. We then restrict ourselves to finitely generated groups. We derive necessary and sufficient conditions for points to lie in $\Lambda_0(G)$ and restate Theorem 2.1 in a form compatible with the present context.

THEOREM 3.1. *If $G$ is a finitely or infinitely generated Kleinian group with $\Lambda_0(G) \neq \emptyset$ then:*

1) *There is a nonvoid perfect set $P \subset \Lambda_0(G)$,*

2) *$\aleph(\Lambda_0(G)) = \aleph_1$, where $\aleph$ denotes the cardinality.*

*Proof.* 2) follows trivially from 1). We will prove 1).

Let $B_i = \complement(\overline{\Omega}_i)$. $B_i$ is open and $\Lambda_0(G) = \cap G_i$. $\Lambda_0(G)$ is either void or dense in $\Lambda(G)$, we assume the latter and therefore $\aleph(\Lambda_0(G)) \geqslant \aleph_0$.

Suppose $a_0, a_1 \in \Lambda_0(G)$ and $a_0 \neq a_1$. Then they have disjoint open neighborhoods $B_0'$, $B_1'$ which are completely contained in $B_1$, since $\overline{\Omega}_1$ is compact (on the sphere). Pick the neighborhoods to have diameters $<1$. Pick distinct points $a_{00}, a_{01} \in B_0'$ and $a_{10}, a_{11} \in B_1'$ and disjoint neighborhoods $B_{00}'$, $B_{01}'$, $B_{10}'$ and $B_{11}'$ each of diameter $<1/2$ and with compact closure in $B_2 \cap B_0'$ (resp. $B_2 \cap B_1'$). Continue the process. For each $n$ let

$$K_n = \bigcup \{B'_{\varepsilon_1, \dots, \varepsilon_n} \colon \varepsilon_1, \dots, \varepsilon_n \, \varepsilon \{0, 1\}\}$$

and let $K = \bigcap K_n$.

Then since $\overline{K}_n \subset K_{n-1} \subset B_{n-1}$, we have $K = \bigcap \overline{K}_n$ and is thus compact. $K \subset \Lambda_0(G)$. We also have for each $n$, every point of $K$ is contained in a $1/n$ neighborhood which contains infinitely many points of $K$, or $K$ is dense in itself and therefore perfect.

Theorem 2.1 may be restated in the following form.

THEOREM 2.1$'$. *If $G$ is a finitely generated Kleinian group and $\lambda \in \Lambda_0(G)$ then either:*

1) *There is a sequence of q-circles $(C_i)$, each lying in $S(G)$, $C_i \subset \mathrm{Bd}\, \Omega_i(G)$ such that $\{\lambda\} = \cap \mathrm{Int}\, C_i$, or*

2) *$\lambda$ lies in the same component of $\Lambda(G)$ as does some $C_i \in S(G)$, $C_i \subset \mathrm{Bd}\, \Omega_i(G)$.*

*Proof.* In Theorem 2 we showed everything stated except that in case 1) $\{\lambda\} = \cap \mathrm{Int}\, C_i$. It is stated there only that $\lambda \in \cap \mathrm{Int}\, C_i$. Equality follows directly from Lemma 2.1.

LEMMA 3.1. *If $\lambda \in L_2(G)$ and $H = H(\lambda)$ is its web subgroup then $\Lambda(H)$ is an $E$-continuum.*

*Proof.* The boundaries of the components of $H$ are separators and Lemma 2.1 shows that these diameters form a null sequence.

LEMMA 3.2.

(i) *$\Lambda(H)$ is locally connected.*

(ii) *If $\lambda \in \Lambda_0(G)$ then for each $\varepsilon > 0$ there is a Jordan curve $J$ such that*

a) $J \subset \Lambda(H)$

b) $\lambda \in \mathrm{Int}\, J$

c) *$d(j, \lambda) < \varepsilon$ for each $j$ lying on $J$.*

(iii) $\Lambda_0(H) = \Lambda_0(G) \cap \Lambda(H) = L_2(G) \cap \Lambda(H)$.

*Proof.*

(i) Follows from Theorem 1.12 and since $\Lambda(H)$ is an $E$-continuum and the boundary of each component of $\complement \Lambda(H) = \Omega(H)$ is a $q$-circle hence is locally connected.

(ii) Follows directly from Theorem 1.13 since if $\lambda \in \Lambda_0(H)$, $\lambda \notin \bigcup \mathrm{Bd}\, \Omega_i(H)$.

(iii) Follows from Theorem 2.3.

THEOREM 3.2. *If $G$ is a finitely generated Kleinian group and $\infty \in \Omega(G)$, then*

(i) $\lambda \in \Lambda_0(G)$ *if and only if there is a nested sequence of Jordan curves* $J_i \subset \Lambda(G)$, *with* diam $(J_i) \to 0$ and $\{\lambda\} = \bigcap \operatorname{Int} J_i$.

(ii) *If $\gamma \in G$ is loxodromic or hyperbolic then the fixed points of $\gamma$ lie in $\Lambda_0(G)$ if and only if there is a Jordan curve $J \subset \Lambda(G)$ which separates the fixed points of $\gamma$.*

*Proof.*

(i) If $\lambda \in \Lambda_0(G)$, then either $\lambda \in L_1(G)$ or $\lambda \in L_2(G)$. In the first case we have by definition that there is a sequence of equivalent $q$-circles whose common interiors contain $\lambda$. That the intersection contains only $\lambda$ is proved in Theorem 2.1′. If $\lambda \in L_2(G)$, then $\lambda$ lies in the corresponding $\Lambda_0(H)$ and the existence of the required Jordan curves follows from the previous lemma.

Conversely suppose the condition is satisfied and $\lambda \in \operatorname{Bd} \Omega_i(G)$. Let $z \in \Omega_i(G)$ and $d = d(z, \lambda)$. Choose $k$ large enough so that $d(J_k) < d$. Now $J_k \cap \Omega_i = \varnothing$, since $J_k \subset \Lambda(G)$ and $\Omega_i$ contains points of both Int $J_k$ and Ext $J_k$, which contradicts $\Omega_i$ being connected.

(ii) Suppose the fixed points $\lambda_1$ and $\lambda_2$ of $\gamma$ lie in $\Lambda_0(G)$. Then by (i) we can choose a Jordan curve $J \subset \Lambda(G)$ which separates $\lambda_1$ and $\lambda_2$. Conversely if $J \subset \Lambda(G)$ separates $\lambda_1$ and $\lambda_2$, then $\gamma^{n_i}(J)$ is a nested sequence of Jordan curves whose common interiors contain either $\lambda_1$ or $\lambda_2$, say $\lambda_1$. It then follows that $\lambda_2$ is contained in the common interiors of $\gamma^{-n_i}(J)$ and as in the latter part of (i) we get $\lambda_1, \lambda_2 \in \Lambda_0(G)$.

We have the immediate corollary.

COROLLARY. *If $G$ is a finitely generated Kleinian group and one of the fixed points of a loxodromic or hyperbolic element of $G$ lies in $\Lambda_0(G)$, then both fixed points lie in $\Lambda_0(G)$.*

## § 4. Function groups

Function groups were the earliest discontinuous groups to be studied. That function groups are almost the only finitely generated Kleinian groups for which $\Lambda_0(G) = \varnothing$ is shown in the following theorem.

THEOREM 4.1. *If $G$ is a finitely generated Kleinian group, then $\Lambda_0(G) = \varnothing$ if and only if either*

1) *$G$ is a function group, or*

2) *$G$ has two components. (In this case $G$ has a subgroup of index 2 which is quasi-Fuchsian, hence is a function group.)*

*Proof.* We may assume $G$ is not elementary, since an elementary group is a function group for which $\Lambda(G)$ is the boundary of the ordinary set, i.e. $\Lambda_0(G) = \varnothing$. One of the implications is trivial. If $G$ is a function group, then $G$ has a component $\Omega_i$ which is invariant and it therefore follows that the limit set $\Lambda(G) = \mathrm{Bd}\,(\Omega_i)$. Thus

$$\Lambda_0(G) = \Lambda(G) - \bigcup_i \mathrm{Bd}\,(\Omega_i) = \emptyset.$$

If $G$ has 2 components and is not a function group, let the stability group of one, hence of both, be $G_1$. It is of index 2, which implies $\Lambda(G) = \Lambda(G_1)$. But $G_1$ is a function group and $\Lambda_0(G_1) = \Lambda_0(G) = \emptyset$.

The converse is proved in the following lemmas.

LEMMA 4.1. *If $G$ is not a function group and there exists a component $\Omega_1$ whose boundary is not a separator, then $\Lambda_0(G) \neq \emptyset$.*

*Proof.*

We assume $\infty \in \Omega_1$. Since $G$ is not a function group, there is an image of $\Omega_1$ contained in a complementary component, $\Omega_{11}$, of $\Omega_1$. The boundary of $\Omega_{11}$ is a separator $C_1 \subset \mathrm{Bd}\,(\Omega_1)$. Since $\mathrm{Bd}\,(\Omega_1)$ does not consist solely of one separator, there is a boundary point $\lambda$ of $\Omega_1$, which lies at positive distance from $C_1$. The images of $\Omega_{11}$ accumulate at $\lambda$, hence there are images of $C_1$ inside a complementary component $\Omega_{12}$ at positive distance from $\Omega_{11}$. We have thus shown that $C_1$ has images of itself in both its interior and exterior. We have proved the induction step in the following argument. We assume we have a finite nest of $k$ disjoint images of $C_1$. Let $C_k$ be that image lying the interiors of all the others. $C_k$ is the image under some $\gamma \in G$ of $C_1$. Since $C_1$ separates two of its images under $G$, so does $C_k$. So Int $C_k$ contains an image of $C_k$, which is therefore an image of $C_1$. We have thus obtained $k+1$ images of $C_1$ in a nest, hence infinitely many in a nest. The intersection of their interiors defines a point in $\Lambda_0(G)$.

The cases in which $L_1(G) \neq \emptyset$ have been considered in the above lemma. We now assume each component of $G$ is simply connected and has a boundary which is a separator. A trivial application of Zoretti's Theorem shows that $\Lambda(G)$ is a continuum, and as in § 3, it is a locally connected $E$-continuum.

LEMMA 4.2. *If $G$ is not a function group and each component of $G$ has a single complementary component, then, either:*

1) *$G$ has exactly two components, or*

2) $\Lambda_0(G) \neq \emptyset$.

*Proof.* If $\Omega(G)$ has exactly two components then the stability group of either—hence both—is a subgroup of index 2 of $G$. This subgroup is then quasi-Fuchsian. If $\Omega(G)$ has more than two components, it has infinitely many. We assume $\infty \in \Omega_1$. We shall first prove the following assertion.

(**A**): If $\Omega_1, ..., \Omega_N$ is any finite collection of components of $\Omega(G)$ and $J$ is any Jordan curve contained in $\Lambda(G)$ such that $\Lambda(G) \cap \operatorname{Int} J \neq \emptyset$ then there exists a limit point $\lambda \in (\operatorname{Int} J) - \bigcup_{i=1}^{N} \operatorname{Bd} \Omega_i$.

*Proof of* (**A**). We suppose (**A**) is false and $\Omega_k$ is some component, distinct from the $\Omega_i$ $i=1, ..., N$, and lying in $\operatorname{Int} J$. First we show that $\operatorname{Bd} \Omega_k \cap \operatorname{Bd} \Omega_i$ must contain a continuum for some $i \in \{1, ..., N\}$. If not, then at least for some $i \in \{1, ..., N\}$, $\operatorname{Bd} \Omega_k \cap \operatorname{Bd} \Omega_i$ is uncountable and closed. Parametrize both $\operatorname{Bd} \Omega_k$ and $\operatorname{Bd} \Omega_i$ via the maps $\varphi_i$ and $\varphi_k$ so that the common points, elements of the intersection, correspond to common values of the parameter. The regions bounded by $\varphi_i(t)$ and $\varphi_k(t)$ for $t$ between the common points form a countable number of disjoint Jordan regions. Each of these regions must contain one of the $\Omega_j$, $j=1, ..., N$ since $\varphi_k(t) \in \Lambda(G)$. This contradiction establishes that $\operatorname{Bd} \Omega_k \cap \operatorname{Bd} \Omega_i$ contains a nontrivial continuum for some $i \in \{1, ..., N\}$.

We now suppose $\operatorname{Bd} \Omega_i \cap \operatorname{Bd} \Omega_k$ contains a non-trivial subcontinuum of $\Lambda(G)$. This continuum is a closed subset of $\operatorname{Bd} \Omega_k$ and $\operatorname{Bd} \Omega_i$, and lies on a $q$-circle. It follows that $\operatorname{Bd} \Omega_i = \operatorname{Bd} \Omega_k$ and therefore $G$ must have exactly the two components $\Omega_i$ and $\Omega_k$, contrary to hypothesis. The existence of the desired limit point follows from this contradiction.

Using (**A**), we can find limit points at positive distance from large components and a fixed Jordan curve contained in $\Lambda(G)$. Let $M_1 = \operatorname{Bd} \Omega_1$ be the fixed Jordan curve, where $\Omega_1$ is the component containing $\infty$. Let $\lambda_1$ be a point in $\Omega(G)$ which lies at a positive distance $d_1$ from $M_1$ and has the following property: there exists a circle $C_1$ of radius $\varepsilon_1 < d_1/8$ about $\lambda_1$ intersecting at least two components and such that $C_1 \cup \operatorname{Int} C_1$ intersects no component of diameter greater than or equal to $d_1/8$. This is possible by (**A**). Let $\{\Omega_i'\}$ be a list of those components intersecting $C_1$ and

$$N_1 = C_1 \cup (\bigcup_i \operatorname{Bd} \Omega_i').$$

Then diam $(N_1) < d_1/2$. Since diam $(\Omega_i')$ forms a null sequence, $N_1$ has property $S$ and therefore, by Theorem 1.14, $N_1$ is locally connected. Then choosing $a = \infty$, $b \in \Omega_i'$ we have, by Theorem 1.15, that the boundary of the complementary component of $N_1$ which contains $\infty$ is a Jordan curve $M_2$. $M_2 \subset \Lambda(G)$ and has diameter bigger than $d_1/4$ and less than $d_1/2$. $C_1 \subset M_2 \cup \operatorname{Int} M_2$; thus $\lambda_1 \in \operatorname{Int} M_2$ and $M_2 \cap M_1 = \emptyset$. Since there exist infinitely many components of $\Omega(G)$ in $\operatorname{Int} M_2$, by using $M_k$ instead of $M_1$ we can repeat the argument inductively. We get a nested sequence of disjoint Jordan curves in $\Lambda(G)$ whose diameters tend to zero. The intersection of their common interiors defines a point in $\Lambda_0(G)$.

## § 5. Examples

It follows easily from Theorem 4.1, that examples of Kleinian groups having non-trivial residual limit sets can be constructed from any finitely generated function group. One need only use Klein's combination theorem. This technique was used by the author [1] to construct a finitely generated group $G$ for which $L_1(G) \neq \emptyset$. We now construct a finitely generated Kleinian group $G$ for which $L_2(G) \neq \emptyset$. Consider the circles

$$C_1 = \{z: d(z, 1) = \sqrt{2}/2\}$$
$$C_2 = \{z: d(z, i) = \sqrt{2}/2\}$$
$$C_3 = \{z: d(z, -i) = \sqrt{2}/2\}$$
$$C_4 = \{z: d(z, -1) = \sqrt{2}/2\}$$

Each circle is tangent to two others. Construct a Fuchsian group of the first kind in each circle with four parabolic generators. The fixed points of the generators lie at the points of tangency and at the points nearest and farthest from the origin on that circle. Using one of Maskit's combination theorems [9] we see that the group $G'$ generated by these four Fuchsian groups is discontinuous. If we take the free product of $G'_2$ with the group generated by the map

$$z \longrightarrow \frac{(2-\sqrt{2})^2}{2} z$$

we get a group $G$. A simple sketch makes it clear that $0 \in L_2(G)$.

The second example is of an infinitely generated group for which the conclusions of Theorem 3.2 do not hold. More precisely we show that there are residual limit points of infinitely generated groups, but none of the required Jordan curves. We first construct an infinitely generated group $G'$ whose limit set consists of non-locally connected continua and no other non-trivial continua. This may be constructed by taking the free product via Klein's Combination Theorem of loxodromic transformations whose isometric circles converge to every point on a topologist's sine curve and a Jordan arc, such that the union $J$ of the two separates the plane. Some fundamental region for $\Omega(G')$ will then have components in both the interior and exterior of that curve. Choose a loxodromic transformation $\gamma$ whose isometric circle $I(\gamma)$ lies in an interior component of $R$ and such that $I(\gamma^{-1})$ lies in an interior component. The free product $G$ of $G'$ and $\{\gamma\}$ is Kleinian by Klein's combination theorem and contains no non-trivial components other than $J$ and its images. $\Lambda_0(G) \neq \emptyset$ since the fixed points of $\gamma$ lie in $\Lambda_0(G)$. It follows that Theorem 3.2 does not hold for infinitely generated groups.

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