THE DETERMINANTAL FORMULA OF SCHUBERT CALCULUS

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As a result of Thom's study of singularities of differentiable mappings there has in recent years been a renewed interest in a result from classical Schubert calculus called the determinantal formula. First I. R. Porteous [9] proved a generalization of an important special case under certain transversality assumptions and later J. N. Damon [1], using the existence of a higher dimensional residue symbol, proved a generalization of the determinantal formula under the same assumptions. Here we shall give a proof of the general form of the determinantal formula without using the residue symbol. Also, recent results on the Cohen-Macaulayness of determinantal varieties allow us to replace the transversality conditions by certain natural codimension conditions.

Our interest in the formula as presented here stems from applications to existence questions of divisors on curves (see [6] and [7]).

The arrangement of the article is as follows: In section one, we introduce the necessary notation and also sketch the proof of the generalized determinantal formula for Schubert schemes. Section two is devoted to the proof of a crucial lemma concerning certain determinants associated to formal power series. In section three we prove the generalized determinantal formula for Schubert schemes. Finally, in section four, the general version of the determinantal formula is proved.

§ 1

Let X be a scheme. If F and G are two locally free sheaves on X, we shall say for short that F is a subsheaf of G and write $F \subseteq G$, whenever F is a subsheaf that is locally a direct summand in G.

Given a sequence $A_1 \subseteq \ldots \subseteq A_d$ of locally free sheaves on X, we denote by $F(A)$ the scheme over X parametrizing flags $C_1 \subset \ldots \subset C_d$ of locally free sheaves $C_i \subseteq A_i$ with rank $(C_i) = i$. More precisely, $F(A)$ is the scheme representing the functor

$T \to \{C_1 \subset \ldots \subset C_d \mid C_i$ locally free of rank i on T and $C_i \subseteq (A)_T\}$, from schemes over X to sets. The universal sequence of locally free sheaves on $F(A)$ we denote by $D_1 \subset \ldots \subset D_d$, where rank $(D_i) = i$. Let E be a locally free sheaf of rank n on X. We denote by $G_d(E)$ the Grassmann bundle over X parametrizing locally free sheaves C of rank d with $C \subseteq E$. The universal sequence of bundles on $G_d(E)$ we denote by

$$O \to D \to \varphi^* E \to Q \to O$$

where φ is the structure morphism $G_d(E) \to X$.

Suppose that $0 < \text{rank}\,(A_i) < \text{rank}\,(A_{i+1})$. Then we denote by $\Omega(A)$ the subscheme of $G_d(E)$ parametrizing locally free sheaves $C \subseteq E$ of rank d such that rank $(A_i \cap C) \geqslant i$ for $i = 1, \ldots, d$. By rank $(A \cap C) \geqslant i$, we mean that the map $\Lambda^{a-i+1} A \to \Lambda^{a-i+1}(E/C)$ induced by the composite map $A \to E \to E/C$ is zero. Let π be the structure morphism $F(A) \to X$. The subsheaf $D_d \subseteq \pi^* E$ defines an X-morphism

$$\alpha\colon F(A) \to G_d(E)$$

such that $\alpha^* D = D_d$. It is easily verified that α maps $F(A)$ onto $\Omega(A)$ and that α is an isomorphism over the open subset of $\Omega(A)$ parametrizing sheaves $C \subseteq E$ of rank d, such that rank $(A_i \cap C) = i$ for $i = 1, \ldots, d$. Moreover α defines a section s of the projection p_2 below

$$\begin{tikzcd}
G_d(E) \times_X F(A) \arrow[r, shift left, "p_2"] \arrow[d, "p_1"] & F(A) \arrow[l, shift left, "s"] \arrow[d, "\pi"] \arrow[dl, "\alpha"'] \\
G_d(E) \arrow[r, "\varphi"] & X
\end{tikzcd}$$

To prove the determinantal formula for $\Omega(A)$ in $G_d(E)$ we first show that $F(A)$ is the zero scheme (via s) of a section of a locally free sheaf on $G_d(E) \times_X F(A)$ of rank $d(n-d)$. The determinantal formula for $F(A)$ in $G_d(E) \times_X F(A)$ is then an immediate consequence of Lemma 1 below and a well known result concerning the relation between the highest Chern class of locally free sheaves and the zeroes of their sections. We then "push down" the determinantal formula for s by p_1 to get the formula for $\Omega(A)$ in $G_d(E)$.

§ 2

Let $\mu_i(t) = \sum_{-\infty}^{\infty} a_{i,j} t^j$ be a formal power series in t for $i = 1, \ldots, d$ (with coefficients in some commutative ring). Given a sequence of integers $f(1), \ldots, f(d)$, define

$$\Delta_{f(1), \ldots, f(d)}(\mu_1, \ldots, \mu_d)$$

to be the determinant of the $d \times d$ matrix $(b_{i,j})$ where

$$b_{i,j} = a_{i,f(i)+j-i}.$$

LEMMA 1. *Assume* $f(1) \leqslant f(2) \leqslant \ldots \leqslant f(d)$, *and let*

$$\mu_i(t) = \prod_{j=1}^{f(i)} (1+\alpha_j t) \Big/ \prod_{j=1}^{i} (1+\beta_j t).$$

Then
$$\Delta_{f(1),\ldots,f(d)}(\mu_1, \ldots, \mu_d) = \prod_{i=1}^{d} \prod_{j=1}^{f(i)} (\alpha_j - \beta_i).$$

Proof. Both sides of the equality of Lemma 1 are homogenous polynomials of degree $\sum_{i=1}^{d} f(i)$ in the α's and β's taken together. Moreover the parts of the homogenous polynomials involving the α's only are both equal to $\prod_{i=1}^{d} \prod_{j=1}^{f(i)} \alpha_j$, as is easily seen by putting the β's equal to zero. Consequently, in order to prove Lemma 1, it is sufficient to show that $\Delta_{f(1),\ldots,f(d)}(\mu_1, \ldots, \mu_d)$ is zero when $\alpha_m = \beta_n$ with $m \leqslant f(n)$.

Let $g = (g_0, \ldots, g_{d-1})$ be the vector whose entries are the coefficients of the polynomial

$$g(t) = \sum_{j=0}^{d-1} g_j t^j = \prod_{i=1}^{d} (1+\beta_i t)/(1+\beta_n t).$$

We shall show that, when $\alpha_m = \beta_n$ with $m \leqslant f(n)$, the vector g lies in the kernel of the linear transformation represented by the matrix $(b_{i,j})$. Since $m \leqslant f(n) \leqslant f(i)$ for $n < i$ we have that $g(t)\mu_i(t)$ is a polynomial of degree strictly less than $(f(i)+d-i)$ for $i=1, \ldots, d$. In particular the coefficient $\sum_{j=1}^{d} g_{d-j} b_{i,j}$ of $t^{f(i)-i+d}$ in $g(t)\mu_i(t)$ is zero. Thus g lies in the kernel of $(b_{i,j})$.

§ 3

We shall in the following assume that the schemes $F(A)$ and $G_d(E)$ over X have a graded intersection theory in the sense of [2] (sections 1, 2 and 3) and that Chern classes exist for locally free sheaves. This will be true, e.g., if X is a quasi-projective and non-singular variety. Indeed $F(A)$ and $G_d(E)$ are smooth and projective over X. Let ψ be a morphism between two such schemes. We denote by ψ^* and ψ_* the inverse and direct-image maps induced by ψ on the corresponding intersection rings. The class of a subscheme Z in the intersection ring we denote by Cycle Z.

Finally, we let

$$c(F) = c_0(F) + c_1(F)t + \ldots$$

denote the Chern polynomial of a locally free sheaf F and we write $c(F-G)$ for $c(F)c(G)^{-1}$.

LEMMA 2. (The determinantal formula for s.) *Consider $F(A)$ as a subscheme of $G_d(E)\times_X F(A)$ via the morphism s. Then we have, in the intersection ring of $G_d(E)\times_X F(A)$,*

$$\text{Cycle } F(A)=\Delta_{(n-d),\ldots,(n-d)}(c(p_1^*Q-p_2^*D_1),\ldots,c(p_1^*Q-p_2^*D_d)).$$

Proof. It is easily verified that s identifies $F(A)$ with the closed subscheme of $G_d(E)\times_X F(A)$ parametrizing pairs (C_1, C_2) of locally free sheaves of rank d such that $p_1^*C_1=p_2^*C_2$. Hence $F(A)$ is the scheme of zeroes of the composite map

$$p_2^*D_d\to p_2^*\pi^*E=p_1^*\pi^*E\to p_1^*Q.$$

That is, $F(A)$ is the scheme of zeroes of a section of the rank $d(n-d)$ bundle $p_2^*D_d\otimes p_1^*Q^\vee$. Since $F(A)$ is of codimension $d(n-d)$ in $G_d(E)\times_X F(A)$ we have by [3] (formula 18 bis or corollary page 153) that

$$\text{Cycle } F(A)=c_{d(n-d)}(p_2^*D_d^\vee\otimes p_1^*Q).$$

Using "the splitting principle" we can write $c_{d(n-d)}(p^*D_d^\vee\otimes p_1^*Q)$ as the coefficient of $t^{d(n-d)}$ in the polynomial

$$\prod_{i=1}^{d}\prod_{j=1}^{n-d}(1+c_1(L_i^\vee\otimes Q_j)t)$$

where Q_j is a "splitting" of Q and $L_j, j=1,\ldots,i$ is a "splitting" of D_i.

Hence
$$\text{Cycle } F(A)=\prod_{i=1}^{d}\prod_{j=1}^{n-d}(c_1(Q_j)-c_1(L_i)).$$

We immediately conclude from Lemma 1 with $f(i)=(n-d)$ that

$$\text{Cycle } F(A)=\Delta_{(n-d),\ldots,(n-d)}(\mu_1,\ldots,\mu_d),$$

where
$$\mu_i(t)=\prod_{j=1}^{n-d}(1+c_1(Q_j)t)\Big/\prod_{j=1}^{i}(1+c_1(L_j)t).$$

Since $\mu_i(t)=c(Q)/c(D_i)$, Lemma 2 is proved.

To "push down" the formula of Lemma 2 to $G_d(E)$ we first note that the structure morphism

$$\pi\colon F(A)\to X$$

can be decomposed into a sequence of morphisms of projective bundles (the Grassmann bundle $G_1(E)$ is canonically isomorphic to the projective bundle $\mathbf{P}(E^\vee)$) in the following way: Let $G_0=X$, $D_0=0$ and $\varphi_0=id_X$. We define G_j together with a structure morphism

$$\pi_j\colon G_j\to X$$

and a locally free subsheaf D'_j of $\pi_j^* A_j$ of rank j inductively by $G_j = G_1(\pi_{j-1}^* A_j / D'_{j-1})$. Let φ_j be the structure morphism $G_j \to G_{j-1}$ and let

$$0 \to L_j \to \pi_{j-1}^* A_j / D'_{j-1} \to Q_j \to 0$$

be the universal sequence of locally free sheaves on G_j. Define π_j to be $\pi_{j-1}\varphi_j$ and D'_j to be the preimage of L_j by the quotient map

$$\pi_j^* A_j \to \pi_j^* A_j / D'_{j-1}.$$

Denote the map $$\varphi_{j+1} \cdots \varphi_{d-1}\varphi_d \colon G_d \to G_j$$

by ψ_j. Then it is clear that the flag

$$\psi_1^* D'_1 \subset \psi_2^* D'_2 \subset \ldots \subset \psi_d^* D'_d$$

on G_d allows us to identify G_d with $F(A)$ such that $\psi_j^* D'_j = D_j$ and $\pi_d = \psi_0 = \pi$.

$$\begin{tikzcd} F(A) \arrow[r, "\psi_j"] \arrow[d, "\pi"'] & G_j \arrow[d, "\varphi_j"] \\ X & G_{j-1} \arrow[l, "\pi_{j-1}"'] \end{tikzcd}$$

Since each G_j is smooth over G_{j-1} of relative dimension $(a_j - j)$, we conclude in particular that $F(A)$ is smooth over X of relative dimension $\sum_{i=1}^{d} (a_i - i)$.

LEMMA 3. *With the above notation we have*

$$(\varphi_i)_*(c(Q_i)) = t^{a_i - i}.$$

Proof. For $j > (a_i - i)$, we already have that $c_j(Q_i)$ is zero because rank $Q_i = (a_i - i)$. Since $(\varphi_i)_*$ preserves dimension, we have $(\varphi_i)_*(c_j(Q_i)) = 0$ for $j < (a_i - i)$.

Let $j = (a_i - i)$. Then it is sufficient to check the formula $(\varphi_i)_*(c_j(Q_i)) = 1$ at a generic point P of G_{i-1}. However, in the fiber over P, the Chern class $c_j(Q_i)$ is represented by a (rational) point and the formula is clear.

LEMMA 4. *Let $R_1, \ldots, R_d$ be locally free sheaves on X and $f(1), \ldots, f(d)$ integers. Then*
$\pi_* \Delta_{\ldots f(i) \ldots}(\ldots c(\pi^* R_i - D_i) \ldots) = \Delta_{\ldots f(i) - a(i) + i \ldots}(\ldots c(R_i - A_i) \ldots)$
where $a(i) = \text{rank } A_i$.

Proof. We shall prove the equation

$$(\psi_{i-1})_* \Delta_{\ldots f(j) \ldots}(\ldots c(\pi^* R_j - D_j) \ldots)$$
$$= \Delta_{\ldots f(i-1), f(i) - a(i) + i \ldots}(\ldots c(\pi_{i-1}^* R_{i-1} - D'_{i-1}), c(\pi_{i-1}^* R_i - \pi_{i-1}^* A_i) \ldots)$$

by descending induction on i. Apply $(\varphi_{i-1})_*$ to the equation. Then by the projection formula the only terms that are "seriously" affected are the coefficients of

$$c(\pi_{i-1}^* R_{i-1} - D'_{i-1}) = c(Q_{i-1})\, c(\pi_{i-1}^* R_{i-1} - \pi_{i-1}^* A_{i-1})$$

and these appear only in the $(i-1)$'st row of "the" $d \times d$-matrix whose (j, k)'th entry is the coefficient of $t^{f(j)-j+k}$ in $c(\pi_{i-1}^* R_j - \varphi_{i-1,j}^* D'_j)$ for $j \leqslant i-1$ and in $c(\pi_{i-1}^* R_j - \pi_{i-1}^* A_j)$ for $j \geqslant i$, where we have put $\varphi_{i-1,j} = \varphi_{i-1}\varphi_{i-2} \cdots \varphi_{j+1}$ for $j < i-1$, and $\varphi_{i,i-1}$ is the identity on G_i. However, by Lemma 2 and the projection formula,

$$(\varphi_{i-1})_*\Big(\sum_{h=j+k} c_j(\pi_{i-1}^* R_{i-1} - \pi_{i-1}^* A_{i-1})\, c_k(Q_{i-1})\Big)$$
$$= c_{h-a(i-1)+i-1}(\pi_{i-2}^* R_{i-1} - \pi_{i-2}^* A_{i-1}).$$

We immediately conclude, using the projection formula, that

$$(\varphi_{i-1})_* \Delta_{\ldots f(i-1), f(i)-a(i)+i\ldots}(\ldots c(\pi_{i-1}^* R_{i-1} - D'_{i-1}), c(\pi_i^* R_i - \pi_i^* A_i)\ldots)$$
$$= \Delta_{\ldots f(i-2), f(i-1)-a(i-1)+i-1\ldots}(\ldots c(\pi_{i-2}^* R_{i-2} - D'_{i-2}), c(\pi_{i-2}^* R_{i-1} - \pi_{i-2}^* A_{i-1})\ldots).$$

Lemma 4 is now the case $i = 1$.

THEOREM 5. *Put $h(i) = (n-d-a(i)+i)$ where $a(i) = \text{rank } A_i$. Then $\Omega(A)$ is of codimension $\sum_{i=1}^d h(i)$ in $G_d(E)$, and*

$$\text{Cycle } \Omega(A) = \Delta_{h(1),\ldots,h(d)}(c(Q - \varphi^* A_1), \ldots, c(Q - \varphi^* A_d)),$$

Proof. We have seen that $F(A)$ is irreducible of relative dimension $\sum_{i=1}^d (a(i)-i)$ over X when X is irreducible. Moreover, α: $F(A) \to \Omega(A)$ is an isomorphism over an open dense subset of $\Omega(A)$. Hence $\dim_X \Omega(A) = \sum_{i=1}^d (a(i)-i)$ and

$$(p_1)_* \text{ Cycle } F(A) = \text{Cycle } \Omega(A).$$

Now apply $(p_1)_*$ to the formula of Lemma 2. We obtain the equation

$$\text{Cycle } \Omega(A) = (p_1)_* \Delta_{\ldots(n-d)\ldots}(\ldots c(p_1^* Q - p_2^* D_i)\ldots).$$

The right hand side of the equation is equal to

$$\Delta_{\ldots h(i)\ldots}(\ldots c(Q - \varphi^* A_i)\ldots)$$

by Lemma 4 applied to p_1 (that is, put X and $F(A)$ equal to $G_d(E)$ and $G_d(E) \times_X F(A)$), and with $R_i = p_2^* Q$ and $f(i) = (n-d)$.

COROLLARY 6. (Porteous' formula). *Let $A \subseteq E$ be a locally free sheaf of* rank a. *Denote* *by* $\sigma_k(A)$ *the subscheme of* $G_d(E)$ *parametrizing locally free sheaves* $C \subseteq E$ *of* rank d *such that* rank $(C \cap A) \geqslant k$. *Then* $\sigma_k(A)$ *is of codimension* $(n-d-a+k)k$ *and*

$$\text{Cycle}\,(\sigma_k(A)) = \Delta_{\ldots(n-d-a+k)\ldots}(\ldots c(Q-\varphi^*A)\ldots)$$

where the index $(n-d-a+k)$ *is repeated* k *times.*

Proof. By "the splitting principle" we can find a map $\beta\colon F \to X$ from a scheme F such that there exist bundles

$$A_1 \subset A_2 \subset \ldots \subset A_{k-1} \subset \beta^*A \subset A_{k+1} \subset \ldots \subset A_d = \beta^*E$$

on F with rank $(A_j) = (a+j-k)$ for $j = 1, \ldots, k-1$ and rank $A_j = (n+j-d)$ for $j = (k+1), \ldots, d$ and such that β^* is injective. Since the condition rank $(A \cap C) \geqslant k$ for a locally free sheaf C clearly implies that rank $(A_i \cap C) \geqslant i$ for all i, we have

$$q^*\,\text{Cycle}\,\sigma_k(A) = \text{Cycle}\,\Omega(A)$$

in the intersection ring of $F \times_X G_d(E)$, where q is the projection onto the second factor. Now apply Theorem 5 to the projection r onto the first factor. We obtain

$$q^*\,\text{Cycle}\,\sigma_k(A) = \Delta_{\ldots h(i)\ldots}(\ldots c(q^*Q-(\varphi q)^*A_i)\ldots)$$

where $h(i) = (n-d-a+k)$ for $i = 1, \ldots, k-1$ and $h(i) = 0$ for $i = (k+1), \ldots, d$. Since q^* is injective we obtain the desired relation in the intersection ring of $G_d(E)$.

§ 4

In order to prove the determinantal formula for "sufficiently general matrices", we need some well known results from commutative algebra. We were, however, unable to find convenient references and shall therefore both state and prove the results below.

Let A be a (noetherian) ring, M an A-module, and I an ideal of A. We denote by depth(I, M) the length of a maximal M-regular sequence contained in I.

LEMMA 7. *Let*

$$M.\colon 0 \to M_0 \to \ldots \to M_n \to 0$$

be a sequence of (finitely generated) free A-modules such that depth $(I, H^i(M.))$ *is zero for all* i. *Then* $H^i(M.)$ *is zero for* $i <$ depth(I, A).

Proof. We proceed by induction on depth (I, A). Let x in I be a regular element of A. Put $\overline{M}. = M. \otimes_A A/(x)$. The short exact sequence

$$0 \to M. \to M. \to \overline{M}. \to 0$$

gives rise to a long exact sequence

$$\dots \to H^i(M.) \xrightarrow{x} H^i(M.) \to H^i(\bar{M}.) \to \dots$$

which shows that depth $(I/(x), H^i(M.))$ is zero for all i. By the induction assumption, $H^i(\bar{M}.) = 0$ for $i <$ (depth $(I, A) - 1$) and consequently, multiplication by x,

$$H^i(M.) \xrightarrow{x} H^i(M.),$$

is injective for $i <$ depth (I, A). Since depth $(I, H^i(M.))$ is zero it follows that $H^i(M.) = 0$.

COROLLARY 8. *Let f: $B \to A$ be a homomorphism of rings. Given a B-module N with support in the variety $V(J)$ of an ideal J of B. Moreover, let*

$$0 \to N_0 \to \dots \to N_n \to N \to 0$$

be a free resolution of N.

Assume that $n =$ depth (I, A) where I is the ideal in A generated by $f(J)$. Then

$$0 \to N_0 \otimes_B A \to \dots \to N_n \otimes_B A \to N \otimes_B A \to 0$$

is a free resolution of $N \otimes_B A$.

Proof. Put $M_i = N_i \otimes A$. Since Supp $N \subset V(J)$ we have Supp $(H^i(M.)) \subset V(I)$ and consequently depth $(I, H^i(M.)) = 0$. We conclude from Lemma 7 that $H^i(M.) = 0$ for $i = 0, \dots, (n-1)$. Corollary 8 follows immediately.

LEMMA 9. *Let f: $X \to Y$ be a morphism of regular schemes and let Z be an equidimensional closed Cohen-Macaulay subscheme of Y. If the intersection theoretic inverse image of Z by f is defined, then*

$$f^* \text{ Cycle } Z = \text{Cycle } f^{-1}Z,$$

and the scheme theoretic invers image $f^{-1}Z$ of Z is Cohen-Macaulay.

Proof. Recall that the intersection theoretic inverse image of Z by f is said to be defined if $f^{-1}Z$ is empty or if

$$\operatorname{codim}(f^{-1}Z, X) = \operatorname{codim}(Z, Y).$$

Assume that $f^{-1}Z$ is nonempty. Let p be a point in X and $q = f(p)$. Put $B = O_{Y,q}$, $A = O_{X,p}$ and $C = O_{Z,q}$. Let J denote the ideal defining Z at q and I the image of J by the map $B \to A$ corresponding to f.

By our dimension assumptions we have

$$\text{depth }(I, A) = \text{depth }(J, B),$$

and since C is Cohen-Macaulay, the homological dimension of the B-module C is equal to depth (J, B). Hence Corollary 8 with $N=C$ implies firstly that $\operatorname{Tor}_i^B(A, C)=0$ for $i>0$, which is the first part of Lemma 9, and secondly that the homological dimension of the A-module $C\otimes_B A$ is at most equal to depth (I, A). The second fact clearly implies that $C\otimes_B A$ is Cohen-Macaulay and finishes the second part of Lemma 9.

THEOREM 10. (The determinantal formula with denominators). *Let X be a nonsingular quasi-projective variety over a field and let u: $F\to G$ be a map between two locally free sheaves on X of* rank *p and q. Moreover, let $A_1\subset\ldots\subset A_d$ be a sequence of locally free subsheaves of F with $d\geqslant(p-q)$ and put $a(i)=$*rank *A_i. Denote by $Z(A)$ the subscheme of X of zeroes of the maps $(i=1, \ldots, d)$*

$$\Lambda^{a(i)-i+1}A_i\to\Lambda^{a(i)-i+1}G$$

induced by u. Then:

(i) *$Z(A)$ is either empty or of codimension at most $\sum_{i=1}^d h(i)$ in X where $h(i)=(q-a(i)+i)$.*
(ii) *If $Z(A)$ is empty or of codimension exactly $\sum_{i=1}^d h(i)$ in X, then,*

$$\text{Cycle } Z(A) = \Delta_{h(1),\ldots,h(d)}(c(G-A_1), \ldots, c(G-A_d)).$$

Proof. Put $E=F\oplus G$. The projection $E\to G$ defines an X morphism

$$\delta\colon X\to G_p(E)$$

such that $\delta^*Q=G$ where Q is the canonical bundle on $G_p(E)$.

By the splitting principle we may assume that there is a sequence of locally free sheaves $A_d\subset A_{d+1}\subset\ldots\subset A_p=E$ on X (we consider F as a subsheaf of E via the graph of u) with rank $A_j=(q+j)$ for $j=(d+1), \ldots, p$. Then $\Omega(A)$ is defined in $G_p(E)$ and, by definition, we have

$$\delta^{-1}\Omega(A)=Z(A).$$

By Theorem 5, the codimension of $\Omega(A)$ in $G_p(E)$ is $\sum_{i=1}^d(p+q-p-a(i)+i)=\sum_{i=1}^d(q-a(i)+i)$. Hence if $Z(A)$ is nonempty we have that the codimension of $Z(A)$ in X is at most $\sum_{i=1}^d h(i)$. Moreover, J. A. Eagon and M. Hochster have proved that $\Omega(A)$ is a Cohen-Macaulay scheme (see [4] Theorem 1, p. 1022 and Corollaries, [5], [6] Theorem 2, p. 7, or [7] Theorem 1, p. 1). Hence, if $Z(A)$ is empty or if codim $(\Omega(A), G_p(E))=$ codim$(Z(A), X)$, we have by Lemma 7 that

$$\delta^* \text{ Cycle } \Omega(A) = \text{Cycle } Z(A).$$

Theorem 8 is now an immediate consequence of Theorem 5. (Note that $h(i)$ is zero for $i=(d+1), \ldots, p$.)

The general version of Porteous' formula can now be obtained from Theorem 10 in the same way Corollary 6 followed from Theorem 5, or we can proceed as in the proof of Theorem 10 and conclude the formula from Corollary 6.

COROLLARY 11. (Porteous' formula). *Let X and u be as in Theorem* 10 *and let* $(p-q) \leqslant k \leqslant p$. *Denote by* $Z_k(u)$ *the scheme of zeroes of the map* $\Lambda^{p-k+1}u$. *Then:*

(i) $Z_k(u)$ *is either empty or of codimension at most* $k(q-p+k)$ *in* X.

(ii) *If* $Z_k(u)$ *is empty or of codimension exactly* $k(q-p+k)$ *in* X *then,*

$$\text{Cycle } Z_k(u) = \Delta_{\dots(q-p+k)\dots}(\dots c(G-F)\dots)$$

where the index $(q-p+k)$ *is repeated* k *times.*

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