THE STRUCTURE OF A VON NEUMANN ALGEBRA WITH A HOMOGENEOUS PERIODIC STATE

(Dedicated to Professor M. Fukamiya on his 60th birthday)

BY

MASAMICHI TAKESAKI

Departement of Mathematics, University of California, Los Angeles, California 90049, USA(1)

Table of Contents

Introduction

When F. J. Murray and J. von Neumann developed the theory of rings of operators in the 1930's, they first classified all factors acting on separable Hilbert spaces into those of type I, type II and type III. By showing that a factor of type I is isomorphic to the algebra $\mathcal{L}(\mathfrak{H})$ of all bounded operators on some Hilbert space $\mathfrak{H}$, they proved that the algebraic type of a factor of type I is completely determined by its dimension. Namely, the factors of type I are classified into those of type I_n, $n = 1, 2, ..., \infty$. According to their theory, we can not only classify the factors of type I, but also understand explicitly the structure of a factor of type I. The situation is much worse for factors of types II and III. Here we have not a complete classification. Furthermore, we had not been able to construct many different factors until quite recently. To obtain infinitely many non-type

(1) The preparation of this paper was supported in part by NSF Grant No. GP28737.

I factors, we had had to wait for Powers' work in 1967, [15]. He showed the existence of continuously many non-isomorphic factors of type III. After that, the construction of non-isomorphic factors proceeded remarkably quickly. In 1968, Araki and Woods introduced a new algebraic invariant, asymptotic ratio set, and used it to partially classify the factors constructed as infinite tensor product of finite factors of type I, [2]. In 1969, McDuff succeeded in constructing continuously many non-isomorphic factors of types II_1 and II_∞, [11] and [12]. Shortly after that, Sakai found an alternative method of constructing continuously many non-isomorphic type II_1 factors as well as type II_∞ factors. Thus we now have many non-isomorphic factors. Unfortunately, most of these constructions rely on algebraic invariants and not structure theorems. Of course, it would be desirable to have structure theorems which could distinguish different factors. We have to admit that the present stage of classification theory as well as structure theory is very far from being complete. Therefore, we should try to obtain a structure theorem for reasonably easy cases. This paper is devoted to getting a structure theorem for von Neumann algebras with a homogeneous periodic state.

At the same time as Powers' work appeared, Tomita established the theory of modular Hilbert algebras (Tomita algebras), [22]. According to this theory, a normal faithful state φ of a von Neumann algebra $\mathcal{M}$ gives rise to a one parameter automorphism group σ_t^φ of $\mathcal{M}$, the modular automorphism group of $\mathcal{M}$ associated with φ. In [18], we have seen that the modular automorphism group σ_t^φ is uniquely determined by φ subject to the Kubo–Martin–Schwinger condition. Furthermore, the factor in question is of type I or type II if and only if σ_t^φ is inner. If φ is a trace, then σ_t^φ is the identity automorphism. Therefore, a very natural class of factors, after the finite ones, consists of those equipped with a state whose associated modular automorphism group is periodic. We develop a structure theory for a certain subclass of this class of factors. Roughly speaking, the factors in question are essentially uniquely expressed as a crossed product of a von Neumann algebra of type II_1 by an endomorphism. Most of the results have been announced in [20] and [21].

In § 1, we decompose a von Neumann algebra $\mathcal{M}$ with a fixed homogeneous periodic state φ, and prove that the algebra is generated by the centralizer $\mathcal{M}_0$ of the state and an isometry which induces an isomorphism θ of the centralizer onto its reduced algebra.

In § 2, we propose a method of constructing a von Neumann algebra $\mathcal{M}$ with a specified decomposition.

In § 3, we compute the new algebraic invariants $S(\mathcal{M})$ and $T(\mathcal{M})$ introduced by Connes, [3] and [4]. As a corollary, we seen in § 5 that the group of all inner automorphisms is not necessarily closed in the group of all automorphisms under any reasonable topology except the uniform topology.

Section 4 is devoted to the comparison of two inner homogeneous periodic states. We prove that the structure theorem obtained in § 1 essentially uniquely determines the factors in question if the relevant state is inner homogeneous.

In § 5, we show, by examples, that we can realize any countable subgroup of the additive group of all real numbers as $T(\mathcal{M})$ for some $\mathcal{M}$.

1. Decomposition of a von Neumann algebra with respect to a homogeneous periodic state

Let $\mathcal{M}$ be a von Neumann algebra. By Aut $(\mathcal{M})$, we denote the group of all automorphisms of $\mathcal{M}$. For a normal positive linear functional φ on $\mathcal{M}$ (most of the time we consider only states), we denote by $G(\varphi)$ the group of all automorphisms of $\mathcal{M}$ leaving φ invariant, that is,

$$G(\varphi) = \{\sigma \in \operatorname{Aut}(\mathcal{M}): \varphi \circ \sigma = \varphi\}.$$

Definition 1.1. We call φ *homogeneous* if $G(\varphi)$ acts ergodically on $\mathcal{M}$, that is, the fixed point algebra $\mathcal{M}^{G(\varphi)}$ of $G(\varphi)$ reduces to the scalar multiples of the identity.

PROPOSITION 1.2. *If φ is homogeneous, then φ is faithful.*

Proof. Let $s(\varphi)$ denote the support of φ. We have then, in general, $s(\varphi\circ\sigma) = \sigma^{-1}(s(\varphi))$ for every $\sigma \in \operatorname{Aut}(\mathcal{M})$. Hence $s(\varphi) = \sigma^{-1}(s(\varphi))$ for every $\sigma \in G(\varphi)$. Therefore $s(\varphi)$ must be a scalar multiple of 1, which means $s(\varphi) = 1$. Hence φ is faithful. Q.E.D.

Definition 1.3. A faithful normal positive linear functional φ on $\mathcal{M}$ is said to be *periodic* if there exists $T > 0$ such that σ_T^φ is the identity automorphism ι of $\mathcal{M}$ where σ_t^φ denotes the modular automorphism group of $\mathcal{M}$ associated with φ. The smallest such $T > 0$ is called the *period* of φ.

We consider a von Neumann algebra $\mathcal{M}$ equipped with a faithful homogeneous periodic normal state φ as our main subject. Let T be the period of φ. Put $\varkappa = e^{-2\pi/T}$, $0 < \varkappa < 1$. Considering the cyclic representation of $\mathcal{M}$ induced by φ, we assume that $\mathcal{M}$ acts on a Hilbert space $\mathfrak{H}$ with a distinguished vector ξ_0 such that

$$\varphi(x) = (x\xi_0 | \xi_0), \; x \in \mathcal{M}, \quad \text{and} \quad [\mathcal{M}\xi_0] = \mathfrak{H},$$

where we denote by $[\mathfrak{M}]$ the closed subspace spanned by $\mathfrak{M}$ for any subset $\mathfrak{M}$ of $\mathfrak{H}$. Making use of the theory of modular Hilbert algebras (which the author proposes to call Tomita algebras), see [18], we associate the self-adjoint non-singular positive operator Δ on $\mathfrak{H}$, called the modular operator, and the unitary involution J such that

6 – 732906 *Acta mathematica* 131. Imprimé le 19 Octobre 1973

$$\left.\begin{aligned} \sigma_t^\varphi(x) &= \Delta^{it}x\Delta^{-it}, \quad x\in\mathcal{M};\\ \Delta^{it}\xi_0 &= \xi_0, \quad J\xi_0 = \xi_0;\\ J\Delta^{it}J &= \Delta^{it}, \quad J\Delta J = \Delta^{-1};\\ J\mathcal{M}J &= \mathcal{M}', \quad J\mathcal{M}'J = \mathcal{M};\\ J\Delta^{\frac{1}{2}}x\xi_0 &= x^*\xi_0, \quad x\in\mathcal{M};\\ J\Delta^{-\frac{1}{2}}y\xi_0 &= y^*\xi_0, \quad y\in\mathcal{M}'. \end{aligned}\right\} \tag{1}$$

By assumption, we have $\Delta^{iT}=1$, so that the spectrum $\mathrm{Sp}(\Delta)$ of Δ is contained in $\{\varkappa^n: n\in\mathbf{Z}\}\cup\{0\}$, where $\mathbf{Z}$ denotes the set of all integers. Let E_n denote the projection of $\mathfrak{H}$ onto the closed subspace

$$\mathfrak{H}_n = \{\xi\in\mathfrak{H}: \Delta\xi = \varkappa^n\xi\}, \quad n\in\mathbf{Z}. \tag{2}$$

We have then the following expressions for Δ and Δ^{it}:

$$\Delta = \sum_{n\in\mathbf{Z}}\varkappa^n E_n, \qquad \Delta^{it} = \sum_{n\in\mathbf{Z}}\varkappa^{int}E_n. \tag{3}$$

The projections E_n are also written as follows:

$$E_n\xi = \frac{1}{T}\int_0^T \varkappa^{-int}\Delta^{it}\xi\,dt, \quad \xi\in\mathfrak{H}. \tag{4}$$

Since $J\Delta J = \Delta^{-1}$, we have

$$J\mathfrak{H}_n = \mathfrak{H}_{-n} \quad\text{and}\quad JE_nJ = E_{-n}, \quad n\in\mathbf{Z}. \tag{5}$$

Set

$$\mathcal{M}_n = \{x\in\mathcal{M}: \sigma_t^\varphi(x) = \varkappa^{int}x\}, \quad n\in\mathbf{Z}. \tag{6}$$

Each $\mathcal{M}_n$, $n\in\mathbf{Z}$, is a σ-weakly closed subspace of $\mathcal{M}$ invariant under σ_t^φ, and $\mathcal{M}_0$ is nothing but the centralizer $\mathcal{M}_\varphi$ of φ in [19]. When $\mathcal{M}$ is finite, H. Umegaki studied $\mathcal{M}_0$ in detail, fixing $\mathcal{M}_0$ first and then considering all states whose centeralizer contains $\mathcal{M}_0$, [23]. For each $n\in\mathbf{Z}$, we consider the integration:

$$\varepsilon_n(x) = \frac{1}{T}\int_0^T \varkappa^{-int}\sigma_t^\varphi(x)\,dt, \quad x\in\mathcal{M}. \tag{7}$$

We have then the following properties:

$$\varepsilon_n(\mathcal{M}) = \mathcal{M}_n; \tag{8}$$

$$\varepsilon_n\circ\varepsilon_m = 0 \quad\text{if } n\neq m,\ \varepsilon_n\circ\varepsilon_n = \varepsilon_n; \tag{9}$$

$$\varepsilon_n(axb) = a\varepsilon_n(x)b \quad\text{if } a, b\in\mathcal{M}_0; \tag{10}$$

$$\varepsilon_n(x)\xi_0 = E_n x\xi_0, \quad x \in \mathfrak{M}; \tag{11}$$

$$x\xi_0 = \sum_{n\in\mathbf{Z}} \varepsilon_n(x)\,\xi_0, \quad x \in \mathfrak{M}. \tag{12}$$

By equality (11), we have $\mathfrak{M}_n\xi_0 \subset \mathfrak{H}_n$.

LEMMA 1.4. *For each $n \in \mathbf{Z}$, we have*

$$\mathfrak{H}_n = [\mathfrak{M}_n\xi_0]. \tag{13}$$

Proof. Let ξ be an arbitrary vector in $\mathfrak{H}_n$. There exists a sequence $\{x_i\}$ in $\mathfrak{M}$ such that $\xi = \lim_{i\to\infty} x_i\xi_0$. Then we have by (11)

$$\xi = E_n\xi = \lim_{i\to\infty} E_n x_i \xi_0 = \lim_{i\to\infty} \varepsilon_n(x_i)\,\xi_0. \qquad \text{Q.E.D.}$$

LEMMA 1.5. *For each n, $m \in \mathbf{Z}$, we have*

$$\mathfrak{M}_n\,\mathfrak{M}_m \subset \mathfrak{M}_{n+m}; \tag{14}$$

$$\mathfrak{M}_n^* = \mathfrak{M}_{-n}; \tag{15}$$

$$\mathfrak{M}_n\mathfrak{H}_m \subset \mathfrak{H}_{n+m}. \tag{16}$$

Proof. We prove only (16) and the others are verified along the same line. For each $x \in \mathfrak{M}_n$ and $\xi \in \mathfrak{H}_m$, we have

$$\begin{aligned}\Delta^{it}x\xi &= \Delta^{it}x\Delta^{-it}\Delta^{it}\xi = \sigma_t^\varphi(x)\Delta^{it}\xi \\ &= \varkappa^{int}x(\varkappa^{imt}\xi) = \varkappa^{i(n+m)t}x\xi. \qquad \text{Q.E.D.}\end{aligned}$$

The following lemma is a special case of the more general result shown recently by E. Størmer, [17]. But, for the sake of completeness, we present a proof.

LEMMA 1.6. *For each $n \in \mathbf{Z}$ and $x \in \mathfrak{M}$, the following two statements are equivalent:*

(i) *x belongs to $\mathfrak{M}_n$, that is, $\sigma_t^\varphi(x) = \varkappa^{\mathrm{int}}x$;*

(ii) *$\varkappa^n\varphi(yx) = \varphi(xy)$ for every $y \in \mathfrak{M}$.*

Proof. (i)$\Rightarrow$(ii): Suppose $\sigma_t^\varphi(x) = \varkappa^{\mathrm{int}}x$, $t \in \mathbf{R}$. For each $y \in \mathfrak{M}$, there exists a bounded function $F(\alpha)$ continuous on and holomorphic in the strip, $0 \leqslant \operatorname{Im}\alpha \leqslant 1$, such that

$$F(t) = \varphi(\sigma_t^\varphi(x)y) \quad \text{and} \quad F(t+i) = \varphi(y\sigma_t^\varphi(x)).$$

We have then

$$F(t) = \varphi(\varkappa^{\mathrm{int}}xy) = \varkappa^{\mathrm{int}}\varphi(xy);$$

$$F(t+i) = \varkappa^{in(t+i)}\varphi(xy) = \varkappa^{-n}\varkappa^{\mathrm{int}}\varphi(xy);$$

hence we have $$\varkappa^{\mathrm{int}}\varphi(yx) = \varphi(y\sigma_t^\varphi(x)) = F(t+i) = \varkappa^{-n}\varkappa^{\mathrm{int}}\varphi(xy).$$

Thus we obtain

$$\varphi(y\sigma_t^\varphi(x)) = \varkappa^{-n}\varphi(\sigma_t^\varphi(x)\,y); \qquad \varkappa^n\varphi(yx) = \varphi(xy),$$

so that (ii) follows.

(ii) $\Rightarrow$ (i): Suppose $\varkappa^n\varphi(yx) = \varphi(xy)$ for every $y \in \mathcal{M}$. We have then

$$F(t) = \varphi(\sigma_t^\varphi(x)\,y) = \varphi(x\sigma_{-t}^\varphi(y)) = \varkappa^n\varphi(\sigma_{-t}^\varphi(y)\,x)$$
$$= \varkappa^n\varphi(y\sigma_t^\varphi(x)),$$

so that we get $F(t+i) = \varkappa^{-n}F(t)$. Put

$$G(\alpha) = \varkappa^{-in\alpha}F(\alpha), \quad 0 \leqslant \mathrm{Im}\ \alpha \leqslant 1.$$

We have then

$$G(t+i) = \varkappa^{-in(t+i)}F(t+i) = \varkappa^{-\mathrm{int}}\varkappa^nF(t+i) = \varkappa^{-\mathrm{int}}F(t) = G(t).$$

The holomorphic function G is bounded on the strip and has period i. Therefore G is constant. Hence the function $F(\alpha)$ is proportional to $\varkappa^{in\alpha}$. Thus we get

$$\varphi(\sigma_t^\varphi(x)\,y) = \varkappa^{int}F(0) = \varepsilon^{int}\varphi(xy),$$

which means that $(y\xi_0|\sigma_t^\varphi(x^*)\,\xi_0) = \varkappa^{int}(y\xi_0|x^*\xi_0)$. Since $\mathcal{M}\xi_0$ is dense in $\mathfrak{H}$, we have $\sigma_t^\varphi(x^*)\,\xi_0 = \varkappa^{-int}x^*\xi_0$, so that $\sigma_t^\varphi(x^*) = \varkappa^{-int}x^*$ because ξ_0 is separating. Thus we get $\sigma_t^\varphi(x) = \varkappa^{int}x$. Q.E.D.

Proposition 1.7. *(i) If $\mathcal{A}$ is a maximal abelian self-adjoint subalgebra of $\mathcal{M}_0$, then $\mathcal{A}$ is maximal abelian in the whole algebra $\mathcal{M}$.*

(ii) The relative commutant $\mathcal{M}_0' \cap \mathcal{M}$ of $\mathcal{M}_0$ is contained in $\mathcal{M}$ as the center $\mathcal{Z}_0$ of $\mathcal{M}_0$.

Proof. Let $\mathcal{A}^c$ denote the relative commutant $\mathcal{A}' \cap \mathcal{M}$ of $\mathcal{A}$ in $\mathcal{M}$. Since σ_t^φ is the identity automorphism on $\mathcal{M}_0$, so it is on $\mathcal{A}$; hence trivially $\sigma_t^\varphi(\mathcal{A}) = \mathcal{A}$ which implies that $\sigma_t^\varphi(\mathcal{A}^c) = \mathcal{A}^c$. Hence $\mathcal{A}^c$ is invariant under σ_t^φ. Therefore we have $\varepsilon_n(\mathcal{A}^c) \subset \mathcal{A}^c$ because ε_n is defined by (7) and the integration is taken under the σ-strong operator topology. Let x be an arbitrary element in $\mathcal{A}^c$. We have then $\varepsilon_n(x) = y \in \mathcal{A}^c \cap \mathcal{M}_n$, so that y^*y belongs to $\mathcal{A}^c \cap \mathcal{M}_0$, so that y^*y is in $\mathcal{A}$. The same is true for yy^*. Therefore $h = (y^*y)^{\frac{1}{2}}$ and $k = (yy^*)^{\frac{1}{2}}$ both belong to $\mathcal{A}$. Let $y = uh = ku$ be the left and right polar decomposition. Since u is in $\mathcal{A}^c$, commute with h and k, so that

$$yy^* = uh^2u^* = huu^*h \leqslant h = y^*y;$$

$$y^*y = u^*k^2u = ku^*uk \leqslant k^2 = yy^*.$$

Thus we get $y^*y=yy^*$. On the other hand, y is in $\mathcal{M}_n$, so that by Lemma 1.6, we have

$$\varphi(yy^*) = \varkappa^n \varphi(y^*y).$$

Hence $\varphi(y^*y)=0$ unless $n=0$, so that $y=0$. Thus $\varepsilon_n(x)=0$ for every non-zero $n\in\mathbf{Z}$. Therefore x falls in $\mathcal{M}_0$. Hence $x\in\mathcal{M}_0\cap\mathcal{A}^c=\mathcal{A}$. Thus $\mathcal{A}$ is maximal abelian in $\mathcal{M}$.

Assertion (ii) follows from the fact $\mathcal{M}_0'\cap\mathcal{M}\subset\mathcal{A}^c$ for any maximal abelian self-adjoint subalgebra $\mathcal{A}$ of $\mathcal{M}_0$. Hence $\mathcal{M}_0'\cap\mathcal{M}\subset\mathcal{A}^c=\mathcal{A}\subset\mathcal{M}_0$. Thus $\mathcal{M}_0'\cap\mathcal{M}$ must be the center $\mathcal{Z}_0$ of $\mathcal{M}_0$. Q.E.D.

For each $n\in\mathbf{Z}$, we define a normal representation $\{\pi_n, \mathfrak{H}_n\}$ of $\mathcal{M}_0$ and a normal anti-representation $\{\pi_n', \mathfrak{H}_n\}$ as follows:

$$\left.\begin{aligned} &\pi_n(a)\xi = a\xi, \quad a\in\mathcal{M}_0,\ \xi\in\mathfrak{H}_n;\\ &\pi_n'(a) = J\pi_{-n}(a^*)J, \quad a\in\mathcal{M}_0. \end{aligned}\right\} \tag{17}$$

By (16), $\mathfrak{H}_n$ is invariant under $\mathcal{M}_0$, so that the representation $\{\pi_n, \mathfrak{H}_n\}$ makes sense. Also equality (5) guarantees that the definition of $\{\pi_n', \mathfrak{H}_n\}$ makes sense. Since $\pi_n'(a)$ is the restriction of Ja^*J to $\mathfrak{H}_n$ and since $Ja^*J\in\mathcal{M}'$, $\pi_n(\mathcal{M}_0)$ and $\pi_n'(\mathcal{M}_0)$ commute. More precisely, we have the following:

LEMMA 1.8. *For each $x\in\mathcal{M}_n$, we have*

$$\pi_n'(a)x\xi_0 = xa\xi_0, \quad a\in\mathcal{M}_0. \tag{18}$$

Proof. We have

$$\begin{aligned}\pi_n'(a)x\xi_0 &= J\pi_{-n}(a^*)Jx\xi_0 = Ja^*Jx\xi_0 = Ja^*\Delta^{\frac12}x^*\xi_0 = Ja^*\varkappa^{-n/2}x^*\xi_0\\ &= \varkappa^{-n/2}Ja^*x^*\xi_0 = \varkappa^{-n/2}\Delta^{\frac12}xa\xi_0 = \varkappa^{-n/2}\varkappa^{n/2}xa\xi_0 = xa\xi_0. \end{aligned}$$ Q.E.D.

Remark. So far, we have not used the homogeneity of the state φ, so that all results obtained above remain true for any periodic state.

Now, we start to discuss the group $G(\varphi)$. We define the unitary representation $\{U(g), \mathfrak{H}\}$ of $G(\varphi)$ by

$$U(g)x\xi_0 = g(x)\xi_0, \quad x\in\mathcal{M},\ g\in G(\varphi).$$

The invariance of φ under $g\in G(\varphi)$ assures that each $U(g)$ extends to a unitary operator $\mathfrak{H}$ which is also denoted by the same symbol $U(g)$. It is obvious that

$$U(g_1g_2) = U(g_1)U(g_2) \text{ and } U(g^{-1}) = U(g)^*$$

for any g_1, g_2 and $g\in G(\varphi)$. For any x, $y\in\mathcal{M}$, we have

$$U(g)xU(g)^*y\xi_0 = U(g)xg^{-1}(y)\xi_0 = g(xg^{-1}(y))\xi_0 = g(x)y\xi_0$$

so that
$$U(g)xU(g)^* = g(x), \quad x \in \mathcal{M}. \tag{19}$$

Since g and σ_t^φ commute by [10] and [19], g leaves $\mathcal{M}_n$ invariant, so that $U(g)$ leaves $\mathfrak{H}_n$ invariant as well. Furthermore, we have

$$\left.\begin{aligned} &U(g)\pi_n(a)U(g)^* = \pi_n \circ g(a);\\ &U(g)\pi_n'(a)U(g)^* = \pi_n' \circ g(a), a \in \mathcal{M}_0, g \in G(\varphi).\end{aligned}\right\} \tag{20}$$

In fact, the first equality follows from (19) and the second one follows from (18) and (19).

LEMMA 1.9. *If $\mathfrak{H}_n \neq \{0\}$, then $\{\pi_n, \mathfrak{H}_n\}$ and $\{\pi_n', \mathfrak{H}_n\}$ are both faithful.*

Proof. Let $\mathfrak{J} = \pi_n^{-1}(0)$. Since π_n is normal, $\mathfrak{J}$ is a σ-weekly closed ideal of $\mathcal{M}_0$. Hence $\mathfrak{J}$ is of the form $\mathfrak{J} = \mathcal{M}_0 z$ for some projection z in $\mathcal{Z}_0$. If x is in $\mathfrak{J}$, then $\pi_n(x) = 0$; so we have, for every $g \in G(\varphi)$,

$$\pi_n \circ g(x) = U(g)\pi_n(x)U(g)^* = 0;$$

hence $g(x)$ falls in $\mathfrak{J}$. Therefore, $g(\mathfrak{J}) \subset \mathfrak{J}$, $g \in G(\varphi)$. Considering g^{-1}, we get $g(\mathfrak{J}) = \mathfrak{J}$, which means $g(z) = z$ for every $g \in G(\varphi)$. By the ergodicity of $G(\varphi)$, $z = 0$ or 1. Since $\pi_n(1) \neq 0$ by the assumption on $\mathfrak{H}_n$, $z = 0$. The assertion for π_n' follows similarly. Q.E.D.

LEMMA 1.10. *For every $n \in \mathbf{Z}$, we have*

$$\mathcal{M}_n \neq \{0\}.$$

Proof. Let $\mathbf{Z}_1 = \{n \in \mathbf{Z}: \mathcal{M}_n \neq \{0\}\}$. We first claim that $\mathbf{Z}_1$ is a subgroup of the additive group $\mathbf{Z}$. Let x be a non-zero element in $\mathcal{M}_n$ for a fixed $n \in \mathbf{Z}_1$. Then x^*x is a non-zero element of $\mathcal{M}_0$ by Lemma 1.5. If $\mathcal{M}_m \neq \{0\}$, then $\mathfrak{H}_m \neq \{0\}$, so that $\pi_m(x^*x) \neq 0$ by Lemma 1.9. Since $\mathcal{M}_m\xi_0$ is dense in $\mathfrak{H}_m$ by Lemma 1.4, there exists an element $y \in \mathcal{M}_m$ such that $\pi_m(x^*x)y\xi_0 \neq 0$; hence $xy \neq 0$. But xy falls in $\mathcal{M}_{n+m}$, so that $\mathcal{M}_{n+m} \neq \{0\}$. Hence $\mathbf{Z}_1$ is additive. By (15) $\mathcal{M}_{-n} \neq \{0\}$ if $n \in \mathbf{Z}_1$. Thus $\mathbf{Z}_1$ is a group.

Let n_0 be the smallest positive integer in $\mathbf{Z}_1$. By the group property of $\mathbf{Z}_1$, we have $\mathbf{Z}_1 = n_0\mathbf{Z}$. Therefore, the spectrum of Δ consists of $\{\varkappa^{n_0 n}: n \in \mathbf{Z}\}$, Hence we have

$$\Delta^{iT/n_0} = \sum_{n \in \mathbf{Z}} \varkappa^{inn_0T/n_0} E_{nn_0} = \sum_{n \in \mathbf{Z}} E_{nn_0} = 1.$$

Unless $n_0 = 1$, this contradicts the fact that T is the period of σ_t^φ. Q.E.D.

LEMMA 1.11. *For each $n \in \mathbf{Z}$, $\{\pi_n(\mathcal{M}_0), \mathfrak{H}_n\}$ admits either a cyclic vector or a separating vector.*

Proof. Let z be the greatest projection in $\mathcal{Z}_0$ such that $\pi_n(z)$ is a cyclic projection. The existence of such a projection is assured by the σ-finiteness of $\mathcal{M}_0$. For each $g \in G(\varphi)$, $\pi_n \circ g(z) = U(g)\pi_n(z) U(g)^*$ is also a cyclic projection. Hence we have $g(z) = z$ for each $g \in G(\varphi)$. The ergodicity of $G(\varphi)$ implies either $z=0$ or 1. If $z=0$, then $\{\pi_n(\mathcal{M}_0), \mathfrak{H}_n\}$ must admit a separating vector. Q.E.D.

LEMMA 1.12. *If ξ is a vector in $\mathfrak{H}_n$, $n \in \mathbf{Z}$, then there exists self-adjoint positive operators h and k affiliated with $\mathcal{M}_0$ and a partial isometry $u \in \mathcal{M}_n$ such that*

$$\xi = uh\xi_0 = ku\xi_0.$$

Prof. Since ξ belongs to $\mathcal{D}(\Delta^{\frac{1}{2}})$, the domain of $\Delta^{\frac{1}{2}}$, if we define an operator a_0 on $\mathcal{M}'\xi_0$ by $ay\xi_0 = y\xi$, $y \in \mathcal{M}'$, then a_0 is preclosed, see [18; § 3]. The preclosed operator a_0 is transformed by Δ^{it} as follows:

$$\Delta^{it} a_0 \Delta^{-it} y\xi_0 = \Delta^{it} a_0 \Delta^{-it} y \Delta^{it} \xi_0 = \Delta^{it}(\Delta^{-it} y \Delta^{it})\xi = y\Delta^{it}\xi = \varkappa^{int} y\xi = \varkappa^{int} a_0 y\xi_0, \quad y \in \mathcal{M}'.$$

Hence we have $\Delta^{it} a_0 \Delta^{-it} = \varkappa^{\mathrm{int}} a_0$. Let a denote the closure of a_0 which is the second adjoint a_0^{**} of a_0. Then we have $\Delta^{it} a \Delta^{-it} = \varkappa^{\mathrm{int}} a$. Suppose $a = uh = ku$ is the left and right polar decomposition of a. Then u is in $\mathcal{M}$, and h and k are affiliated with $\mathcal{M}$. The equality:

$$\varkappa^{\mathrm{int}} uh = \Delta^{it} uh \Delta^{-it} = \Delta^{it} u \Delta^{-it} \Delta^{it} h \Delta^{-it}$$

together with the unicity of the polar decomposition, implies that

$$\sigma_t^{\varphi}(u) = \Delta^{it} u \Delta^{-it} = \varkappa^{int} u;$$
$$\Delta^{it} h \Delta^{-it} = h.$$

Hence h is affiliated with the fixed point algebra $\mathcal{M}_0$ of σ_t^{φ}. Similarly k is affiliated with $\mathcal{M}_0$. Now, we have

$$\xi = a_0\xi_0 = a\xi = uh\xi_0 = ku\xi_0. \qquad \text{Q.E.D.}$$

LEMMA 1.13. *For $n \geq 1$, $\{\pi_n(\mathcal{M}_0)\, \mathfrak{H}_n\}$ (resp. $\{\pi_{-n}(\mathcal{M}_0), \mathfrak{H}_{-n}\}$) does not admit a separating vector (resp. a cyclic vector).*

Proof. Let ξ be a separating vector for $\{\pi_n(\mathcal{M}_0), \mathfrak{H}_n\}$. By Lemma 1.12, ξ is of the form $\xi = uh\xi_0$ with u a partial isometry in $\mathcal{M}_n$ and h a self-adjoint positive operator affiliated with $\mathcal{M}_0$. By definition, $\pi_n(a)\xi = 0$, $a \in \mathcal{M}_0$, implies $a=0$, so that $au=0$ implies $a=0$. But $(1-uu^*)u=0$ and $1-uu^*$ is in $\mathcal{M}_0$, so that $uu^*=1$. By Lemma 1.6, we get

$$1 \geq \varphi(u^*u) = \varkappa^{-n}\varphi(uu^*) = \varkappa^{-n} > 1,$$

a contradiction.

Suppose ξ is a cyclic vector for $\{\pi_{-n}(\mathcal{M}_0), \mathfrak{H}_{-n}\}$. It is then separating for $\pi'_{-n}(\mathcal{M}_0)$. By Lemma 1.12, we have $\eta = kv\xi_0$ with v a partial isometry in $\mathcal{M}_{-n}$ and k a self-adjoint positive operator affiliated with $\mathcal{M}_0$. Since $\pi'_{-n}(a)\eta = 0$ implies $a=0$ for any $a \in \mathcal{M}_0$, $kva\eta = 0$ implies, by Lemma 1.8, $a=0$; hence $va=0$ implies $a=0$. Therefore, we get $v^*v=1$ since $v(1-v^*v)=0$ and $v^*v \in \mathcal{M}_0$. But Lemma 1.6 yields the following:

$$1 \geqslant \varphi(vv^*) = \varkappa^{-n}\varphi(v^*v) = \varkappa^{-n} > 1,$$

a contradiction. Q.E.D.

As an immediatiate consequence of Lemmas 1.11 and 13, we get the following:

COROLLARY 1.14. *For $n \geqslant 1$, $\{\pi_n(\mathcal{M}_0), \mathfrak{H}_n\}$ (resp. $\{\pi_{-n}, \mathfrak{H}_{-n}\}$) admits a cyclic (resp. separating) vector.*

LEMMA 1.15. *The subspace $\mathcal{M}_1$ of $\mathcal{M}$ contains an isometry u such that for $n \geqslant 1$,*

$$\mathcal{M}_n = \mathcal{M}_0 u^n;$$

$$\mathcal{M}_{-n} = u^{*n}\mathcal{M}_0.$$

Proof. By Corollary 1.14, there exists a cyclic vector ξ for $\{\pi_1, \mathfrak{H}_1\}$. As seen already, ξ is of the form $\xi = ku\xi_0$ with u a partial isometry in $\mathcal{M}_1$ and k a self-adjoint positive operator affiliated with $\mathcal{M}_0$. Being cyclic for $\pi_1(\mathcal{M}_0)$, ξ is separating for $\pi'_1(\mathcal{M}_0)$. Hence $\pi'_1(a)\xi = 0$ implies $a=0$ for any $a \in \mathcal{M}_0$, so that if $ua=0$, $a \in \mathcal{M}_0$, then $\pi'_1(a)\xi = kua\xi = 0$; hence $a=0$. From the facts that $u(u^*u-1)=0$ and that $u^*u \in \mathcal{M}_0$, it follows that $u^*u=1$. Hence u is an isometry. Let $n \geqslant 1$. We have then $(u^n)^*u^n = 1$. If x is in $\mathcal{M}_n$, then $x(u^*)^n$ belongs to $\mathcal{M}_0$ by Lemma 1.5, so that

$$x = x(u^{*n}u^n) = (xu^{*n})u^n \in \mathcal{M}_0 u^n.$$

Thus $\mathcal{M}_n \subset \mathcal{M}_0 u^n$. Lemma 1.5 yields that $\mathcal{M}_0 u^n \subset \mathcal{M}_n$. Thus we get $\mathcal{M}_n = \mathcal{M}_0 u^n$. Since $\mathcal{M}_{-n} = \mathcal{M}_n^*$, we have $\mathcal{M}_{-n} = u^{*n}\mathcal{M}_0$. Q.E.D.

COROLLARY 1.16. *The von Neumann algebra $\mathcal{M}$ is generated by $\mathcal{M}_0$ and any isometry u in $\mathcal{M}_1$. More precisely, in the pre-Hilbert space structure induced by the state φ, $\mathcal{M}$ is decomposed into a "direct sum" as follows:*

$$\mathcal{M} \text{ “}=\text{” } \ldots \oplus u^{*n}\mathcal{M}_0 \oplus \ldots \oplus u^*\mathcal{M}_0 \oplus \mathcal{M}_0 \oplus \mathcal{M}_0 u \oplus \ldots \oplus \mathcal{M}_0 u^n \oplus \ldots.$$

For each $z \in \mathcal{M}$, we denote

$$x(n) = \varepsilon_n(x) \in \mathcal{M}_n, \quad n \in \mathbf{Z}. \tag{21}$$

We have then

$$x\xi_0 = \sum_{n \in \mathbf{Z}} x(n)\,\xi_0.$$

LEMMA 1.17. *For each $x, y \in \mathcal{M}$, we have*

$$(xy)(n)\,\xi_0 = \sum_{m\in\mathbf{Z}} x(m)\,y(n-m)\,\xi_0, \tag{22}$$

where the summation is taken in the strong topology in $\mathfrak{H}$.

Proof. First we assume that y is in $\mathcal{M}_m$. We then have

$$\begin{aligned}
xy\xi_0 &= x(\varkappa^{-m/2}\Delta^{\frac{1}{2}}y\xi_0) = \varkappa^{-m/2}x\Delta^{\frac{1}{2}}y\xi_0 \\
&= \varkappa^{-m/2}JJxJJ\Delta^{\frac{1}{2}}y\xi_0 = \varkappa^{-n/2}J(JxJ)\,y^*\,\xi_0 \\
&= \varkappa^{-m/2}Jy^*(JxJ)\,\xi_0 \quad \text{since} \quad JxJ\in\mathcal{M}', \\
&= \varkappa^{-m/2}Jy^*Jx\xi_0 \quad \text{since} \quad J\xi_0 = \xi_0, \\
&= \varkappa^{-m/2}Jy^*J\sum_{l\in\mathbf{Z}} x(l)\,\xi_0 \\
&= \sum_{l\in\mathbf{Z}}\varkappa^{-m/2}Jy^*Jx(l)\,\xi_0 = \sum_{l\in\mathbf{Z}}\varkappa^{-m/2}x(l)\,Jy^*J\xi_0 = \sum_{l\in\mathbf{Z}}x(l)\,\varkappa^{-m/2}(\Delta^{\frac{1}{2}}y\xi_0) = \sum_{l\in\mathbf{Z}}x(l)\,y\xi_0.
\end{aligned}$$

Therefore, we have, for a general $y\in\mathcal{M}$,

$$xy\xi_0 = x\sum_{m\in\mathbf{Z}} y(m)\,\xi_0 = \sum_{m\in\mathbf{Z}} xy(m)\,\xi_0 = \sum_{m\in\mathbf{Z}}\sum_{l\in\mathbf{Z}} x(l)\,y(m)\,\xi_0,$$

hence

$$(xy)(n)\,\xi_0 = E_n xy\xi_0 = \sum_{m\in\mathbf{Z}}\sum_{l\in\mathbf{Z}} E_n x(l)\,y(m)\,\xi_0 = \sum_{l\in\mathbf{Z}} x(l)\,y(n-l)\,\xi_0.$$

Q.E.D.

LEMMA 1.18. *If u and v are isometries in $\mathcal{M}_n$, $n\geq 1$, then there exists a partial isometry w in $\mathcal{M}_0$ such that*

$$v = wu \ \text{ and } \ u = w^*v.$$

Proof. Let $w = vu^*$. Then w is in $\mathcal{M}_0$, and we have

$$wu = vu^*u = v;$$

$$w^*v = (vu^*)^*v = uv^*v = u.$$

Furthermore, we get

$$w^*w = uv^*vu^* = uu^*;$$

$$ww^* = vu^*uv^* = vv^*.$$

Hence w is a partial isometry in $\mathcal{M}_0$ with the intial projection uu^* and the final projection vv^*. Q.E.D.

LEMMA 1.19. *Let u be an isometry in $\mathcal{M}_n$, $n\geq 1$, and let $e = uu^*$. Then e has uniform relative dimension in $\mathcal{M}_0$, more precisely, we have*

$$e^\natural = \varkappa^n 1.$$

Proof. By Lemma 1.18, there exists, for each $g\in G(\varphi)$, a partial isometry $w(g)\in\mathcal{M}_0$ such that $g(u)=w(g)u$. Hence we have

$$\begin{aligned} g(e) &= g(uu^*) = g(u)g(u)^* \\ &= w(g)uu^*w(g)^* = w(g)ew(g)^*. \end{aligned}$$

By the ergodicity of $G(\varphi)$ on $\mathcal{M}_0$, we can find nets

$$\{\lambda_i^\alpha : i=1,2,\dots,n_\alpha\}\subset[0,1] \text{ and } \{g_i^\alpha : i=1,2,\dots,n_\alpha\}\subset G(\varphi)$$

such that

$$\varphi(e)\,1 = w-\lim_\alpha \sum_{i=1}^{n_\alpha}\lambda_i^\alpha g_i^\alpha(e) = w-\lim_\alpha \sum_{i=1}^{n_\alpha}\lambda_i^\alpha w(g_i^\alpha)ew(g_i^\alpha)^* = e^\natural$$

On the other hand, we have by Lemma 1.6,

$$\varphi(e)=\varphi(uu^*)=\varkappa^n\varphi(u^*u)=\varkappa^n\varphi(1)=\varkappa^n. \qquad \text{Q.E.D.}$$

We now introduce a positive linear map Ad (a) of $\mathcal{M}$ into $\mathcal{M}$ for each fixed $a\in\mathcal{M}$ as follows:

$$\mathrm{Ad}\,(a)(x)=axa^*,\quad x\in\mathcal{M}. \tag{23}$$

If u is unitary, then Ad (u) is an inner automorphism, and if u is an isometry then Ad (u) is an isomorphism of $\mathcal{M}$ onto $e\mathcal{M}e$ with $e=uu^*$.

LEMMA 1.20. *Let e be a projection of a von Neumann algebra $\mathcal{N}$ such that the central support of e is* 1. *If θ is an isomorphism of $\mathcal{N}$ onto $e\mathcal{N}e$, then there exists uniquely an automorphism $\tilde\theta$ of the center $\mathcal{Z}$ of $\mathcal{N}$ such that $\theta(a)=\tilde\theta(a)e$ for any $a\in\mathcal{Z}$.*

Proof. The center of $e\mathcal{N}e$ is $\mathcal{Z}e$, and the map: $a\in\mathcal{Z}\mapsto ae\in\mathcal{Z}e$ is an isomorphism of $\mathcal{Z}$ onto $\mathcal{Z}e$ by the assumption on the central support of e. Since θ is an isomorphism of $\mathcal{N}$ onto $e\mathcal{N}e$ as assumed, we have $\theta(\mathcal{Z})=\mathcal{Z}e$. Hence we can define an automorphism $\tilde\theta$ of $\mathcal{Z}$ so that $\theta(a)=\tilde\theta(a)e$, $a\in\mathcal{Z}$. Q.E.D.

By Lemma 1.5, if u is an isometry in $\mathcal{M}_1$, then Ad (u) leaves $\mathcal{M}_0$ invariant, so we denote by θ_u the restriction of Ad (u) to $\mathcal{M}_0$. Since uu^* is of uniform relative dimension in $\mathcal{M}_0$ by Lemma 1.20, the central support of uu^* must be 1. Hence θ_u induces an auto morphism $\tilde\theta_u$ on $\mathcal{Z}_0$ by Lemma 1.21.

LEMMA 1.21. (i) *For any isometries u and v in $\mathcal{M}_1$, we have*

$$\tilde\theta_u(a)=\tilde\theta_v(a),\, a\in Z_0.$$

(ii) *The inverse $\tilde\theta_u^{-1}$ is given by*

$$\tilde\theta_u^{-1}(a)=u^*au,\, a\in Z_0.$$

Proof. By Lemma 1.19, v is of the form $v=wu$ with w a partial isometry in $\mathcal{M}_0$. Hence we have

$$\theta_v(x)=vxv^*=wuxu^*w^*=w\theta_u(x)w^*,\quad x\in\mathcal{M}_0.$$

Let $e=uu^*$ and $f=vv^*$. We have then, for each $a\in\mathcal{Z}_0$,

$$\begin{aligned}\tilde{\theta}_v(a)f=\theta_v(a)&=w\theta_u(a)w^*\\&=w\tilde{\theta}_u(a)ew^*=\tilde{\theta}(a)wew^*=\tilde{\theta}_u(a)f,\end{aligned}$$

so that $\tilde{\theta}_u(a)=\tilde{\theta}_v(a)$, $a\in\mathcal{Z}_0$. Let $b=\tilde{\theta}_u(a)$, $a\in\mathcal{Z}_0$. We have then $be=\theta_u(a)=uau^*$, so that

$$a=u^*uau^*u=u^*beu=u^*b(eu)=u^*bu,$$

which means that $\tilde{\theta}_u^{-1}(b)=u^*au$. Q.E.D.

Thus, the automorphism $\tilde{\theta}_u$ of $\mathcal{Z}_0$ does not depend on the choice of the isometry u in $\mathcal{M}_1$, so that we denote it simply by $\tilde{\theta}$.

PROPOSITION 1.22. *The center $\mathcal{Z}$ of $\mathcal{M}$ is precisely the fixed point subalgebra of $\mathcal{Z}_0$ under $\tilde{\theta}$. Hence $\mathcal{M}$ is a factor if and only if $\tilde{\theta}$ is ergodic.*

Proof. The center $\mathcal{Z}$ of $\mathcal{M}$ is obviously contained in the relative commutant $\mathcal{M}_0^c$ of $\mathcal{M}_0$ in $\mathcal{M}$ which is the center $\mathcal{Z}_0$ of $\mathcal{M}_0$ by Proposition 1.7. (ii). Since $\mathcal{M}$ is generated by $\mathcal{M}_0$ and an isometry u in $\mathcal{M}_1$ by Corollary 1.16, the center $\mathcal{Z}$ is the set of all elements in $\mathcal{Z}_0$ commuting with u. An element $a\in\mathcal{Z}_0$ commutes with u if and only if $a=u^*au=\tilde{\theta}^{-1}(a)$; hence if and only if a is a fixed point of $\tilde{\theta}$. Q.E.D.

LEMMA 1.23. *For any isometry $u\in\mathcal{M}_n$, $n\geqslant 1$, we have.*

$$\varphi\circ\theta_u(x)=\varkappa^n\varphi(x),\quad x\in\mathcal{M}_0;$$

$$\varphi\circ\tilde{\theta}(a)=\varphi(a),\quad a\in\mathcal{Z}_0.$$

Proof. For each $x\in\mathcal{M}_0$, we have, by Lemma 1.6,

$$\varkappa^n\varphi(x)=\varkappa^n\varphi(xu^*u)=\varphi(uxu^*)=\varphi\circ\theta_u(x).$$

Choose an isometry u in $\mathcal{M}_1$ and let $e=uu^*$. We have for each $a\in\mathcal{Z}_0$

$$\varkappa\varphi(a)=\varphi(\theta_u(a))=\varphi(\tilde{\theta}_u(a)e)=\varphi(\tilde{\theta}_u(a)e^\natural)=\varkappa\varphi(\tilde{\theta}_u(a)\qquad\text{by Lemma 1.19.}$$

Thus $\varphi(a)=\varphi(\tilde{\theta}(a))$. Q.E.D.

LEMMA 1.24. *If u is an isometry in $\mathcal{M}_n$, $n\geqslant 1$, then we have*

$$\theta_u(x)^\natural = \varkappa^n \tilde{\theta}^n(x^\natural),\ x\in \mathcal{M}_0.$$

Proof. For each $a\in Z_0$, we have

$$\begin{aligned}\varphi(\theta_u(x)^\natural a) = \varphi(\theta_u(x)\,a) &= \varphi(uxu^* a)\\ &= \varkappa^n \varphi(xu^* au)\\ &= \varkappa^n \varphi(x\tilde{\theta}^{-n}(a)) \qquad \text{by Lemma 1.21,}\\ &= \varkappa^n \varphi(x^\natural \tilde{\theta}^{-n}(a))\\ &= \varkappa^n \varphi(\tilde{\theta}^n(x^\natural)\,a),\end{aligned}$$

so that $\theta_u(x)^\natural = \varkappa^n \tilde{\theta}^n(x^\natural)$. Q.E.D.

LEMMA 1.25. *For any pair p, q of projections in $\mathcal{M}_0$, and an integer n, the following two statements are equivalent:*

(i) *There exists a partial isometry $v\in\mathcal{M}_n$ such that*

$$p = v^* v \quad \textit{and} \quad q = vv^*;$$

(ii) $q^\natural = \varkappa^n \tilde{\theta}^n(p^\natural)$.

Proof. (i) $\Rightarrow$ (ii): Considering v^* if $n\leqslant -1$, we may assume $n\geqslant 1$. Let u be an isometry in $\mathcal{M}_n$. Put $w = vu^* \in \mathcal{M}_0$. We have then

$$ww^* = vu^*uv^* = vv^* = q; \qquad w^*w = uv^*vu^* = upu^* = \theta_u(p).$$

Hence we have, by Lemma 1.24,

$$q^\natural = \theta_u(p)^\natural = \varkappa^n \tilde{\theta}^n(p^\natural).$$

(ii) $\Rightarrow$ (i): Interchanging p and q if $n\leqslant -1$, we may assume $n\geqslant 1$. Choose an isometry u in $\mathcal{M}_n$. We have then $\theta_u(p)^\natural = \varkappa^n\tilde{\theta}^n(p^\natural) = q^\natural$ by Lemma 1.24. Hence there exists a unitary w in $\mathcal{M}_0$

$$q = w\theta_u(p)\,w^* = wupu^*w^*.$$

Put $v = wup$. It is then clear that v is a partial isometry in $\mathcal{M}_n$ such that $vv^* = q$ and

$$v^*v = pu^*w^*wup = pu^*up = p. \qquad \text{Q.E.D.}$$

LEMMA 1.26. *The von Neumann algebra $\mathcal{M}_0$ is of type* II_1.

Proof. Let u be an isometry in $\mathcal{M}_1$, and let $e_n = u^n u^{*n}$, $n\geqslant 1$. We have then $e_n^\natural = \varkappa^n 1$

by Lemma 1.19. Hence $\mathcal{M}_0$ does not have a direct summand of type I, so that it is of type II. Q.E.D.

We are now in the position to state our first result which has been proved already.

THEOREM 1.27. *Suppose $\mathcal{M}$ is a von Neumann algebra equipped with a faithful homogeneous periodic state φ of period $T>0$. Let $\varkappa = e^{-2\pi/T}$. The centralizer $\mathcal{M}_0$ of φ is of type II_1 and there exists an isometry u with the following properties:*

(i) *$\mathcal{M}$ is generated by $\mathcal{M}_0$ and u;*

(ii) Ad (u) *induces an isomorphism of $\mathcal{M}_0$ onto $e\mathcal{M}_0 e$ where $e=uu^*$;*

(iii) *For each positive integer n,*

$$\mathcal{M}_n = \mathcal{M}_0 u^n \text{ and } \mathcal{M}_{-n} = u^{*n}\mathcal{M}_0,$$

where $\mathcal{M}_n$ is defined by (6)*;*

(iv) *The algebra $\mathcal{M}$ is written as:*

$$\mathcal{M}'' ='' \dots \oplus u^{*n}\mathcal{M}_0 \oplus \dots \oplus u^*\mathcal{M}_0 \oplus \mathcal{M}_0 \oplus \mathcal{M}_0 u \oplus \dots \oplus \mathcal{M}_0 u^n \oplus \dots,$$

where "$=$" means that for any $x \in \mathcal{M}$ there exists a sequence $\{x(n)\}$ such that $x(n) \in \mathcal{M}_n$ and $x = \sum_{n\in\mathbf{Z}} x(n)$ under the Hilbert space metric topology induced by φ;

(v) *The isomorphism θ of $\mathcal{M}_0$ onto $e\mathcal{M}_0 e$ induces an automorphism $\tilde{\theta}$ of the center $\mathcal{Z}_0$ of $\mathcal{M}_0$ such that the center $\mathcal{Z}$ of $\mathcal{M}$ is precisely the fixed point subalgebra of $\mathcal{Z}_0$ under $\tilde{\theta}$;*

(vi) *For any pair p, q of projections in $\mathcal{M}_0$ there exists a partial isometry v in $\mathcal{M}_n$ such that $p=v^*v$ and $q=vv^*$ if and only if $\varkappa^n\tilde{\theta}^n(p^\natural) = \tilde{\theta}^n(q^\natural)$.*

Concerning the natural question to what extent the couple $(\mathcal{M}_0, u)$ in Theorem 1.27 determines the structure of the couple $(\mathcal{M}, \varphi)$, we have the following:

THEOREM 1.28. *Suppose $\mathcal{M}$ and $\bar{\mathcal{M}}$ are two von Neumann algebras equipped with faithful homogeneous periodic states φ and $\bar{\varphi}$ respectively. Necessary and sufficient conditions that there exists an isomorphism σ of $\bar{\mathcal{M}}$ onto $\mathcal{M}$ such that $\bar{\varphi} = \varphi\circ\sigma$ are given by the following:*

(i) *$T=\bar{T}$, where T and $\bar{T}$ are the period of φ and $\bar{\varphi}$ respectively;*

(ii) *There exist an isomorphism σ_0 of $\bar{\mathcal{M}}_0$ onto $\mathcal{M}_0$ and a partial isometry w in $\mathcal{M}_0$ such that*

$$\bar{\varphi}_0 = \varphi_0 \circ \sigma_0;$$
$$\theta \circ \sigma_0(x) = w^* \sigma_0 \circ \bar{\theta}(x) w;$$
$$\sigma_0 \circ \bar{\theta}(x) = w\theta \circ \sigma_0(x) w^*, \quad x \in \bar{\mathcal{M}}_0,$$

where φ and $\bar{\varphi}_0$ mean the restrictions of φ and $\bar{\varphi}$ to $\mathcal{M}_0$ and $\bar{\mathcal{M}}_0$ respectively, and θ and $\bar{\theta}$ mean the isomorphisms of $\mathcal{M}_0$ and $\bar{\mathcal{M}}_0$ considered in Theorem 1.27 corresponding to φ and $\bar{\varphi}$ respectively.

Proof. We keep the same notations as before, putting bars on top of the symbols corresponding to $\bar{\mathcal{M}}$ and $\bar{\varphi}$. For example, $\bar{\mathfrak{H}}$ means the representation Hilbert space of $\bar{\mathcal{M}}$ and $\bar{\xi}_0$ means the cyclic vector in $\bar{\mathfrak{H}}$ such that $\bar{\varphi}(x) = (x\bar{\xi}_0 | \bar{\xi}_0)$, $x \in \bar{\mathcal{M}}$.

Suppose there exists an isomorphism σ of $\bar{\mathcal{M}}$ onto $\mathcal{M}$ with $\bar{\varphi} = \varphi \circ \sigma$. We have then $\sigma_t^{\bar{\varphi}} = \sigma^{-1} \circ \sigma_t^{\varphi} \circ \sigma$, $t \in \mathbf{R}$, so that we have

$$T = \bar{T}, \ \mathcal{M}_n = \sigma(\bar{\mathcal{M}}_n), \quad n \in \mathbf{Z}.$$

Let u and $\bar{u}$ be isometrics in $\mathcal{M}_1$ and $\bar{\mathcal{M}}_1$ respectively such that

$$\theta(x) = uxu^*, \quad x \in \mathcal{M}_0;$$
$$\bar{\theta}(x) = \bar{u}x\bar{u}^*, \quad x \in \bar{\mathcal{M}}_0.$$

Since $\sigma(\bar{u})$ is an isometry in $\mathcal{M}_1$, there exists, by Lemma 1.8, a partial isometry w in $\mathcal{M}_0$ with $\sigma(\bar{u}) = wu$ and $u = w^*\sigma(\bar{u})$. Hence we have, for every $x \in \bar{\mathcal{M}}_0$,

$$w\theta \circ \sigma(x) w^* = wu\sigma(x) u^* w^* = \sigma(\bar{u})\sigma(x)\sigma(\bar{u}^*) = \sigma(\bar{u}x\bar{u}^*) = \sigma \circ \bar{\theta}(x).$$

Thus, statements (i) and (ii) are verified.

Suppose conditions (i) and (ii) hold. We claim first that wu is an isometry in $\mathcal{M}_1$. But this is seen by the following:

$$1 \geqslant (wu)^*(wu) = u^*w^*wu \geqslant u^*w^*\sigma_0 \circ \bar{\theta}(1) wu = u^*\theta \circ \sigma_0(1) u = u^*\theta(1) u = 1.$$

Let $v = wu$. We have then $\sigma_0 \circ \bar{\theta}(x) = v\sigma_0(x) v^*$, $x \in \bar{\mathcal{M}}_0$, and $\sigma_0 \bar{\theta}^n(x) = v^n \sigma_0(x) v^{*n}$. For each integer $n \geqslant 0$, we define linear maps U_n of $\bar{\mathcal{M}}_0 \bar{u}^n \bar{\xi}_0$ into $\mathcal{M}_0 u^n \xi_0$ and U_{-n} of $\bar{u}^{*n} \bar{\mathcal{M}}_0 \bar{\xi}_0$ into $u^{*n} \mathcal{M}_0 \xi_0$ as follows:

$$U_n x \bar{u}^n \bar{\xi}_0 = \sigma_0(x) v^n \xi_0,$$
$$U_{-n} \bar{u}^{*n} x \bar{\xi}_0 = v^{*n} \sigma_0(x) \xi_0, \quad x \in \bar{\mathcal{M}}_0.$$

Since $v = wu$ is an isometry in $\mathcal{M}_1$, U_n and U_{-n} are both subjective. Furthermore, U_n and U_{-n} are isometries. In fact, we have for any $x, y \in \bar{\mathcal{M}}_0$,

$$(U_n x\bar{u}^n\bar{\xi}_0 | U_n y\bar{u}^n\bar{\xi}_0) = (\sigma_0(x)v^n\xi_0 | \sigma_0(y)v^n\xi_0) = (v^{*n}\sigma_0(y^*x)v^n\xi_0 | \xi_0)$$

$$= \varphi_0\circ\theta_{v^n}(\sigma_0(y^*x)) = \varkappa^n\varphi_0\circ\sigma_0(y^*x) \qquad \text{by Lemma 1.23}$$

$$= \varkappa^n\bar{\varphi}_0(y^*x) = \varphi_0\circ\theta_{\bar{u}^n}(y^*x) = (x\bar{u}^n\bar{\xi}_0 | y\bar{u}^n\bar{\xi}_0);$$

$$(U_{-n}\bar{u}^{*n}x\bar{\xi}_0 | U_{-n}\bar{u}^{*n}y\bar{\xi}_0) = (v^{*n}\sigma_0(x)\xi_0 | v^{*n}\sigma_0(y)\xi_0) = (\sigma_0(y^*)v^n v^{*n}\sigma_0(x)\xi_0 | \xi_0)$$

$$= (\sigma_0(y^*)\sigma_0\circ\bar{\theta}^n(1)\sigma_0(x)\xi_0 | \xi_0) = (\sigma_0(y^*)\sigma_0(\bar{u}^n\bar{u}^{*n})\sigma_0(x)\xi_0 | \xi_0)$$

$$= \varphi_0\circ\sigma_0(y^*\bar{u}^n\bar{u}^{*n}x) = \bar{\varphi}(y^*\bar{u}^n\bar{u}^{*n}x) = (\bar{u}^{*n}x\bar{\xi}_0 | \bar{u}^{*n}y\bar{\xi}_0).$$

Thus U_n and U_{-n} extend to isometries of $\bar{\mathfrak{H}}_n$ onto $\mathfrak{H}_n$ and $\bar{\mathfrak{H}}_{-n}$ onto $\mathfrak{H}_{-n}$, which we denote by U_n and U_{-n} again. Let $U = \sum_{n\in\mathbf{Z}}^{\oplus} U_n$. We have then a unitary U of $\bar{\mathfrak{H}}$ onto $\mathfrak{H}$ which sends $\bar{\mathfrak{H}}_n$ onto $\mathfrak{H}_n$ and $\bar{\xi}_0$ into ξ_0. For any $x, y \in \mathcal{M}_0$ and $n \geq 1$, we have

$$Uxy\bar{u}^n\bar{\xi}_0 = U_n xy\bar{u}^n\bar{\xi}_0 = \sigma_0(xy)v^n\xi_0 = \sigma_0(x)\sigma_0(y)v^n\xi_0 = \sigma_0(x)Uy\bar{u}^n\bar{\xi}_0;$$

$$Ux\bar{u}^{*n}y\bar{\xi}_0 = U\bar{u}^{*n}\bar{u}^n x\bar{u}^{*n}y\bar{\xi}_0 = U\bar{u}^{*n}\bar{\theta}^n(x)y\bar{\xi}_0 = v^{*n}\sigma_0\circ(\bar{\theta}^n(x)y)\xi_0$$

$$= v^{*n}\sigma_0\circ\bar{\theta}^n(x)\sigma_0(y)\xi_0 = v^{*n}v^n\sigma_0(x)v^{*n}\sigma_0(y)\xi_0$$

$$= \sigma_0(x)v^{*n}\sigma_0(y)\xi_0 = \sigma_0(x)U\bar{u}^{*n}y\bar{\xi}_0.$$

Thus we have $$UxU^* = \sigma_0(x), \quad x \in \mathcal{M}_0.$$

Furthermore, we get, for each $y \in \bar{\mathcal{M}}_0$ and $n \geq 0$,

$$U\bar{u}y\bar{u}^n\bar{\xi}_0 = U\bar{u}y\bar{u}^*\bar{u}^{n+1}\bar{\xi}_0 = U\bar{\theta}(y)\bar{u}^{n+1}\bar{\xi}_0 = \sigma_0\circ\bar{\theta}(y)v^{n+1}\xi_0$$

$$= v\sigma_0(y)v^*v^{n+1}\xi_0 = v\sigma_0(y)v^n\xi_0 = vUy\bar{u}^n\bar{\xi}_0,$$

$$U\bar{u}\bar{u}^{*n}y\bar{\xi}_0 = U\bar{u}^{*(n-1)}y\bar{\xi}_0 = v^{*(n-1)}\sigma_0(y)\xi_0 = vv^{*n}\sigma_0(y)\xi_0 = vU\bar{u}^{*n}y\bar{\xi}_0 \quad \text{for } n \geq 1.$$

Thus we have $$U\bar{u}U^* = v.$$

By Corollary 1.16, $\bar{\mathcal{M}}$ (resp. $\mathcal{M}$) is generated by $\bar{\mathcal{M}}_0$ and $\bar{u}$ (resp. $\mathcal{M}_0$ and v). Therefore we have

$$\mathcal{M} = U\bar{\mathcal{M}}U^*.$$

Hence the unitary U induces a spatial isomorphism σ of $\bar{\mathcal{M}}$ onto $\mathcal{M}$ which extends σ_0. Since $U\bar{\xi}_0 = \xi_0$, we have $\bar{\varphi} = \varphi\circ\sigma$. Q.E.D.

2. Construction of a von Neumann algebra with a specified decomposition

By the results in § 1, especially Theorems 1.27 and 1.28, a von Neumann algebra $\mathcal{M}$ equipped with a homogeneous periodic state φ is essentially determined by the centralizer $\mathcal{M}_0$ of φ and an isomorphism θ of $\mathcal{M}_0$ onto $e\mathcal{M}_0 e$, where e is a projection with $e^\natural = \varkappa 1$. The natural question now is whether or not we can construct a von Neumann algebra $\mathcal{M}$ equipped with a homogeneous periodic state φ whoes decomposition is described by a given von Neumann algebra $\mathcal{M}_0$ of type II_1 and a given isomorphism θ of $\mathcal{M}_0$ onto $e\mathcal{M}_0 e$ where e is a projection in $\mathcal{M}_0$ with $e^\natural = \varkappa 1$. We shall answer this question afirmatively in this section.

Suppose now $\mathcal{M}_0$ is a von Neumann algebra of type II_1 and θ is an isomorphism of $\mathcal{M}_0$ onto $e\mathcal{M}_0 e$ where e is a projection in $\mathcal{M}_0$ with $e^\natural = \varkappa 1$, $0<\varkappa<1$. Let $\mathcal{Z}_0$ denote the center of $\mathcal{M}_0$. By Lemma 1.20 there exists an automorphism $\tilde{\theta}$ of $\mathcal{Z}_0$ such that $\theta(a) = \tilde{\theta}(a)e$ for each $a \in \mathcal{Z}_0$.

LEMMA 2.1. *For every* $x \in \mathcal{M}_0$, *we have*

$$\theta(x)^\natural = \varkappa\tilde{\theta}(x^\natural). \tag{1}$$

Proof. Let $x^{\natural\prime} = (1/\varkappa)\,\tilde{\theta}^{-1}(\theta(x)^\natural)$. We have then, for any $x, y \in \mathcal{M}_0$

$$(xy)^{\natural\prime} = \frac{1}{\varkappa}\tilde{\theta}^{-1}(\theta(xy)^\natural) = \frac{1}{\varkappa}\tilde{\theta}^{-1}([\theta(x)\,\theta(y)]^\natural) = \frac{1}{\varkappa}\tilde{\theta}^{-1}([\theta(y)\,\theta(x)]^\natural) = \frac{1}{\varkappa}\tilde{\theta}^{-1}(\theta(yx)^\natural) = (yx)^{\natural\prime};$$

$$(x^*x)^\natural = \frac{1}{\varkappa}\tilde{\theta}^{-1}(\theta(x^*x)^\natural) \geqslant 0.$$

For any $a \in Z_0$, we have

$$(ax)^{\natural\prime} = \frac{1}{\varkappa}\tilde{\theta}^{-1}(\theta(ax)^\natural) = \frac{1}{\varkappa}\tilde{\theta}^{-1}([\theta(a)\theta(x)]^\natural) = \frac{1}{\varkappa}\tilde{\theta}^{-1}([\tilde{\theta}(a)\,e\theta(x)]^\natural)$$

$$= \frac{1}{\varkappa}\tilde{\theta}^{-1}(\tilde{\theta}(a)\,[e\theta(x)]^\natural) = \frac{1}{\varkappa}\tilde{\theta}^{-1}(\tilde{\theta}(a)\,\theta(x)^\natural) = a\frac{1}{\varkappa}\tilde{\theta}^{-1}(\theta(x)^\natural) = ax^{\natural\prime};$$

$$1^{\natural\prime} = \frac{1}{\varkappa}\tilde{\theta}^{-1}(\theta(1)^\natural) = \frac{1}{\varkappa}\tilde{\theta}^{-1}(e^\natural) = \frac{1}{\varkappa}\tilde{\theta}^{-1}(\varkappa 1) = 1.$$

Thus, the map: $x \mapsto x^{\natural\prime}$ is the center valued trace of $\mathcal{M}_0$, hence $x^{\natural\prime} = x^\natural$ by the unicity of the center valued trace. Hence $\theta(x)^\natural = \varkappa\tilde{\theta}(x^\natural)$. Q.E.D.

Suppose there exists a $\tilde{\theta}$-invariant faithful normal state φ_0 on $\mathcal{Z}_0$. We extend φ_0 to a faithful normal trace on $\mathcal{M}_0$, putting

$$\varphi_0(x) = \varphi_0(x^\natural), \quad x \in \mathcal{M}_0. \tag{2}$$

It follows then from Lemma 2.1 that

$$\varphi_0 \circ \theta(x) = \varkappa \varphi_0(x), \quad x \in \mathcal{M}_0. \tag{3}$$

For each integer $n \geq 1$, we write

$$e_n = \theta^n(1) = \theta(e_{n-1}),\ e_0 = 1. \tag{4}$$

For each integer $n \geq 0$, we consider the vector spaces $\mathfrak{A}_n$ and $\mathfrak{A}_{-n}$ defined by

$$\left.\begin{aligned} \mathfrak{A}_n &= \mathcal{M}_0 e_n, \quad n = 0, 1, \ldots; \\ \mathfrak{A}_{-n} &= e_n \mathcal{M}_0, \quad n = 1, 2, \ldots. \end{aligned}\right\} \tag{5}$$

Let $\mathfrak{A}$ denote the algebraic direct sum of $\{\mathfrak{A}_n\}$, that is,

$$\mathfrak{A} = \sum_{n \in \mathbf{Z}}{}'^{\oplus} \mathfrak{A}^n. \tag{6}$$

In $\mathfrak{A}_0$, we consider the same involutive algebra structure as in $\mathcal{M}_0$. But we denote by $\eta(x)$ the element in $\mathfrak{A}_0$ corresponding to $x \in \mathcal{M}_0$. Let α be the element in $\mathfrak{A}_1$ corresponding to $e_1 \in \mathcal{M}_0 e_1$, and $\alpha^\#$ the element in $\mathfrak{A}_{-1}$ corresponding to $e_1 \in e_1 \mathcal{M}_0$. For $n \geq 1$, let α^n be the element in $\mathfrak{A}_n$ corresponding to $e_n \in \mathcal{M}_0 e_n$ and $\alpha^{\#n}$ denote the element in $\mathfrak{A}_{-n}$ corresponding to $e_n \in e_n \mathcal{M}_0$. We denote by $\eta(x)\alpha^n$ (resp. $\alpha^{\#n}\eta(x)$) the elements in $\mathfrak{A}_n$ (resp. $\mathfrak{A}_{-n}$) corresponding to $xe_n \in \mathcal{M}_0 e_n$ (resp. $e_n x \in e_n \mathcal{M}_0$). We first define the product of $\mathfrak{A}_n$ and $\mathfrak{A}_m$, $m \in \mathbf{Z}$, as follows:

$$\left.\begin{aligned}
&\eta(x)\,\alpha^n \eta(y)\,\alpha^m = \eta(x\theta^n(y))\,\alpha^{n+m}; && \\
&\alpha^{\#n}\eta(x)\,\alpha^{\#m}\eta(y) = \alpha^{\#(n+m)}\eta(\theta^m(x)\,y); && \\
&\eta(x)\,\alpha^n \alpha^{\#m}\eta(y) = \eta(x\theta^{(n-m)}(y))\,\alpha^{n-m} && \text{if} \quad n \geq m; \\
&\eta(x)\,\alpha^n \alpha^{\#m}\eta(y) = \alpha^{\#(m-n)}\eta(\theta^{(m-n)}(x)\,y) && \text{if} \quad n \leq m; \\
&\alpha^{\#n}\eta(x)\,\eta(y)\,\alpha^m = \eta(\theta^{-n}(xy))\,\alpha^{m-n} && \text{if} \quad n \leq m; \\
&\alpha^{\#n}\eta(x)\,\eta(y)\,\alpha^m = \alpha^{\#(n-m)}\eta(\theta^{-m}(xy)) && \text{if} \quad n \geq m,
\end{aligned}\right\} \tag{7}$$

where $n, m \geq 0$. We remark here that in the last two equalities xy falls in $e_n \mathcal{M}_0 e_n$ or $e_m \mathcal{M}_0 e_m$ according to whether $n \leq m$ or $n \geq m$, so that $\theta^{-n}(xy)$ or $\theta^{-m}(xy)$ makes sense. Extending the product defined by (7), we make $\mathfrak{A}$ an algebra over the complex number field $\mathbf{C}$. Namely, we denote by $\xi(n)$ the $\mathfrak{A}_n$-component of any $\xi \in \mathfrak{A}$, then for any ξ, $\eta \in \mathfrak{A}$,

$$(\xi\eta)(n) = \sum_{m \in \mathbf{Z}} \xi(m)\,\eta(n-m), \tag{8}$$

where the product of $\xi(m)$ and $\xi(n-m)$ are given by (7). We define the involution $\sharp$ in $\mathfrak{A}$ by

$$\left.\begin{aligned}(\eta(x)\,\alpha^n)^\sharp &= \alpha^{\sharp n}\eta(x^*);\\ (\alpha^{\sharp n}\eta(x))^\sharp &= \eta(x^*)\,\alpha^n;\\ \xi^\sharp(n) &= \xi(-n)^\sharp.\end{aligned}\right\} \tag{9}$$

Thus we obtain the involutive algebra $\mathfrak{A}$. We remark that $\mathfrak{A}$ is generated by $\mathfrak{A}_0$ and α algebraically as an involutive algebra under the relation:

$$\alpha^\sharp\alpha = \eta(1),\, \alpha\alpha^\sharp = \eta(e_1); \qquad \alpha\eta(x) = \eta(\theta(x))\,\alpha,\, x\in\mathcal{M}_0. \tag{10}$$

Since $\mathfrak{A}$ is determined by $\mathcal{M}_0$ and θ, we denote it by $\mathfrak{A}(\mathcal{M}_0, \theta)$ if necessary.

LEMMA 2.2. *Suppose σ_0 is a *-homomorphism of $\mathcal{M}_0$ into an involutive algebra $\mathfrak{B}$. If $\mathfrak{B}$ is generated algebraically as an involutive algebra by the image $\sigma(\mathcal{M}_0)$ and an element β such that $\beta^*\beta = 1$, $\beta\beta^* = \sigma(e_1)$ and $\beta\sigma(x) = \sigma\circ\theta(x)\beta$ for every $x\in\mathcal{M}_0$, then there exists uniquely a *-homomorphism σ of $\mathfrak{A}$ onto $\mathfrak{B}$ such that*

$$\sigma\eta(x) = \sigma_0(x), \quad x\in\mathcal{M}_0;$$

$$\sigma(\alpha) = \beta.$$

Proof. The unicity of σ follows from the fact that $\mathfrak{A}$ is generated by $\mathfrak{A}_0$ and α. For a $\xi\in\mathfrak{A}$, let

$$\xi(n) = \eta(x_n)\,\alpha^n, \quad x_n\in\mathcal{M}_0 e_n,$$

$$\xi(-n) = \alpha^{\sharp n}\eta(x_{-n}), \quad x_{-n}\in e_n\,\mathcal{M}_0$$

for $n\geqslant 0$. We write $\xi\sim\{x_n\}$ since ξ is determined by $\{x_n\}$. We define σ by

$$\sigma(\xi) = \sum_{n=0}^{\infty}\sigma_0(x_n)\,\beta^n + \sum_{n=1}^{\infty}\beta^{*n}\sigma_0(x_{-n}).$$

It is easily seen that σ is the desired *-homomorphism. Q.E.D.

COROLLARY 2.3. *The algebra $\mathfrak{A}(\mathcal{M}_0, \theta)$ admits a one complex parameter automorphism group $\Delta(\omega)$, $\omega\in\mathbf{C}$ such that*

$$\Delta(\omega)\xi = \xi, \quad \xi\in\mathfrak{A}_0;$$

$$\Delta(\omega)\xi = \varkappa^{n\omega}\xi, \quad \xi\in\mathfrak{A}_n, \quad n\in\mathbf{Z}.$$

Proof. Let $\beta_t = \varkappa^{it}\alpha$, $t\in\mathbf{R}$. The algebra $\mathfrak{A}$ is generated by $\mathfrak{A}_0$ and β_t. Applying Lemma 2.2 to the map η: $x\in\mathcal{M}_0\mapsto\eta(x)\in\mathfrak{A}_0$ and β_t, we obtain a *-endomorphism $\Delta(it)$ of $\mathfrak{A}$ for each $t\in\mathbf{R}$. Since

$$\Delta(i(s+t))\alpha=\beta_{s+t}=\varkappa^{i(s+t)}\alpha=\varkappa^{is}\varkappa^{it}\alpha=\varkappa^{is}\Delta(it)\alpha=\Delta(it)(\varkappa^{is}\alpha)=\Delta(it)\Delta(is)\alpha,$$

we have $\Delta(i(s+t))=\Delta(is)\Delta(it)$. Hence $\{\Delta(it): t\in\mathbf{R}\}$ is a one parameter group of antomorphisms of $\mathfrak{A}$. Putting

$$\Delta(\omega)\alpha=\varkappa^{\omega}\alpha,\quad \omega\in\mathbf{C},$$

we obtain the complexification $\Delta(\omega)$, $\omega\in\mathbf{C}$, of $\Delta(it)$, $t\in\mathbf{R}$. It is then easily seen that

$$(\Delta(\omega)\xi)^{\#}=\Delta(-\bar{\omega})\xi^{\#},\ \xi\in\mathfrak{A},\ \omega\in\mathbf{C},$$

where $\bar{\omega}$ means, of course, the complex conjugate of ω. Q.E.D.

We equip the algebra $\mathfrak{A}$ with an inner product as follows; For each $\xi\sim\{x_n\}$ and $\eta\sim\{y_n\}$ in $\mathfrak{A}$, that is

$$\xi=\sum_{n=1}^{\infty}\alpha^{\#n}\eta(x_{-n})+\eta(x_0)+\sum_{n=1}^{\infty}\eta(x_n)\alpha^n;$$

$$\eta=\sum_{n=1}^{\infty}\alpha^{\#n}\eta(y_{-n})+\eta(y_0)+\sum_{n=1}^{\infty}\eta(y_n)\alpha^n,$$

we define

$$(\xi|\eta)=\sum_{n=1}^{\infty}\varphi_0(y_{-n}^{*}x_{-n})+\varphi_0(y_0^{*}x_0)+\sum_{n=1}^{\infty}\varkappa^{-n}\varphi_0(y_n^{*}x_n). \tag{12}$$

LEMMA 2.4. *If we denote by ξ_0 the identity $\eta(1)$ of $\mathfrak{A}$, then we have*

$$(\xi|\eta)=(\eta^{\#}\xi|\xi_0),\ \xi,\quad \eta\in\mathfrak{A}.$$

Proof. Since both sides of the equality are sesquilinear forms of ξ and η, it is sufficient to show the equality for $\xi\in\mathfrak{A}_n$ and $\eta\in\mathfrak{A}_m$, n, $m\in\mathbf{Z}$. From the definition, it follows that $\mathfrak{A}_n$ and $\mathfrak{A}_m$ are orthogonal if $n\neq m$. Hence we may assume that ξ and η are in the same $\mathfrak{A}_n$ because $\eta^{\#}\xi\in\mathfrak{A}_{n-m}$ if $\xi\in\mathfrak{A}_n$ and $\eta\in\mathfrak{A}_m$, and $\mathfrak{A}_{n-m}$ is orthogonal to $\mathfrak{A}_0$ if $n\neq m$. Let $\xi=\eta(x)\alpha^n$ and $\eta=\eta(y)\alpha^n$ with x, $y\in\mathcal{M}_0e_n$. We have then

$$(\eta^{\#}\xi|\xi_0)=(\alpha^{\#n}\eta(y^*x)\alpha^n|\xi_0)=(\eta(\theta^{-n}(y^*x))|\xi_0)=\varphi_0(\theta^{-n}(y^*x))=\varkappa^{-n}\varphi_0(y^*x)=(\xi|\eta).$$

If $\xi=\alpha^{\#n}\eta(x)$ and $\eta=\alpha^{\#n}\eta(y)$ for some x, $y\in e_n\mathcal{M}_0$, then we have

$$(\eta^{\#}\xi|\xi_0)=(\eta(y^*)\alpha^n\alpha^{\#n}\eta(x)|\xi_0)=(\eta(y^*x)|\xi_0)=\varphi_0(y^*x)=(\xi|\eta).$$ Q.E.D.

LEMMA 2.5. *For each ξ, η, $\zeta\in\mathfrak{A}$, we have*

$$(\xi\eta|\zeta)=(\eta|\xi^{\#}\zeta).$$

Proof. By Lemma 2.4, we have

$$(\xi\eta|\zeta)=(\zeta^{\#}(\xi\eta)|\xi_0)=((\xi^{\#}\zeta)^{\#}\eta|\xi_0)=(\eta|\xi^{\#}\zeta).$$ Q.E.D.

Let $\mathfrak{H}$ denote the completion of $\mathfrak{A}$ and $\mathfrak{H}_n$ the closure of $\mathfrak{A}_n$ in $\mathfrak{H}$. We have then

$$\mathfrak{H} = \sum_{n\in\mathbf{Z}}^{\oplus} \mathfrak{H}_n.$$

THEOREM 2.6. *The involutive algebra $\mathfrak{A}$, together with the one complex parameter automorphism group $\Delta(\omega)$, $\omega \in \mathbf{C}$, and the inner product is a Tomita algebra, (a modular Hilbert algebra).*

Proof. By the existence of the identity ξ_0, $\mathfrak{A}_q = \mathfrak{A}$ so that $\mathfrak{A}$ is non-degenerate.

Since $\alpha^{\#}\alpha = \xi_0$ and $\pi(\alpha)^* = \pi(\alpha^{\#})$ by Lemma 2.5, the left multiplication operator $\pi(\alpha)$ by α is an isometry, so that it is bounded). Since $\mathfrak{A}$ is generated by $\mathfrak{A}_0$ and α, in order to verify that the left multiplication in $\mathfrak{A}$ is bounded, it is sufficient to show that the left multiplication in $\mathfrak{A}$ given by an element of $\mathfrak{A}_0$ is bounded, For each $a \in \mathcal{M}_0$ and $x \in \mathcal{M}_0 e_n$, $n \geq 1$ we have

$$\begin{aligned}\|\eta(a)\eta(x)\alpha^n\|_p^2 &= \|\eta(ax)\alpha^n\|^2 = (\eta(ax)\alpha^n \,|\, \eta(ax)\alpha^n) \\ &= \varkappa^{-n}\varphi_0(x^*a^*ax) \leq \|a\|^2 \varkappa^{-n}\varphi_0(x^*x) \\ &= \|a\|^2 \|\eta(x)\alpha^n\|^2.\end{aligned}$$

For $x \in e_n \mathcal{M}_0$, we get

$$\begin{aligned}\|\eta(a)\alpha^{\#n}\eta(x)\|^2 &= \|\alpha^{\#n}\alpha^n\eta(a)\alpha^{\#n}\eta(x)\|^2 = \|\alpha^{\#n}\eta(\theta^n(a)x)\|^2 = (\alpha^{\#n}\eta(\theta^n(a)x) \,|\, \alpha^{\#n}\eta(\theta^n(a)x)) \\ &= \varphi_0(x^*\theta^n(a^*a)x) \leq \|a\|^2\varphi_0(x^*x) = \|a\|^2 \|\alpha^{\#n}\eta(x)\|^2.\end{aligned}$$

Therefore, the map $\pi(\eta(a))$: $\xi \in \mathfrak{A}_n \mapsto \eta(a)\xi \in \mathfrak{A}_n$ is bounded uniformly for $n \in \mathbf{Z}$; hence it is extended to a bounded operator on $\mathfrak{H}$.

By Corollary 2.3. if we denote by E_n the projection of $\mathfrak{H}$ onto $\mathfrak{H}_n$, then we have

$$\Delta(\omega) = \sum_{n\in\mathbf{Z}} \varkappa^{n\omega} E_n \text{ on } \mathfrak{A}.$$

Hence $\Delta(t)$, $t \in \mathbf{R}$, is essentially self-adjoint, and we get

$$(\Delta(\omega)\xi \,|\, \eta) = (\xi \,|\, \Delta(\bar{\omega})\eta), \quad \xi, \eta \in \mathfrak{A}.$$

Since the summation

$$(\Delta(\omega)\xi \,|\, \eta) = \sum_{n\in\mathbf{Z}} \varkappa^{n\omega}(\xi(n) \,|\, \eta(n))$$

has only finitely many terms for every $\xi, \eta \in \mathfrak{A}$, the function: $\omega \in \mathbf{C} \to (\Delta(\omega)\xi \,|\, \eta)$ is holomorphic on the whole plane $\mathbf{C}$.

Now, we finish the proof with the following calculation for $\xi \sim \{x_n\}$ and $\eta \sim \{y_n\}$ in $\mathfrak{A}$:

$$\begin{aligned}(\Delta(1)\xi^{\#}|\eta^{\#}) &= \sum_{n\in\mathbf{Z}}(\Delta(1)\xi(-n)^{\#}|\eta(-n)^{\#})\\
&= \sum_{n=1}^{\infty}(\Delta(1)\eta(x_{-n}^{*})\alpha^{n}|\eta(y_{-n}^{*})\alpha^{n}) + (\eta(x_0^{*})|\eta(y_0^{*})) + \sum_{n=1}^{\infty}(\Delta(1)\alpha^{\#n}\eta(x_n^{*})|\alpha^{\#n}\eta(y_n^{*}))\\
&= \sum_{n=1}^{\infty}\varkappa^{n}(\eta(x_{-n}^{*})\alpha^{n}|\eta(y_{-n}^{*})\alpha^{n}) + \varphi_0(y_0x_0^{*}) + \sum_{n=1}^{\infty}\varkappa^{-n}(\alpha^{\#n}\eta(x_n^{*})|\alpha^{\#}\eta(y_n^{*}))\\
&= \sum_{n=1}^{\infty}\varkappa^{n}\varkappa^{-n}\varphi_0(y_{-n}x_{-n}^{*}) + \varphi_0(y_0x_0^{*}) + \sum_{n=1}^{\infty}\varkappa^{-n}\varphi_0(y_nx_n^{*})\\
&= \sum_{n=1}^{\infty}\varphi(y_{-n}x_{-n}^{*}) + \varphi_0(y_0x_0^{*}) + \sum_{n=1}^{\infty}\varkappa^{-n}\varphi_0(y_nx_n^{*}) = (\eta|\xi).\end{aligned}$$

Q.E.D.

Let $\mathcal{M}$ denote the left von Neumann algebra $\mathcal{L}(\mathfrak{A})$ of $\mathfrak{A}$ and φ the state on $\mathcal{M}$ defined by

$$\varphi(x) = (x\xi_0|\xi_0), \quad x\in\mathcal{M}. \tag{13}$$

Since ξ_0 is the identity of the Tomita algebra $\mathfrak{A}$, the modular automorphism group σ_t^{φ} of $\mathcal{M}$ associated with φ is given by

$$\sigma_t^{\varphi}(x) = \Delta^{it}x\Delta^{-it}, \quad x\in\mathcal{M}, \tag{14}$$

where Δ denotes the non-singular self-adjoint positive operator given by

$$\Delta = \sum_{n\in\mathbf{Z}}\varkappa^{n}E_n, \tag{15}$$

which is the closure of $\Delta(1)$. We denote by $\mathcal{A}$ the image $\pi(\mathfrak{A})$ of $\mathfrak{A}$ in $\mathcal{M}$. Of course, the map: $\xi\in\mathfrak{A}\mapsto\pi(\xi)\in\mathcal{A}$ is a *-isomorphism.

LEMMA 2.7. *The map: $x\in\mathcal{M}_0\mapsto\pi(\eta(x))\in\mathcal{M}$ is a normal isomorphism of $\mathcal{M}_0$ into $\mathcal{M}$.*

Proof. Let $\{x_i\}_{i\in I}$ be a bounded net in $\mathcal{M}_0$ converging σ-strongly to zero. We have then $\lim\varphi(x_i^{*}x_i)=0$, which means that $\{\eta(x_i)\}_{i\in I}$ converges to zero in $\mathfrak{H}$. For any $\xi\in\mathfrak{A}$, we have

$$\lim_i \pi(\eta(x_i))\xi = \lim_i \pi'(\xi)\eta(x_i) = 0$$

where we recall that every element in $\mathfrak{A}$ is right bounded as well as left bounded, $\mathfrak{A}$ being a Tomita algebra. Since $\|\pi(\eta(x_i))\|\leqslant\|x_i\|$ as seen in the proof of Theorem 2.6, and since $\mathfrak{A}$ is dense in $\mathfrak{H}$, $\{\pi(\eta(x_i))\}_{i\in I}$ converges strongly to zero in $\mathcal{M}$. Q.E.D.

We may therefore identify $\mathcal{M}_0$ with its image in $\mathcal{M}$ under the isomorphism in the lemma.

LEMMA 2.8. *The von Neumann subalgebra $\mathcal{M}_0$ of $\mathcal{M}$ is the centralizer of the state φ.*

Proof. For any $x \in \mathfrak{M}_0$, we have

$$\sigma_t^\varphi(x)\,\xi_0 = \Delta^{it} x \Delta^{-it}\xi_0 = \Delta^{it} x\xi_0 = \Delta^{it}\eta(x) = \eta(x) = x\xi_0,$$

so that $\sigma_t^\varphi(x) = x$, $t \in \mathbf{R}$. Hence $\mathfrak{M}_0$ is contained in the centralizer $\mathfrak{M}_\varphi$ of φ. Suppose an $x \in \mathfrak{M}$ is fixed under σ_t^φ. We see then that $x\xi_0$ is fixed by Δ^{it}, so that $x\xi_0$ falls in $\mathfrak{H}_0$. Since x is bounded, $x\xi_0$ is left bounded with respect to the unimodular Hilbert algebra $\mathfrak{A}_0$, (in this case there is no difference between left and right boundedness). Hence there exists $a \in \mathfrak{M}_0$ such that $x\xi_0 = \eta(a) = a\xi_0$ because $\mathfrak{M}_0$ acting on $\mathfrak{H}_0$ is the left algebra of $\mathfrak{A}_0$. Hence we have $x = a$, ξ_0 being separating for $\mathfrak{M}$. Thus x falls in $\mathfrak{M}_0$. Therefore, $\mathfrak{M}_0$ is the fixed point subalgebra of $\mathfrak{M}$ under σ_t^φ. Q.E.D.

From (15), it fillows that Δ^{it} is periodic. Namely, putting

$$T = -\frac{2\pi}{\log \varkappa}, \tag{16}$$

we have $\Delta^{iT} = 1$. Therefore, we have $\sigma_T^\varphi = \iota$, so that φ is periodic.

Let $u = \pi(\alpha)$. As seen in the proof of Theorem 2.6, u is an isometry in $\mathfrak{M}$, and we have

$$\sigma_t^\varphi(u) = \varkappa^{it} u, \quad t \in \mathbf{R},$$

because $\sigma_t^\varphi(u)\xi_0 = \Delta^{it}\alpha = \varkappa^{it}\alpha = \varkappa^{it}u\xi_0$. Let

$$\mathfrak{M}_n = \pi(\mathfrak{A}_n), \quad n \in \mathbf{Z}.$$

LEMMA 2.9. *The subspace $\mathfrak{M}_n$, $n \in \mathbf{Z}$, is precisely the set of all elements $x \in \mathfrak{M}$ such that $\sigma_t^\varphi(x) = \varkappa^{int}x$.*

Proof. It is verified as before that for any $x \in \mathfrak{M}_n$ we have $\sigma_t^\varphi(x) = \varkappa^{int}x$. Conversely, if $\sigma_t^\varphi(x) = \varkappa^{int}x$, then we consider an element $y = xu^{*n}$ if $n \geqslant 0$ or $y = u^n x$ if $n < 0$. We have then $\sigma_t^\varphi(y) = y$, so that y falls in $\mathfrak{M}_0$: hence $x = yu^n$ if $n \geqslant 0$ or $x = u^{*n}y$ if $n < 0$. Thus x falls in $\mathfrak{M}_n$. Q.E.D.

THEOREM 2.10. *The von Neumann algebra $\mathfrak{M}$ admits the decomposition described in Theorem 1.27. Namely, statements* (i)–(vi) *in Theorem 1.27 hold for $\mathfrak{M}$ and u with no alteration.*

Proof. Except statement (vi) we have verified all the statements, so we will prove only (vi). In Lemma 2.1, replacing θ by θ^n, $n \geqslant 1$,

$$\theta^n(x)^\natural = \varkappa^n \tilde{\theta}^n(x^\natural), \; x \in \mathfrak{M}_0, \; n \geqslant 1.$$

Therefore, the proof of Lemma 1.25 works without any alteration. Q.E.D.

The fact that $\mathcal{M}$ is of type III will be verified later, Corollary 3.6, together with the observation of the value of t for which the modular automorphism σ_t^φ is inner.

The whole construction of $\mathcal{M}$ depends apparently on the choice of the faithful normal $\tilde{\theta}$-invariant state φ_0 on $\mathcal{Z}_0$. We will see that this dependence is superficial. Namely, the resulting algebra $\mathcal{M}$ is uniquely determined by $(\mathcal{M}_0, \theta)$ up to isomorphism. We take therefore another $\tilde{\theta}$-invariant faithful normal state ψ_0 on $\mathcal{Z}_0$. It is then extended to a faithful normal trace on $\mathcal{M}_0$ by $\psi_0(x) = \psi_0(x^\natural)$, $x \in \mathcal{M}_0$, as in the case of φ_0. We then construct a Tomita algebra $\mathfrak{B}$ based on $\{\mathcal{M}_0, \theta, \psi_0\}$. It is obvious that the only difference between $\mathfrak{A}$ and $\mathfrak{B}$ appears in their inner products, and that $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic as involutive algebras. To distinguish objects related to $\mathfrak{A}$ or $\mathfrak{B}$, we attach the subscripts $\mathfrak{A}$ and $\mathfrak{B}$ to the relevant notations, such as $\pi_{\mathfrak{A}}$, $\pi_{\mathfrak{B}}$, $\eta_{\mathfrak{A}}$, $\eta_{\mathfrak{B}}$ and so on. Let $\mathcal{B} = \pi_{\mathfrak{B}}(\mathfrak{B})$. Clearly $\mathcal{A}$ and $\mathcal{B}$ are isomorphic as involutive algebras. We denote by $\mathfrak{K}$ the completion of $\mathfrak{B}$ and by $\mathfrak{K}_n$ the closure of $\mathfrak{B}_n$ the subspace of $\mathfrak{B}$ corresponding to $\mathfrak{A}_n$. Let β be the element in $\mathfrak{B}$ corresponding to α.

PROPOSITION 2.11. *There exists a unitary operator U of $\mathfrak{K}$ onto $\mathfrak{H}$ such that*

$$U\mathcal{L}(\mathfrak{B})U^* = \mathcal{L}(\mathfrak{A}); \quad U\pi_{\mathfrak{B}}(\eta_{\mathfrak{B}}(x))U^* = \pi_{\mathfrak{A}}(\eta_{\mathfrak{A}}(x)), x \in \mathcal{M}_0;$$

$$U\pi_{\mathfrak{B}}(\beta)U^* = \pi_{\mathfrak{B}}(\alpha); \quad U\Delta_{\mathfrak{B}}U^* = \Delta_{\mathfrak{A}}.$$

Proof. By the Radon-Nikodym Theorem for traces, there exists a vector ξ^ψ in $[\eta_{\mathfrak{A}}(\mathcal{Z}_0)]$ such that

$$\psi_0(x) = (\eta_{\mathfrak{A}}(x)\xi^\psi | \xi^\psi), \quad x \in \mathcal{M}_0.$$

Since ψ_0 is a faithful trace on $\mathcal{M}_0$, we have $\mathfrak{H}_0 = [\mathfrak{A}_0\xi^\psi]$. Hence we get $\mathfrak{H} = [\mathfrak{A}\xi^\psi]$.

Let ϱ (resp. $\bar{\varrho}$) be the canonical isomorphism of $\mathfrak{B}$ (resp. $\mathcal{B}$) onto $\mathfrak{A}$ (resp. $\mathcal{A}$). We claim that

$$\psi(x) = (\bar{\varrho}(x)\xi^\psi | \xi^\psi), \quad x \in \mathcal{B}. \tag{17}$$

For each $x \in \mathcal{M}_0$, we have

$$\psi(\pi_{\mathfrak{B}}(\eta_{\mathfrak{B}}(x))) = \psi_0(x) = (\pi_{\mathfrak{A}}(\eta_{\mathfrak{A}}(x))\xi^\psi | \xi^\psi) = (\bar{\varrho}\circ\pi_{\mathfrak{B}}(\eta_{\mathfrak{B}}(x))\xi^\psi | \xi^\psi).$$

For $n \geq 1$, we get

$$\psi(\pi_{\mathfrak{B}}(\eta_{\mathfrak{B}}(x)\beta^n)) = 0; \qquad (\bar{\varrho}(\pi_{\mathfrak{B}}(\eta_{\mathfrak{B}}(x)\beta^n))\xi^\psi | \xi^\psi) = (\pi_{\mathfrak{A}}(\eta_{\mathfrak{A}}(x)\alpha^n)\xi^\psi | \xi^\psi) = 0;$$

$$\psi(\pi_{\mathfrak{B}}(\beta^{\sharp n}\eta_{\mathfrak{B}}(x))) = 0; \qquad (\bar{\varrho}(\pi_{\mathfrak{B}}(\beta^{\sharp n}\eta_{\mathfrak{B}}(x)))\xi^\psi | \xi^\psi) = \pi_{\mathfrak{A}}(\alpha^{\sharp n}\eta_{\mathfrak{A}}(x))\xi^\psi | \xi^\psi) = 0.$$

By linearity, equality (17) holds. Therefore, if we define an operator U of $\mathfrak{B}$ onto $\pi_{\mathfrak{A}}(\mathfrak{A})\xi^\psi$ by:

$$U\eta = \varrho(\eta)\xi^\psi, \quad \eta \in \mathfrak{B}$$

then the operator U is extended to a unitary operator of $\mathfrak{K}$ onto $\mathfrak{H}$ which is also denoted by U. The unitary operator U implements the isomorphism $\bar{\varrho}$ of $\mathcal{B}$ onto $\mathcal{A}$, because for any $\xi, \eta \in \mathfrak{B}$, we have

$$U\pi_{\mathfrak{B}}(\xi)\,\eta = U\xi\eta = \varrho(\xi\eta)\,\xi^{\varphi} = \varrho(\xi)\,\varrho(\eta)\,\xi^{\varphi} = \pi_{\mathfrak{A}}(\varrho(\xi))\,U\eta = \bar{\varrho}\circ\pi_{\mathfrak{B}}(\xi)\,U\eta.$$

Thus, $\bar{\varrho}$ is extended to the spacial isomorphism of $\mathcal{L}(\mathfrak{B})$ onto $\mathcal{L}(\mathfrak{A})$ implemented by the unitary operator U. Since U maps $\mathfrak{K}_n$ onto $\mathfrak{H}_n$ for each $n \in \mathbf{Z}$, we have

$$U\Delta_{\mathfrak{B}}U^* = \Delta_{\mathfrak{A}}. \qquad \text{Q.E.D.}$$

Thus, the von Neumann algebra $\mathcal{L}(\mathfrak{A})$, denoted by $\mathcal{M}$, together with its decomposition, is determined uniquely by the pair $\{\mathcal{M}_0, \theta\}$ up to isomorphism. Hence we denote it by $\mathcal{R}(\mathcal{M}_0, \theta)$ if necessary. Using this notation, Theorem 1.27 is restated as follows:

THEOREM 2.12. *A von Neumann algebra $\mathcal{M}$ with a homogeneous periodic state is isomorphic to $\mathcal{R}(\mathcal{M}_0, \theta)$ for some pair $(\mathcal{M}_0, \theta)$ of $\mathcal{M}_0$ and θ.*

The von Neumann algebra $\mathcal{R}(\mathcal{M}_0, \theta)$ is characterized by the following result:

THEOREM 2.13. *Let $\mathcal{N}$ be a von Neumann algebra with a periodic faithful normal state ψ of period $T = -2\pi/\log \varkappa$. Let $\mathcal{N}_0$ be the centralizer of ψ. For the existence of an isomorphism σ of $\mathcal{R}(\mathcal{M}_0, \theta)$ onto $\mathcal{N}$ for some $\mathcal{M}_0$ and θ such that $\sigma(\mathcal{M}_0) = \mathcal{N}_0$ and $\varphi = \psi\circ\sigma$, it is necessary and sufficient that $\mathcal{N}$ is generated by $\mathcal{N}_0$ and an isometry v such that $\sigma_t^{\varphi}(v) = \varkappa^{it}v$, $t \in \mathbf{R}$, and $(vv^*)^{\natural} = \varkappa 1$ in $\mathcal{N}_0$.*

Proof. The necessity of the condition has been verified already; so we have only to prove the sufficiency.

We take $\mathcal{N}_0$ as $\mathcal{M}_0$ and $\theta(x) = vxv^*$. We have then

$$\psi(vxv^*) = \varkappa\psi(xv^*v) = \varkappa\psi(x).$$

We take the restriction ψ_0 of ψ to $\mathcal{N}_0$ as φ_0. It is straight forward to prove that $\mathcal{N}$ is isomorphic to $\mathcal{R}(\mathcal{M}_0, \theta)$ with this choice. Q.E.D.

To see when the state φ on $\mathcal{R}(\mathcal{M}_0, \theta)$ is homogeneous, we provide the following:

THEOREM 2.14. *Let σ_0 be an automorphism of $\mathcal{M}_0$. The following statements are equivalent:*

(i) *There exists an automorphism σ such that $\sigma(x) = \sigma_0(x)$, $x \in \mathcal{M}_0$,* and $\varphi\circ\sigma = \varphi$;

(ii) *There exists a partial isometry w in $\mathcal{M}_0$ such that*

$$\theta\circ\sigma_0(x)=w^*(\sigma_0\circ\theta)(x)w,$$

$$\sigma_0\circ\theta(x)=w\theta\circ\sigma_0(x)w^*,\quad x\in\mathcal{M}_0;$$

$$\varphi_0\circ\sigma_0=\varphi_0.$$

Noticing that any partial isometry in a finite von Neumann algebra extends to a unitary element in the algebra, we may say that condition (ii) requires that θ and σ_0 commute modulo inner automorphisms.

The proof follows the same lines as the proof of Theorem 1.28, so we omit it.

An automorphism of $\mathcal{M}_0$ satisfying condition (ii) in the theorem is called *admissible*. If $\tilde{\theta}$ is ergodic, then the condition $\varphi_0\circ\sigma_0=\varphi_0$ follows automatically from the other conditions.

COROLLARY 2.15. *The state φ on $\mathcal{R}(\mathcal{M}_0,\theta)$ is homogeneous if and only if the group of all admissible automorphisms of $\mathcal{M}_0$ acts ergodically on the center $\mathcal{Z}_0$ of $\mathcal{M}_0$.*

As in § 1, we define a representation π_n and an anti-representation π_n' of $\mathcal{M}_0$ on $\mathfrak{H}_n$ for $n\in\mathbf{Z}$ as follows:

$$\pi_n(a)\xi=a\xi,\quad \xi\in\mathfrak{H}_n;\qquad \pi_n'(a)=J\pi_{-n}(a^*)J,\quad a\in\mathcal{M}_0. \tag{18}$$

We recall the definition of projection e_n, $n\geqslant 0$, as $e_n=\theta^n(1)$. We define $e_{-n}=Je_nJ$ for $n\geqslant 1$ We put

$$\mathfrak{K}_n=e_n\mathfrak{H}_0,\quad n\in\mathbf{Z}.$$

With these notations, we have the following;

THEOREM 2.16. *For each $n\geqslant 1$, we have*

$$\{\pi_n,\mathfrak{H}_n\}\cong\{\pi_0,\mathfrak{K}_{-n}\};$$

$$\{\pi_{-n},\mathfrak{H}_{-n}\}\cong\{\theta^n,\mathfrak{K}_n\},$$

where $\{\pi_0,\mathfrak{K}_{-n}\}$ means the restriction of π_0 to the invariant subspace $\mathfrak{K}_{-n}$.

Proof. By definition, we have, for $n\geqslant 1$, $\mathfrak{H}_n=[\mathcal{M}_0u^n\xi_0]$, where $u=\pi(\alpha)$ as before, and

$$\mathfrak{K}_{-n}=Je_n\mathfrak{H}_0=Je_n[\mathcal{M}_0\xi_0]=J[e_n\mathcal{M}_0\xi_0]=[\mathcal{M}_0e_n\xi_0].$$

Let V_{-n} be the operator of $\mathcal{M}_0e_n\xi_0$ onto $\mathcal{M}_0u_n\xi_0$ defined by $V_{-n}xe_n\xi_0=\varkappa^{n/2}xu^n\xi_0$. We have then

$$\begin{aligned}(V_{-n}xe_n\xi_0\,|\,V_{-n}ye_n\xi_0)&=\varkappa^n(xu^n\xi_0\,|\,yu^n\xi_0)=\varkappa^n(u^{*n}y^*xu^n\xi_0\,|\,\xi_0)\\&=\varkappa^n\varphi_0\circ\theta^{-n}(e_ny^*xe_n)=\varphi_0(e_ny^*xe_n)=(xe_n\xi_0\,|\,ye_n\xi_0).\end{aligned}$$

Hence V_{-n} extends to a unitary operator of $\mathfrak{K}_{-n}$ onto $\mathfrak{H}_n$. It is easy to see that V_{-n} intertwines π_0 and π_n.

Next, we have, for $n \geqslant 1$,

$$\mathfrak{H}_{-n} = [u^{*n}\mathcal{M}_0\xi_0];$$

$$\mathfrak{K}_n = [e_n \mathcal{M}_0\xi_0].$$

Let $V_n e_n x\xi_0 = u^{*n}x\xi_0$, $x \in \mathcal{M}_0$. We have then

$$(V_n e_n x\xi_0 | V_n e_n y\xi_0) = (u^{*n}x\xi_0 | u^{*n}y\xi_0) = (y^* e_n x\xi_0 | \xi_0)$$
$$= \varphi_0((e_n y)^* e_n x) = (e_n x\xi_0 | e_n y\xi_0).$$

Hence V_n extends to an isometry of $\mathfrak{K}_n$ onto $\mathfrak{H}_{-n}$. Furthermore, we have, for each a, $x \in \mathcal{M}$,

$$V_n\theta^n(a)e_n x\xi_0 = u^{*n}\theta^n(a)e_n x\xi_0 = u^{*n}u^n a u^{*n}e_n x\xi_0 = au^{*n}e_n x\xi_0 = \pi_{-n}(a)V_n e_n x\xi_0.$$

Hence we have $V_n\theta^n(a)V_n^* = \pi_{-n}(a)$. Q.E.D.

COROLLARY 2.17. *For each n, the representation* $\{\pi_n, \mathfrak{H}_n\}$ *is of uniform multiplicity. More precisely, the coupling operator of* $\{\pi_n(\mathcal{M}_0), \mathfrak{H}_n\}$ *is* $\varkappa^n 1$.

3. The algebraic invariants $S(\mathcal{M})$ and $T(\mathcal{M})$ of Connes

In this section, we compute the algebraic invariants $S(\mathcal{M})$ and $T(\mathcal{M})$ introduced recently by A. Connes, see [3] and [4], for the von Neumann algebra $\mathcal{R}(\mathcal{M}_0, \theta)$.

For each faithful normal state ω of $\mathcal{M}$, ($\mathcal{M}$ is assumed to be σ-finite), we denote by Δ_ω the modular operator associated with ω. Of course, Δ_ω is defined on the representation space $\mathfrak{H}_\omega$ of the cyclic representation π_ω of $\mathcal{M}$ induced by ω. However, if we fix a faithful normal state φ on $\mathcal{M}$, and if we fix a Hilbert space $\mathfrak{H}$, on which $\mathcal{M}$ acts, with a cyclic vector ξ_0 such that $\varphi(x) = (x\xi_0 | \xi_0)$, then any other normal state ω of $\mathcal{M}$ is of the form $\omega(x) = (x\xi_\omega | \xi_\omega)$ for some unit vector ξ_ω in the closure of $\mathcal{M}_+\xi_0$ by the Radon Nikodym Theorem [18; Theorem 15.1]; therefore, the cyclic representation π_ω of $\mathcal{M}$ is realized on $\mathfrak{H}$ as follows:

$$\pi_\omega(a)x\xi_\omega = ax\xi_\omega, \quad a, x \in \mathcal{M}.$$

Thus the modular operator Δ_ω associated with a faithful normal state ω is also defined on the same Hilbert space $\mathfrak{H}$.

Definition 3.1. The spectrum of Δ_ω is called the *spectrum* of a faithful normal state ω and is denoted by Sp (ω).

According to Størmer's recent result, [17], we can define Sp (ω) without direct use of Δ_ω.

THEOREM 3.2. (E. Størmer) *For a faithful normal state ω on a von Neumann algebra $\mathcal{M}$ and a non-negative real number λ, the following statements are equivalent:*

(i) *λ falls in* Sp (ω);

(ii) *For any $\varepsilon>0$, there exists non-zero $x\in\mathcal{M}$ such that*

$$|\lambda\omega(yx)-\omega(xy)|\leqslant\varepsilon\omega(y^*y)^{\frac{1}{2}}. \tag{1}$$

For the proof, we refer to Størmer's original paper [17].

We now consider the algebraic invarients $S(\mathcal{M})$ and $T(\mathcal{M})$ defined by Connes, [3] and [4].

Definition 3.3. The algebraic invariant $S(\mathcal{M})$ of a σ-finite von Neumann algebra $\mathcal{M}$ is defined by

$$S(\mathcal{M})=\cap\,\mathrm{Sp}\,(\omega),$$

where ω runs over all faithful normal states on $\mathcal{M}$. We call $S(\mathcal{M})$ the *modular spectrum* of $\mathcal{M}$.

From the definition it is obvious that $S(\mathcal{M})$ is an algebraic invariant. Recently A. Connes proved that if $\mathcal{M}$ is a factor then non-zero elements in $S(\mathcal{M})$ form a closed subgroup of the multiplicative group $\mathbf{R}_+^*$ of all positive real numbers.

THEOREM 3.4. *Let $\mathcal{M}=\mathcal{R}(\mathcal{M}_0,\theta)$ for a couple $(\mathcal{M}_0,\theta)$ as before. Suppose $\mathcal{M}$ is a factor, equivalently, $\tilde{\theta}$ is ergodic. We conclude the following:*

(i) *If $\tilde{\theta}$ is periodic in the sense that $\tilde{\theta}^m=\iota$ for some integer $m\neq 0$, then*

$$S(\mathcal{M})=\{\kappa^{n_0n}\colon n\in\mathbf{Z}\}\cup\{0\},$$

where n_0 is the smallest positive integer with $\tilde{\theta}^{n_0}=\iota$. The assumption holds if and only if $\dim\mathcal{Z}_0=n_0$.

(ii) *If $\tilde{\theta}$ is not periodic, equivalently if $\tilde{\theta}$ is properly ergodic, then*

$$S(\mathcal{M})=\{0,1\}.$$

Before going into the proof, we recall Connes' result [7], by which we can compute $S(\mathcal{M})$ out of a fixed faithful normal state ω. For a projection p in $\mathcal{M}$ and a faithful normal state ω on $\mathcal{M}$, we define a faithful normal state ω_p on $p\mathcal{M}p$ by

$$\omega_p(x) = \frac{1}{\omega(p)}\,\omega(x), \quad x \in p\mathcal{M}p.$$

The point is that ω_p may be regarded as a state on $\mathcal{M}$ itself in the case that $\mathcal{M}$ is a factor of type III because $\mathcal{M} \cong p\mathcal{M}p$. Connes' result says that $S(\mathcal{M}) = \cap\,\{\mathrm{Sp}\,(\omega_p)$: p runs over all non-zero projections in $\mathcal{Z}_\omega$, the center of the centralizer $\mathcal{M}_\omega\}$ if $\mathcal{M}$ is a factor. With the aid of this result, we should compute only $\cap\,\{\mathrm{Sp}\,(\varphi_p)$: p runs over all non-zero projections in $\mathcal{Z}_0\}$.

Proof of Theorem 3.4. Suppose $\tilde{\theta}$ is periodic. Since $\tilde{\theta}$ is ergodic, $\mathcal{Z}_0$ must be finite dimensional. Let $n_0 = \dim \mathcal{Z}_0$ and p be a minimal projection in $\mathcal{Z}_0$. It is easily seen that $\{\tilde{\theta}^n(p)\colon\ 0 \leqslant n \leqslant n_0 - 1\}$ orthogonal and $\sum_{n=0}^{n_0-1} \tilde{\theta}^n(p) = 1$. We consider φ_p on $p\mathcal{M}p$. Since $\sigma_t^\varphi(p) = p$, the centralizer of φ_p is $p\mathcal{M}p \cap \mathcal{M}_0 = p\mathcal{M}_0 p$. Hence the centralizer of φ_p is a factor because the center of $p\mathcal{M}_0 p$ is $\mathcal{Z}_0 p = \mathbf{C}p$. Therefore we have $S(\mathcal{M}) = \mathrm{Sp}\,(\varphi_p)$ by Connes' result as quoted above. The decomposition of $p\mathcal{M}p$ with respect to φ_p is obviously given by $\{p\mathcal{M}_n p\colon\ n \in \mathbf{Z}\}$. We claim that $p\mathcal{M}_n p \neq \{0\}$ if and only if $n \in n_0\mathbf{Z}$. In fact, for $n \geqslant 1$, with the isometry $u \in \mathcal{M}_1$ defined in § 2 so that $uxu^* = \theta(x)$, $x \in \mathcal{M}_0$, we have

$$p\mathcal{M}_n p = p\mathcal{M}_0 u^n p = \mathcal{M}_0 p u^n p = \mathcal{M}_0 p u^n p u^{*n} u^n = \mathcal{M}_0 p \tilde{\theta}^n(p) u^n;$$

$$p\mathcal{M}_{-n} p = p u^{*n} \mathcal{M}_0 p = p u^{*n} p \mathcal{M}_0 = u^{*n} u^n p u^{*n} p \mathcal{M}_0 = u^{*n} \tilde{\theta}^n(p) p \mathcal{M}_0.$$

Hence $p\mathcal{M}_n p \neq \{0\}$ if and only if $p\tilde{\theta}^n(p) \neq 0$; if and only if $p = \tilde{\theta}^n(p)$; if and only if $n \in n_0\mathbf{Z}$. Therefore, in this case, we have $\mathrm{Sp}\,(\varphi_p) = \{\varkappa^{n_0 n}\colon\ n \in \mathbf{Z}\} \cup \{0\}$.

Suppose now $\mathcal{Z}_0$ is not finite dimensional. By the ergodicity of $\tilde{\theta}$, it follows that $\tilde{\theta}^n \neq \iota$ for any non-zero integer n. Hence for an integer n there exists a non-zero projection $p \in \mathcal{Z}_0$ such that $p\tilde{\theta}^n(p) = 0$. As before, we have $p\mathcal{M}_n p = \{0\}$. Hence $pJpJ\mathfrak{H}_n = \{0\}$. Therefore the spectrum of $pJpJ\Delta$ does not contain $\varkappa^n$. Since the modular operator associated with φ_p is given by $pJpJ\Delta$, we have $\mathrm{Sp}\,(\varphi_p) = \mathrm{Sp}\,(pJpJ\Delta)$, so that $\varkappa^n \notin \mathrm{Sp}\,(\varphi_p)$. Therefore we get $S(\mathcal{M}) = \cap\,\mathrm{Sp}\,(\varphi_p) = \{0, 1\}$. Q.E.D.

Let $\mathrm{Int}\,(\mathcal{M})$ be the group of all inner automorphisms of $\mathcal{M}$. Connes proved in [5] that for a fixed $t \in \mathbf{R}$, $\sigma_t^\omega \in \mathrm{Int}\,(\mathcal{M})$ for some faithful normal state ω on $\mathcal{M}$ implies that $\sigma_t^\psi \in \mathrm{Int}\,(\mathcal{M})$ for any other faithful normal state ψ on $\mathcal{M}$. Therefore, the set

$$T(\mathcal{M}) = \{t \in \mathbf{R}\colon \sigma_t^\varphi \in \mathrm{Int}\,(\mathcal{M})\} \tag{3}$$

is an algebraic invariant of $\mathcal{M}$.

THEOREM 3.5. *Let $\mathcal{M} = \mathcal{R}(\mathcal{M}_0, \theta)$. For each $t \in \mathbf{R}$, the following statements are equivalent:*

(i) $\varkappa^{it}$ *is a point spectrum of* $\tilde{\theta}$. *Namely, there is a unitary* v *in* $\mathcal{Z}_0$ *with* $\tilde{\theta}(v)=\varkappa^{it}v$;
(ii) $t\in T(\mathcal{M})$.

Proof. (i)$\Rightarrow$(ii): Consider the inner automorphism $\mathrm{Ad}\,(v^*)$ of $\mathcal{M}$ induced by v^*. We have then $\sigma_t^{\varphi}(x)=x=\mathrm{Ad}(v^*)(x)$ for any $x\in\mathcal{M}_0$ and

$$\mathrm{Ad}\,(v^*)u=v^*uv=v^*uvu^*u=v^*\theta(v)u=v^*\tilde{\theta}(v)u=v^*(\varkappa^{it}v)u=\varkappa^{it}u=\sigma_t^{\varphi}(u).$$

Hence $\mathrm{Ad}\,(v^*)$ and σ_t^{φ} coincide on the generators $\mathcal{M}_0$ and u, so that $\mathrm{Ad}\,(v^*)=\sigma_t^{\varphi}$ is inner.

(ii)$\Rightarrow$(i): Let v be a unitary in $\mathcal{M}$ such that $\mathrm{Ad}\,(v^*)=\sigma_t^{\varphi}$. For any $x\in\mathcal{M}_0$, we have

$$v^*xv=\sigma_t^{\varphi}(x)=x,$$

so that v is in the relative commutant $\mathcal{M}_0'\cap\mathcal{M}=\mathcal{Z}_0$ of $\mathcal{M}_0$. Hence v is a unitary in $\mathcal{Z}_0$.

Now, we have

$$\varkappa^{it}u=\sigma_t^{\varphi}(u)=v^*uv=v^*uvu^*u=v^*\theta(v)u=v^*\tilde{\theta}(v)u;\quad \varkappa^{it}vu=\tilde{\theta}(v)u,$$

so that we get

$$\varkappa^{it}ve_1=\varkappa^{it}vuu^*=\tilde{\theta}(v)uu^*=\tilde{\theta}(v)e_1,$$

which implies $\varkappa^{it}v=\tilde{\theta}(v)$. Q.E.D.

COROLLARY 3.6. *Let* $\mathcal{M}=\mathcal{R}(\mathcal{M}_0,\theta)$. *If the center* $\mathcal{Z}_0$ *of* $\mathcal{M}_0$ *has separable predual* $(\mathcal{Z}_0)_*$, *then* $\mathcal{M}$ *is always of type III.*

Proof. Since the predual of $\mathcal{Z}_0$ is separable, the ergodic automorphism $\tilde{\theta}$ has at most a countable point spectrum, which can not exhaust the real line. Hence σ_t^{φ} is outer for some $t\in\mathbf{R}$; thus $\mathcal{M}$ is of type III, by [18; Theorem 14.2].

COROLLARY 3.7. *Let* $\mathcal{M}=\mathcal{R}(\mathcal{M}_0,\theta)$. *If the ergodic automorphism* $\tilde{\theta}$ *of* $\mathcal{Z}_0$ *does not have point spectrum except 1, then we have*

$$T(\mathcal{M})=T\mathbf{Z},$$

where $T=-2\pi/\log\varkappa$.

4. Comparison of periodic states with the same period

Concerning a decomposition of a factor $\mathcal{M}$, a natural question is whether the decomposition in Theorem 1.27 is unique in some sense. To attack this problem, we have to compare two periodic states φ and ψ on the same factor $\mathcal{M}$.

Suppose φ and ψ are two faithful normal states on a von Neumann algebra. Following Connes' ideas, we consider the 2×2 matrix algebra $\mathcal{D}=\mathcal{M}\otimes\mathcal{L}(\mathfrak{H}_2)$ over $\mathcal{M}$, where $\mathfrak{H}_2$

denotes a 2-dimensional Hilbert space. Let $\{e_{i,j}: i, j=1,2\}$ be a system of matrix units in $\mathcal{L}(\mathfrak{H}_2)$. Every $x \in \mathcal{D}$ is of the form:

$$x = x_{11} \otimes e_{11} + x_{12} \otimes e_{12} + x_{21} \otimes e_{21} + x_{22} \otimes e_{22}, \tag{1}$$

where $x_{i,j} \in \mathcal{M}$, $i, j=1,2$ we define a faithful state χ on $\mathcal{D}$ by

$$\chi(x) = \tfrac{1}{2}(\varphi(x_{11}) + \psi(x_{22})). \tag{2}$$

Connes showed in [5] that there exists a strongly continuous one parameter family $\{u_t\}$ of unitaries in $\mathcal{M}$ such that

$$\sigma_t^\chi(1 \otimes e_{12}) = u_t \otimes e_{12}; \qquad \sigma_t^\psi(x) = u_t^* \sigma_t^\varphi(x) u_t, x \in \mathcal{M}. \tag{3}$$

Noticing that $x \otimes e_{12} = (x \otimes e_{11})(1 \otimes e_{12})$, $x \in \mathcal{M}$, we have, for each x in $\mathcal{D}$ given by (1),

$$\sigma_t^\chi(x) = \sigma_t^\varphi(x_{11}) \otimes e_{11} + \sigma_t^\varphi(x_{12}) u_t \otimes e_{12} + u_t^* \sigma_t^\varphi(x_{21}) \otimes e_{21} + u_1^* \sigma_t^\varphi(x_{22}) u_t \otimes e_{22}. \tag{4}$$

Lemma 4.1. *The one parameter family $\{u_t\}$ satisfies the cocycle equality:*

$$u_{s+t} = \sigma_t^\varphi(u_s) u_t = \sigma_s^\varphi(u_t) u_s. \tag{5}$$

Proof. Cocycle equality (5) follows from the simple calculation:

$$\begin{aligned} u_{s+t} \otimes e_{12} = \sigma_{s+t}^\chi(1 \otimes e_{12}) &= \sigma_s^\chi \circ \sigma_t^\chi(1 \otimes e_{12}) \\ &= \sigma_s^\chi(u_t \otimes e_{12}) = \sigma_s^\chi((u_t \otimes e_{11})(1 \otimes e_{12})) \\ &= (\sigma_s^\varphi(u_t) \otimes e_{11})(u_s \otimes e_{12}) = \sigma_s^\varphi(u_t) u_s \otimes e_{12}. \end{aligned}$$

Q.E.D.

We consider a one parameter family ϱ_t of isometrics of $\mathcal{M}$ onto $\mathcal{M}$ defined by

$$\varrho_t(x) = \sigma_t^\varphi(x) u_t, \quad x \in \mathcal{M}. \tag{6}$$

Lemma 4.2. *The family $\{\varrho_t\}$ enjoys the following properties:*

(i) $\varrho_{s+t} = \varrho_s \circ \varrho_t$;

(ii) *For each fixed $x \in \mathcal{M}$, the map: $t \in \mathbf{R} \mapsto \varrho_t(x) \in \mathcal{M}$ is σ-strongly continuous.*

Proof. Property (ii) follows from the facts that $t \mapsto u_t$ is strongly continuous, and that the product is jointly continuous on bounded parts of $\mathcal{M}$ with respect to the σ-strong topology.

We have, for any $x \in \mathcal{M}$,

$$\varrho_{s+t}(x) = \sigma_{s+t}^\varphi(x) u_{s+t} = \sigma_s^\varphi \circ \sigma_t^\varphi(x) \sigma_s^\varphi(u_t) u_s = \sigma_s^\varphi(\sigma_t^\varphi(x) u_t) u_s = \varrho_s \circ \varrho_t(x).$$

Q.E.D.

Suppose φ and ψ have the same period, say $T>0$, and suppose $\mathcal{M}$ is a factor. Let $\varkappa = e^{-2\pi/T}$, $0<\varkappa<1$, as before. We have then, for any $x \in \mathcal{M}$,

$$x = \sigma_T^\psi(x) = u_T^* \sigma_T^\varphi(x) u_T = u_T^* x u_T. \tag{7}$$

Hence u_T and x commute. Since $\mathcal{M}$ is a factor, there exists α with $\varkappa < \alpha \leqslant 1$ such that $u_T = \alpha^{iT} 1$. Hence we have the following:

LEMMA 4.3. *Let $\mathcal{M}$ be a factor. If φ and ψ have the same period, say $T>0$, then there exists α with $\varkappa < \alpha \leqslant 1$ such that*

$$\varrho_T(x) = \alpha^{it} x, \quad x \in \mathcal{M},$$

where $\varkappa = e^{-2\pi/T}$.

We assume throughout the rest of this section that (i) $\mathcal{M}$ is a factor and (ii) φ and ψ have the same period $T>0$. We fix the above α and $\varkappa$. By Lemma 4.3, we obtain a periodic one parmeter isometry group $\{\varrho_s'\}$ on $\mathcal{M}$ by the following

$$\varrho'(x) = \alpha^{-it} \varrho_t(x) = \alpha^{-it} \sigma_t^\varphi(x) u_t, x \in \mathcal{M}.$$

We have then $\varrho_T' = \iota$. For each $n \in \mathbf{Z}$,

$$\mathcal{M}_n^{\varphi,\psi} = \{x \in \mathcal{M} : \varrho_t'(x) = \varkappa^{int} x\}. \tag{8}$$

By (6) and (7), we have also

$$\mathcal{M}_n^{\varphi,\psi} = \{x \in \mathcal{M} : \varrho_t(x) = \alpha^{it} \varkappa^{int} x\} = \{x \in \mathcal{M} : \sigma_t^\varphi(x) u_t = \alpha^{it} \varkappa^{int} x\}. \tag{8'}$$

LEMMA 4.4. *For some $n \in \mathbf{Z}$, $\mathcal{M}_n^{\varphi,\psi} \neq \{0\}$.*

Proof. For each $x \in \mathcal{M}$ and $\omega \in \mathcal{M}_*$, the predual of $\mathcal{M}$, we consider a function $f_{x,\omega}(t) = \langle \varrho_t'(x), \omega \rangle$. The function $f_{x,\omega}$ has the period T and is non-zero for some x and ω. Hence for some $n \in \mathbf{Z}$, we have

$$a_n(x, \omega) = \frac{1}{T} \int_0^T \varkappa^{-int} f_{x,\omega}(t)\, dt \neq 0.$$

It is clear that the map: $\omega \in \mathcal{M}_* \mapsto a_n(x, \omega)$ is linear, and bounded because $|f_{x,\omega}(t)| \leqslant \|x\| \|\omega\|$. Hence there exists $a_n \in \mathcal{M}$ such that $a_n(x, \omega) = \langle a_n, \omega \rangle$. We claim that a_n falls in $\mathcal{M}_n^{\varphi,\psi}$. In fact, we have, for every $\omega \in \mathcal{M}_*$,

$$\begin{aligned} \langle \varrho_s'(a_n), \omega \rangle = \langle a_n, \omega \circ \varrho_s' \rangle &= \frac{1}{T} \int_0^T \varkappa^{-int} \langle \varrho_t'(x), \omega \circ \varrho_s' \rangle\, dt \\ &= \frac{1}{T} \int_0^T \varkappa^{-int} \langle \varrho_{s+t}'(x), \omega \rangle\, dt = \frac{1}{T} \int_0^T \varkappa^{-int(t-s)} \langle \varrho_t'(x), \omega \rangle\, dt \\ &= \varkappa^{ins} \frac{1}{T} \int_0^T \varkappa^{-nt} \langle \varrho_t'(x), \omega \rangle\, dt = \varkappa^{ins} \langle a_n, \omega \rangle. \end{aligned}$$

Therefore, $\varrho_s'(a_n) = \varkappa^{ins} a_n$, so that $a_n \in \mathcal{M}_n^{\varphi,\psi} \neq \{0\}$. Q.E.D.

By $\{\mathcal{M}_n^\varphi : n \in \mathbf{Z}\}$ and $\{\mathcal{M}_n^\psi : n \in \mathbf{Z}\}$, we denote the decompositions of $\mathcal{M}$ with respect to φ and ψ respectively. For example, $\mathcal{M}_0^\varphi$ (resp. $\mathcal{M}_0^\psi$) is the centralizer of φ (resp. ψ). We naturally understand the notations $\mathcal{Z}_0^\varphi$, and $\mathcal{Z}_0^\psi$ and so on.

LEMMA 4.5. *For each l, m, $n \in \mathbf{Z}$, we have*

$$\mathcal{M}_m^\varphi \mathcal{M}_l^{\varphi,\psi} \mathcal{M}_n^\psi \subset \mathcal{M}_{m+l+n}^{\varphi,\psi}.$$

Proof. For each $a \in \mathcal{M}_n^\varphi$, $b \in \mathcal{M}_m^\psi$ and $x \in \mathcal{M}_l^{\varphi,\psi}$, we have

$$\begin{aligned}\varrho_t'(axb) = \alpha^{-it}\sigma_t^\varphi(axb)\,u_t &= \alpha^{-it}\sigma_t^\varphi(a)\,\sigma_t^\varphi(x)\,\sigma_t^\varphi(b)\,u_t \\ &= \varkappa^{imt}a\alpha^{-it}\sigma_t^\varphi(x)\,u_t u_t^*\sigma_t^\varphi(b)\,u_t = \varkappa^{imt}a\varrho_t'(x)\,\sigma_t^\psi(b) \\ &= \varkappa^{imt}a(\varkappa^{ilt}x)\,(\varkappa^{int}b) = \varkappa^{i(m+l+n)t}\,axb. \qquad \text{Q.E.D.}\end{aligned}$$

LEMMA 4.6. *If $\mathcal{M}_1^\varphi$ and $\mathcal{M}_1^\psi$ contain isometries respectively, then $\mathcal{M}_n^{\varphi,\psi} \neq \{0\}$ for all $n \in \mathbf{Z}$.*

Proof. Let $u \in \mathcal{M}_1^\varphi$ and $v \in \mathcal{M}_1^\psi$ be isometries. We choose first an $n \in \mathbf{Z}$ with $\mathcal{M}_n^{\varphi,\psi} \neq \{0\}$. We have then, for $k \geq 1$,

$$\begin{aligned}&\mathcal{M}_{n+k}^{\varphi,\psi} \supset u^k\mathcal{M}_n^{\varphi,\psi} \neq \{0\}, \\ &\mathcal{M}_{n-k}^{\varphi,\psi} \supset \mathcal{M}_n^{\varphi,\psi}v^{*k} \neq \{0\}. \qquad \text{Q.E.D.}\end{aligned}$$

We now further assume that $\mathcal{M}_1^\varphi$ and $\mathcal{M}_1^\psi$ contains isometries u_φ and u_ψ respectively such that $(u_\varphi u_\varphi^*)^\natural = \varkappa 1$ and $(u_\psi u_\psi^*)^\natural = \varkappa 1$. Therefore, by Theorem 2.13, $\mathcal{M}$ has two discriptions $\mathcal{M} = \mathcal{R}(\mathcal{M}_0^\varphi, \theta_\varphi)$ and $\mathcal{M} = \mathcal{R}(\mathcal{M}_0^\psi, \theta_\psi)$ with respect to φ and ψ respectively. Let u_φ (resp. u_φ) be an isometry in $\mathcal{M}_1^\varphi$ (resp. $\mathcal{M}_1^\psi$) such that

$$\theta_\varphi(x) = u_\varphi x u_\varphi^*,\ x \in \mathcal{M}_0^\varphi; \qquad \theta_\psi(x) = u_\psi x u_\psi^*,\ x \in \mathcal{M}_0^\psi. \tag{9}$$

LEMMA 4.7. *For each $n \in \mathbf{Z}$, if x is an element of $\mathcal{M}_n^{\varphi,\psi}$, then we have*

$$\alpha\varkappa^n\psi(yx) = \varphi(xy), \quad y \in \mathcal{M}. \tag{10}$$

Proof. First of all, we observe

$$\sigma_t^\chi(x \otimes e_{12}) = \sigma_t^\varphi(x)\,u_t \otimes e_{12} = \alpha^{it}\varrho_t'(x) \otimes e_{12} = (\alpha\varkappa^n)^{it}x \otimes e_{12}.$$

Therefore, we get, by Lemma 1.6,

$$\alpha\varkappa^n\psi(yx) = 2\alpha\varkappa^n\chi((y \otimes e_{21})\,(x \otimes e_{12})) = 2\chi((x \otimes e_{12})\,(y \otimes e_{21})) = \varphi(xy). \qquad \text{Q.E.D.}$$

LEMMA 4.8. *If $\mathcal{M}_0^\varphi$ and $\mathcal{M}_0^\psi$ are both factors, then for any pair of non-zero projections $p \in \mathcal{M}_0^\varphi$ and $q \in \mathcal{M}_0^\psi$, we have*

$$p\mathcal{M}_n^{\varphi,\psi}q \neq \{0\}.$$

Proof. For a fixed $q \in \mathcal{M}_0^\psi$, let

$$\mathcal{J}_q^\varphi = \{x \in \mathcal{M}_0^\varphi : x\mathcal{M}_n^{\varphi,\psi}q = 0\};$$

for a fixed $p \in \mathcal{M}_0^\varphi$, let

$$\mathcal{J}_p^\psi = \{x \in \mathcal{M}_0^\psi : p\mathcal{M}_n^{\varphi,\psi}x = 0\}.$$

Both $\mathcal{J}_q^\varphi$ and $\mathcal{J}_p^\psi$ are σ-weakly closed ideals of $\mathcal{M}_0^\varphi$ and of $\mathcal{M}_0^\psi$ respectively, so that they are either $\{0\}$ or all of $\mathcal{M}_0^\varphi$ or $\mathcal{M}_0^\psi$ respectively. But if $\mathcal{J}_q^\varphi = \mathcal{M}_0^\varphi$, then $\mathcal{M}_n^{\varphi,\psi}q = \{0\}$, so that $\mathcal{J}_1^\psi$ contains q. Hence if $q \neq 0$, then $\mathcal{J}_1^\psi = \mathcal{M}_0^\psi$. But this is impossible because $\mathcal{M}_n^{\varphi,\psi} \neq \{0\}$ and $1 \in \mathcal{M}_0^\psi$. Therefore $\mathcal{J}_q^\varphi = \{0\}$ for any non-zero q which means that $p\mathcal{M}_n^{\varphi,\psi} \neq \{0\}$ for any non-zero p. Q.E.D.

LEMMA 4.9. *For each $x \in \mathcal{M}_n^{\varphi,\psi}$, if*

$$x = vh = kv, \quad h = (x^*x)^{\frac{1}{2}}, \quad k = (xx^*)^{\frac{1}{2}}$$

is the left and right polar decomposition of x, then v is in $\mathcal{M}_n^{\varphi,\psi}$ and $h \in \mathcal{M}_0^\psi$ and $k \in \mathcal{M}_0^\varphi$.

Proof. We have

$$\varkappa^{int}vh = \alpha^{it}\sigma_t^\varphi(vh)\,u_t = \alpha^{it}\sigma_t^\varphi(v)\,u_t u_t^*\sigma_t^\varphi(h)\,u_t = \alpha^{it}\sigma_t^\varphi(v)\,u_t\sigma_t^\psi(h);$$

$$\varkappa^{int}kv = \alpha^{it}\sigma_t^\varphi(kv)\,u_t = \alpha^{it}\sigma_t^\varphi(k)\,\sigma_t^\varphi(v)\,u_t.$$

Hence the unity of the polar decomposition yields our assertions. Q.E.D.

Definition 4.10. A faithful mormal state φ on a factor $\mathcal{M}$ is said to be *inner homogeneous* if $\mathcal{M}_\varphi' \cap \mathcal{M} = \{\lambda 1\}$.(1)

Now, we can compare two inner homogeneous periodic states as follows.

THEOREM 4.12. *Suppose φ and ψ are faithful inner homogeneous periodic states on a factor $\mathcal{M}$. We conclude then the following:*

(i) *The periods of φ and ψ are same;*

(ii) *There exist isometries u and v in $\mathcal{M}$ such that*

(1) After finishing this work, Dr. A. Connes kindly informed the author that he has succeeded in proving the existence of an inner homogeneous periodic state on a factor $\mathcal{M}$ with

$$S(\mathcal{M}) = \{\lambda^n : n \in \mathbf{Z}\} \cup \{0\},\ 0 < \lambda < 1.$$

He also mentions in the letter that such a factor $\mathcal{M}$ is written as the crossed product of a II_∞-factor by an automorphism, which is closely related to our structure theorem.

$$\psi(x) = \frac{1}{\varphi(uu^*)} \varphi(uxu^*), \ x \in \mathcal{M};$$

$$\varphi(x) = \frac{1}{\psi(vv^*)} \psi(vxv^*), \ x \in \mathcal{M};$$

$$p = uu^* \in \mathcal{M}_0^{\varphi} \quad and \quad f = vv^* \in \mathcal{M}_0^{\psi};$$

(iii) *There exists a unitary* $u \in \mathcal{M}$ *such that*

$$\psi(x) = \varphi(uxu^*), \quad x \in \mathcal{M},$$

if and only if the state χ *on* $\mathcal{D}$ *defined above has the same period* T *as* φ *and* ψ.

Proof. By Theorem 3.5, φ and ψ have the same period $T > 0$. Let n be a fixed integer. Let $\{u_i\}_{i \in I}$ be a maximal family of partial isometries in $\mathcal{M}_n^{\varphi,\psi}$ such that $p_i = u_i u_i^*$ and $q_i = u_i^* u_i$ are both orthogonal in $\mathcal{M}_0^{\varphi}$ and in $\mathcal{M}_0^{\psi}$ respectivey. Let $u = \sum_{i \in I} u_i$. Obviously u is a partial isometry in $\mathcal{M}_n^{\varphi,\psi}$. Let $p = uu^*$ and $q = u^*u$. If $p \neq 1$ and $q \neq 1$, then $(1-p)\mathcal{M}_n^{\varphi,\psi}(1-q) \neq \{0\}$ by Lemma 4.8. Take a non-zero $x \in (1-p)\mathcal{M}_n^{\varphi,\psi}(1-q)$. Let $x = vh = kv$ be the left and right polar decomposition of x. By Lemma 4.9, the partial isometry v is in $\mathcal{M}_n^{\varphi,\psi}$, and $vv^* \leqslant (1-p)$ and $v^*v \leqslant (1-q)$. Hence $\{u_i\}_{i \in I} \cup \{v\}$ is properly larger than $\{p_i\}_{i \in I}$, which contradicts the maximality. Therefore, either $p = 1$ or $q = 1$.

However, by Lemma 4.7, we have

$$\alpha\varkappa^n \psi(q) = \alpha\varkappa^n \psi(u^*u) = \varphi(uu^*) = \varphi(p).$$

Hence if $\alpha\varkappa^n < 1$, then $p \neq 1$, so that $q = 1$. If $\alpha\varkappa^n > 1$, $q \neq 1$, so that $p = 1$. Therefore if $n \geqslant 1$, then u is an isometry, and if $n \leqslant -1$, then u is a co-isometry, that is, u^* is an isometry because the inequality $\varkappa < \alpha \leqslant 1$ implies $1 < \varkappa^n \alpha$ for $n \leqslant -1$. If $\alpha\varkappa^n = 1$, then u is a unitary. However, $\alpha\varkappa^n = 1$ implies that $n = 0$ and $\alpha = 1$.

Choosing $n = 0$, we have an isometry u in $\mathcal{M}_0^{\varphi,\psi}$. Hence we have

$$\alpha\psi(x) = \alpha\psi(xu^*u) = \varphi(uxu^*),$$

so that

$$\psi(x) = \frac{1}{\alpha} \varphi(uxu^*).$$

It is obvious that $\alpha = \varphi(uu^*)$. Choosing $n = -1$, we hve a co-isometry $u \in \mathcal{M}_{-1}^{\varphi,\psi}$, so that

$$\alpha\varkappa^{-1} \psi(u^*xu) = \varphi(xuu^*) = \varphi(x).$$

Hence with an isometry $v = u^*$, we get

$$\varphi(x)=\frac{\alpha}{\varkappa}\psi(vxv^*).$$

Putting $x=1$, we have $\varkappa/\alpha=\psi(vv^*)$.

If $\sigma_T^\chi=\iota$, then we have $\alpha=1$, so that an isometry $u\in\mathcal{M}_0^{\varphi,\psi}$ must be a unitary. Now, suppose there exists a unitary u in $\mathcal{M}$ such that $\psi(x)=\varphi(uxu^*)$, $x\in\mathcal{M}$. We claim that $u\otimes e_{12}$ is in the centralizer $\mathcal{D}_\chi$ of χ. Let x be an element of $\mathcal{M}$. We have then

$$\chi((x\otimes e_{11})(u\otimes e_{12}))=\chi(xu\otimes e_{12})=0;$$

$$\chi((u\otimes e_{12})(x\otimes e_{11}))=\chi(0)=0;$$

$$\chi((x\otimes e_{12})(u\otimes e_{12}))=\chi(0)=0;$$

$$\chi((u\otimes e_{12})(x\otimes e_{12}))=\chi(0)=0;$$

$$\chi((x\otimes e_{22})(u\otimes e_{12}))=\chi(0)=0;$$

$$\chi((u\otimes e_{12})(x\otimes e_{22}))=\chi(ux\otimes e_{12})=0;$$

$$\begin{aligned}\chi((x\otimes e_{21})(u\otimes e_{12}))&=\chi(xu\otimes e_{22})=\tfrac{1}{2}\psi(xu)=\tfrac{1}{2}\varphi(ux)\\&=\chi(ux\otimes e_{11})=\chi((u\otimes e_{12})(x\otimes e_{21})).\end{aligned}$$

Thus $u\otimes e_{12}$ falls in $\mathcal{D}_\chi$, so that

$$u\otimes e_{12}=\sigma_t^\chi(u\otimes e_{12})=\sigma_t^\varphi(u)\,u_t\otimes e_{12}.$$

Hence we have $u=\sigma_t^\varphi(u)u_t$, and then $u_t=\sigma_t^\varphi(u^*)u$. Thus $u_T=\sigma_T^\varphi(u^*)u=u^*u=1$. By equality (4) for σ_t^χ, we have $\sigma_T^\chi=\iota$. Q.E.D.

Rephrasing the theorem, we have the following:

COROLLARY 4.13. *If φ and ψ are inner homogeneous periodic states on a factor $\mathcal{M}$, then there exists a projection $p\in\mathcal{M}_0^\varphi$ and $q\in\mathcal{M}_0^\psi$ such that*

$$\{\mathcal{M},\psi\}\cong\{p\mathcal{M}p,\varphi_p\};$$

$$\{\mathcal{M},\varphi\}\cong\{q\mathcal{M}q,\psi_q\}.$$

Therefore, the collection $\{p\mathcal{M}p,\varphi_p\}$ exhausts all possible inner homogeneous periodic states up to isomorphism.

THEOREM 4.14. *Let φ be an inner homogeneous periodic state on a factor $\mathcal{M}$. Let $\varkappa=e^{-2\pi/T}$, where T is the period of φ. Let p and q be two projections in $\mathcal{M}_0^\varphi$. For isometries u and v in $\mathcal{M}$ with $p=uu^*$ and $q=vv^*$, put*

$$\psi(x)=\varphi_p(uxu^*),\ \omega(x)=\varphi_q(vxv^*),\quad x\in\mathcal{M}.$$

The following statements are equivalent:

(i) $\varphi(q)=\varkappa^n\varphi(p)$ *for some* $n\in\mathbf{Z}$;

(ii) *There exists a unitary* $w\in\mathcal{M}$ *such that*

$$\omega(x)=\psi(wxw^*),\quad x\in\mathcal{M}.$$

Proof. (i) $\Rightarrow$ (ii): By Lemma 1.25 there exists a partial isometry $\tilde{u}$ in $\mathcal{M}_n^\varphi$ such that $\tilde{u}^*\tilde{u}=p$ and $\tilde{u}\tilde{u}^*=q$. Let $w=v^*\tilde{u}u$. Clearly w is a unitary in $\mathcal{M}$, and we have

$$\begin{aligned}\psi(w^*xw)&=\frac{1}{\varphi(p)}\varphi(uw^*xwu^*)=\frac{1}{\varphi(p)}\varphi(\tilde{u}^*vxv^*\tilde{u})\\&=\frac{1}{\varkappa^n\varphi(p)}\varphi(\tilde{u}\tilde{u}^*vxv^*)\qquad\text{by Lemma 1.6,}\\&=\frac{1}{\varphi(q)}\varphi(vxv^*)=\omega(x).\end{aligned}$$

(ii) $\Rightarrow$ (i): Let w be a unitary in $\mathcal{M}$ such that $\omega(x)=\psi(wxw^*)$. By definition, it follows that

$$\frac{1}{\varphi(p)}\varphi(uwxw^*u^*)=\frac{1}{\varphi(q)}\varphi(vxv^*),\ x\in\mathcal{M}.$$

Replacing x by v^*xv, we get

$$\frac{1}{\varphi(q)}\varphi(qxq)=\frac{1}{\varphi(p)}\varphi(uwv^*xvw^*u^*).$$

Let $\tilde{u}=uwv^*$. We have then

$$\tilde{u}^*\tilde{u}=vw^*u^*uwv^*=q;\qquad \tilde{u}\tilde{u}^*=uwv^*vw^*u^*=p;$$

$$\frac{1}{\varphi(q)}\varphi(qxq)=\frac{1}{\varphi(p)}\varphi(\tilde{u}x\tilde{u}^*),\ x\in\mathcal{M}.$$

Replacing x by $qx\tilde{u}$, we get

$$\varphi(\tilde{u}xp)=\frac{\varphi(p)}{\varphi(q)}\varphi(qx\tilde{u}),\ x\in\mathcal{M};$$

hence for any $x\in\mathcal{M}$, we have

$$\varphi(\tilde{u}x)=\varphi(p\tilde{u}x)=\varphi(\tilde{u}xp)=\frac{\varphi(p)}{\varphi(q)}\varphi(qx\tilde{u})=\frac{\varphi(p)}{\varphi(q)}\varphi(x\tilde{u}q)=\frac{\varphi(p)}{\varphi(q)}\varphi(x\tilde{u}).$$

Thus we have $\varphi(p)/\varphi(q)=\varkappa^n$ for some $n\in\mathbf{Z}$. Q.E.D.

Thus, the unitary equivalence class of all inner homogeneous periodic states is parameterized by the half-open interval $(\varkappa, 1]$. We write $\psi\sim\varphi_\lambda$, $\varkappa<\lambda\leqslant 1$, when $\{\mathcal{M},\psi\}$ is isomorphic to $\{p\mathcal{M}p,\varphi_p\}$ with $\varphi(p)=\lambda$.

COROLLARY 4.15. *Let φ be an inner homogeneous periodic state on a factor $\mathcal{M}$. For an inner homogeneous periodic state ψ with $\psi \sim \varphi_\lambda$, $\varkappa < \lambda \leqslant 1$, where $\varkappa = e^{-2\pi/T}$ for the period T of φ, the following two statements are equivalent:*

(i) *There exists $\sigma \in \mathrm{Aut}(\mathcal{M})$ such that $\psi = \varphi \circ \sigma$:*

(ii) *There exists a projection $p \in \mathcal{M}_0^\varphi$ with $\varphi(p) = \lambda$, an isomorphism ϱ_0 of $\mathcal{M}_0^\varphi$ onto $p\mathcal{M}_0^\varphi p$ and a partial isometry w in $\mathcal{M}_0^\varphi$ such that*

$$\varrho_0 \circ \theta(x) = w\theta \circ \varrho_0(x) w^*;$$

$$\theta \circ \varrho_0(x) = w^* \varrho_0 \circ \theta(x) w, \quad x \in \mathcal{M}_0^\varphi,$$

where θ denotes an isomorphism of $\mathcal{M}_0^\varphi$ onto $e\mathcal{M}_0^\varphi e$ for a projection $e \in \mathcal{M}_0^\varphi$ with $\varphi(e) = \varkappa$ such that $\mathcal{M} = \mathcal{R}(\mathcal{M}_0, \theta)$.

Proof. (i) $\Rightarrow$ (ii): Let u be an isometry in $\mathcal{M}$ with $p = uu^*$ such that $\psi(x) = \varphi_p(uxu^*)$ and $\varphi(p) = \lambda$. It is then clear that $\sigma^{-1}(\mathcal{M}_0^\varphi) = \mathcal{M}_0^\psi = u^* p \mathcal{M}_0^\varphi pu$. Let $\varrho(x) = u\sigma^{-1}(x)u^*$, $x \in \mathcal{M}$. Then ϱ is an isomorphism of $\mathcal{M}$ onto $p\mathcal{M}p$. We have, for each $x \in \mathcal{M}$,

$$\varphi_p \circ \varrho(x) = \varphi_p(u\sigma^{-1}(x)u^*) = \psi(\sigma^{-1}(x)) = \varphi(x);$$

hence ϱ is an isomorphism of $\{\mathcal{M}, \varphi\}$ onto $\{p\mathcal{M}p, \varphi_p\}$. Therefore, Theorem 1.28 implies the conclusion.

(ii) $\Rightarrow$ (i): By Theorem 1.28, there exists an isomorphism ϱ of $\{\mathcal{M}, \varphi\}$ onto $\{p\mathcal{M}p, \varphi_p\}$ which extends ϱ_0. Let $\sigma(x) = \varrho^{-1}(uxu^*)$. We have then

$$\varphi \circ \sigma(x) = \varphi \circ \varrho^{-1}(uxu^*) = \varphi_p(uxu^*) = \psi(x) \qquad \text{Q.E.D.}$$

COROLLARY 4.16. *Let $\mathcal{M}_0$ and $\mathcal{N}_0$ be two II_1-factors. Let $e \in \mathcal{M}_0$ and $f \in \mathcal{N}_0$ be projections. Let θ and ϱ be isomorphisms of $\mathcal{M}_0$ onto $e\mathcal{M}_0 e$ and $\mathcal{N}_0$ onto $f\mathcal{N}_0 f$. Let $\mathcal{M} = \mathcal{R}(\mathcal{M}_0, \theta)$ and $\mathcal{N} = \mathcal{R}(\mathcal{N}_0, \varrho)$. For $\mathcal{M}$ and $\mathcal{N}$ to be isomorphic, it is necessary and sufficient that*

(i) *$\varphi_0(e) = \psi_0(f)$, where φ_0 and ψ_0 are the canonical trace of $\mathcal{M}_0$ and $\mathcal{N}_0$ respectively;*

(ii) *There exists a projection p in $\mathcal{M}_0$, an isomorphism σ of $\mathcal{N}_0$ onto $p\mathcal{M}_0 p$ and a partial isometry w in $\mathcal{M}_0$ such that*

$$w\theta \circ \sigma(x) w^* = \sigma \circ \varrho(x), \quad \theta \circ \sigma(x) = w^* \sigma \circ \varrho(x) w, \quad x \in \mathcal{N}_0.$$

5. Examples

In order to obtain more insight into the objects, we consider various examples of factors equipped with homogeneous periodic states. Throughout this sections, we assume that the von Neumann algebras in question have separable preduals. Hence they have faithful normal representations on separable Hilbert spaces.

Let $\mathcal{A}$ be an abelian von Neumann algebra, finite dimensional or infinite dimensional. Let θ_1 be an ergodic automorphism of $\mathcal{A}$. Suppose $\mathcal{A}$ admits a faithful θ_1-invariant normal state μ. This assumption implies that $\mathcal{A}$ is either finite dimensional or isomorphic to $L^\infty(0, 1)$, the algebra of all essentially bounded measurable functions over the unit interval $(0, 1)$ with respect to the Lebesque measure. Let $\mathcal{F}$ be a II_1-factor. Let f be a projection in $\mathcal{F}$ such that $\mathcal{F} \cong f\mathcal{F}f$. Hence $\tau(f)$ is in the fundamental group $\mathfrak{G}(\mathcal{F})$ of $\mathcal{F}$ in the sense of Murray and von Neumann, where τ means the canonical trace of $\mathcal{F}$. Let $\varkappa = \tau(f)$, $0 < \varkappa < 1$. Let θ_p be an isomorphism of $\mathcal{F}$ onto $f\mathcal{F}f$. Now we consider the tensor product $\mathcal{M}_0 = \mathcal{A} \overline{\otimes} \mathcal{F}$ of $\mathcal{A}$ and $\mathcal{F}$. The center $\mathcal{Z}_0$ of $\mathcal{M}_0$ is given by $\mathcal{Z}_0 = \mathcal{A} \otimes 1$. Let $e = 1 \otimes f$, $\varphi_0 = \mu \otimes \tau$ and $\theta = \theta_1 \otimes \theta_2$. In this setting, we have $\dot{\theta} = \theta_1 \otimes \iota$, so that $\dot{\theta}$ is ergodic on the center $\mathcal{Z}_0$. We construct a factor $\mathcal{R}(\mathcal{M}_0, \theta)$, say $\mathcal{M}$, based on $\mathcal{M}_0$ and θ according to the process described in § 2. Let $T = -2\pi/\log \varkappa$. As Theorems 3.4 and 3.5 mention, we conclude the following:

(i) If dim. $\mathcal{A} = n_0$, then $S(\mathcal{M}) = \{\varkappa^{n_0 n} \colon n \in \mathbf{Z}\}$ and $T(\mathcal{M}) = (T/n_0)\mathbf{Z}$;

(ii) If dim. $\mathcal{A} = \infty$, then $S(\mathcal{M}) = \{0, 1\}$ and $T(\mathcal{M}) = \{t \in \mathbf{R} \colon \varkappa^{it}$ is in the point spectrum of $\theta_1\}$. We consider an automorphism $g_0 = \theta_1 \otimes \iota$ of $\mathcal{M}_0$. Obviously g_0 and θ commute and g_0 leaves φ_0 invariant, so that g_0 by Theorem 2.14 extends to an automorphism g of $\mathcal{M}$ with $\varphi \circ g = \varphi$. Since g_0 is ergodic on $\mathcal{Z}_0$, the automorphism group generated by g and modular automorphism group $\{\sigma_t^\varphi\}$ acts ergodically on $\mathcal{M}$, hence $G(\varphi)$ is ergodic on $\mathcal{M}$. Therefore, the periodic state φ is homogeneous. For any given countable subgroup of the torus group $\{\lambda \in \mathbf{C} \colon |\lambda| = 1\}$, there exists an ergodic automorphism θ_1 on $\mathcal{A} = L^\infty(0, 1)$ whose point spectrum is precisely the given group. Hence we have the following:

THEOREM 5.1. *For any countable subgroup G of the additive group $\mathbf{R}$, there exists a factor $\mathcal{M}$ equipped with a periodic homogeneous state such that $S(\mathcal{M}) = \{0, 1\}$ and $T(\mathcal{M}) = G$.*

For the proof, we need to consider an ergodic automorphism of $\mathcal{A} = L^\infty(0, 1)$ whose point spectrum is precisely $\{\varkappa^{it} \colon t \in G\}$.

COROLLARY 5.2. *In the group* Aut $(\mathcal{M})$ *of all automorphisms of a factor $\mathcal{M}$, the group* Int $(\mathcal{M})$ *of all inner automorphisms of $\mathcal{M}$ is not necessarily closed under a topology in* Aut $(\mathcal{M})$ *which makes the function*: $t \in \mathbf{R} \to \sigma_t^\varphi \in$ Aut $(\mathcal{M})$ *continuous.*

Therefore, Int $(\mathcal{M})$ is not necessarily closed under any reasonable topology in Aut $(\mathcal{M})$ except the uniform topology.

We will examine an infinite tensor product of 2×2 matrix algebras. Let $\{M_n \colon n = 1, 2, \ldots\}$ be a sequence of 2×2-matrix algebras. We fix λ, $0 < \lambda < \frac{1}{2}$. Put $\varkappa = \lambda/(1 - \lambda)$. For each $n = 1, 2, \ldots$, we define a state ω_λ^n on M_n by:

$$\omega_\lambda^n\begin{pmatrix}a, & b\\ c, & d\end{pmatrix}=\lambda a+(1-\lambda)\,b.$$

The modular automorphism group σ_t^n of M_n associated with ω_λ is the inner automorphism group induced by the one parameter unitary matrix group in M_n:

$$h_n^{it}=\begin{pmatrix}\lambda^{it}, & 0\\ 0, & (1-\lambda)^{it}\end{pmatrix}\in M_n.$$

It is obvious that each ω_λ^n has the period $T=-2\pi/\log\varkappa$. Let $\mathcal{M}^\lambda$ denote the tensor product $\Pi_{n=1}^\infty\otimes(M_n,\omega_\lambda^n)$ of M_n with respect to the reference states $\{\omega_\lambda^n\}$. Let $\omega_\lambda=\Pi_{n=}^\infty\ \otimes\omega_\lambda^n$. The von Neumann algebra $\mathcal{M}^\lambda$ is considered as the one generated by the image of the infinite C^*-algebra tensor product $\Pi_{n=1}^\infty\hat{\otimes}M_n$ under the cyclic representation induced by ω_λ. R. Powers proved that for $\lambda\neq\lambda'$, $0<\lambda$, $\lambda'<\frac{1}{2}$, $\mathcal{M}^\lambda$ and $\mathcal{M}^{\lambda'}$ are non-isomorphic factors. Let σ_t be the modular automorphism group of $\mathcal{M}^\lambda$ associated with ω_λ. In the following sense, $\{\sigma_t\}$ is the tensor product of σ_t^n:

$$\sigma_t(x_1\otimes\ldots\otimes x_n\otimes 1\otimes\ldots\otimes 1\otimes\ldots)=\sigma_t^1(x_1)\otimes\ldots\otimes\sigma_t^n(x_n)\otimes 1\otimes\ldots.$$

Hence ω_λ has period T. For each $n\geqslant 1$, we define an automorphism g_n of $\mathcal{M}^\lambda$ by

$$g_n(x_1\otimes x_2\otimes\ldots\otimes x_m\otimes 1\otimes\ldots)=x_n\otimes x_1\otimes\ldots\otimes x_{n-1}\otimes x_{n+1}\otimes\ldots\otimes 1x_m\otimes 1\otimes\ldots.$$

Each g_n leaves ω_λ invariant. Hence $g_n\in G(\omega_\lambda)$. Since the fixed points of $\{g_n\colon n=1,2,\ldots,\}$ consist only of scalar multiples of the identity, ω_λ is certainly homogeneous. Let $M^n=M_1\otimes\ldots\otimes M_n$. Then g_n is decomposed as $g_n=\bar{g}_n\otimes\iota$ according to the tensor product decomposition:

$$\mathcal{M}^\lambda=M^n\otimes\prod_{k=n+1}^{\infty}\otimes\{M_k,\omega_\lambda^k\}.$$

Since M^n is certainly a factor of type I (actually type I_{2^n}), $\bar{g}_n$ is inner, so that there exists a unitary $\bar{u}_n\in M^n$ such that $\bar{g}_n(x)=\bar{u}_n x\bar{u}_n^*$, $x\in M^n$. Therefore g_n is inner. In fact, $u_n=\bar{u}_n\otimes 1$ gives rise to the automorphism g_n. Therefore, ω_λ is inner homogeneous. Hence the centralizer $\mathcal{M}_0^\lambda$ of ω_λ is a II_1-factor. We claim that $\mathcal{M}_0^\lambda$ is a hyperfinite II_1-factor. Let M_0^n be the centralizer of ω_λ^n in M^n. Of course, M_0^n is finite dimensional. Identifying M^n and $M^n\otimes 1$, we regard M^n as a subalgebra of $\mathcal{M}^\lambda$. Since the restriction of ω_λ to M^n is $\omega_\lambda^1\otimes\ldots\otimes\omega_\lambda^n$, M_0^n is a subalgebra of $\mathcal{M}_0^\lambda$. We note here that $\varepsilon_0(M^n)=M_0^n$, where ε_0 is the mapping defined by (1.7) with respect to σ_t (hence ω_λ). Let x be an element in $\mathcal{M}_0^\lambda$. There exists a bounded sequence $\{x_j\}$ in $\bigcup_{n=1}^\infty M^n$ converging σ-strongly to x. Since ε_0 is σ-strongly continuous, $\{\varepsilon_0(x_j)\}$ converges σ-strongly to $\varepsilon_0(x)=x$. Since $\varepsilon(x_j)\in\bigcup_{n=1}^\infty M_0^n$ we have proved that $\mathcal{M}_0^\lambda$ is approximated by the union of the increasing sequence $\{M_0^n\}$ of finite dimensional subalgebras. Therefore, Murray and von Neumann's Theorem [14; Theorem XII] implies that $\mathcal{M}_0^\lambda$ is a hyperfinite II_1-factor. Thus, the state ω_λ of $\mathcal{M}^\lambda$ is periodic and inner homogeneous. Therefore, there exists an isomorphism θ_λ of $\mathcal{M}_0^\lambda$ onto $e_\lambda \mathcal{M}_0^\lambda e_\lambda$, where e_λ is a projection in $\mathcal{M}_0^\lambda$ such that $\tau(e_\lambda) = \lambda/(1-\lambda) = \varkappa$ for the canonical trace of $\mathcal{M}_0^\lambda$, such that $\mathcal{M}^\lambda = \mathcal{R}(\mathcal{M}_0^\lambda, \theta_\lambda)$.

Since the explicit construction of θ_λ requires a more precise analysis of the group measure space construction of a hyperfinite II_1-factor, we will publish it elsewhere independently.

6. Remark on the one parameter group ρ_t

In § 4, we introduced a one parameter group ϱ_t of isometries of $\mathcal{M}$ onto $\mathcal{M}$ for two faithful normal states φ and ψ on $\mathcal{M}$. We state here some interesting properties of ϱ_t without proof. Since ϱ_t depends on φ and ψ. we denote it by $\varrho_t^{\varphi,\psi}$ Then $\varrho_t^{\varphi,\psi}$ enjoys the following properties:

(i)
$$\varrho_t^{\varphi,\psi}(xy) = \varrho_t^{\varphi,\omega}(x)\,\varrho_t^{\omega,\psi}(y)$$

for any x, $y \in \mathcal{M}$ and any faithful normal states φ, ψ and ω;

(ii) For any x, $y \in \mathcal{M}$ there exists a bounded function $F(\alpha)$ continuous on and holomorphic in the strip, $0 \leqslant \mathrm{Im}\,\alpha \leqslant 1$, such that

$$F(t) = \varphi(\varrho_t^{\varphi,\psi}(x)\,y),$$
$$F(t+i) = \psi(y\varrho_t^{\varphi,\psi}(x));$$

(iii) $\varrho_t^{\varphi,\psi}$ is a unique one parameter group of transformations on $\mathcal{M}$ such that (ii) holds.

Further analysis of $\varrho_t^{\varphi,\psi}$ will be published elsewhere.

References

[1]. Araki, H., A classification of factors, II. *Publications of Research Institute for Math. Sciences, Kyoto University*, Ser. A4 (1968), 585–593.

[2]. —— Araki, H. & Woods, J., A classification of factors. *Publications of Research Institute for Math. Sciences, Kyoto University*, Ser. A4 (1968), 51–130.

[3]. Connes, A., Un nouvel invariant pour les algèbres de von Neumann. *C. R. Acad. Paris*, Ser. A273 (1971), 900–903.

[4]. —— Calcul des deux invariants d'Araki et Woods par la théorie de Tomita et Takesaki. *C. R. Acad. Paris*, Ser. A274 (1972), 175–177.

[5]. —— Groupe modulairé d'une algèbre de von Neumann de genre denombrable. *C. R. Acad. Paris*, Ser. A274 (1972), 1923–1926.

[6]. —— États présque periodiques sur une algèbre de von Neumann. *C. R. Acad. Paris*, Ser. A274 (1972), 1402–1405.

[7]. —— Letters to the author.

[8]. DIXMIER, J., *Les algèbres d'operateurs dans l'espace hilbertien*. Gauthier-Villars, Paris (1957).

[9]. GRIFFIN, E., Some contributions to the theory of rings of operators. *Trans. Amer. Math. Soc.*, 75 (1953), 471–504.

[10]. HERMAN, R. & TAKESAKI, M., States and automorphism groups of operator algebras. *Comm. Math. Phys.*, 19 (1970), 142–160.

[11]. McDUFF, D., A countable infinity of II_1 factors. *Ann. Math.*, 90 (1969), 361–371.

[12]. —— Uncountably many II_1 factors. *Ann. Math.*, 90 (1969), 372–377.

[13]. MURRAY, F. J. & VON NEUMANN, J., On rings of operators. *Ann. Math.*, 37 (1936), 116–229.

[14]. —— Rings of operators, IV. *Ann. Math.*, 44 (1943), 716–808.

[15]. POWERS, R., Representations of uniformly hyperfinite algebras and their associated von Neumann rings. *Ann. Math.*, 86 (1967), 138–171.

[16]. SAKAI, S., An uncountable number of II_1 and II_∞ factors. *J. Functional Analysis*, 5 (1970), 236–246.

[17]. STØRMER, E., Spectra of states, and asymptotically abelian C^*-algebras. To appear.

[18]. TAKESAKI, M., *Tomita's theory of modular Hilbert algebras and its applications*. Lecture Notes in Mathematics 128, Springer-Verlag, (1970).

[19]. —— States and automorphisms of operator algebras standard representations and the Kubo-Martin-Schwinger boundary condition, *Summer Rencontres in Mathematics and Physics, Battelle Seattle* (1971)., Lecture Notes in Physics 20, Springer-Verlag, 205–246.

[20]. —— Periodic and homogeneous states on a von Neumann algebra, I. To appear in *Bull. Amer. Math. Soc.* 79 (1973), 202–206.

[21]. —— Periodic and homogeneous states on a von Neumann algebra, II. To appear in *Bull. Amer. Math. Soc.*

[22]. TOMITA, M., Standard forms of von Neumann algebras. *Vth Functional analysis Symposium of the Math. Soc. of Japan, Sendai* (1967).

[23]. UMEGAKI, H., Conditional expectation in operator algebra, IV (Entropy and information), *Kōdai Math. Sem. Rep.*, 14 (1962), 59–85.

Received September 21, 1972