A NOTE ON THE CAPILLARY PROBLEM

BY

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The purpose of this note is to extend the result of Theorem 3 in the preceding paper [2] to a configuration not amenable to the methods of that reference.

§ 1

Consider an open set $\mathcal{N}$ in Euclidean n-space, bounded in part by a surface Σ of class $C^{(2)}$ whose mean curvature H^{Σ} does not change sign; the sign of H^{Σ} is chosen to be positive when the curvature vector points into $\mathcal{N}$. As in [2], we use the symbols Σ and $\mathcal{N}$ to denote also the area and volume of these sets.

Let $u(\mathbf{x})$ be a solution in $\mathcal{N}$ of

$$\operatorname{div}\left(\frac{1}{W}\nabla u\right) = nH(\mathbf{x}), \qquad W^2 = 1 + |\nabla u|^2, \tag{1}$$

where $H(\mathbf{x})$ is prescribed in $\mathcal{N}$, and set

$$\mathbf{T}u = \frac{1}{W}\nabla u \tag{2}$$

in $\mathcal{N}$. Denote by $\boldsymbol{\nu}$ the exterior directed unit normal on Σ. We observe first that *if $H(\mathbf{x})$ is bounded on one side, then for any open subset $\Sigma^* \subset \Sigma$, the quantity $\int_{\Sigma^*} \mathbf{T}u \cdot \boldsymbol{\nu}\, d\sigma$ is uniquely defined as a limit of integrals over surfaces converging in any uniform way to Σ^* from within $\mathcal{N}$.* To see this, it suffices to suppose $H > 0$ and to show the result when Σ^* is the part of Σ lying interior to a (small) sphere S centered on Σ, so that Σ^* and a part $S^* \subset S$ together bound $\mathcal{N}^* \subset \mathcal{N}$. Integrating (1) by parts in the subregion cut off by an approximating surface Γ^*, and passing to the limit as $\Gamma^* \to \Sigma^*$, we find

$$\lim_{\Gamma^* \to \Sigma^*} \int_{\Gamma^*} \mathbf{T}u \cdot \boldsymbol{\nu}\, d\sigma = \int_{N^*} H(\mathbf{x})\, d\mathbf{x} - \int_{S^*} \mathbf{T}u \cdot \boldsymbol{\nu}\, d\sigma.$$

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Here the first term on the right exists because $H>0$, the second exists because $|\mathbf{T}u\boldsymbol{\nu}|<1$; thus, $\int_{\Sigma^*}\mathbf{T}u\boldsymbol{\nu}\,d\sigma$ is defined by the term on the left.[1]

With this definition in mind we shall prove the following result:

THEOREM 1. *Let $u(\mathbf{x})$ be a bounded solution of (1) in $\mathcal{N}$. If $H^{\Sigma}\geqslant(n/(n-1))H_0\geqslant 0$ and $H(\mathbf{x})\leqslant H_0$ in $\mathcal{N}$, then $\int_{\Sigma^*}\mathbf{T}u\cdot\boldsymbol{\nu}\,d\sigma<\Sigma^*$ on any open subset $\Sigma^*\subset\Sigma$. If $|H^{\Sigma}|\leqslant(n/(n-1))H_0$ and $H(\mathbf{x})\geqslant H_0$ in $\mathcal{N}$, then $\int_{\Sigma}\mathbf{T}u\cdot\boldsymbol{\nu}\,d\sigma>-\Sigma^*$ on any open subset $\Sigma^*\subset\Sigma$.*

Remark. In [2] it was assumed, chiefly for reasons of notational simplicity, that $\mathbf{T}u\boldsymbol{\nu}$ defines, as a limit from within $\mathcal{N}$, a function $\cos\gamma$ almost everywhere on Σ. Under this additional hypothesis Theorem 1 states that there is no bounded solution in $\mathcal{N}$ for which $\cos\gamma=1$ (resp. $\cos\gamma=-1$) on any open subset of Σ. It is in this sense that Theorem 1 completes the result of Theorem 3 in [2].

Theorem 1 has a geometrical content, which one sees by noting that $H(\mathbf{x})$ is the mean curvature of the surface $\mathcal{S}$ defined by $u(\mathbf{x})$, that $((n-1)/n)H^{\Sigma}$ is the mean curvature of the (vertical) cylindrical wall Z over Σ, and that $\mathbf{T}u\cdot\boldsymbol{\nu}]_{\Sigma}$ is the cosine of the angle γ between $\mathcal{S}$ and Z at points of contact. Thus, under the given hypotheses, the theorem restricts the ways in which $\mathcal{S}$ can (in a limiting sense) be tangent to Z.

Theorem 1 is best possible in a number of ways. It is false for unbounded solutions, as follows from a construction due to Spruck [5], which we shall also use in the proof. It is false if $H(\mathbf{x})>((n-1)/n)H^{\Sigma}$ (respectively $H(\mathbf{x})<((n-1)/n)H^{\Sigma}$), as follows from the example of § 2.3 in [2]. If $H(\mathbf{x})<H^0<((n-1)/n)H^{\Sigma}$ (respectively $H(\mathbf{x})>H^0>((n-1)/n)|H^{\Sigma}|$), then there even exists γ_0, $0<\gamma_0<\pi$, so that there is no solution $u(\mathbf{x})$ in $\mathcal{N}$ for which $\lim_{\mathbf{x}\to\Sigma}\mathbf{T}u\cdot\boldsymbol{\nu}\geqslant\cos\gamma_0$ (respectively $\lim_{\mathbf{x}\to\Sigma}\mathbf{T}u\cdot\boldsymbol{\nu}\leqslant\cos\gamma_0$), see Theorem 3 in [2]. We show later by example that in the class of all bounded solutions, no such γ_0 can be found under the present hypotheses; nevertheless, we do show (Theorem 2) that if $|u(\mathbf{x})|<M$ in $\mathcal{N}$, then there does exist a γ_0, depending on M, thus strengthening Theorem 1 for any class of equi-bounded solutions of (1).

The underlying heuristic content of Theorem 1 is that if two surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$ are tangent at a common point $\mathbf{p}$, if the mean curvature vector $\mathbf{h}_2$ of $\mathcal{S}_2$ at $\mathbf{p}$ is either directed oppositely to the corresponding vector $\mathbf{h}_1$ of $\mathcal{S}_1$ or satisfies $|\mathbf{h}_2|<|\mathbf{h}_1|$, then $\mathcal{S}_2$ cannot lie on the side of $\mathcal{S}_1$ into which $\mathbf{h}_1$ points. This much is geometrically evident. If $|\mathbf{h}_2|=|\mathbf{h}_1|$, then nothing can be said unless information is known on these vectors in an entire neighborhood of $\mathbf{p}$; however, it is still easy to give conditions that lend themselves to analysis. A related problem for which it is less evident what happens appears when the surfaces

[1] We note that since $|\mathbf{T}u\cdot\boldsymbol{\nu}|<1$, there holds $\int_{\mathcal{N}^*}H(\mathbf{x})\,d\mathbf{x}<\infty$ whenever a solution of (1) can be defined in $\mathcal{N}^*$.

have a manifold Γ of contact that serves as a boundary for one of them (say S_2) so that S_2 is not known to be continuable across Γ as a surface satisfying the hypotheses. It is this type of situation that we discuss here, for the special case in which S_1 is a cylinder and S_2 projects simply onto the base of S_1.

The problem has appeared previously, incidental to other investigations; the result of Theorem 1 was given by Jenkins and Serrin [3, Lemma 3] for minimal surfaces, later—with the same method of proof—by Spruck [5, Lemma 4.2] for surfaces of constant mean curvature, in the case $n=2$ and under a hypothesis that the intersection manifold Γ is continuous over the base plane.

The conceptually interesting new feature of the present note is that no hypothesis at all is introduced on Γ; in fact, the surfaces are not explicitly required to contact. Thus, assuming only that $u(\mathbf{x})$ is a bounded solution of (1) in $\mathcal{N}$, and that Σ satisfies the (necessary) curvature inequality, one can infer information on the angle of contact (defined in a limiting sense) between the two surfaces. This is the natural setting for the capillary problem, which is the physical motivation for this note and for the two papers appearing with it. In that problem, a fluid surface is to be determined by the condition that it makes a prescribed angle with bounding walls; no reference to boundary curves appears in the problem's formulation.

Before proving Theorem 1, we derive a preliminary result:

LEMMA 1. *Let $\mathcal{N}$ and Σ be as above, with $H^\Sigma \geqslant (n/(n-1))H_0$. Suppose there is a point $\mathbf{p}\in\Sigma$ at which $H^\Sigma(\mathbf{p}) > (n/(n-1))H_0$. Then there exists (locally) a surface Υ of constant mean curvature H_0^Υ, such that $\mathbf{p}\in\Upsilon$, $H^\Sigma(\mathbf{p}) > H_0^\Upsilon > (n/(n-1))H_0$, and $\Upsilon \cap \mathcal{N} = \phi$.*

Proof. We adopt a coordinate frame with $\mathbf{p}$ at the origin, so that the x_n axis is normal to Σ at $\mathbf{p}$ and points into $\mathcal{N}$, and so that the other coordinate axes coincide with the principal directions on Σ.

Denote by $\lambda_1, \ldots, \lambda_{n-1}$ the principal curvatures of Σ at $\mathbf{p}$. Thus, $\Sigma_1^{n-1}\lambda_j = (n-1)H^\Sigma(\mathbf{p})$. For any H_0^Υ in the indicated range, set $\varepsilon = H^\Sigma(\mathbf{p}) - H_0^\Upsilon > 0$, $\eta_i = \lambda_i - \varepsilon$. Then $\eta_i < \lambda_i$, $i=1,\ldots,n-1$, and $\Sigma_1^{n-1}\eta_i = (n-1)H_0^\Upsilon$. We consider the initial value problem for the equation

$$\sum_{i=1}^{n-1}\left(\frac{1}{W}\varphi_{x_i}\right)_{x_i} = (n-1)H_0^\Upsilon \qquad W^2 = 1 + |\nabla\varphi|^2, \tag{3}$$

with initial data

$$\begin{aligned} \varphi(x_1, \ldots, x_{n-2}, 0) &= \frac{1}{2}\sum_1^{n-2}\eta_i x_i^2 \\ \frac{\partial\varphi}{\partial x_{n-1}}(x_1, \ldots, x_{n-2}, 0) &= 0 \end{aligned} \tag{4}$$

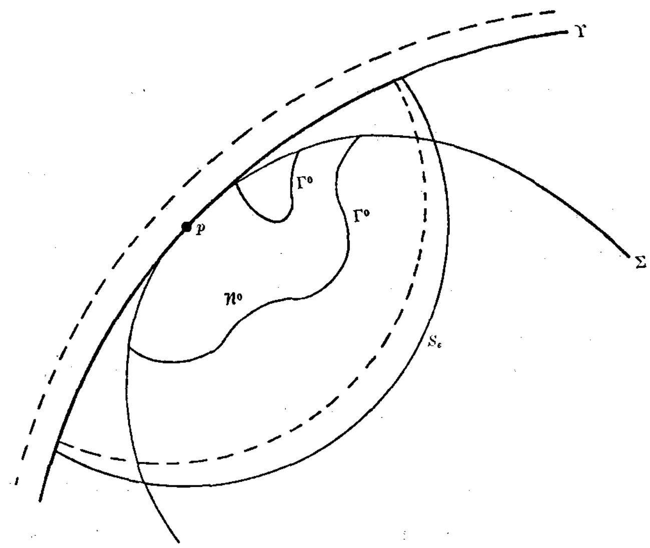

Fig. 1

on the hyperplane $x_{n-1}=0$. The existence of a solution of (3, 4) in a neighborhood of $\mathbf{x}=\{x_j\}=0$ is assured by the Cauchy-Kowaleski theorem [4].

At $\mathbf{x}=0$ there holds $(\partial/\partial x_j)\varphi=0$, $j=1, ..., n-1$, and $(\partial^2/(\partial x_i \partial x_j))\varphi=0$ if $i \neq j$. Thus, for $j=1, ..., n-2$, η_j is the jth principal curvature of the surface Υ: $\varphi(\mathbf{x})$ at $\mathbf{x}=0$. But by (3), H_0^Υ is the mean curvature of Υ. Hence η_{n-1} is the $(n-1)$th principal curvature at $\mathbf{x}=0$; in some neighborhood of $x=0$, the surface Υ satisfies the requirements of the lemma.

Proof of Theorem 1. It suffices to consider the case $H^\Sigma \geqslant (n/(n-1))H_0 \geqslant 0$, as the other is analogous. We suppose the theorem false, and restrict attention to the given open subset, which we again denote by Σ.

If $H^\Sigma \equiv (n/(n-1))H_0$, we identify Σ with Υ and select an arbitrary point $\mathbf{p}$ on this surface. Otherwise there is a point $\mathbf{p} \in \Sigma$ at which $H^\Sigma(\mathbf{p}) > (n/(n-1))H_0$. By Lemma 1, there is a surface Υ through $\mathbf{p}$ and exterior to $\mathcal{N}$, such that H^Υ is constant and satisfies $H^\Sigma(\mathbf{p}) > H^\Upsilon > (n/(n-1))H_0$. Let $S_\varepsilon(\mathbf{p})$ be a sphere of radius ε centered at $\mathbf{p}$ and bounding, with part of Υ, a region $\mathcal{M}^*$ with $\mathcal{N}^* = \mathcal{M}^* \cap \mathcal{N} \neq \phi$ (see Fig. 1). We denote the two parts of the boundary of $\mathcal{N}^*$ by $\Sigma^* \subset \Sigma$ and by $\Gamma^* \subset S_\varepsilon(\mathbf{p})$, and define $u(\mathbf{p}) = \limsup_{\mathbf{x}\to\mathbf{p}} u(\mathbf{x})$, $\mathbf{x} \in \mathcal{N}$.

We use in a basic way a theorem of Spruck [5, § 9]: *if ε is sufficiently small, there exists a solution $v(\mathbf{x})$ of*

$$\operatorname{div}\left(\frac{1}{W}\nabla v\right) = (n-1)H^\Upsilon, \qquad W^2 = 1 + |\nabla v|^2 \tag{5}$$

in $\mathcal{M}^*$, such that $\lim_{x\to\Gamma^*} v(\mathbf{x}) = 0$ *and* $\lim_{x\to\Upsilon} v(\mathbf{x}) = +\infty$.

We proceed in several steps:

(i) Let $\hat{v} = v - c$, where c is chosen so that on Γ^*, $\hat{v} < glb_{x \in N^*} u(\mathbf{x})$. Note that $\hat{v}$ is again a solution of (1) in $\mathcal{N}^*$, and $\hat{v}(\mathbf{p}) = \infty$.

(ii) Let $\hat{v}_\delta(\mathbf{x}) = \hat{v}(\mathbf{x} + \boldsymbol{\nu}\delta)$, where $\boldsymbol{\nu}$ is the unit exterior normal to Σ at $\mathbf{p}$, and choose δ sufficiently small that there still holds $\hat{v}_\delta(\mathbf{p}) > u(\mathbf{p})$. $\hat{v}_\delta(\mathbf{x})$ is defined in a region $\mathcal{N}_\delta^*$ obtained by displacing $\mathcal{N}^*$ a distance δ along $\boldsymbol{\nu}$.

(iii) The function $w(\mathbf{x}) = \hat{v}_\delta(\mathbf{x}) - u(\mathbf{x})$ is defined in $\mathcal{N}_\delta^* \cap \mathcal{N}^*$. Denote by $\mathcal{N}^0$ the open component in $\mathcal{N}_\delta^*$ having $\mathbf{p}$ as a boundary point, and in which $w > 0$. $\mathcal{N}^0$ is bounded by $\Sigma^0 \subset \Sigma^*$ and by a closed set $\Gamma^0 \subset \mathcal{N}_\delta^* \cap \mathcal{N}^*$ (see Fig. 1).

(iv) We now consider the formal identity

$$\int_{\mathcal{N}^0} w \left[\operatorname{div}\left(\frac{1}{W} \nabla \hat{v}_\delta\right) - \operatorname{div}\left(\frac{1}{W} \nabla u\right)\right] d\mathbf{x} = \int_{\Sigma^0} w [\mathbf{T}\hat{v}_\delta - \mathbf{T}u] \cdot \boldsymbol{\nu}\, d\sigma$$

$$- \int_{\mathcal{N}^0} \left(\frac{1}{W} \nabla \hat{v}_\delta - \frac{1}{W} \nabla u\right) \cdot \nabla w \, dx. \tag{6}$$

Here the integral over Σ^0 is to be understood as a limit of integrals over surfaces approximating Σ^0 from within $\mathcal{N}^0$, and the integral over Γ^0 does not appear since $w = 0$ on that set.(1)

From the construction of $w(\mathbf{x})$, we find that the integral on the left in (6) is non-negative. The integrand in the other integral over $\mathcal{N}^0$ can be expressed as a positive definite quadratic form in the components of ∇w; thus this integral is positive.

To study the integral over Σ^0, we choose as approximating surfaces a family Σ_n^0 that, in suitable local parameters, tends to Σ^0 together with the first order derivatives of the position vector. The convergence will then also be in area, and we find, since $|\mathbf{T}u| < 1$, that if the element $d\sigma_n \subset \Sigma_n^0$ corresponds to $d\sigma \subset \Sigma^0$, then

$$\oint_{\Sigma_n^0} \mathbf{T}u \cdot \boldsymbol{\nu}\, d\sigma_n = \oint_{\Sigma^0} (\mathbf{T}u \cdot \boldsymbol{\nu})_{\Sigma_n^0}\, d\sigma + \varepsilon_n$$

where $\varepsilon_n \to 0$ as $n \to \infty$.

By definition, $\lim_{n\to\infty} \oint_{\Sigma_0^n} \mathbf{T}u \cdot \boldsymbol{\nu}\, d\sigma_n = \oint_{\Sigma^0} \mathbf{T}u \cdot \boldsymbol{\nu}\, d\sigma$, and since by hypothesis $\oint_{\Sigma^*} \mathbf{T}u \cdot \boldsymbol{\nu}\, d\sigma$ Σ^* and $\Sigma^0 \subset \Sigma^*$, there follows, using again $|\mathbf{T}u| < 1$, $\oint_{\Sigma^0} \mathbf{T}u \cdot \boldsymbol{\nu}\, d\sigma = \Sigma^0$. Thus,

$$\lim_{n\to\infty} \oint_{\Sigma^0} (\mathbf{T}u \cdot \boldsymbol{\nu})_{\Sigma_n^0}\, d\sigma = \Sigma^0. \tag{7}$$

We have by the above construction that on Σ_n^0, $|\mathbf{T}\hat{v}_\delta| \leqslant \cos\gamma_0 < 1$ for all sufficiently large n. Let μ_n be the measure of the set on Σ^0, on which $(\mathbf{T}u \cdot \boldsymbol{\nu})_{\Sigma_n^0} < \cos\gamma_0$. The inequality

(1) The difficulties arising from possible irregularities of Γ° can be overcome by a simple approximation procedure, based on the fact that Γ° is a level set of $w(\mathbf{x})$; cf. footnote (2) in [1].

$|\mathbf{T}u| < 1$ now implies, with (7), that $\mu_n \to 0$. It follows then from $0 < w < M$ in Ω_0 that

$$\oint_{\Sigma_n^0} w[\mathbf{T}\hat{v}_\delta - \mathbf{T}u]\cdot \mathbf{v}\, d\sigma < (2+\varepsilon)\, M\mu_n$$

and finally, from (6)

$$0 < \int_{\mathcal{N}^0} [\mathbf{T}\hat{v}_\delta - \mathbf{T}u]\cdot \nabla w\, d\mathbf{x} \leqslant 0.$$

This contradiction establishes the theorem.

§ 2

The above proof yields somewhat more information than is stated in Theorem 1. To achieve the contradiction it suffices to have

$$\cos \gamma_0 = \max_{\Sigma^0} \mathbf{T}\hat{v}_\delta \cdot \mathbf{v} \leqslant \inf_{\Sigma^* \subset \Sigma^0} \frac{1}{\Sigma^*} \int_{\Sigma^*} \mathbf{T}u \cdot \mathbf{v}\, d\sigma,$$

(the integral on the right being understood, as before, in a limiting sense), so that the theorem will be improved as soon as a uniform bound on the left-hand term can be obtained. Clearly this bound depends, for given Σ, only on ε and on the distance δ that $\mathcal{N}^*$ can be displaced while maintaining the inequality $\hat{v}_\delta(\mathbf{p}) > u(\mathbf{p})$. ε can be chosen depending only on the geometry, that is, to permit the Spruck construction in a region $\mathcal{M}^*$ of the desired form. Assuming the geometry fixed, the maximum permissible δ depends only on the bound for $|u(\mathbf{x})|$ in $\mathcal{N}$. This remark proves one of the inequalities of the following result, the other one being obtainable, as before, analogously.

THEOREM 2. *If $H^\Sigma \geqslant (n/(n-1))H_0 \geqslant 0$ and $H(\mathbf{x}) \leqslant H_0$ in $\mathcal{N}$, then there exists γ_0, $0 < \gamma_0 < \pi$, depending only on $\mathcal{N}$ and on M, such that for any solution $u(\mathbf{x})$ of (1) in $\mathcal{N}$ satisfying $|u(\mathbf{x})| < M$ there holds $\cos\gamma_0 \geqslant \inf_{\Sigma^* \subset \Sigma}(1/\Sigma^*)\int_{\Sigma^*}\mathbf{T}u\cdot\mathbf{v}\,d\sigma$. If $|H^\Sigma| \leqslant (n/(n-1))H_0$ and $H(\mathbf{x}) \geqslant H_0$ in $\mathcal{N}$, then there exists γ_1, $0 < \gamma_1 < \pi$, such that if $u(\mathbf{x})$ satisfies (1) in $\mathcal{N}$ and $|u(\mathbf{x})| < M$, then $\cos\gamma_1 \leqslant \sup_{\Sigma^* \subset \Sigma}(1/\Sigma^*)\int_{\Sigma^*}\mathbf{T}u\cdot\mathbf{v}\,d\sigma$.*

Remark. If, as is done in [2], it is assumed that $\mathbf{T}u\cdot\mathbf{v}$ defines, as a limit from within $\mathcal{N}$, a function $\cos\gamma$ almost everywhere on Σ, then the result of the theorem becomes that there cannot hold $\cos\gamma \geqslant \cos\gamma_0$ (resp. $\cos\gamma \leqslant \cos\gamma_1$) on Σ.

§ 3

The restriction $|u(\mathbf{x})| < M$ of Theorem 2 is necessary. This can be seen by consideration of the properties of the Spruck surface; alternatively, we present here a more elementary example.

Let $\mathcal{N}$ be the disk $x^2+y^2<1$, and Σ the arc $x^2+y^2=1$, $y>0$. The cylindrical surface Z in Euclidean 3-space, that projects onto Σ, has mean curvature $H=\frac{1}{2}=\frac{1}{2}H^{\Sigma}$. If Z is rotated a (small) angle α about the x-axis, in such a way that points on the positive y-axis are turned away from the positive z-axis, we obtain a surface Z^{α}: $u^{\alpha}(c, y)$ defined over $\mathcal{N}$ and satisfying the equation

$$\operatorname{div}\left(\frac{1}{W}\nabla u^{\alpha}\right)=1=2H.$$

Let $\mathbf{p}^{\alpha}$ be a point on $Z^{\alpha}\cap Z$. $\mathbf{p}^{\alpha}$ lies at the intersection of a generator on Z through Σ, and of the image generator on Z^{α}. It follows that the angle γ^{α} between Z and Z^{α} at $\mathbf{p}^{\alpha}$ satisfies $0<\gamma^{\alpha}\leqslant\alpha$, uniformly on Σ, and hence $\lim_{\alpha\to 0}\boldsymbol{\nu}\cdot \mathbf{T}u^{\alpha}]_{(x,y)}=1$ uniformly for $(x, y)\in\Sigma$. The surface Z^{α}: $u^{\alpha}(x, y)$ is however defined and bounded in N (although of course not equibounded in α) for all $\alpha>0$.

§ 4

We remark finally that the sense of Theorems 1 and 2 is to give conditions under which $|\mathbf{T}u\boldsymbol{\nu}|$ cannot be close to unity on a boundary set of significant size. That the size of the boundary set must enter into the result is easily seen from the example at the end of § 3.3 in [2], which shows that under the hypotheses of the theorems any value of $\mathbf{T}u\cdot\boldsymbol{\nu}$ in the closed interval $[-1, 1]$ can be obtained at an isolated boundary point.

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