K-THEORY OBSTRUCTIONS TO THE EXISTENCE OF VECTOR FIELDS

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1. Introduction

In the paper [7] with M. F. Atiyah, we showed how to apply K-theory for computing the top dimensional obstruction to the existence of r linearly independent vector fields on an oriented manifold X. The purpose of this note is to extend the method of [7] to apply also for the other higher obstructions.

Following classical obstruction theory as developed for example in Steenrod's book [16, part 3] we fix a triangulation of the n-dimensional oriented closed manifold X, and construct the vector fields successively over the q-skeleton X^q. Assume that the set $\mathbf{u} = \{u_1, ..., u_r\}$ is defined and linearly independent over X^{q-1}. As is well-known this gives rise to a natural obstruction cocycle

$$\mathfrak{o}(\mathbf{u}) \in C^q(X, (\pi_{q-1}(V_{n,r}))^t)$$

in the cochain complex of X with coefficients in the local coefficient system which restricted to a q-simplex σ^q of X is the $(q-1)$-th homotopy group of the Stiefel manifold of r-frames in the tangent space at the 1-st vertex of σ^q. As X is assumed to be oriented this coefficient system is actually trivial. The cohomology class

$$\{\mathfrak{o}(\mathbf{u})\} \in H^q(X, \pi_{q-1}(V_{n,r}))$$

is the obstruction to deforming $\mathbf{u}$ (relative to X^{q-2}) into a set which has an extension over X^q. As an example of our results we shall prove the following theorem.

THEOREM 1.1. *Let X be a manifold as above of dimension $n = 4k - s \geq 6$, and let $\mathbf{u} = \{u_1, u_2, u_3\}$ be three linearly independent vector fields over X^{n-2}. Then for $s \neq 3$ we have $\{\mathfrak{o}(\mathbf{u})\} = 0$ in $H^{n-1}(X, \pi_{n-2}(V_{n,3}))$.*

If $s = 3$ then $\pi_{n-2}(V_{n,3}) = \mathbf{Z}/4$, and assuming $H_1(X, \mathbf{Z})$ has no 2-torsion we have

$$\{\mathfrak{o}(\mathbf{u})\} = (-1)^{k-1} L_{k-1}(p_1, \ldots, p_{k-1})$$

in $H^{4(k-1)}(X, \mathbf{Z}/4)$, *where* $L_{k-1}(p_1, \ldots, p_{k-1})$ *is the Hirzebruch L-polynomial in the Pontrjagin classes.*

Combining with the results of [7] and the fact (see Massey [14]) that $\delta^* w_{4k-4} = 0$ for an oriented $(4k-1)$-manifold, we get the following table of necessary and sufficient conditions for the existence of 3 linearly independent vector fields

Table 1

$\dim X \geq 7$	
$4k$	$w_{4k-2}(X) = 0$, $E(X) = 0$, $S(X) \equiv 0 \bmod 8$
$4k+1$	$\delta^* w_{4k-2}(X) = 0$, $L_k(p_1, \cdots, p_k) \equiv 0 \bmod 4$, $R(X) = 0$.
$4k+2$	$w_{4k}(X) = 0$, $E(X) = 0$,
$4k+3$	No condition

In case $\dim X = 4k+1$ we must assume that $H_1(X, \mathbf{Z})$ has no 2-torsion, but apart from that X is only assumed to be oriented. In Table 1 $w_i(X)$ is of course the i-th Stiefel Whitney class, δ^* is the Bockstein homomorphism, $E(X)$ denotes the Euler characteristic, $S(X)$ the Hirzebruch signature and $R(X)$ the real semi-characteristic.

Many of the results of Table 1 were already proved under more restrictive hypotheses by E. Thomas (see [17] and [18]). Notice that Table 1 extends the classical result of Stiefel that every oriented 3-manifold is parallelizable.

In general, we shall apply the homomorphism

$$\theta^s: \pi_{q-1}(V_{n,r}) \to KR^{4k-q}(P_{r+s-1}, P_{s-1}) \tag{1.2}$$

defined in [7], and we want to calculate

$$\theta^s\{\mathfrak{o}(\mathbf{u})\} \in H^q(X, KR^{4k-q}(P_{r+s-1}, P_{s-1})).$$

Using Poincaré duality it is in favorable cases enough to calculate the cup-product

$$\langle \bar{x} \cup \theta^s\{\mathfrak{o}(\mathbf{u})\}, [X] \rangle \tag{1.3}$$

for all classes $\bar{x} \in H^{n-q}(X)$. Our main result is an expression for (1.3) in terms of the index-homomorphism in K-theory, and this in turn is expressible in terms of characteristic classes (Theorem 4.4). Taking $\bar{x} = 1$ in (1.3) we of course recapture the results of [7]. In fact this note is a straight forward extension of the method developed there.

In section 2 we recall the main properties of the basic K-theoretic characteristic class defined in [7] and in section 3 we derive the general formula for the expression (1.3). This we apply in a few interesting cases in section 4 and in particular prove Theorem 1.1.

The author is pleased to thank Professor M. F. Atiyah, whose ideas are the basis of this paper.

2. Notation

As in [7] we shall use Real K-theory in the sense of Atiyah [3] for spaces with involution (called Real spaces). Recall that a Real vector bundle over a Real space X is a complex vector bundle with an anti-linear involution covering the involution on X. Let $KR(X)$ denote the Grothendieck group of all such Real vector bundles. Also if C_s denotes the Clifford algebra on s generators (see Atiyah-Bott-Shapiro [6]), $M_s(X)$ denotes the Grothendieck group of $\mathbf{Z}/2$-graded Real Clifford modules over X (see [3]). The corresponding K-theories are denoted M_s^*. Notice that $M_s^*(X)$ is in a natural way a module over $KR^*(X)$.

We shall freely use the notation of K-theory defined for locally compact spaces in the sense of [8].

As mentioned in [7, section 3] there are natural homomorphisms of cohomology theories

$$\gamma_r^s: M_s^*(X) \to KR^*(X \times (P_{r+s-1}, P_{s-1})),$$

where X is any Real space and P_l denotes the real projective space of dimension l (with trivial involution). In particular for $s=0$ and X compact, γ_r^0 is simply the map, which sends a pair of Real vector bundles (E^+, E^-) into $E^+-(E^-\otimes H)$, where H is the Hopf bundle. For s arbitrary it is easy to verify that γ_r^s is a module homomorphism with respect to the module structures over $KR^*(X)$.

Now consider an ordinary real oriented vector bundle E of dimension n over a compact space X (in the applications X is a manifold and $E=TX$ is the tangent bundle). Assume first $n=4k$ and choose a metric on E. Then the exterior algebra $\Lambda^*(E)$ is a fibrewise module for the bundle of Clifford algebras $C^*(E)$ (see [5]). In particular left multiplication by the volume section ω yields an endomorphism L_ω of $\Lambda^*(E)$ satisfying $(L_\omega)^2=1$. Therefore $\Lambda^*(E)$ splits into the bundles of eigenspaces $\Lambda_+^*(E)$ and $\Lambda_-^*(E)$ for L_ω,

$$\Lambda^*(E) = \Lambda_+^*(E) \oplus \Lambda_-^*(E).$$

Complexifying and pulling back over the total space of E, we can consider $\Lambda_\pm^*(E)$ as Real vector bundles over the Real space iE, which is E with antipodal involution along the fibres.

For $v \in E_x$, where $x \in X$, left Clifford multiplication by iv (where $i \in \mathbf{C}$ denotes the imaginary unit) defines a homomorphism

$$iL_v\colon \Lambda_+^*(E) \to \Lambda_-^*(E),$$

and in this way we define a homomorphism φ of Real vector bundles over iE,

$$\varphi: \Lambda_+^*(E) \to \Lambda_-^*(E).$$

It is well-known that the triple

$$(\Lambda_+^*(E), \Lambda_-^*(E), \varphi) \tag{2.1}$$

defines an element of $KR(iE)$, which, in the case of $E = TX$ for X a manifold, gives $(-1)^k$ times the symbol of the signature operator (see Atiyah [5]).

Now both $\Lambda_+^*(E)$ and $\Lambda_-^*(E)$ are $\mathbf{Z}/2$-graded (into "even" and "odd"), so the above construction actually yields an element

$$\beta^0(E) \in M_0(iE).$$

In fact

$$\beta^0(E) = (\beta^+(E), \beta^-(E))$$

where

$$\beta^+(E) = (\Lambda_+^{\mathrm{ev}}(E), \Lambda_-^{\mathrm{odd}}(E), \varphi^+)$$

$$\beta^-(E) = (\Lambda_-^{\mathrm{odd}}(E), \Lambda_+^{\mathrm{ev}}(E), \varphi^-),$$

where φ^+ and φ^- are the restrictions of φ.

If E has s linearly independent sections, then both $\Lambda_+^*(E)$ and $\Lambda_-^*(E)$ are actually $\mathbf{Z}/2$-graded C_s-modules (by Clifford multiplication with the sections on the right), so the triple (2.1) yields an element in $M_s(iE)$.

In particular the bundle $E \oplus \mathbf{R}^s$, where E now has dimension $n = 4k - s$, gives rise to an element

$$\beta^s(E) \in M_s^s(iE).$$

In [7] the above construction is relativized with regard to a set $\mathbf{u} = \{u_1, \ldots, u_r\}$ of linearly independent sections over some closed subset $Y \subseteq X$, thus yielding a class

$$\alpha^s(E; \mathbf{u}) \in KR^s((iE(X - Y) \times (P_{r+s-1} - P_{s-1}). \tag{2.2}$$

(In [7] this class is denoted $\alpha_E^s(u_1, \ldots, u_r)$.) If $j\colon (X, \emptyset) \to (X, Y)$ is the natural map then we put

$$\alpha_r^s(E) = j^* \alpha^s(E; \mathbf{u}) = \gamma_r^s(\beta^s(E)), \tag{2.3}$$

in the group $KR^s(iE \times (P_{r+s-1} - P_{s-1}))$. In particular for $s = 0$

$$j^*\alpha^0(E;\mathbf{u})=\beta^+(E)-H\otimes\beta^-(E),$$

in $KR(iE\times P_{r-1})$.

The homomorphism (1.2) is simply defined using the class (2.2) for the case $(X, Y)=(B^q, S^{q-1})$, the ball and sphere in $\mathbf{R}^q$, and $E=X\times\mathbf{R}^n$.

Finally we shall use various forms of the topological index map ind. We return to the case of $E=TX$ the tangent bundle of an oriented manifold. For example

$$\text{ind}: M_s^*(iTX)\to M_s^*(\text{pt})$$

is defined as follows (compare [8]).

Embed X in $\mathbf{R}^{n+l}$ with normal bundle N of dimension l. This defines a Thom map

$$g: (\mathbf{R}^{n+l})^+\to N^+, \tag{2.4}$$

where $+$ denotes one point compactification. (Collapse everything outside a tubular neighbourhood of X.) Also define a Thom isomorphism

$$\Phi_N: M_s^j(iTX)\to M_s^j(iTX\oplus iN\oplus N)=M_s^{j+n+l}(N)\quad (j\in\mathbf{Z})$$

by multiplication with the Thom class for the Real vectorbundle $N\otimes\mathbf{C}=N\oplus iN$. Then

$$\text{ind}=\Sigma^{-(n+l)}\circ g^*\circ\Phi_N,$$

where Σ is the suspension isomorphism. Analogously there is an index map

$$\text{ind}: KR^*(iTX\times(P_{r+s-1}-P_{s-1})\to KR^*(P_{r+s-1}-P_{s-1}). \tag{2.5}$$

Since the homomorphism γ_r^s is a module homomorphism over KR^*, it commutes with Thom isomorphisms, and it follows that the index homomorphism commutes with the homomorphism γ_r^s.

3. The general formula

We now turn to the situation described in section 1 of an oriented closed manifold X with a triangulation, and a set $\mathbf{u}=\{u_1, ..., u_r\}$ of linearly independent vector fields defined over X^{q-1}, the $(q-1)$-skeleton of X. In this section we want to relate the obstruction class

$$\theta^s\{\mathfrak{o}(\mathbf{u})\}\in H^q(X, KR^{4k-q}(P_{r+s-1}-P_{s-1})), \tag{3.1}$$

to the characteristic class

$$\alpha_r^s(TX)\in KR^s(iTX\times(P_{r+s-1}-P_{s-1})) \tag{3.2}$$

of section 2.

Again we embed X in $\mathbf{R}^{n+l}$ with normal bundle N. For any abelian group A we then have a Thom isomorphism

$$\Psi_N\colon H^j(X, A) \to H^{j+l}(N^+, A), \quad j \in \mathbf{N}.$$

As in section 2 we also have a Thom isomorphism Φ_N for $N \otimes \mathbf{C}$ in KR-theory, and instead of (3.1) and (3.2) we shall consider the two elements

$$\Psi_N(\theta^s\{\mathfrak{o}(\mathbf{u})\}) \in H^{q+l}(N^+, KR^{n+s-q}(P_{r+s-1} - P_{s-1})), \tag{3.3}$$

$$\Phi_N(\alpha_r^s(TX)) \in KR^{s+n+l}(N \times (P_{r+s-1} - P_{s-1}). \tag{3.4}$$

The triangulation of X induces a cell decomposition of the Thom space N^+, and we can consider the Atiyah-Hirzebruch spectral sequence E_t^{**} (see e.g. Dold [11]) for N^+ and the cohomology theory

$$h^*(\cdot) = KR^*(\cdot \times (P_{r+s-1} - P_{s-1})).$$

Recall that in this spectral sequence

$$E_1^{j,k} = h^{j+k}((N^+)^j, (N^+)^{j-1}) = C^j(N^+, h^k(\mathrm{pt}))$$

$$E_2^{j,k} = \tilde{H}^j(N^+, h^k(\mathrm{pt}))$$

$$E_\infty^{j,k} = h^{j+k}(N^+)_j / h^{j+k}(N^+)_{j+1}$$

where $\qquad h^m(N^+)_j = \ker [h^m(N^+) \to h^m((N^+)^{j-1})].$

With this notation we now have

PROPOSITION 3.5. *Suppose there exist r linearly independent vector fields $\mathbf{u} = \{u_1, \ldots, u_r\}$ over X^{q-1}. Then the class* (3.3) *in $E_2^{q+l,\, n+s-q}$ is an infinite cycle. Furthermore the class* (3.4) *lies in $h^{s+n+l}(E)_{q+l}$ and represents* (3.3) *in $E_\infty^{q+l,\, n+s-q}$.*

Proof. Consider the class (2.2) for the pair (X, X^{q-1})

$$\alpha^s(TX; \mathbf{u}) \in h^s(iTX \,|\, X - X^{q-1})$$

or better

$$\Phi_N(\alpha^s(TX; \mathbf{u})) \in h^{n+s+l}(N \,|\, X - X^{q-1}). \tag{3.6}$$

The class (3.4) is clearly the image of the class (3.6) under the natural map, and as the composite map

$$h^*(N \,|\, X - X^{q-1}) \to h^*(N) \to h^*(N \,|\, X^{q-1})$$

is zero, we obviously have that the class (3.4) lies in $h^*(N)_{q+l}$. Therefore the restriction of (3.6) to $N \,|\, X^q$,

$$\Phi_N(\alpha^s(TX, \mathbf{u})) \in h^*(N \mid X^q - X^{q-1}) = C^{q+l}(N^+, h^*(\mathrm{pt})) \tag{3.7}$$

is clearly an infinite cycle of $E_1^{q+l,*}$ represented in $E_\infty^{q+l,*}$ by the class (3.4). Finally it is an easy excision argument to show that the element (3.7) is in fact the cycle $\Psi_N\theta^s\{\mathfrak{o}(\mathbf{u})\}$.

In order to derive a formula for (1.3) we first make the following observation. A class

$$\bar{z} \in E_2^{n+l,*} = H^{n+l}(N^+, h^*(\mathrm{pt}))$$

is always an infinite cycle. In fact the Thom map (2.4) induces an isomorphism

$$g^*: E_t^{n+l,*} \to H^{n+l}((\mathbf{R}^{n+l})^+, h^*(\mathrm{pt})) \cong h^*(\mathrm{pt})$$

for $t \geqslant 2$. Therefore if $z \in h^*(N^+)_{n+l}$ represents $\bar{z}$ in E_∞, then we can compute $\bar{z}$ evaluated on the fundamental class $[N]$ by the formula

$$\langle \bar{z}, [N] \rangle = \Sigma^{-(n+l)} g^*(z) \tag{3.8}$$

where $g^*: h^*(N^+) \to h^*((\mathbf{R}^{n+l})^+)$. In [7] we actually considered the case $q=n$, where $\bar{z}$ can be chosen as $\Psi_N\theta^s\{\mathfrak{o}(\mathbf{u})\}$. In the general case we shall also consider the Atiyah-Hirzebruch spectral sequence $'E_t^{p,q}$ for X and KR^*-theory. The pairing

$$KR^* \otimes h^* \to h^*$$

induces a pairing of spectral sequences (see [11]) such that the pairing

$$'E_2^{**} \otimes E_2^{**} \to E_2^{**}$$

is the cup-product induced by the natural pairing of coefficients, and such that the pairing

$$'E_\infty^{j,*} \otimes E_\infty^{k,*} \to E_\infty^{j+k,*}$$

is induced by the pairing

$$KR^*(X)_j \otimes h^*(N)_k \to h^*(N)_{k+l}.$$

With this notation we can now prove a general formula for the expression (1.3).

THEOREM 3.9. *Suppose there exist r linearly independent vector fields $\mathbf{u} = \{u_1, \ldots, u_r\}$ over X^{n-1} and suppose $\bar{x} \in H^{n-q}(X, KR^i(\mathrm{pt}))$ is an infinite cycle in the spectral sequence $'E_\infty^{**}$. If $x \in KR^{i+n-q}(X)_{n-q}$ represents $\bar{x}$ in $'E_\infty^{**}$ then*

$$\langle \bar{x} \cup \theta^s\{\mathfrak{o}(\mathbf{u})\}, [X] \rangle = \mathrm{ind}\,(x \cdot \alpha_r^s(TX)) \tag{3.10}$$

where the cup-product is induced by the pairing

$$KR^i(\mathrm{pt}) \otimes KR^{n+s-q}(P_{r+s-1}, P_{s-1}) \to KR^{n+s-q+1}(P_{r+s-1}, P_{s-1}) \tag{3.11}$$

and where ind *is the index map* (2.5).

Proof. According to Proposition 3.5, $\Psi_N\theta^s\{\mathfrak{o}(\mathbf{u})\}$ is represented in $E_\infty^{q+l,\,n+s-q}$ by $\Phi_N\alpha_r^s(TX)$, and therefore

$$\bar{x} \cup \Psi_N\theta^s\{\mathfrak{o}(\mathbf{u})\}$$

is represented by

$$x\cdot\Phi_N\alpha_r^s(TX)$$

in $E_\infty^{n+l,\,i+n+s-q}$. It then follows from (3.8) that

$$\langle \bar{x}\cup\Psi_N\theta^s\{\mathfrak{o}(\mathbf{u})\}, [N]\rangle = \Sigma^{-(n+l)}g^*(x\cdot\Phi_N\alpha_r^s(TX))$$

or equivalently

$$\langle \Psi_N(\bar{x}\cup\theta^s\{\mathfrak{o}(\mathbf{u})\}), [N]\rangle = \operatorname{ind}(\bar{x}\cdot\alpha_r^s(TX))$$

by the definition of the index homomorphism (see section 2). This proves the theorem.

Remark 1. For $q=n$ and $\bar{x}=1$, Theorem 3.9 is exactly [7, Theorem 2.20].

Remark 2. It follows from (2.3) that

$$\operatorname{ind}(x\cdot\alpha_r^s(TX)) = \gamma_r^s \operatorname{ind}(x\cdot\beta^s(TX)) \tag{3.12}$$

where

$$x\cdot\beta^s(TX)\in M^{i+n+s-q}(iTX).$$

4. Special cases

In this section we shall specialize Theorem 3.9 in certain cases, and in particular we shall prove Theorem 1.1.

As in [7, Proposition 5.6] one can easily prove the following statement concerning the homomorphism (1.2).

PROPOSITION 4.1. *Assume $q\leqslant 2(n-r)-1$ or $q=n+3-r\geqslant 6$. Then the homomorphism*

$$\theta^s\colon \pi_{q-1}(V_{n,r})\to KR^{n+s-q}(P_{r+s-1}, P_{s-1})$$

is a monomorphism for $n-4\leqslant q\leqslant n+3-r$ and an epimorphism for $n-3\leqslant q\leqslant n+4-r$. Hence θ^s is an isomorphism for $n-3\leqslant q\leqslant n+3-r$.

It is now easy to prove at least the first part of Theorem 1.1. Notice that according to Proposition 4.1, θ^s is an isomorphism for $q=n-1$ and $r=3$ provided $n\geqslant 6$, so it is enough to calculate the class

$$\theta^s\{\mathfrak{o}(\mathbf{u})\}\in H^{n-1}(X, KR^{s+1}(P_{s+2}, P_{s-1})).$$

Here $KR^{s+1}(P_{s+2}, P_{s-1})=\mathbf{Z}/2$, $\mathbf{Z}/2\oplus\mathbf{Z}/2$, 0 or $\mathbf{Z}/4$ for $s\equiv 0$, 1, 2 or 3 mod 4 respectively. Furthermore it is straight forward to check that

$$KR^{-1}(\mathrm{pt})\otimes KR^{s+1}(P_{s+2}, P_{s-1}) \to KR^{s}(P_{s+2}, P_{s-1})$$

is an isomorphism for $s \not\equiv 3 \bmod 4$. An arbitrary class $\bar{x} \in H^1(X, \mathbf{Z}/2)$ can be represented as an infinite cycle in the spectral sequence $'E^{**}$ by an element $x = L-1 \in KR(X)_1$, where L is the line bundle with $w_1(L)=\bar{x}$. It follows from Theorem 3.9 and Poincaré duality th a $\theta^s\{\mathfrak{o}(\mathfrak{u})\}=0$ for $s\not\equiv 3 \bmod 4$ once we have shown that $\operatorname{ind}(x\cdot\alpha_r^s(TX))=0$. On the other hand $x\cdot\beta^s(TX)\in M_s^s(iTX)$ so

$$\operatorname{ind}(x\cdot\beta^s(TX))\in M_s^s(\mathrm{pt})$$

which is torsionfree for $s\not\equiv 3 \bmod 4$ whereas x has finite order. Hence by (3.12)

$$\operatorname{ind}(x\cdot\alpha_r^s(TX)) = \gamma_r^s(\operatorname{ind}(x\cdot\beta^s(TX)))=0$$

for $s\not\equiv 3 \bmod 4$.

The second part of Theorem 1.1 we shall prove in a more general context. Thus we shall make formula (3.10) more explicit under the following assumptions. Suppose $q=4l<n$ and suppose $\bar{x}\in H^{n-q}(X, \mathbf{Z})$ is an infinite cycle represented by $x\in KR^{n-q}(X)_{n-q}$ Then (3.11) takes the form

$$KR^0(\mathrm{pt})\otimes KR^{n+s-q}(P_{r+s-1}, P_{s-1}) \to KR^{n+s-q}(P_{r+s-1}, P_{s-1})$$

which is the usual $\mathbf{Z}$-module structure on $KR^{n+s-q}(P_{r+s-1}, P_{s-1})$, hence the cupproduct in (3.10) is the usual one. Choose $s>0$ such that $n+s-q\equiv 0 \bmod 8$. Then

$$KR^{n+s-q}(P_{r+s-1}, P_{s-1}) \cong KR(P_{r+s-1}, P_{s-1})$$

and it follows (e.g. from Atiyah-Bott-Shapiro [6] or Adams [1]) that this group is either cyclic of order a power of 2 or is the direct sum of such a group and an infinite cyclic group. Let a_k denote the well-known series of 2-powers 1, 2, 4, 4, 8, 8, 8, 8 etc. (see Atiyah-Bott-Shapiro [6]).

LEMMA 4.2. *Assume $q=4l<n$ and choose s such that $n-q+s\equiv 0$ mod 8. Then the image of*

$$\theta^s\colon \pi_{q-1}(V_{n,r}) \to KR^{n-q+s}(P_{r+s-1}, P_{s-1})$$

is contained in the unique cyclic subgroup of order $2a_{r-(n-q)}$. Furthermore the natural map

$$KR^{n-q+s}(P_{r+s-1}, P_{s-1}) \to KR^{n-q+s}(P_{r+s-1})$$

restricted to the torsion group is injective.

Proof. The natural inclusion

$$j\colon V_{q,r-(n-q)} \to V_{n,r}$$

gives rise to the commutative diagram

$$(4.3)\qquad \begin{array}{ccc} \pi_{q-1}(V_{q,r-(n-q)}) & \longrightarrow & \pi_{q-1}(V_{n,r}) \\ \downarrow{\scriptstyle \theta^{s+n-q}} & & \downarrow{\scriptstyle \theta^s} \\ KR^{n+s-q}(P_{r+s-1}, P_{s+(n-q)-1}) & \longrightarrow & KR^{n+s-q}(P_{r+s-1}, P_{s-1}) \end{array}$$

where the top map is onto. Therefore the image of θ^s is contained in the image of the bottom horizontal map of (4.3), and it is a straight forward calculation (using e.g. Adams [1]) to show that the image of this map is the cyclic subgroup of order $2a_{r-(n-q)}$. The same calculation shows the second statement of the lemma.

In the following we let θ denote the map

$$\theta\colon \pi_{q-1}(V_{n,r}) \to \mathbf{Z}/2a_{r-n+q}$$

defined by the commutative diagram

$$\begin{tikzcd} \pi_{q-1}(V_{n,r}) \arrow[r, "\theta^s"] \arrow[dr, "\theta"'] & KR^{n+s-q}(P_{r+s-1}, P_{s-1}) \\ & \mathbf{Z}/2a_{r-(n-q)} \arrow[u, "\cong"'] \end{tikzcd}$$

The generator σ of $\mathbf{Z}/2a_{r-(n-q)}$ is chosen such that under the periodicity isomorphism composed with the natural map

$$KR(P_{r+s-1}, P_{s-1}) \to KR(P_{r+s-1})$$

σ maps to $-(H-1) \in KR(P_{r+s-1})$.

Remark. Using [7, Proposition 5.13] for the map θ^{s+n-q} in the diagram (4.3) one can actually show that θ in certain cases maps onto the subgroup generated by σ. Also, using Proposition 4.1 it follows that θ is injective if

$$r \leqslant \min(n-q+3,\, n-3).$$

Now we let $c\colon KR^* \to K^*$ denote complexification, and $\mathrm{ph}\colon KR^* \to H^*(-, \mathbf{Q})$ denotes the composite of c and the Chern character ch. We shall also use the following characteristic class defined by Atiyah-Singer [9]. Let E denote a $2k$-dimensional real oriented vectorbundle; then $\mathfrak{L}(E)$ is defined by the formal factorization

$$\mathfrak{L}(E) = \textstyle\prod_{i=1}^{k} \frac{1}{2}x_i/\tanh \frac{1}{2}x_i = \sum_j \mathfrak{L}_j(p_1, \ldots, p_k)$$

where the Pontrjagin classes $p_j = p_j(E)$ are formally the elementary symmetric polynomials in $x_1^2 \ldots x_k^2$. It is the class $2^k \mathfrak{L}(TX)$ which naturally occurs in the calculations by Atiyah-Singer [9] of the index of the signature operator. The class $\mathfrak{L}(E)$ is stable and can therefore also be defined for odd-dimensional bundles.

With this notation we now have the following theorem.

THEOREM 4.4. *Let X be a closed oriented manifold of dimension n. Suppose there exist r linearly independent vector fields $\mathbf{u} = \{u_1, \ldots, u_r\}$ over X^{q-1}, where $q = 4l$, and suppose*

$\bar{x} \in H^{n-q}(X, \mathbf{Z})$ *is an infinite cycle in the spectral sequence* $'E_*^{**}$. *If* $x \in KR^{n-q}(X)_{n-q}$ *represents* $\bar{x}$ *in* $'E_\infty^{**}$ *then*

$$\langle \bar{x} \cup \theta\{\mathfrak{o}(\mathbf{u})\}, [X]\rangle = (-1)^l \langle \mathrm{ph}\, x \cup 2^{2l}\mathfrak{L}(TX), [X]\rangle \bmod 2a_{r-(n-q)} \tag{4.5}$$

Remark 1. Notice that

$$2^{2l}\mathfrak{L}(TX) = \textstyle\sum_j 2^{2(l-j)} L_j$$

where $L_j = L_j(p_1, \ldots, p_k)$ is the Hirzebruch L-class (see [13]). Furthermore according to Atiyah [4]

$$\mathrm{ph}\, x = \bar{x}_0 + \bar{x}_1 + \bar{x}_2 + \ldots$$

where $\bar{x}_0 = \bar{x}$ and $\bar{x}_i \in H^{4i+n-q}(X, \mathbf{Q})$ are classes such that $2^{2i}\bar{x}_i$ are 2-integral (that is, they have only odd denominators). Hence the reduction mod $2a_{r-(n-q)}$ makes good sense.

Remark 2. If the vector fields exist and are linearly independent over all of X then $\mathfrak{o}(\mathbf{u}) = 0$ and (4.5) reduces to a divisibility theorem of K. H. Mayer [15].

Proof of Theorem 4.4. We want to calculate

$$\langle \bar{x} \cup \theta\{\mathfrak{o}(\mathbf{u})\}, [X]\rangle \in \mathbf{Z}/2a_{r-(n-q)}. \tag{4.6}$$

It follows from Lemma 4.2 that it is enough to calculate the image in the group

$$\widetilde{KR}^{n+s+q}(P_{r+s-1}) \cong \mathbf{Z}/a_{r+s} \tag{4.7}$$

From (3.10) it follows that (4.6) equals ind $(x\alpha_r^s(TX))$ and therefore the image in (4.7) equals

$$\mathrm{ind}\,(x \cdot \alpha_{r+s}^0(E)) \in KR^{n-q+s}(P_{r+s-1})$$

where $E = TX \oplus \mathbf{R}^s$. Now

$$\alpha_{r+s}^0(E) = \beta^+(E) - \beta^-(E) \otimes H$$

as elements of $KR(i(TX \oplus \mathbf{R}^s) \times P_{r+s-1})$. As ind $(x \cdot \alpha_{r+s}^0(E))$ lies in $\widetilde{KR}^{n-q+s}(P_{r+s-1})$ we get

$$\mathrm{ind}\,(x\alpha_{r+s}^0(E)) = \mathrm{ind}\,(-x\beta^-(E)) \cdot (H-1)$$

and

$$-\mathrm{ind}\,(x \cdot \beta^-(E)) \in KR^{n-q+s}(\mathrm{pt}) \cong \mathbf{Z}.$$

This integer of course equals

$$-\mathrm{ind}\,(cx \cdot c\beta^-(E)) \in K^{n-q+s}(\mathrm{pt}) \cong \mathbf{Z}, \tag{4.8}$$

where c denotes complexification. Since $-\beta^-(E)$ is represented by the triple

$$-\beta^-(E) = (\Lambda_+^{\mathrm{ev}}(E), \Lambda_-^{\mathrm{odd}}(E), \varphi)$$

and since
$$\beta^+(E)\oplus\beta^-(E) = (\Lambda_+(E), \Lambda_-(E), \varphi),$$
$$\beta^+(E)\oplus(-\beta^-(E)) = (\Lambda^{\mathrm{ev}}(E), \Lambda^{\mathrm{odd}}(E), \varphi),$$

we can use the calculations of Atiyah-Singer [9] to compute (4.8). The result is

$$-\operatorname{ind}(x\beta^-(E)) = (-1)^{k+1}\langle \mathrm{ph}\, x \cup 2^{2k-1}\mathcal{L}(TX\oplus\mathbf{R}^s), [X]\rangle. \tag{4.9}$$

Therefore the image of (4.6) in the group (4.7) is simply the reduction mod a_{r+s} of the expression (4.9). Considered as an element in the subgroup of order $2a_{r+n-q}$ it must be divided by

$$a_{r+s}/2a_{r+n-q} = \tfrac{1}{2}\cdot 2^{\frac{1}{2}(n+s-q)} = 2^{2k-1}\cdot 2^{-2l}$$

where we have used the fact that s was chosen such that $n+s-q$ is divisible by 8. The sign is determined by our choice of generator, and we have thus proved the theorem.

As an application we can now prove the second part of Theorem 1.1 (that is for $n=4k-3$). In this case $\theta^s=\theta$ is our isomorphism of $\pi_{n-2}(V_{n,3})$ onto $\mathbf{Z}/4$. Every class $\bar{x}\in H^1(X, \mathbf{Z})$ is induced by a map $f_x\colon X\to S^1$. If $\lambda\in KR^1(S^1)$ is the canonical generator, then clearly $x=f_x^*(\lambda)$ is an element of $KR^1(X)_1$, and obviously $\bar{x}$ is represented in $'E_\infty^{**}$ by x. As ph $x=\bar{x}$ it follows from Theorem 4.4 that

$$\langle \bar{x}\cup\theta\{\mathfrak{o}(\mathbf{u})\}, [X]\rangle = (-1)^{k-1}\langle \bar{x}\cup L_{k-1}(X), [X]\rangle \bmod 4,$$

for all integral classes $\bar{x}$. Therefore we conclude (see Lemma 4.10 below)

$$\theta\{\mathfrak{o}(\mathbf{u})\} \equiv (-1)^{k-1}L_{k-1}(X)\in H^{n-1}(X, \mathbf{Z}/4)$$

modulo the reduction of integral 2-torsion classes. By Poincaré duality $H^{n-1}(X, \mathbf{Z})$ is isomorphic to $H_1(X, \mathbf{Z})$, so as we have assumed that there is no 2-torsion there, this ends the proof.

We have here implicitly used the following well-known consequence of Poincaré duality and the universal coefficient theorem.

LEMMA 4.10. *Let p be a prime and X an oriented manifold of dimension n. Then $x\in H^i(X, \mathbf{Z}/p^l)$ is the reduction of an integral class if and only if $\langle x\cup y, [X]\rangle=0$ for all p-torsion classes $y\in H^{n-i}(X, \mathbf{Z})$. In particular x is the reduction of an integral p-torsion class if and only if $\langle x\cup y, [X]\rangle=0$ for all integral classes $y\in H^{n-i}(X, \mathbf{Z})$.*

In the proof of Theorem 1.1 above we were so fortunate that every class in $H^1(X, \mathbf{Z})$ survive in the spectral sequence. Geometrically an integral cohomology class in dimension one is dual to the fundamental class of a codimension one submanifold with trivial normal bundle in X. More generally formula (4.5) is applicable in case $\bar{x}$ is the dual to the "fundamental class" $f_*[M]$ of a singular manifold (M, f) of X with a Spin-structure on the normal bundle of f. More explicitly let M be a q-dimensional oriented compact manifold and $f: M \to X$ a differentiable map. The pair (M, f) is called a singular manifold of X and the normal bundle $N(f)$ of f is an oriented vector bundle over M such that $TM \oplus N(f)$ is stably equivalent to f^*TX. We shall now use a characteristic class $\mathcal{C}$ in rational cohomology for oriented vector bundles E defined by the formal factorization

$$\mathcal{C}(E) = \textstyle\prod_i \cosh \frac{1}{2} x_i \tag{4.11}$$

where $p(E) = \prod_i (1 + x_i^2)$ is the total Pontrjagin class.

COROLLARY 4.12. *Let X be as above and let $\mathbf{u} = \{u_1, ..., u_r\}$ be a set of linearly independent vector fields over the $(q-1)$-skeleton where $q = 4l$. Let furthermore (M^q, f) be a singular manifold in X with a Spin-structure for the normal bundle $N(f)$. Let $\bar{x} \in H^{n-q}(X, \mathbf{Z})$ be the dual class of $f_*[M] \in H_q(X, \mathbf{Z})$. Then*

$$\langle \bar{x} \cup \theta\{\mathfrak{v}(\mathbf{u})\}, [X] \rangle = (-1)^l \langle 2^{2l} \mathcal{C}(N(f)) \mathcal{L}(M), [M] \rangle \bmod 2a_{r-(n-q)} \tag{4.13}$$

Proof. Embed M in a high dimensional Euclidean space, $i: M \to \mathbf{R}^m$. Then the embedding $f \times i: M \to X \times \mathbf{R}^m$ has normalbundle $N(f)$, and taking a tubular neighbourhood of M, we get in the usual way a map g from the m-th suspension of X to the Thom complex of the bundle $N(f)$:

$$g: \Sigma^m X \to N(f)^+$$

It is easy to see that if $U \in H^*(N(f))$ denotes the Thom-class then g^*U is the m-th suspension of the class $\bar{x}$ dual to the class $f_*[M] \in H_*(X)$. By assumption $N(f)$ is a Spin-bundle, hence has a Thom class $\lambda_{N(f)} \in KR^*(N(f))$ (explicitly defined in [2] or [6]). Obviously $\bar{x}$ is represented in $'E_\infty^{*,*}$ by $x = g^* \lambda_{N(f)} \in KR^*(X)$. It is now a straightforward calculation with characteristic classes to deduce (4.13) from (4.5).

Remark. Theorem 3.9 of course also applies for the primary obstructions, i.e. the Stiefel-Whitney classes and (3.10) thus expresses a relation between the Stiefel-Whitney classes and the index of certain operators with coefficients in vector bundles. For example one immediately recover the well-known result of Wu and Massey (see Massey [14]) that for an oriented manifold X of dimension n, $w_{n-1}(X) = 0$ if $n \not\equiv 1 \bmod 4$. For $n = 4k+1$ using [7, Lemma 4.3] one gets the formula due to Atiyah (see Atiyah-Singer [10, Proposition 3.4] and also [12, Corollary 2.7]) for the semi-characteristic of a double covering.

Finally let us mentioned that the method of this paper of course also applies for the normal bundle instead of the tangentbundle or more generally for any oriented bundle

E over X such that there is a Spin-bundle F with $E \oplus F$ stably isomorphic to N (if $E = TX$ take $F = N \oplus N$). We can assume that E has dimension n. Then just using another Thom isomorphism in KR-theory we can repeat the whole argument. For example in formula (4.5) we just have to replace $\mathcal{L}(TX)$ by $\mathcal{C}(E)\hat{\mathcal{A}}(TX)$ where $\mathcal{C}(E)$ is defined by (4.11) and $\hat{\mathcal{A}}(TX)$ is the usual class

$$\hat{\mathcal{A}} = \prod_i [x_i/(e^{x_i/2} - e^{-x_i/2})].$$

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Received August 13, 1973