STOCHASTIC INTEGRALS IN THE PLANE

BY

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§ 0. Introduction

Let W be the random measure in $\mathbf{R}_+^2$, the positive quadrant of the plane, which assigns to each Borel set A a Gaussian random variable of mean zero and variance $m(A)$, where m is Lebesgue measure, and which assigns independent random variables to disjoint sets (see [6], [14], [15], [19] and [20]). It is natural to construct stochastic integrals with respect to W (see [1], [4], [7], [10], [13], [15], [18] and [20]) but one can do more. Define a process $W=\{W_z, z\in\mathbf{R}_+^2\}$ by $W_z=W(R_z)$, where R_z is the rectangle whose lower left hand corner is the origin and whose upper right hand corner is z. W is called the two-parameter Wiener process. It is a continuous process, and if we write $z=(s, t)$ and fix $s, t\to W_{st}$ is a Brownian motion; likewise, $s\to W_{st}$ is also a Brownian motion. Since the theory of stochastic integration with respect to Brownian motion is well-known, this opens the possibility of stochastic line integrals; we will see that one can integrate along all sufficiently smooth curves in $\mathbf{R}_+^2$.

The question that motivated this study was that of holomorphic processes, and this question still forms the goal of the present article. A process Φ is holomorphic if it has a derivative ϕ, in the sense that $\Phi_z=\Phi_0+\int_0^z\phi\,\partial W$, where the line integral is taken over any sufficiently smooth curve connecting 0 and z. These processes turn out to have a structure which is in some ways remarkably like that of classical holomorphic functions of a complex variable, even though they are real, not complex, valued. For instance, if Φ is holomorphic, so is its derivative ϕ, and there is even an analogue of the power series expansion.

These processes are treated in § 9. The earlier chapters are concerned with diverse questions. One of the foremost preoccupations is simply to develop a stochastic calculus. Thus, after the various line and surface integrals have been defined, we show in § 6 that the interplay between line and surface integrals is expressed by an analogue of Green's theorem, as in the classical case. An immediate application of this is a proof of the existence and continuity of the local time for W by means of an appropriate version of Tanaka's formula [11].

No special effort has been made to achieve maximum generality. In particular, we have not tried to pass beyond the square integrable case in our integrals. On the other hand, we have treated integration in greater generality than is needed for our study of holomorphic processes, partially in the hope of discovering more about martingales having $\mathbf{R}_+^2$ as a parameter set.

The theory of martingales with a partially ordered parameter set is still in its primitive state. We should distinguish between two cases: the Brownian case, in which the fields $\mathcal{F}_z$ are generated by W, and the general case, in which the $\mathcal{F}_z$ satisfy only (F1)–(F4) below. In the Brownian case, Wong and Zakai [18] have proved that any square integrable martingale can be written as a sum of two stochastic integrals. (We give a different proof of this in § 3.) This allows us to reduce many problems to direct calculation. For instance, we show in § 3 that all martingales bounded in $L \log L$ are continuous. (However, an example shows that there are L^1-bounded martingales which are everywhere discontinuous, so that the question of martingale continuity is evidently more delicate than in the classical case.) In fact, in the general case, the question of whether L^2-bounded, or even bounded, martingales have a version which is right-continuous and has left limits is open.

One final question which deserves mention here is that of the characterization of square integrable strong martingales. This important class of martingales crops up early in our story, for certain types of integrals can be defined only for strong martingales. In the Brownian case we can characterize them completely: they are the class of square integrable martingales which can be written in the form $M_z = \int_{R_z} \phi dW$ (Theorem 8.1). On the other hand, strong martingales have path-independent variation (see § 8 for the definition of this concept, which was introduced by Wong and Zakai [18]). All indications at our disposal suggest that path-independent variation is another characterization of the strong martingales, but our results are incomplete in this direction (Theorem 8.2).

The reader will notice that the techniques used throughout the article are rather closely tied to the cartesian coordinates in the plane, whereas it would seem that one should be able to integrate in a coordinate-free manner. This is true to a certain extent, but one usually wants to integrate random, rather than deterministic functions, and this requires something like the following.

1°. There is a partial ordering $\prec$ in some subset $\Gamma \subset \mathbf{R}^2$. If A and $B \subset \Gamma$, we say $A \prec z$ if $x \prec z$ for all $x \in A$, and we say $z \prec B$ if $z \prec y$ for all $y \in B$. $A \prec B$ means $A \prec z$ for all $z \in B$.

2°. There exists a family of σ-fields $\{\mathcal{F}_z, z \in \Gamma\}$ such that

(a) if $z \prec z'$ then $\mathcal{F}_z \subset \mathcal{F}_{z'}$;

(b) if $A \prec z$, then $W(A)$ is $\mathcal{F}_z$-measurable;

(c) if $z \prec B$, then $W(B)$ is independent of $\mathcal{F}_z$.

Notice that 2° is satisfied if we take $\mathcal{F}_z = \sigma\{W(A), A \prec z\}$. With such a family of fields, one can hope to imitate Ito's development of the stochastic integral to define the integral of $\mathcal{F}_z$-adapted processes with respect to W.

While the partial ordering does not determine a coordinate system, it may suggest one, and vice-versa. For instance, in polar coordinates, one might use the partial ordering "$(r, \theta) \prec (r', \theta')$ iff $r \leqslant r'$ and $\theta \leqslant \theta'$". We will not try to give such a general treatment, however, and we will treat nothing more exotic than Cartesian coordinates in $\mathbf{R}^2_+$. We will always use "$\prec$" for the partial order

$$(s, t) \prec (s', t) \quad \text{iff} \quad s \leqslant s' \quad \text{and}\ t \leqslant t'.$$

We also write

$$(s, t) \prec\prec (s', t') \quad \text{if}\ s < s' \quad \text{and}\ t < t'.$$

There are two other partial orders compatible with cartesian coordinates which we shall find useful, corresponding to positive cones equal to the right half-plane and the upper half plane respectively. Accordingly, if $\mathcal{F}_z$ is a family of σ-fields satisfying 2° (a), we define

$$\mathcal{F}^1_{st} = \mathcal{F}_{s\infty} \overset{\text{def}}{=} \bigvee_v \mathcal{F}_{sv}$$

and

$$\mathcal{F}^2_{st} = \mathcal{F}_{\infty t} \overset{\text{def}}{=} \bigvee_u \mathcal{F}_{ut}.$$

We will usually reserve z, ζ, η, and ξ for points of $\mathbf{R}^2_+$, while s, t, u, v, σ and τ usually refer to real variables. This notation reveals an ambivalent attitude toward $\mathbf{R}^2_+$. When we integrate over it, it is of course just the positive quadrant of the plane. But when it is the parameter set of a martingale, it becomes two-dimensional time—definitely a more mysterious object.

§ 1. Square integrable martingales

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\{\mathcal{F}_z, z \in \mathbf{R}^2_+\}$ be a family of sub-σ-fields of $\mathcal{F}$ satisfying

(F1) if $z \prec z'$ then $\mathcal{F}_z \subset \mathcal{F}_{z'}$;
(F2) $\mathcal{F}_0$ contains all null sets of $\mathcal{F}$;
(F3) for each z, $\mathcal{F}_z = \bigcap_{z \ll z'} \mathcal{F}_{z'}$;
(F4) for each z, $\mathcal{F}^1_z$ and $\mathcal{F}^2_z$ are conditionally independent given $\mathcal{F}_z$.

All except (F4) are self-explanatory. The following condition is easily seen to be equivalent to (F4): for all bounded random variables (r.v.'s) X and all $z \in \mathbf{R}_+^2$,

$$E\{X \mid \mathcal{F}_z\} = E\{E\{X \mid \mathcal{F}_z^1\} \mid \mathcal{F}_z^2\}^{(1)}.$$

In particular, if $X = I_\Lambda$, where $\Lambda \in \mathcal{F}_z^1 \cap \mathcal{F}_z^2$, then $E\{I_\Lambda \mid \mathcal{F}_z\} = I_\Lambda$ and so $\Lambda \in \mathcal{F}_z$, which implies $\mathcal{F}_z^1 \cap \mathcal{F}_z^2 \subset \mathcal{F}_z$, hence $\mathcal{F}_z^1 \cap \mathcal{F}_z^2 = \mathcal{F}_z$, by (F1).

Here are two examples of families of fields which satisfy (F4):

(a) Let $\{\mathcal{F}_s^1, s \in \mathbf{R}_+\}$ and $\{\mathcal{F}_t^2, t \in \mathbf{R}_+\}$ be two independent families of sub-σ-fields of $\mathcal{F}$. If $z = (s, t)$, put $\mathcal{F}_z = \mathcal{F}_s^1 \vee \mathcal{F}_t^2$.

(b) Let $\{X(A)$: A a rectangle in $\mathbf{R}_+^2\}$ be a process such that if $A_1, \ldots, A_n$ are disjoint rectangles, then $X(A_1), \ldots, X(A_n)$ are independent. Put $\mathcal{F}_z = \sigma\{X(A), A \prec z\}$.

In the first six sections, except the third, $\{\mathcal{F}_z\}$ will be a fixed family satisfying (F1)–(F4). If $\{\mathcal{G}_z, z \in \mathbf{R}_+^2\}$ is a family of σ-fields and $X = \{X_z, z \in \mathbf{R}_+^2\}$ is a stochastic process, we say X is *$\mathcal{G}_z$-adapted* if X_z is $\mathcal{G}_z$-measurable for all z. If X is $\mathcal{F}_z$-adapted, we shall simply say X is *adapted*. X is said to be *measurable* if $(z, \omega) \to X_z(\omega)$ is $\mathcal{B} \times \mathcal{F}$ measurable, where $\mathcal{B}$ is the class of Borel sets on $\mathbf{R}_+^2$.

DEFINITION. *A process $M = \{M_z, z \in \mathbf{R}_+^2\}$ is a martingale if*

(1) *M is adapted;*

(2) *for each z, M_z is integrable;*

(3) *for each $z \prec z'$,*

$$E\{M_{z'} \mid \mathcal{F}_z\} = M_z.$$

"Martingale" always means "martingale relative to $\{\mathcal{F}_z\}$". When discussing martingales relative to other fields, we shall always specify the fields.

Let us introduce a notation for rectangles. Suppose $z = (s, t)$ and $z' = (s', t')$. If $z \prec\prec z'$, $(z, z']$ will denote the rectangle $(s, s'] \times (t, t']$. We denote the rectangle $(0, z]$ by R_z. A martingale is often thought of as having orthogonal increments. In two dimensions, the relevant increments are the increments over rectangles. The increment of X over the rectangle $A = ((s, t), (s', t')]$ is

$$X(A) = X_{s't'} - X_{st'} - X_{s't} + X_{st}. \tag{1.1}$$

If X_{st} were a two-dimensional distribution function, (1.1) would give the measure of A, and it is often more convenient for us to speak in the language of measures. Accordingly,

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(1) In the sequel, equations between r.v.'s are to be interpreted a.s., unless the contrary is explicitly mentioned.

we say that the process X *induces a measure* (also denoted by X) on rectangles by the formula (1.1). (This gives, in fact, a finitely additive measure on the algebra of finite unions of half-open rectangles.) Similarly, a measure μ on rectangles induces a process X by

$$X_z = \mu(R_z), \quad z \in \mathbf{R}^2_+. \tag{1.2}$$

There are several notions of orthogonal increments in two-dimensional time because there are several relevant families of fields. To take this into account, we introduce the following definitions.

DEFINITION.

Let $X = \{X_z, z \in \mathbf{R}^2_+\}$ *be a process such that* X_z *is integrable for each* z.

(a) X *is a weak martingale if*

(1) X *is adapted;*

(2) $E\{X((z, z'])\,|\,\mathcal{F}_z\} = 0$ *for each* $z \prec\prec z'$.

(b) X *is an i-martingale* $(i = 1, 2)$ *if*

(1) X *is* $\mathcal{F}^i_z$*-adapted;*

(2) $E\{X((z, z'])\,|\,\mathcal{F}^i_z\} = 0$ *for each* $z \prec\prec z'$.

(c) X *is a strong martingale if*

(1) X *is adapted;*

(2) X *vanishes on the axes;*

(3) $E\{X((z, z'])\,|\,\mathcal{F}^1_z \vee \mathcal{F}^2_z\} = 0$ *for each* $z \prec\prec z'$.

Thanks to hypothesis (F4) we have the following proposition:

PROPOSITION 1.1. *A martingale is both a* 1- *and a* 2*-martingale.*

Proof. Suppose X is a martingale and let $A = ((s, t), (s', t')]$, where $s < s'$ and $t < t'$. Write $X(A) = (X_{s't'} - X_{s't}) - (X_{st'} - X_{st})$. By (F4),

$$E\{X_{s't'} - X_{s't}\,|\,\mathcal{F}^2_{s't}\} = E\{X_{s't'} - X_{s't}\,|\,\mathcal{F}_{s't}\} = 0.$$

Similarly,

$$E\{X_{st'} - X_{st}\,|\,\mathcal{F}^2_{st}\} = 0.$$

Since $\mathcal{F}^2_{s't} = \mathcal{F}^2_{st}$, $E\{X(A)\,|\,\mathcal{F}^2_{st}\} = 0$. By symmetry, $E\{X(A)\,|\,\mathcal{F}^1_{st}\} = 0$. qed

Notice that if $\{X_{so}, \mathcal{F}^1_{so}, s \in \mathbf{R}_+\}$ and $\{X_{ot}, \mathcal{F}^2_{ot}, t \in \mathbf{R}_+\}$ are martingales, the converse is also true. Indeed, X being both a 1- and a 2-martingale, it is adapted, by (F4); hence if $s < s'$, we have, setting $A = ((s, o), (s', t)]$,

$$E\{X_{s't} - X_{st}\,|\,\mathcal{F}_{st}\} = E\{X(A)\,|\,\mathcal{F}_{st}\} = E\{E\{X(A)\,|\,\mathcal{F}^1_{st}\}\,|\,\mathcal{F}_{st}\} = 0.$$

Similarly, if $t < t'$,

$$E\{X_{st'} - X_{st} \mid \mathcal{F}_{st}\} = 0,$$

which shows that X is a martingale.

In the following, the notions of martingale, strong martingale, etc., will be used for processes of the form $\{X_z, z \prec z_0\}$ without further comment.

If we may anticipate, the Wiener process $W = \{W_{st}\}$ is a strong martingale, the process $J = \{J_{st}\}$ introduced in § 6 is a martingale but not a strong martingale, while the product $JW = \{(JW)_{st}\}$ is a weak martingale but not a martingale.

Both martingales and strong martingales play major rôles in what follows, while weak martingales are peripheral, occuring mainly in the decomposition theorem (see Theorem. 1.5).

The theory of martingales with parameter set $\mathbf{R}^2_+$ is underdeveloped territory at the time of this writing, but enough is known to enable us to follow the usual construction of the Ito integral, at least superficially. Let us say that a process $\{X_z\}$ is *right-continuous* if for a.e. ω, $\lim_{\substack{z' \to z \\ z < z'}} X_{z'}(\omega) = X_z(\omega)$ for all $z \in \mathbf{R}^2_+$, and that it has *left limits* if, for a.e. ω, $\lim_{\substack{z' \to z \\ z' \ll z}} X_{z'}(\omega)$ exists for all $z \in (\mathbf{R}_+ - \{0\})^2$. We denote the limit by X_{z-}. The maximal inequality in our case (see [2]) becomes:

THEOREM 1.2. *Let $\{M_z, z \in \mathbf{R}^2_+\}$ be a right-continuous martingale. Then for $\lambda > 0$,*

(a) $\lambda P\{\sup_z |M_z| \geq \lambda\} \leq \dfrac{e}{e-1} + \dfrac{e}{e-1} \sup_z E\{|M_z| \log^+ |M_z|\};$

(b) $E\{\sup_z |M_z|^p\} \leq \left(\dfrac{p}{p-1}\right)^{2p} \sup_z E\{|M_z|^p\}, \quad p > 1.$

One consequence, also proved in [2], is that a martingale $\{M_z\}$ which is bounded in $L \log L$ must converge a.s. as $z \to \infty$ to a limit M_∞, and $M_z = E\{M_\infty \mid \mathcal{F}_z\}$. A second consequence is the following lemma, whose proof is exactly the same as in one dimension.

LEMMA 1.3. *Let $\{M^n\}$ be a sequence of right-continuous square integrable martingales. Suppose $\sup_z E\{(M_z^{n+1} - M_z^n)^2\} < 2^{-n}$. Then with probability one the sequence M_z^n converges uniformly in z as $n \to \infty$.*

For $p \geq 1$, let $\mathcal{M}^p$ be the class of all right-continuous martingales $M = \{M_z, z \in \mathbf{R}^2_+\}$ such that $M_z = 0$ on the axes and $E\{|M_z|^p\} < \infty$ for all z. Let $\mathcal{M}^p_c$ (resp. $\mathcal{M}^p_S$) denote the class of continuous (resp. strong) martingales in $\mathcal{M}^p$. For our purposes, it will usually be sufficient to work with bounded subsets of $\mathbf{R}^2_+$, the extension to all of $\mathbf{R}^2_+$ then being routine. Accordingly, let $\mathcal{M}^p(z_0)$ be the class of right-continuous martingales $M = \{M_z, z \prec z_0\}$ such that $M=0$ on the axes and $E\{|M_{z_0}|^p\}<\infty$. We are mainly interested in the case $p=2$. Give $\mathcal{M}^2(z_0)$ the norm and inner product

$$\|M\| = (E\{M_{z_0}^2\})^{\frac{1}{2}} \quad \text{and} \quad (M, N) = E\{M_{z_0} N_{z_0}\}.$$

As above, $\mathcal{M}^p_c(z_0)$ and $\mathcal{M}^p_S(z_0)$ will denote the continuous and strong martingales, respectively, in $\mathcal{M}^p(z_0)$.

Proposition 1.4. *$\mathcal{M}^2(z_0)$ with this norm is a Hilbert space. $\mathcal{M}^2_c(z_0)$ and $\mathcal{M}^2_S(z_0)$ are both closed subspaces.*

Proof. We must check that $\mathcal{M}^2(z_0)$ is complete and that $\mathcal{M}^2_c(z_0)$ and $M^2_S(z_0)$ are closed. Let $\{M^n\}$ be a Cauchy sequence. We may suppose, by taking a subsequence if necessary, that $\|M^{n+1}-M^n\|^2 \leqslant 2^{-n}$. Then, by Lemma 1.3, M^n converges uniformly in $z \prec z_0$ to a process M. If $\Lambda \in \mathcal{F}_z$ and $z \prec z'$,

$$\int_\Lambda M_{z'}\,dP = \lim_{n\to\infty} \int_\Lambda M^n_{z'}\,dP = \lim_{n\to\infty} \int_\Lambda M^n_z\,dP = \int_\Lambda M_z\,dP,$$

where we can go to the limit under the integrals because $\{M^n_z\}$ and $\{M^n_{z'}\}$, being L^2-convergent subsequences, are uniformly integrable. Thus M is a right-continuous martingale, hence $\mathcal{M}^2(z_0)$ is complete. The same argument applied to $M^n(A)$, where $A=(z, z']$, and a $\Lambda \in \mathcal{F}^1_z \vee \mathcal{F}^2_z$ shows that M is a strong martingale if the M^n are. Finally, M is continuous if the M^n are, by uniform convergence. qed

Unhappily, the Meyer submartingale decomposition theorem in two-dimensional time is true only in a weakened form. We must give two versions, one for martingales and one for strong martingales.

Definiton. *A process $X=\{X_z, z \in \mathbf{R}^2_+\}$ is an increasing process if*

(1) *X is right-continuous and adapted;*
(2) *$X_z=0$ on the axes;*
(3) *$X(A) \geqslant 0$ for each rectangle $A \subset \mathbf{R}^2_+$.*

Theorem 1.5. *Let $M \in \mathcal{M}^2(z_0)$. There exists an increasing process $\mathbf{A}=\{\mathbf{A}_z, z \prec z_0\}$ such that $\{M^2_z - \mathbf{A}_z, z \prec z_0\}$ is a weak martingale.*

Proof. For simplicity, assume $z_0=(1, 1)$ and divide R_{z_0} into rectangles whose corners are at the points $z_{ij}=(2^{-m}i, 2^{-n}j)$, $i=0, \ldots, 2^m$, $j=0, \ldots, 2^n$. Let $\Delta_{ij}=(z_{ij}, z_{i+1,\,j+1}]$. Define $\mathbf{A}^{mn}_{z_{ij}}$ by $\mathbf{A}^{mn}_{z_{io}}=\mathbf{A}^{mn}_{z_{oj}}=0$ and

$$\mathbf{A}^{mn}_{z_{ij}}(\Delta_{ij}) = E\{(M^2)(\Delta_{ij}) \mid \mathcal{F}_{z_{ij}}\}. \tag{1.3}$$

By Proposition 1.1, this is positive. Let $z=(s,t)\in R_{z_0}$ be dyadic (i.e. s and t are dyadic rationals). We claim that $\mathbf{A}_z^{mn}$, which is defined for m and n sufficiently large, converges weakly when m and then n tend to ∞. For $u\leqslant 1$, set

$$\mathbf{B}_{ut}^{n}=\sum_{j=0}^{2^nt-1} E\{M^2_{u,2^{-n}(j+1)}-M^2_{u,2^{-n}j}\,|\,\mathfrak{F}_{u,2^{-n}j}\}.$$

Since $\{M^2_{uv}, \mathfrak{F}_{uv}, v\leqslant 1\}$ is a positive submartingale, we know [16] [13] that $\mathbf{B}^n_{ut}$ converges weakly when $n\to\infty$ to a limit $\mathbf{B}^\infty_{ut}$. On the other hand, if $u<u'\leqslant 1$, by (F4),

$$\begin{aligned}E\{\mathbf{B}^n_{u't}-\mathbf{B}^n_{ut}\,|\,\mathfrak{F}_{ut}\}&=E\Bigg\{\sum_{j=0}^{2^nt-1}\Big(E\{M^2_{u',2^{-n}(j+1)}-M^2_{u',2^{-n}j}\,|\,\mathfrak{F}_{u',2^{-n}j}\}\\&\qquad-E\{M^2_{u,2^{-n}(j+1)}-M^2_{u,2^{-n}j}\,|\,\mathfrak{F}_{u,2^{-n}j}\}\Big)\,\Big|\,\mathfrak{F}_{ut}\Bigg\}\\&=\sum_{j=0}^{2^nt-1}E\Big\{(M^2)\Big(((u,2^{-n}j),(u',2^{-n}(j+1))]\Big)\,\Big|\,\mathfrak{F}_{u,2^{-n}j}\Big\}\geqslant 0.\end{aligned}\tag{1.4}$$

Thus $\{\mathbf{B}^n_{ut}, \mathfrak{F}_{ut}, u\leqslant 1\}$ is a positive submartingale. It follows that $\{\mathbf{B}^\infty_{ut}, \mathfrak{F}_{ut}, u\leqslant 1\}$ is also a positive submartingale, hence, again by [16] [13],

$$\sum_{i=0}^{2^ms-1} E\{\mathbf{B}^\infty_{2^{-m}(i+1),t}-\mathbf{B}^\infty_{2^{-m}i,t}\,|\,\mathfrak{F}_{2^{-m}i,t}\}$$

converges weakly when $m\to\infty$. But by (1.4)

$$\sum_{i=0}^{2^ms-1} E\{\mathbf{B}^n_{2^{-m}(i+1),t}-\mathbf{B}^n_{2^{-m}i,t}\,|\,\mathfrak{F}_{2^{-m}i,t}\}=\mathbf{A}^{mn}_{st}.$$

Since the operation of taking weak limits commutes with the conditional expectation, we conclude that the iterated weak limit $\mathbf{A}^\infty_z=\lim_{m\to\infty}\lim_{n\to\infty}\mathbf{A}^{mn}_z$ exists for dyadic $z\prec z_0$. If $D\subset R_{z_0}$ is a rectangle with dyadic vertices, a passage to the limit in (1.3) gives us

$$A^\infty(D)\geqslant 0;\quad E\{(M^2)(D)\,|\,\mathfrak{F}_z\}=E(\mathbf{A}^\infty(D)\,|\,\mathfrak{F}_z\}.\tag{1.5}$$

For each $z\prec z_0$, define

$$\mathbf{A}_z=\inf\{\mathbf{A}^\infty_{z'\wedge z_0}, z\prec\prec z', z' \text{ dyadic}\}.$$

This defines an increasing process which satisfies (1.5). qed

The question of uniqueness of the increasing process constructed above is delicate. Indeed, it is closely related to that of the existence, for a bounded martingale, of a version which has left limits and we cannot show, in general, that such a version exists. However, we conjecture the existence and we can prove it in the Brownian case, where all $L \log L$-bounded martingales are even continuous. Under this condition, it is easily seen that the process $\mathbf{A}$ of Theorem. 1.5 is unique. As it happens, the question of uniqueness is unimportant for our purposes, so we propose simply to ignore it in this article. We will just agree, if no other precisions are given, to denote by $\langle M\rangle = \{\langle M\rangle_z, z\in \mathbf{R}^2_+\}$ any increasing process $\mathbf{A}$ such that $M^2-\mathbf{A}$ is a weak martingale. In the same spirit, if $M, N\in \mathcal{M}^2(z_0)$, we denote by $\langle M, N\rangle = \{\langle M, N\rangle_z, z\in \mathbf{R}^2_+\}$ any process $\mathbf{B}$ which is the difference of two increasing processes and such that $MN-\mathbf{B}$ is a weak martingale, e.g. $\langle M, N\rangle = \frac{1}{2}(\langle M+N\rangle - \langle M\rangle - \langle N\rangle)$. Accordingly, relations such as $\langle M\rangle = \mathbf{A}$ or $\langle M, N\rangle = \mathbf{B}$ will signify that $\mathbf{A}$ and $\mathbf{B}$ are possible choices of $\langle M\rangle$ and $\langle M, N\rangle$ respectively.

We will say that two martingales M and N are orthogonal if MN is a weak martingale. We write $M\perp N$.

PROPOSITION 1.6. *Let* $M, N\in \mathcal{M}^2(z_0)$. *Then*

(a) $E\{MN(D)\,|\,\mathcal{F}_z\} = E\{M(D)N(D)\,|\,\mathcal{F}_z\}$ *for each rectangle* $D=(z, z']\subset R_{z_0}$;

(b) $M\perp N$ *iff* $E\{M(D)N(D)\,|\,\mathcal{F}_z\}=0$ *for each rectangle* $D=(z, z']\subset R_{z_0}$.

Proof. Since (b) is an immediate consequence of (a), we prove (a) only. If $z=0$, there is nothing to be shown. Suppose then that $0\prec\prec z\prec\prec z'$. Divide the rectangle $(0, z']$ into four disjoint subrectangles $A=(0, z]$, $D=(z, z']$, B and C. A little algebra gives

$$MN(D) = M(D)N(D)+M(A)N(D)+M(D)N(A)+M(D)N(B)+M(B)N(D)$$
$$+M(C)N(D)+M(D)N(C)+M(C)N(B)+M(B)N(C).$$

It is then easy to see that the conditional expectations, relative to $\mathcal{F}_z$, of all the terms of the right-hand side, starting from the second, vanish. qed

Theorem 1.5 holds for both ordinary and strong martingales. If we begin with a strong martingale, we might hope that the increasing process has better properties, e.g. that $M^2-\langle M\rangle$ is a martingale, rather than just a weak martingale. We will attack this from a slightly different viewpoint.

Let $M\in \mathcal{M}^2(z_0)$. We know that for each fixed t, $\{M_{st}, \mathcal{F}_{st}, s\leqslant s_0\}$ is a martingale. Thus let $\{\mathbf{A}^1_{st}, s\leqslant s_0\}$ be the unique one-parameter increasing process which is predictable relative to the family $\{\mathcal{F}_{st}, s\leqslant s_0\}$ and such that $\{M^2_{st}-\mathbf{A}^1_{st}, \mathcal{F}_{st}, s\leqslant s_0\}$ is a martingale. ("Predictable" here has its usual sense: $(s, \omega)\to \mathbf{A}^1_{st}(\omega)$ is measurable with respect to the σ-field on $\mathbf{R}_+\times\Omega$ generated by all left-continuous $\mathcal{F}_{st}$-adapated processes.) It follows that

$\{\mathbf{A}^1_{st}, s \leqslant s_0\}$ is the unique one-parameter increasing process which is predictable relative to the larger fields $\{\mathcal{F}^1_{st}, s \leqslant s_0\}$ and such that $\{M^2_{st} - \mathbf{A}^1_{st}, \mathcal{F}^1_{st}, s \leqslant s_0\}$ is a martingale. Indeed, both M_{st} and $\mathbf{A}^1_{st}$ are $\mathcal{F}_{st}$-measurable for all $s \leqslant s_0$, so by (F4), if $s < s' \leqslant s_0$,

$$E\{M^2_{s't} - \mathbf{A}^1_{s't} \mid \mathcal{F}^1_{st}\} = E\{M^2_{s't} - \mathbf{A}^1_{s't} \mid \mathcal{F}_{st}\} = M^2_{st} - \mathbf{A}^1_{st}.$$

We will denote $\{\mathbf{A}^1_{st}, s \leqslant s_0, t \leqslant t_0\}$ by $\mathbf{A}^1$. The process $\mathbf{A}^2$ is defined in an analogous manner.

Proposition 1.7. *If* $M \in \mathcal{M}^2_S(z_0)$, *then for each rectangle* $D \subset R_{z_0}$ *and each* $z \prec D$,

(a) $E\{M(D)^2 \mid \mathcal{F}^i_z\} = E\{(M^2)(D) \mid \mathcal{F}^i_z\} \quad (i=1,2)$;

(b) $\mathbf{A}^i(D) \geqslant 0$ $(i=1,2)$.

Proof. Let $D = ((s,t), (s't')]$ and set $\Delta_1 = M_{s't} - M_{st}$ and $\Delta_2 = M_{s't'} - M_{st'}$. Then

$$E\{(M^2)(D) \mid \mathcal{F}^1_{st}\} = E\{\Delta_2^2 - \Delta_1^2 \mid \mathcal{F}^1_{st}\},$$

for, by Proposition 1.1,

$$E\{M_{s't'} M_{st'} \mid \mathcal{F}^1_{st}\} = M^2_{st'} \quad \text{and} \quad E\{M_{s't} M_{st} \mid \mathcal{F}^1_{st}\} = M^2_{st}.$$

Noting that $M(D) = \Delta_2 - \Delta_1$, this equals

$$E\{2\Delta_1 M(D) + M(D)^2 \mid \mathcal{F}^1_{st}\}.$$

But since M is a strong martingale,

$$E\{\Delta_1 M(D) \mid \mathcal{F}^1_{st}\} = E\{\Delta_1 E\{M(D) \mid \mathcal{F}^1_{st} \vee \mathcal{F}^2_{st}\} \mid \mathcal{F}^1_{st}\} = 0$$

which proves (a) for $i=1$.

To prove (b), note that (a) implies that $\{M^2_{st'} - M^2_{st}, \mathcal{F}^1_{st}, s \leqslant s_0\}$ is a submartingale, while $\{(M^2_{st'} - M^2_{st}) - (\mathbf{A}^1_{st'} - \mathbf{A}^1_{st}), \mathcal{F}^1_{st}, s \leqslant s_0\}$ is a martingale. But $\{\mathbf{A}^1_{st'} - \mathbf{A}^1_{st}\}$ is a predictable process of bounded variation, and hence must be the increasing process of the decomposition of the submartingale, which proves (b) for $i=1$. The proofs for $i=2$ are similar. qed

The process $\{\mathbf{A}^1_{st}\}$ is right-continuous and increasing as a function of s for fixed t, but since we defined it separately for each t, we cannot expect it to have nice properties in t for a fixed s. However, in the case of a strong martingale we can use part (b) above to replace $\mathbf{A}^1$ and $\mathbf{A}^2$ respectively by their right-continuous versions

$$\inf\{\mathbf{A}^1_{s,t'\wedge t_0};\ t < t',\ t' \text{ rational}\} \quad \text{and} \quad \inf\{\mathbf{A}^2_{s'\wedge s_0, t};\ s < s',\ s' \text{ rational}\}.$$

It is easily seen that this amounts to a standard modification, so we can and do assume in this case that $\mathbf{A}^1$ and $\mathbf{A}^2$ are right-continuous increasing processes of two parameters.

Henceforth we will denote $\mathbf{A}^1$ and $\mathbf{A}^2$ by $[M]^1$ and $[M]^2$ respectively. Furthermore, if $M, N \in \mathcal{M}^2(z_0)$, we define

$$[M, N]^i = \tfrac{1}{2}([M+N]^i - [M]^i - [N]^i) \quad (i=1, 2).$$

We will need the notion of predictability for two parameter processes. Let $\{\mathcal{G}_z, z \in \mathbf{R}^2_+\}$ be an increasing right-continuous family of σ-fields. Consider the space $\mathbf{R}^2_+ \times \Omega$. We define a σ-field $\mathcal{P}_{\mathcal{G}}$ of subsets of this space, called the σ-field of *$\mathcal{G}_z$-predictable sets*: $\mathcal{P}_{\mathcal{G}}$ is the σ-field generated by sets of the form

$$(z, z'] \times \Lambda, \quad \text{where } z \prec\prec z' \quad \text{and } \Lambda \in \mathcal{G}_z.$$

A process $X = \{X_z, z \in \mathbf{R}^2_+\}$ is *$\mathcal{G}_z$-predictable* if $(z, \omega) \to X_z(\omega)$ is $\mathcal{P}_{\mathcal{G}}$-measurable. Let us compare this with the usual definition: if $\{\mathcal{H}_s, s \geq 0\}$ is a right-continuous family of σ-fields, the σ-field $\mathcal{Q}_{\mathcal{H}}$ of $\mathcal{H}_s$-predictable subsets of $\mathbf{R}_+ \times \Omega$ is the σ-field generated by sets of the form $(s, s'] \times \Lambda$, where $\Lambda \in \mathcal{H}_s$. Write $\mathbf{R}^2_+ \times \Omega = \mathbf{R}_+ \times (\mathbf{R}_+ \times \Omega)$. It is an easy exercise to show that if $\mathcal{G}_z = \mathcal{F}^1_z$ and $\mathcal{H}_s = \mathcal{F}^1_{so}$, then $\mathcal{P}_{\mathcal{G}} = \mathcal{B} \times \mathcal{Q}_{\mathcal{H}}$, where $\mathcal{B}$ is the Borel field of $\mathbf{R}_+$.

PROPOSITION 1.8. *If $M \in \mathcal{M}^2_S(z_0)$, then $[M]^i$ is the unique $\mathcal{F}^i_z$-predictable increasing process such that*

$$E\{M(D)^2 \mid \mathcal{F}^i_z\} = E\{(M^2)(D) \mid \mathcal{F}^i_z\} = E\{[M]^i(D) \mid \mathcal{F}^i_z\}, \tag{1.6}$$

for each rectangle $D = (z, z'] \subset R_{z_0}$ $(i=1, 2)$.

Proof. (1.6) follows from Proposition 1.7 and the fact that $M^2 - [M]^i$ is an i-martingale. If we fix t, we know that $[M]_{\cdot t}$ is predictable relative to $\{\mathcal{F}^1_{st}\}$, i.e. $\mathcal{Q}_{\mathcal{F}^1}$-measurable. As $t \to [M]^1_{st}$ is right-continuous, it follows that $(s, t, \omega) \to [M]^1_{st}(\omega)$ is $\mathcal{B} \times \mathcal{Q}_{\mathcal{F}^1} = \mathcal{P}_{\mathcal{F}^1}$-measurable, i.e. $\mathcal{F}^1_z$-predictable. If $\mathbf{B}$ is a second $\mathcal{F}^1_z$-predictable increasing process satisfying (1.6), it follows that

$$E\{M^2_{s't} - M^2_{st} \mid \mathcal{F}^1_{st}\} = E\{\mathbf{B}_{s't} - \mathbf{B}_{st} \mid \mathcal{F}^1_{st}\},$$

i.e. $\{M^2_{st} - \mathbf{B}_{st}, \mathcal{F}^1_{st}, s \leq s_0\}$ is a martingale. By uniqueness of the Meyer decomposition, $\mathbf{B} = [M]^1$. qed

Note that for a strong martingale M, either $[M]^1$ or $[M]^2$ can serve as the process $\langle M \rangle$ above. We can ask if $[M]^1 = [M]^2$. In general, there is no reason that it should. If, on the other hand, M is strong, it appears the two are equal. We have not succeeded in establishing this equality in general, but the following theorem covers the majority of applications we have in mind.

THEOREM 1.9. *Let* $M \in \mathfrak{M}_S^2(z_0)$. *Either of the following two conditions implies that* $[M]^1 = [M]^2$.

(a) *The fields* $\mathfrak{F}_z$ *are those generated by* W.

(b) M *is continuous and* $E\{M_{z_0}^4\} < \infty$.

Proof. (a) This follows from the uniqueness of $\langle M \rangle$, but it can also be seen directly. Let X be a bounded continuous martingale. We claim that if $z = (s, t) \in R_{z_0}$,

$$E\{X_z [M]_z^k\} = E\left\{\int_{R_z} X_\zeta \, d[M]_\zeta^k\right\}, \quad (k = 1, 2). \tag{1.7}$$

Suppose z is dyadic and let z_{ij} and Δ_{ij} be as in the proof of Theorem 1.5. Write the right-hand side as a limit of sums of the form

$$\sum_{i,j} E\{X_{z_{ij}}[M]^k(\Delta_{ij})\} = \sum_{i,j} E\{(X_{z_{ij}} - X_{z_{i+1,\,j+1}})\,[M]^k(\Delta_{ij})\} + \sum_{i,j} E\{X_{z_{i+1,\,j+1}}[M]^k(\Delta_{ij})\}. \tag{1.8}$$

The first sum on the right hand side is majorized by $E\{\sup_{i,j} |X_{z_{ij}} - X_{z_{i+1,\,j+1}}|\,[M]_z^k\}$, and this tends to zero as $m, n \to \infty$ by continuity of X. Since X is a martingale and $[M]^k(\Delta_{ij})$ is $\mathfrak{F}_{z_{i+1,\,j+1}}$-measurable, $E\{X_{z_{i+1,\,j+1}}[M]^k(\Delta_{ij})\} = E\{X_z [M]^k(\Delta_{ij})\}$, so the second sum on the right-hand side of (1.8) equals $E\{X_z [M]_z^k\}$, proving (1.7). On the other hand, the left-hand side of (1.8) equals (thanks to (1.6))

$$\sum_{i,j} E\{X_{z_{ij}}(M^2)(\Delta_{ij})\},$$

which is independent of k. Evidently

$$E\{X_z [M]_z^1\} = E\{X_z [M]_z^2\}. \tag{1.9}$$

But if (a) holds, all bounded martingales are continuous (see § 3), hence we can choose X_z to be any bounded $\mathfrak{F}_z$-measurable r.v. Then (1.9) implies that $[M]_z^1 = [M]_z^2$.

(b) If $z = (s, t) \in R_{z_0}$ is dyadic, we know by [16] [13], that $[M]^1$ is a weak limit—even an L^1-limit since M is continuous—of

$$\overset{m}{\mathbf{A}}{}_z^1 = \sum_{i=0}^{2^n s - 1} E\{(M_{2^{-m}(i+1),\,t} - M_{2^{-m}i,\,t})^2 \mid \mathfrak{F}_{2^{-m}i,\,t}\}.$$

By Proposition 1.7, this is equal

$$\sum_{i=0}^{2^m s - 1} \sum_{j=0}^{2^n t - 1} E\{M(\Delta_{ij})^2 \mid \mathfrak{F}_{2^{-m}i,\,t}\}.$$

Thus, it is enough to prove that

$$E\left\{\left[\overset{m}{\mathbf{A}}{}^1_z-\sum_{i=0}^{2^ms-1}\sum_{j=0}^{2^nt-1}E\{M(\Delta_{ij})^2|\mathfrak{F}_{z_{ji}}\}\right]^2\right\}$$

$$=E\left\{\left[\sum_{i=0}^{2^ms-1}\sum_{j=0}^{2^nt-1}(E\{M(\Delta_{ij})^2|\mathfrak{F}^1_{z_{ij}}\}-E\{M(\Delta_{ij})^2|\mathfrak{F}_{z_{ij}}\})\right]^2\right\} \tag{1.10}$$

converges to zero when $m, n\to\infty$, since then the same argument will apply to

$$\overset{n}{\mathbf{A}}{}^2_z=\sum_{j=0}^{2^nt-1}E\{(M_{s,2^{-n}(j+1)}-M_{s,2^{-n}j})^2|\mathfrak{F}_{s,2^{-n}j}\},$$

permitting us to conclude that

$$E\left\{\left[\overset{n}{\mathbf{A}}{}^2_z-\sum_{i=0}^{2^ms-1}\sum_{j=0}^{2^nt-1}E\{M(\Delta_{ij})^2|\mathfrak{F}_{z_{ij}}\}\right]^2\right\}$$

also converges to zero, hence that $\overset{m}{\mathbf{A}}{}^1_z$ and $\overset{n}{\mathbf{A}}{}^2_z$ have the same limit, implying that $[M]^1_z=[M]^2_z$.

Set

$$d_{ij}=E\{M(\Delta_{ij})^2|\mathfrak{F}^1_{z_{ij}}\}-E\{M(\Delta_{ij})^2|\mathfrak{F}_{z_{ij}}\}.$$

By $(F4)$, d_{ij} is $\mathfrak{F}_{z_{i,j+1}}$-measurable, and if $j<j'$,

$$E\{d_{ij}d_{i'j'}\}=E\{d_{ij}E\{d_{i'j'}|\mathfrak{F}_{z_{ij}\vee z_{i'j'}}\}\}=0.$$

Thus the right-hand side of (1.10) is

$$\sum_{i,j}E\{d^2_{ij}\}+2\sum_{i,j}(\sum_{i'>i}E\{d_{ij}d_{i'j}\}). \tag{1.11}$$

We will show that both these terms tend to zero. The first term is majorized by

$$2\sum_{i,j}E\{M(\Delta_{ij})^4\}\leqslant 2E\{\sup_{i,j}M(\Delta_{ij})^2\sum_{i,j}M(\Delta_{ij})^2\}$$

$$\leqslant 2[E\{\sup_{i,j}M(\Delta_{ij})^4\}E\{(\sum_{i,j}M(\Delta_{ij})^2)^2\}]^{1/2}.$$

But

$$E\{\sup_{i,j,m,n}M(\Delta_{ij})^4\}\leqslant \text{const.}\,E\{\sup_{z'<z}M^4_{z'}\}\leqslant\text{const.}\,E\{M^4_z\},$$

by Theorem 1.2. Hence, since M is continuous, the first factor above tends to zero as m, $n\to\infty$. The second factor is bounded by const. $E\{M^4_z\}<\infty$, according to Burkholder's inequality extended to the case of two parameters [12]. It follows that the first term of (1.11) tends to zero.

Passing to the second, set $\delta_{ij} = (z_{i+1,\,j}, z_{2^m s,\,j+1}]$. Then

$$\begin{aligned}\sum_{i,j}\Big(\sum_{i'>i} E\{d_{ij}d_{i'j}\}\Big) &= \sum_{i,j} E\{d_{ij}E\{\sum_{i'>i} d_{i'j} \,|\, \mathcal{F}^1_{z_{ij}}\}\}\\ &= \sum_{i,j} E\{d_{ij}(E\{M(\delta_{ij})^2 \,|\, \mathcal{F}^1_{z_{ij}}\} - E\{M(\delta_{ij})^2 \,|\, \mathcal{F}_{z_{ij}}\})\}\\ &= \sum_{i,j} E\{E\{M(\Delta_{ij})^2 \,|\, \mathcal{F}^1_{z_{ij}}\}[E\{M(\delta_{ij})^2 \,|\, \mathcal{F}^1_{z_{ij}}\} - E\{M(\delta_{ij})^2 \,|\, \mathcal{F}_{z_{ij}}\}]\}.\end{aligned}$$

Let

$$H = \sup_{i,j}(E\{\sup_{k,l} M(\delta_{kl})^2 \,|\, \mathcal{F}^1_{z_{ij}}\} + E\{\sup_{k,l} M(\delta_{kl})^2 \,|\, \mathcal{F}_{z_{ij}}\}).$$

Then the last term is dominated in absolute value by

$$E\{H\sum_{i,j} M(\delta_{ij})^2\} \leqslant [E\{H^2\}E\{(\sum_{i,j} M(\delta_{ij})^2)^2\}]^{1/2} \leqslant \text{const.}\ [E\{\sup_{i,j} M(\delta_{ij})^4\}E\{(\sum_{i,j} M(\delta_{ij})^2)^2\}]^{1/2}.$$

But, as before,

$$E\{\sup_{i,j,m,n} M(\delta_{ij})^4\} \leqslant \text{const.}\ E\{M_z^4\} < \infty.$$

Hence, since M is continuous, the first factor tends to zero as $m, n \to \infty$. The second factor being bounded by const. $E\{M_z^4\} < \infty$, it follows that the second term of (1.11) tends to zero.

§2. Surface integrals

We are going to define two different types of integrals in this section, the first analogous to the familiar Ito integral and the second a kind of multiple Wiener integral.

If M and N are right-continuous square integrable strong martingales, then $[M, N]^i$ is the unique $\mathcal{F}^i_z$-predictable process which is the difference of two increasing processes and such that $MN - [M, N]^i$ is an i-martingale ($i = 1, 2$). On the other hand if M and N are martingales, the process $\langle M, N\rangle$ may not be unique. Recall we defined that to be any process which is the difference of two increasing processes and for which $MN - \langle M, N\rangle$ is a weak martingale. This lack of uniqueness, while annoying, is not serious. The processes $\langle M\rangle$ and $\langle M, N\rangle$ will be used principally for their expectations, and for these, we have the following result:

PROPOSITION 2.1. *Let ϕ be a positive $\mathcal{F}_z$-predictable process and let $\mathbf{A}$ and $\mathbf{B}$ be increasing processes such that for each rectangle $D = (z, z'] \subset \mathbf{R}^2_+$,*

$$E\{\mathbf{A}(D) \,|\, \mathcal{F}_z\} = E\{\mathbf{B}(D) \,|\, \mathcal{F}_z\}.$$

Then

$$E\left\{\int_{R_{z_0}} \phi\, d\mathbf{A}\right\} = E\left\{\int_{R_{z_0}} \phi\, d\mathbf{B}\right\} \quad \text{for each } z_0 \in \mathbf{R}^2_+. \tag{2.1}$$

Proof. Let $K=(\zeta, \zeta']$ be a rectangle and suppose $\phi_z=\alpha I_K(z)$, where α is bounded and $\mathcal{F}_\zeta$-measurable. Then

$$E\left\{\int_{R_{z_0}} \phi\, d\mathbf{A}\right\}=E\{\alpha\mathbf{A}(K\cap R_{z_0})\}=E\{\alpha E\{\mathbf{A}(K\cap R_{z_0})\,|\,\mathcal{F}_\zeta\}\}$$

$$=\{\alpha E\{\mathbf{B}(K\cap R_{z_0})\,|\,\mathcal{F}_\zeta\}\}=E\left\{\int_{R_{z_0}} \phi\, d\mathbf{B}\right\}.$$

But (2.1) remains true for sums of such functions and since these generate the $\mathcal{F}_z$-predictable processes, the theorem follows by a monotone class argument. qed

Let $M\in\mathcal{M}^2(z_0)$ and let $A=(z_1, z_2]$. We define the stochastic integral $\phi\cdot M$ of a function $\phi_z=\alpha I_A(z)$, where α is bounded and $\mathcal{F}_z$-measurable, by

$$\phi\cdot M_z=\alpha M(A\cap R_z),\quad z\prec z_0. \tag{2.2}$$

Notice that a stochastic integral is a process, not a random variable. It has the following properties:

$$\phi\cdot M\in\mathcal{M}^2(z_0);\ \ \phi\cdot M\in\mathcal{M}^2_c(z_0) \text{ if } M\in\mathcal{M}^2_c(z_0) \text{ and } \phi\cdot M\in\mathcal{M}^2_S(z_0) \text{ if } M\in\mathcal{M}^2_S(z_0). \tag{2.3}$$

If ϕ and ψ are of the above form and if M and N are in $\mathcal{M}^2(z_0)$, then

$$\langle\phi\cdot M, \psi\cdot N\rangle_z=\int_{R_z}\phi\psi\, d\langle M, N\rangle,\quad z\prec z_0. \tag{2.4}$$

In particular,

$$\|\phi\cdot M_z\|^2=E\left\{\int_{R_z}\phi^2 d\langle M\rangle\right\}.$$

The property (2.3) follows by inspection. Let us check (2.4). Suppose A_1 and A_2 are disjoint rectangles with lower left-hand corners z_1 and z_2 respectively. Let α_1 and α_2 be bounded and $\mathcal{F}_{z_1}$- and $\mathcal{F}_{z_2}$-measurable, respectively. Let $\phi_z=\alpha_1 I_{A_1}(z), \psi_z=\alpha_2 I_{A_2}(z)$. First, $\phi\cdot M$ and $\psi\cdot N$ are orthogonal if $A_1\cap A_2=\varnothing$. Indeed, if B is any rectangle, say $B=(z, z']$:

$$E\{\phi\cdot M(B)\psi\cdot N(B)\,|\,\mathcal{F}_z\}=E\{\alpha_1\alpha_2 M(B\cap A_1)N(B\cap A_2)\,|\,\mathcal{F}_z\}.$$

Now $B\cap A_1$ and $B\cap A_2$ are disjoint and it is easy to see that they can be separated by either a horizontal or a vertical line. Suppose for instance the separating line is horizontal, $B\cap A_1$ is below and $B\cap A_2$ is above the line. If z'' is the lower left-hand corner of $B\cap A_2$, then α_1, α_2 and $M(B\cap A_1)$ are $F^2_{z''}$-measurable, hence

$$E\{\alpha_1\alpha_2 M(B\cap A_1)E\{N(B\cap A_2)\,|\,\mathcal{F}^2_{z''}\}\,|\,\mathcal{F}_z\}=0.$$

By Propositon 1.6, $\phi\cdot M$ and $\psi\cdot N$ are orthogonal. Thus, (2.4) follows since $\phi\psi=0$. Next, suppose $A_1=A_2=A$ and calculate $E\{\phi\cdot M(B)\psi\cdot N(B)\,|\,\mathcal{F}_z\}$. We have, by Proposition 1.6,

$$\begin{aligned} E\{\phi\cdot M(B)\,\psi\cdot N(B)\,|\,\mathcal{F}_z\} &= E\{(\phi\cdot M)\,(\psi\cdot N)\,(B)\,|\,\mathcal{F}_z\} \\ &= \alpha_1\alpha_2 E\{M(B\cap A)\,N(B\cap A)\,|\,\mathcal{F}_z\} = \alpha_1\alpha_2 E\{MN(B\cap A)\,|\,\mathcal{F}_z\} \\ &= \alpha_1\alpha_2 E\left\{\int_{B\cap A} d\langle M,N\rangle\,|\,\mathcal{F}_z\right\} = E\left\{\int_B \phi\psi\, d\langle M,N\rangle\,|\,\mathcal{F}_z\right\}. \end{aligned}$$

This proves (2.4) in case A_1 and A_2 are identical. The general case follows by dividing A_1 and A_2 into sub-rectangles which are either disjoint or identical.

We say ϕ is a *simple function* if there exists a finite number of rectangles $A_i=(z_i, z_i']$ and bounded r.v.'s α_i, such that α_i is $\mathcal{F}_{z_i}$-measurable and

$$\phi_z=\sum_i \alpha_i I_{A_i}(z).$$

If ϕ is simple, we define

$$\phi\cdot M_z=\sum_i \alpha_i M(A_i\cap R_z).$$

It is immediate that if ϕ and ψ are simple, they satisfy (2.3) and (2.4), and that $\phi\cdot M=\{\phi\cdot M_z\}$ is a linear function of ϕ. Notice that a simple function is $\mathcal{F}_z$-predictable.

Let $\mathcal{L}^2_M(z_0)$ be the class of all $\mathcal{F}_z$-predictable (of all adapted measurable—if $M=W$) processes $\phi=\{\phi_z, z\prec z_0\}$ such that $E\{\int_{R_{z_0}}\phi^2 d\langle M\rangle\}<\infty$ and $\mathcal{L}^2_M$ be that of $\mathcal{F}_z$-predictable (adapted measurable—if $M=W$) processes $\phi=\{\phi_z, z\in\mathbf{R}^2_+\}$ for which $E\{\int_{R_z}\phi^2 d\langle M\rangle\}<\infty$ for all $z\in\mathbf{R}^2_+$. By Proposition 2.1, the definition of $\mathcal{L}^2_M(z_0)$ and $\mathcal{L}^2_M$ do not depend on the particular choice of $\langle M\rangle$.

With the obvious identifications, $\mathcal{L}^2_M(z_0)$ is a Hilbert space under the norm $(E\{\int_{R_{z_0}}\phi^2 d\langle M\rangle\})^{\frac{1}{2}}$. It is not hard to see that the simple functions form a dense subset of $\mathcal{L}^2_M(z_0)$. The map $\phi\to\phi\cdot M$ of simple functions into $\mathcal{M}^2(z_0)$ is linear and (by (2.4)) preserves the norm. Thus it can be extended by continuity into a linear norm-preserving map of $\mathcal{L}^2_M(z_0)$ into $\mathcal{M}^2(z_0)$. We will often denote the random variable $\phi\cdot M_z$ by $\int_{R_z}\phi dM$. We will also write $\int_A \phi dM$ and $M(A)$ instead of $(\phi I_A)\cdot M_{z_0}$ and $I_A\cdot M_{z_0}$ respectively (A Borel subset of R_{z_0}). To summarize:

THEOREM 2.2. *Let* $M\in\mathcal{M}^2(z_0)$ *and let* $\phi\in\mathcal{L}^2_M(z_0)$. *Then*

(a) $\phi\cdot M\varepsilon\mathcal{M}^2(z_0)$ *and* (*by Proposition* 1.4) $\phi\cdot M\in\mathcal{M}^2_c(z_0)$ (*resp.* $\mathcal{M}^2_S(z_0)$) *if* $M\in\mathcal{M}^2_c(z_0)$ (*resp.* $\mathcal{M}^2_S(z_0)$);

(b) $\phi\cdot M$ *is linear in* ϕ;

(c) *if* ϕ *and* ψ *are in* $\mathcal{L}^2_M(z_0)$ *and* $\mathcal{L}^2_N(z_0)$, *respectively, then*

$$\langle\phi\cdot M, \psi\cdot N\rangle_z=\int_{R_z}\phi\psi d\langle M,N\rangle; \tag{2.5}$$

$$\|\phi \cdot M_z\|^2 = E\left\{\int_{R_z} \phi^2 d\langle M\rangle\right\} \tag{2.6}$$

Remark. One can extend the integral to $M \in \mathcal{M}^2$ and $\phi \in \mathcal{L}^2_M$ by choosing a sequence $z_n \to \infty$ and defining $\phi \cdot M = \lim \phi_n \cdot M$, where $\phi_n = \phi \cdot I_{R_{z_n}}$. We will use this extension without further comment.

Note. Henceforth we will adopt the following convention: each time the word "*predictable*" is used in expressing conditions of integrability (for the types of stochastic integral introduced above and hereafter), it is to be replaced by "*adapted measurable*" if $M = W$.

It will be useful to be able to integrate ϕ adapted to larger fields than $\mathcal{F}_z$. We say ϕ is *weakly predictable* if it is either $\mathcal{F}^1_z$- or $\mathcal{F}^2_z$-predictable. We can integrate weakly predictable ϕ, but we pay a price, losing some of the nice properties of the integrals of $\mathcal{F}_z$-predictable processes.

Let $M \in \mathcal{M}^2_S(z_0)$. We will extend the integral so that we can integrate $\mathcal{F}^1_z$-predictable processes. We proceed as before: if $A = (z_1, z_1']$ and α is bounded and $\mathcal{F}^1_{z_1}$-measurable, set $\phi_z = \alpha I_A(z)$ and define $\phi \cdot M$ by (2.2). By inspection, we have

$$(\phi \cdot M)_{st} \text{ is right-continuous in } s \text{ and is continuous if } M \text{ is;} \tag{2.7}$$

$$\phi \cdot M \text{ is a 1-martingale.} \tag{2.8}$$

Suppose $\psi_z = \beta I_B(z)$, where $B = (z_2, z_2']$ and β is bounded and $\mathcal{F}^1_{z_2}$-measurable. If $N \in \mathcal{M}^2_S(z_0)$ we have

$$[\phi \cdot M, \phi \cdot N]^1_z = \int_{R_z} \phi\psi d[M, N]^1. \tag{2.9}$$

(The first member has been defined only for martingales. This is the only place where we use it for 1-martingales. The definition is the same, except that $\mathcal{F}_z$ is replaced by $\mathcal{F}^1_z$.) In particular,

$$\|\phi \cdot M_z\|^2 = E\left\{\int_{R_z} \phi^2 d[M]^1\right\}. \tag{2.10}$$

The proof of (2.9) is the same as that of (2.4), except that $\mathcal{F}_z$ and $\mathcal{F}^2_z$ are replaced by $\mathcal{F}^1_z$ and $\mathcal{F}^1_z \vee \mathcal{F}^2_z$ respectively.

If ϕ is a $\mathcal{F}^1_z$-adapted simple function, we define $\phi \cdot M$ in the obvious way and $\phi \cdot M$ again satisfies (2.7)–(2.10). It remains to pass to the limit.

It is here that our previous approach breaks down, for while $\{(\phi \cdot M)_{st}\}$ is a martingale in s for fixed t, it may not be a martingale in t for fixed s. (If $\phi_z = \alpha I_A(z)$, $\{(\phi \cdot M)_{st}\}$ will be a martingale in t, relative to its natural fields. However this is no longer true for simple functions.) Consequently the maximal theorem which allowed us to pass to the limit uniformly is no longer valid. However, we do have the following:

$$\lambda^2 P\{\sup_{s \leqslant s_0} |(\phi \cdot M)_{st}| \geqslant \lambda\} \leqslant E\{(\phi \cdot M)^2_{s_0 t}\} \leqslant E\left\{\int_{R_{z_0}} \phi^2 d[M]^1\right\}.$$

It follows that if $\{\phi_n\}$ is a sequence of simple functions such that $E\{\int_{R_{z_0}} (\phi_{n+1} - \phi_n)^2 d[M]^1\} < 2^{-n}$, we have:

$$\begin{gathered}\text{for each } t,\ (\phi_n \cdot M)_{st} \text{ converges uniformly in } s \text{ with probability one} \\ \text{(the exceptional set may depend on } t\text{).}\end{gathered} \tag{2.11}$$

Now, if ϕ is $\mathcal{F}^1_z$-predictable and $E\{\int_{R_{z_0}} \phi^2 d[M]^1\} < \infty$, we can find a sequence of simple functions $\{\phi_n\}$ such that $E\{\int_{R_{z_0}} (\phi - \phi_n)^2 d[M]^1\} < 2^{-n}$. We then define

$$\phi \cdot M_z = \begin{cases} \lim_{n \to \infty} \phi_n \cdot M_z & \text{if the limit exists,} \\ 0 & \text{otherwise.} \end{cases}$$

It is now easy to check that the properties (2.7)–(2.10) remain true under a passage to the limit, giving us:

THEOREM 2.3. *Suppose M and N are in $\mathcal{M}^2_S(z_0)$ and suppose ϕ and ψ are $\mathcal{F}^1_z$-predictable processes such that $E\{\int_{R_{z_0}} \phi^2 d[M]^1\}$ and $E\{\int_{R_{z_0}} \psi^2 d[N]^1\}$ are finite. Then* (2.7)–(2.10) *hold.*

Remark. We have only defined the integrals of $\mathcal{F}^1_z$-predictable ϕ, but of course the $\mathcal{F}^2_z$-predictable processes are handled in exactly the same way.

We want to consider yet another stochastic integral, which was introduced by Wong and Zakai [18] for W. This is not an integral over $\mathbf{R}^2_+$, but over $\mathbf{R}^2_+ \times \mathbf{R}^2_+$.

Let us introduce another order relation in $\mathbf{R}^2_+$, complementary to "$\prec$". If $z = (s, t)$ and $z' = (s', t')$, we say $z \curlywedge z'$ if $s \leqslant s'$ and $t \geqslant t'$, and that $z \underset{\wedge}{\curlywedge} z'$ if $s < s'$ and $t > t'$. ("$\curlywedge$" is the relation "$\prec$" turned clockwise 90°.)

PROPOSITION 2.4. *Suppose $M \in \mathcal{M}^2(z_0)$ and let $A = (z_1, z_1']$ and $B = (z_2, z_2']$ be rectangles such that if $z \in A$ and $z' \in B$, then $z \curlywedge z'$. Define the process X by*

$$X_z = \alpha M(A \cap R_z) M(B \cap R_z), \quad z \prec z_0,$$

where α is bounded and $\mathcal{F}_{z_1 \vee z_2}$-measurable. Then:

(a) *X is a right-continuous martingale, which is continuous if M is.*

Suppose X is square integrable and M is a strong martingale. Then:

(b) *M and X are orthogonal and*

$$\langle X\rangle_z = \alpha^2 \iint_{R_z\times R_z} I_A(\zeta)\, I_B(\xi)\, d[M]^2_\zeta\, d[M]^1_\xi. \tag{2.12}$$

Proof. Suppose $z=(s,t)\prec\prec z'=(s',t')$ and let $D=(z,z']$. Notice that $X(D)=\alpha M(A')M(B')$, where $A'=A\cap(R_{s't'}-R_{s't})$ and $B'=B\cap(R_{s't'}-R_{st'})$. Suppose z'' is the lower left-hand corner of A'. Then both α and $M(B')$ are $\mathcal{F}^2_{z''}$-measurable, so

$$E\{X(D)\,|\,\mathcal{F}^2_z\}=E\{E\{X(D)\,|\,\mathcal{F}^2_{z''}\}\,|\,\mathcal{F}^2_z\}=E\{\alpha M(B')\,E\{M(A')\,|\,\mathcal{F}^2_{z''}\}\,|\,\mathcal{F}^2_z\}=0.$$

A similar argument shows $E\{X(D)\,|\,\mathcal{F}^1_z\}=0$, hence X is a martingale.

Let us calculate

$$E\{X(D)\,M(D)\,|\,\mathcal{F}_z\}=E\{\alpha M(A')\,M(B')\,M(D)\,|\,\mathcal{F}_z\}. \tag{2.13}$$

Write $M(D)=M(A'\cap D)+M(B'\cap D)+M(D-A'-B')$. Notice that α, $M(A'\cap D)$ and $M(A')$ are $\mathcal{F}^1_{z_2}$-measurable, hence,

$$E\{\alpha M(A')\,M(B')\,M(A'\cap D)\,|\,\mathcal{F}_z\}=E\{\alpha M(A')\,M(A'\cap D)\,E\{M(B')\,|\,\mathcal{F}^1_{z_3}\}\,|\,\mathcal{F}_z\}=0.$$

Using the fact that M is a strong martingale, similar arguments show

$$E\{\alpha M(A')\,M(B')\,M(B'\cap D)\,|\,\mathcal{F}_z\}=E\{\alpha M(A')\,M(B')\,M(D-A'-B')\,|\,\mathcal{F}_z\}=0.$$

Thus (2.13) vanishes, and Proposition 1.6 implies XM is a weak martingale, proving the first part of (b).

Let us also calculate $E\{X(D)^2\,|\,\mathcal{F}_z\}=E\{\alpha^2 M(A')^2 M(B')^2\,|\,\mathcal{F}_z\}$. If z''' is the lower left-hand corner of B' and if $z_3=z''\vee z'''$, this equals

$$E\{\alpha^2 E\{M(A')^2 M(B')^2\,|\,\mathcal{F}_{z_3}\}\,|\,\mathcal{F}_z\}.$$

But $\mathcal{F}^1_{z_3}$ and $\mathcal{F}^2_{z_3}$ are conditionally independent given $\mathcal{F}_{z_3}$, so

$$\begin{aligned} E\{M(A')^2M(B')^2\,|\,\mathcal{F}_{z_3}\} &= E\{M(A')^2\,|\,\mathcal{F}_{z_3}\}\,E\{M(B')^2\,|\,\mathcal{F}_{z_3}\}\\ &= E\left\{\int_{A'} d[M]^2\,|\,\mathcal{F}_{z_3}\right\}E\left\{\int_{B'} d[M]^1\,|\,\mathcal{F}_{z_3}\right\}=E\left\{\int_{A'} d[M]^2\int_{B'} d[M]^1\,|\,\mathcal{F}_{z_3}\right\}. \end{aligned}$$

Thus, noting that $\mathcal{F}_z\subset\mathcal{F}_{z_3}$,

$$E\{X(D)^2\,|\,\mathcal{F}_z\}=E\left\{\alpha^2\int_{B'}\int_{A'} d[M]^2 d[M]^1\,|\,\mathcal{F}_z\right\}. \tag{2.14}$$

Checking with the definition of A' and B', we see that if we define $\mathbf{A}_z$ to be equal to the right-hand side of (2.12), the right-hand side of (2.14) can be written

$$E\{\mathbf{A}(D)\,|\,\mathcal{F}_z\},$$

where $\mathbf{A}$ is the process $\{A_z\}$. Thus $X^2-\mathbf{A}$ is a weak martingale. qed

Let $M\in\mathcal{M}^4_S(z_0)$, so that in particular $E\{M^4_{z_0}\}<\infty$. This is simply to assure ourselves that products such as $M(A)M(B)$ are square integrable. To simplify notation, assume $z_0=(1,1)$. We want to define the integral $\psi\cdot MM$ for a suitably large class of processes.

Fix an integer n and divide R_{z_0} into rectangles $\Delta_{ij}=(z_{ij},z_{i+1,\,j+1}]$, where $z_{ij}=(2^{-n}i,\,2^{-n}j)$. If i, j, k and l are positive integers with $i<k\leqslant n$ and $l<j\leqslant n$, define

$$\psi_{ijkl}(\zeta,\xi)=\alpha I_{\Delta_{ij}}(\zeta)\,I_{\Delta_{kl}}(\xi), \tag{2.15}$$

where α is bounded and $\mathcal{F}_{z_{kj}}$-measurable. Define

$$\psi_{ijkl}\cdot MM_z=\alpha M(\Delta_{ij}\cap R_z)M(\Delta_{kl}\cap R_z),\quad z\in R_{z_0}.$$

By Proposition 2.4, $\psi_{ijkl}\cdot MM=\{\psi_{ijkl}\cdot MM_z\}$ is a martingale and

$$\langle\psi_{ijkl}\cdot MM\rangle_z=\iint_{R_z\times R_z}\psi^2_{ijkl}(\zeta,\xi)\,d[M]^2_\zeta\,d[M]^1_\xi. \tag{2.16}$$

Furthermore, if $m<q\leqslant n$ and $r<p\leqslant n$, let $\psi_{mpqr}(\zeta,\xi)=\beta I_{\Delta_{mp}}(\zeta)\,I_{\Delta_{qr}}(\xi)$, where β is bounded and $\mathcal{F}_{z_{qp}}$-measurable. Then

$$\langle\psi_{ijkl}\cdot MM,\psi_{mpqr}\cdot MM\rangle_z=\iint_{R_z\times R_z}\psi_{ijkl}(\zeta,\xi)\,\psi_{mpqr}(\zeta,\xi)\,d[M]^2_\zeta\,d[M]^1_\xi. \tag{2.17}$$

In particular $\psi_{ijkl}\cdot MM$ and $\psi_{mpqr}\cdot MM$ are orthogonal if $(i,j,k,l)\neq(m,p,q,r)$. The proof of (2.17) is immediate from (2.16) and the definition of $\langle\cdot,\cdot\rangle$.

We say ψ is a *simple function* if it is a finite sum of funtions of the form ψ_{ijkl} for some n. For simple functions ψ we define $\psi\cdot MM$ to be the sum of the corresponding $\psi_{ijkl}\cdot MM$. One easily checks that this definition is independent of the particular representation of ψ as a sum. From Propositions 2.4 and (2.17)

$$\psi\cdot MM\in\mathcal{M}^2(z_0)\text{ and is continuous if }M\text{ is;} \tag{2.18}$$

$$\langle\psi\cdot MM,\chi\cdot MM\rangle_z=\iint_{R_z\times R_z}\psi(\zeta,\xi)\,\chi(\zeta,\xi)\,d[M]^2_\zeta\,d[M]^1_\xi. \tag{2.19}$$

Let $\mathcal{P}$ be the σ-field on $\mathbf{R}^2_+\times\mathbf{R}^2_+\times\Omega$ generated by the simple functions. We call $\mathcal{P}$ the field of *predictable sets*—there will be no confusion with the class of $\mathcal{F}_z$-predictable sets we have defined before, since the latter are subsets of $\mathbf{R}^2_+\times\Omega$. We say that a process $\psi=\{\psi(\zeta,\xi):\,\zeta,\xi\in\mathbf{R}^2_+\}$ is *predictable* if it is $\mathcal{P}$-measurable as a function of (ζ,ξ,ω). We say that ψ is *adapted* if $\psi(\zeta,\xi)$ is $\mathcal{F}_{\zeta\vee\xi}$-measurable for each ζ, ξ. In the following, these notions will be used for processes of the form $\psi=\{\psi(\zeta,\xi):\,\zeta,\xi\prec z_0\}$ without further comment.

Let $\mathcal{L}^2_{MM}(z_0)$ be the class of all processes $\psi = \{\psi(\zeta, \xi)\colon \zeta, \xi \prec z_0\}$ satisfying

(1) ψ is predictable;

(2) $\psi(\zeta, \xi) = 0$ unless $\zeta \curlywedge \xi$;

(3) $E\{\iint_{R_{z_0} \times R_{z_0}} \psi^2(\zeta, \xi)\, d[M]^2_\zeta\, d[M]^1_\xi\} < \infty$;

and let $\mathcal{L}^2_{MM}$ be the class of all processes ψ on $\mathbf{R}^2_+ \times \mathbf{R}^2_+$ satisfying (1), (2) and (3) for all $z_0 \in \mathbf{R}^2_+$. We give $\mathcal{L}^2_{MM}(z_0)$ the scalar product

$$(\psi, \chi) = E\left\{\iint_{R_{z_0} \times R_{z_0}} \psi(\zeta, \xi)\, \chi(\zeta, \xi)\, d[M]^2_\zeta\, d[M]^1_\xi\right\}.$$

Then, with the obvious identifications, $\mathcal{L}^2_{MM}(z_0)$ is a Hilbert space and the simple functions form a dense subset. The map $\psi \to \psi \cdot MM$ of simple functions into $\mathcal{M}^2(z_0)$ preserves the norm by (2.19), hence it can be extended by continuity to a linear map from $\mathcal{L}^2_{MM}(z_0)$ into $\mathcal{M}^2(z_0)$. To summarize:

THEOREM 2.5. *Let M be a right-continuous strong martingale for which $E\{M^4_{z_0}\} < \infty$. Then the mapping $\psi \to \psi \cdot MM$ defined above is a norm-preserving linear map of $\mathcal{L}^2_{MM}(z_0)$ into $\mathcal{M}^2(z_0)$ which satisfies* (2.18) *and* (2.19). *Furthermore, $\psi \cdot MM$ is orthogonal to M.*

Remarks. 1°. The fact that $\psi \cdot MM$ is continuous, if M is, follows from Proposition 1.4. Moreover, $\psi \cdot MM \perp M$ is a consequence of Proposition 2.4 (b).

2°. In general, $\psi \cdot MM$ is not a strong martingale.

3°. We will often denote $\psi \cdot MM_z$ by $\iint_{R_z \times R_z} \psi\, dM\, dM$. We will also write $\iint_{A \times B} \psi\, dM\, dM$ and $\iint_{A \times B} \psi\chi\, d[M]^2\, d[M]^1$ instead of $(\psi I_{A \times B}) \cdot MM_{z_0}$ and $\iint_{A \times B} \psi(\zeta, \xi)\, \chi(\zeta, \xi)\, d[M]^2_\zeta\, d[M]^1_\xi$ respectively (A, B Borel subsets of R_{z_0}).

4°. One can extend the integral to $M \in \mathcal{M}^4_S$ and $\psi \in \mathcal{L}^2_{MM}$ and we will use this extension without further comment.

We can get some insight into the integral $\psi \cdot MM$ by considering it as an iterated integral of the form

$$\psi \cdot MM_z = \int_{R_z} \left(\int_{R_z} \psi(\zeta, \xi)\, dM_\xi \right) dM_\zeta.$$

Here, $\psi \cdot MM_z$ is the integral, first of an $\mathcal{F}^1_\xi$-adapted process, then of an $\mathcal{F}^2_\zeta$-adapted process. To make this rigorous, we must prove a type of stochastic Fubini's theorem. We will confine ourselves to the case of a martingale $M \in \mathcal{M}^4_S(z_0)$ such that $[M]^1$ and $[M^2]$ are *deterministic*, i.e. independent of ω. (Incidentally, this is another case where we can prove that $[M]^1 = [M]^2$.) Then we have the following theorem:

Theorem 2.6. *If $\psi \in \mathfrak{L}^2_{MM}(z_0)$, then $\psi(\zeta, \cdot)$ is $\mathcal{F}^1_\xi$-predictable and $E\{\int_{R_{z_0}} \psi^2(\zeta,\xi)\, d[M]^1_\xi\} < \infty$ for $d[M]^2 -$ a.e. $\zeta \prec z_0$. Furthermore, we can define a process $\{I(\zeta), \zeta \in R_{z_0}\}$ such that*

(a) *$I(\zeta)$ is $\mathcal{F}^2_\zeta$-predictable;*

(b) $E\{\int_{R_{z_0}} I(\zeta)^2\, d[M]^2_\zeta\} = E\{\iint_{R_{z_0}\times R_{z_0}} \psi^2(\zeta, \xi)\, d[M]^2_\zeta\, d[M]^1_\xi]\}$;

(c) $I(\zeta) = \int_{R_{z_0}} \psi(\zeta, \xi)\, dM_\xi$ for $d[M]^2$ — a.e. ζ;

(d) $\int_{R_{z_0}} I(\zeta)\, dM_\zeta = \psi \cdot MM_{z_0}$.

Remark. If we interchange 1 and 2 above, (d) becomes the "stochastic Fubini's theorem":

$$\int_{R_{z_0}} \left(\int_{R_{z_0}} \psi(\zeta, \xi)\, dM_\xi \right) dM_\zeta = \int_{R_{z_0}} \left(\int_{R_{z_0}} \psi(\zeta, \xi)\, dM_\zeta \right) dM_\xi = \psi \cdot MM_{z_0}. \tag{2.20}$$

Proof. Let us suppose ψ is of the form (2.15). If we adopt the notation of (2.15), we can let

$$I(\zeta) = \alpha I_{\Delta_{ij}}(\zeta)\, M(\Delta_{kl}). \tag{2.21}$$

One sees by inspection that $I(\zeta)$ is $\mathcal{F}^2_\zeta$-predictable and that its integral is given by

$$I \cdot M_{z_0} = \alpha M(\Delta_{ij})\, M(\Delta_{kl}) = \psi \cdot MM_{z_0}. \tag{2.22}$$

Furthermore,

$$E\left\{\int_{R_{z_0}} I^2(\zeta)\, d[M]^2_\zeta\right\} = E\{\alpha^2 [M]^2(\Delta_{ij})\, M(\Delta_{kl})^2\}.$$

Both α and $[M]^2(\Delta_{ij})$ are $\mathcal{F}^1_{z_{kl}}$-measurable, while $E\{M(\Delta_{kl})^2 \mid \mathcal{F}^1_{z_{kl}}\} = E\{[M]^1(\Delta_{kl}) \mid \mathcal{F}^1_{z_{kl}}\}$, so if we condition first by $\mathcal{F}^1_{z_{kl}}$, the above becomes

$$= E\{\alpha^2 [M]^2(\Delta_{ij})\, [M]^1(\Delta_{kl})\} = E\left\{\iint_{R_{z_0}\times R_{z_0}} \psi^2(\zeta, \xi)\, d[M]^2_\zeta\, d[M]^1_\xi\right\}.$$

Thus (a)–(d) hold for ψ of the form (2.15) and, by an easy extension, for simple ψ.

In general, if $\psi \in \mathfrak{L}^2_{MM}(z_0)$, there exist simple ψ_n such that

$$E\left\{\iint_{R_{z_0}\times R_{z_0}} (\psi_n(\zeta, \xi) - \psi(\zeta, \xi))^2 d[M]^2_\zeta\, d[M]^1_\xi\right\} \to 0.$$

By taking a subsequence, if necessary, we can suppose that, for $d[M]^2$—a.e. ζ,

$$E\left\{\int_{R_{z_0}} (\psi_n(\zeta, \xi) - \psi(\zeta, \xi))^2 d[M]^1_\xi\right\} < 2^{-n}, \tag{2.23}$$

for all large enough n. Let $I_n(\zeta)=\psi_n(\zeta,\ \cdot\ )\cdot M_{z_0}$ and define

$$I(\zeta)=\begin{cases}\lim_{n\to\infty} I_n(\zeta) & \text{if the limit exists,}\\ 0 & \text{otherwise.}\end{cases}$$

Then (c) holds and since each I_n is $\mathcal{F}^2_\zeta$-predictable, so is I and (a) is satisfied. Moreover, from (b), which holds for simple ψ, we have

$$E\left\{\int_{R_{z_0}} (I_m(\zeta)-I_n(\zeta))^2 d[M]^2_\zeta\right\}=E\left\{\iint_{R_{z_0}\times R_{z_0}} (\psi_m(\zeta,\xi)-\psi_n(\zeta,\xi))^2 d[M]^2_\zeta d[M]^1_\xi\right\}.$$

Since $\{\psi_n\}$ is a Cauchy sequence in $\mathcal{L}^2_{MM}(z_0)$, $\{I_n\}$ is also a Cauchy sequence and its limit must be I. Thus we can pass to the limit to get (b). Furthermore, by Theorem 2.3,

$$I\cdot M_{z_0}=\lim_{n\to\infty} I_n\cdot M_{z_0},$$

where the limit takes place in L^2. At the same time we have

$$I_n\cdot M_{z_0}=\psi_n\cdot MM_{z_0}$$

and

$$\psi\cdot MM_{z_0}=\lim_{n\to\infty}\psi_n\cdot MM_{z_0},$$

where the limit again is in L^2, which implies that $I\cdot M_{z_0}=\psi\cdot MM_{z_0}$.

§ 3. The representation of square integrable martingales

It is well-known that every square-integrable martingale relative to the natural fields of Brownian motion can be written as a constant plus a stochastic integral. This is an immediate consequence of Ito's orthogonal decomposition of a square integrable functional of a normal random measure into multiple Wiener integrals [8] and the remark of Ito (see [8], Theorem 5.1) that in the particular case of Brownian motion, these integrals become iterated stochastic integrals. Such a decomposition is no longer possible in the case of the two-parameter Wiener process, at least if by stochastic integral one means the stochastic integral of an adapted function. However, it is possible if one allows stochastic integrals of functions which are $\mathcal{F}^1_z$- or $\mathcal{F}^2_z$-adapted. More precisely, we have the following theorem, which was recently proved by Wong and Zakai. For this section, the fields $\mathcal{F}_z$ are those generated by W: $\mathcal{F}_z=\sigma\{W_\xi,\ \xi\prec z\}$.

THEOREM 3.1. (Wong and Zakai) *If $M=\{M_z,\ \mathcal{F}_z,\ z\in\mathbf{R}^2_+\}$ is a square integrable martingale, then for each $z\in\mathbf{R}^2_+$,*

$$M_z=M_0+\phi\cdot W_z+\psi\cdot WW_z, \tag{3.1}$$

where $\phi\in\mathcal{L}^2_W$ and $\psi\in\mathcal{L}^2_{WW}$.

This was proved in [18]. Because we will need it in what follows, we thought it worthwhile to give an elementary proof here, based on Green's formula (6.8) and the completeness of the Hermite polynomials.

We begin with a simple lemma, which is a special case of a result (not given here) on products of stochastic integrals.

If A is a rectangle, $\mathfrak{J}_A$ (resp. $\mathfrak{J}_{AA}$) is the class of functions $\phi \in \mathcal{L}^2_W$ (resp. $\psi \in \mathcal{L}^2_{WW}$) such that for each $\xi \in A$ (resp. $\zeta, \xi \in A$) $\phi(\xi)$ is $\mathcal{G}_\xi$-measurable (resp. $\psi(\zeta, \xi)$ is $\mathcal{G}_{\zeta \vee \xi}$-measurable), where $\mathcal{G}_\xi = \sigma\{W(A \cap R_{\xi'}), \xi' \prec \xi\}$.

Lemma 3.2. *Let A_1 and A_2 be two disjoint rectangles contained in R_{z_0} such that $A_1 \cup A_2$ is a rectangle A. If $\phi_i \in \mathfrak{J}_{A_i}$ and $\psi_i \in \mathfrak{J}_{A_i A_i}$ $(i=1, 2)$, there exist $\phi \in \mathfrak{J}_A$ and $\psi \in \mathfrak{J}_{AA}$ such that*

$$\left(\int_{A_1} \phi_1 \, dW + \iint_{A_1 \times A_1} \psi_1 \, dW \, dW\right)\left(\int_{A_2} \phi_2 \, dW + \iint_{A_2 \times A_2} \psi_2 \, dW \, dW\right)$$

$$= \int_A \phi \, dW + \iint_{A \times A} \psi \, dW \, dW.$$

Proof. We must verify that each of the four terms of the product on the left-hand side can be written in the form of the right-hand side. We will only consider the fourth term. The verification in the other three cases is similar, and in fact simpler. We have

$$\iint_{A_1 \times A_1} \psi_1 \, dW \, dW \iint_{A_2 \times A_2} \psi_2 \, dW \, dW = \iint_{A \times A} \psi \, dW \, dW, \tag{3.2}$$

where $\psi = \psi' + \psi''$, with ψ' and ψ'' defined by the following formulas in the case where A_1 is to the left of A_2:

$$\begin{aligned} \psi'(\zeta, \xi) &= \begin{cases} \psi_2(\zeta, \xi) \displaystyle\iint_{(A_1 \cap Q_\zeta) \times (A_1 \cap Q_\xi)} \psi_1 \, dW \, dW & \text{if } \zeta, \xi \in A_2, \\ 0 & \text{otherwise}, \end{cases} \\ \psi''(\zeta, \xi) &= \begin{cases} \displaystyle\int_{A_1} \psi_1(\zeta, \zeta') \, dW_{\zeta'} \int_{A_2 \cap Q_\zeta} \psi_2(\xi', \xi) \, dW_{\xi'} & \text{if } \zeta \in A_1 \text{ and } \xi \in A_2, \\ 0 & \text{otherwise}, \end{cases} \end{aligned} \tag{3.3}$$

where Q_ζ is the strip bounded by the axes and by the horizontal line which passes through ζ.

To prove this for simple functions it is enough to consider ψ_1 and ψ_2 of the form

$$\psi_i(\zeta, \xi) = \alpha_i I_{B_i \times B_i'}(\zeta, \xi),$$

where B_i, B_i' are rectangles contained in A_i, such that $B_i \curlywedge B_i'$(1) and having lower left-

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(1) $B_i \curlywedge B_i'$ iff $\zeta \in B_i$ and $\xi \in B_i'$ implies $\zeta \curlywedge \xi$.

hand corners at z_i and z_i', respectively, and where α_i is a bounded $\mathcal{G}_{z_i \vee z_i'}$-measurable r.v. ($i=1, 2$). In this case,

$$\iint_{A\times A} \psi\, dW\, dW = \alpha_1 \alpha_2\, W(B_1')\, W(B_2') \left[\int_{B_1} W(B_2 \cap Q_z)\, dW_z + \int_{B_2} W(B_1 \cap Q_z)\, dW_z \right]$$

$$= \alpha_1 \alpha_2\, W(B_1')\, W(B_2')\, W(B_1)\, W(B_2) = \iint_{A_1\times A_1} \psi_1\, dW\, dW \iint_{A_2\times A_2} \psi_2\, dW\, dW.$$

For the general case, we consider two sequences of simple functions $\{\psi_1^n\}$ and $\{\psi_2^n\}$ converging to ψ_1 and ψ_2 in $\mathcal{F}_{A_1 A_1}$ and $\mathcal{F}_{A_2 A_2}$ respectively. If we define ψ^n by (3.3), starting with ψ_1^n and ψ_2^n, we have by the foregoing that

$$\iint_{A_1\times A_1} \psi_1^n\, dW\, dW \iint_{A_2\times A_2} \psi_2^n\, dW\, dW = \iint_{A\times A} \psi^n\, dW\, dW.$$

Each term on the left-hand side converges in L^2 to the corresponding term on the left-hand side of (3.2). On the right-hand side, an easy calculation shows that

$$E\left\{\iint_{A\times A} (\psi - \psi^n)^2(\zeta, \xi)\, d\zeta\, d\xi\right\} \leqslant \text{const.} \left[E\left\{\iint_{R_{z_0}\times R_{z_0}} (\psi_1 - \psi_1^n)^2(\zeta, \xi)\, d\zeta\, d\xi\right\} + E\left\{\iint_{R_{z_0}\times R_{z_0}} (\psi_2 - \psi_2^n)^2(\zeta, \xi)\, d\zeta\, d\xi\right\}\right],$$

which implies that the right-hand side also converges to the right-hand side of (3.2). qed

Proof of Theorem 3.1. Notice that the representation is unique (up to negligible sets), since if $M = \phi \cdot W + \psi \cdot WW = \phi' \cdot W + \psi' \cdot WW$, then

$$0 = E\{[(\phi - \phi') \cdot W_z + (\psi - \psi') \cdot WW_z]^2\}$$

$$= E\left\{\int_{R_z} (\phi - \phi')^2(\xi)\, d\xi + \iint_{R_z\times R_z} (\psi - \psi')^2(\zeta, \xi)\, d\zeta\, d\xi\right\}.$$

It is enough to prove that if $z_0 \in \mathbf{R}_+^2$ and if $X \in L^2$ is an $\mathcal{F}_{z_0}$-measurable r. v., then there exist $\phi \in \mathcal{L}_W^2(z_0)$ and $\psi \in \mathcal{L}_{WW}^2(z_0)$ such that

$$X = E\{X\} + \int_{R_{z_0}} \phi\, dW + \iint_{R_{z_0}\times R_{z_0}} \psi\, dW\, dW. \tag{3.4}$$

Indeed, if M is a square integrable martingale, let $z_n = (n, n)$. Then there exist ϕ_n and ψ_n such that (3.4) holds with z_0, X, ϕ and ψ replaced by z_n, M_{z_n}, ϕ_n and ψ_n respectively. Taking conditional expectations, if $z \prec z_n$

$$M_z = E\{M_{z_n} \mid \mathcal{F}_z\} = E\{M_{z_n}\} + \phi_n \cdot W_z + \psi_n \cdot WW_z.$$

By uniqueness, with probability one, $\phi_n = \phi_{n-1}$ and $\psi_n = \psi_{n-1}$ a.e. on $R_{z_{n-1}}$ and $R_{z_{n-1}} \times R_{z_{n-1}}$ respectively, so that (3.1) holds with $\phi(\xi) = \phi_n(\xi)$, if $\xi \in R_{z_n}$, and $\psi(\zeta, \xi) = \psi_n(\zeta, \xi)$, if $\zeta, \xi \in R_{z_n}$.

Let $z_0 \in \mathbf{R}^2_+$ and divide R_{z_0} into mn congruent subrectangles $A_{ij} = (z_{ij}, z_{i+1,j+1}]$ $(i = 1, \ldots, m, j = 1, \ldots, n)$. Fix i and j for the moment and set

$$\hat{W}_z = W((z_{ij}, z_{ij} + z]).$$

Then $\{\hat{W}_z, z \in \mathbf{R}^2_+\}$ is a two-parameter Wiener process and if $H_p(x, t)$ is the p^{th} Hermite polynomial, an application of Green's formula (Theorem 6.1) gives, for $p \geqslant 1$,

$$H_p(\hat{W}_w, |A_{ij}|) = \int_{R_w} H_{p-1}(\hat{W}_\xi, |R_\xi|)\, d\hat{W}_\xi + \int_{R_w} H'_{p-1}(\hat{W}_\xi, |R_\xi|)\, d\hat{J}_\xi,$$

where $w = z_{i+1,j+1} - z_{ij}$ and $|A|$ is the area of A. The left-hand side is just $H_p(W(A_{ij}), |A_{ij}|)$ and the first term on the right-hand side is

$$\int_{A_{ij}} H_{p-1}(A_{ij} \cap R_\xi, |A_{ij} \cap R_\xi|)\, dW_\xi.$$

The second term on the right-hand side can be written

$$\iint_{A_{ij} \times A_{ij}} I_{\{\zeta \curlywedge \xi\}}(\zeta, \xi) H'_{p-1}(W(A_{ij} \cap R_{\zeta \vee \xi}), |A_{ij} \cap R_{\zeta \vee \xi}|)\, dW_\zeta\, dW_\xi.$$

Thus, for each i, j and $p \geqslant 1$, we have

$$H_p(W(A_{ij}), |A_{ij}|) = \int_{A_{ij}} \phi^{ij}\, dW + \iint_{A_{ij} \times A_{ij}} \psi^{ij}\, dW\, dW, \tag{3.5}$$

where $\phi^{ij} \in \mathfrak{J}_{A_{ij}}$ and $\psi^{ij} \in \mathfrak{J}_{A_{ij} A_{ij}}$. Since the Hermite polynomials form a complete orthogonal system in $L^2(\mathbf{R}, \exp(-x^2/2t)\, dx)$, if f is in that space for $t = |A_{ij}|$, it follows that $f(W(A_{ij}))$ can be written in the form of a constant (which comes from the term $H_0 \equiv 1$) plus a term of the type given in (3.5). Consequently, if for each i, j, $f_{ij} \in L^2(\mathbf{R}, \exp(-x^2/2|A_{ij}|)\, dx)$, we have

$$\prod_{i=1}^{m} \prod_{j=1}^{n} f_{ij}(W(A_{ij})) = \prod_{i=1}^{m} \prod_{j=1}^{n} \left(c_{ij} + \int_{A_{ij}} \phi^{ij}\, dW + \iint_{A_{ij} \times A_{ij}} \psi^{ij}\, dW\, dW\right).$$

Now this is a sum of a constant plus terms of the form

$$\text{const.} \prod \left(\int_{A_{ij}} \phi^{ij}\, dW + \iint_{A_{ij} \times A_{ij}} \psi^{ij}\, dW\, dW\right),$$

where the product is over some subset of $\{(i, j)\colon 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n\}$. But each of these products can be represented in the form (3.4). This can be seen, using Lemma 3.2, by induction first on the rectangles belonging to a single column of the subdivision of R_{z_0}, and then on these columns. Hence (3.4) is true for the X in L^2 of the particular form $\Pi_{i=1}^{m} \Pi_{j=1}^{n} f_{ij}(W(A_{ij}))$. The passage to the X in L^2 of the form $f(W(A_{11}), ..., W(A_{mn}))$ and, afterwards, to the general $\mathcal{F}_{z_0}$-measurable r.v.'s X in L^2 is then routine. qed

One result which can be deduced either from Theorem 3.1 or directly (see [14]) is the zero-one law.

COROLLARY 3.3 *The field* $\bigcap_{0 \ll z} \mathcal{F}_z^1 \vee \mathcal{F}_z^2$ *is trivial.*

Here is another immediate consequence of Theorem 3.1. Recall that, in this section, the fields $\mathcal{F}_z$ are those generated by W.

COROLLARY 3.4. *If* M *is a martingale such* $E\{|M_z|\log^+|M_z|\} < \infty$ *for all* $z \in \mathbf{R}_+^2$, *then* M *has a continuous version.*

Proof. This holds for square integrable M since the stochastic integrals in (3.1) are continuous (Theorems 2.2 and 2.5). The extension to the M for which $E\{|M_z|\log^+|M_z|\} < \infty$ is immediate thanks to the maximal inequality (Theorem 1.2). qed

Note. We can not extend Corollary 3.4 to L^1-bounded martingales. In fact, such martingales can have oscillatory discontinuities and do not necessarily have a right continuous version, as the following example shows. This is based on known examples and the observation that one can construct independent two-(space) dimensional Brownian motions $\{B_s\}$ and $\{\hat{B}_t\}$ such that $\{(B_s, \hat{B}_t)\}$ is $\mathcal{F}_{st}$-adapted. To do this, define

$$B_s = \sqrt{2}\,(W_{s,1/2} - W_{1,1/2},\ W_{s,1} - W_{s,1/2} - W_{1,1} + W_{1,1/2}), \quad s \geqslant 1,$$

$$\hat{B}_t = \sqrt{2}\,(W_{1/2,t} - W_{1/2,1},\ W_{1,t} - W_{1/2,t} - W_{1,1} + W_{1/2,1}), \quad t \geqslant 1.$$

Then $\{(B_s, \hat{B}_t)\}$ is $\mathcal{F}_{st}$-adapted. Let f be defined on the product of the unit circle with itself, and let

$$\sigma = \inf\{s \geqslant 1: |B_s| = 1\}, \tau = \inf\{t \geqslant 1: |\hat{B}_t| = 1\}.$$

If f is integrable, then

$$M_{st} = E\{f(B_\sigma, \hat{B}_\tau) \mid \mathcal{F}_{st}\}, (1, 1) \prec (s, t),$$

is a martingale and if h is the biharmonic function on the product of the unit disc with itself which has boundary values f, then

$$M_{st} = h(B_s, \hat{B}_t) \quad \text{for } 1 \leqslant s \leqslant \sigma \quad \text{and } 1 \leqslant t \leqslant \tau.$$

We know we can choose f such that

$$\limsup_{s \uparrow \sigma,\, t \uparrow \tau} h(B_s, \hat{B}_t) = \infty \quad \text{and} \quad \liminf_{s \uparrow \sigma,\, t \uparrow \tau} h(B_s, \hat{B}_t) = -\infty.$$

This shows that $\lim_{s\uparrow\sigma, t\uparrow\tau} M_{st}$ doesn't exist. (See [17], [9] and [2], which also gives some further references.)

Now we can use this to construct worse examples in various ways. For instance, let us notice that we can define countable families of independent two-dimensional Brownian motions $\{B_s^n, n=1, 2, ..., s \geqslant 1\}$ and $\{\hat{B}_t^n, n=1, 2, ..., t \geqslant 1\}$ by essentially the same trick as above and in such a way that the processes $\{(B_s^n, \hat{B}_t^n), s, t \geqslant 1\}_{n=1}^{\infty}$ are independent and $\mathfrak{F}_{st}$-adapted. Let

$$\sigma_n = \inf\{s \geqslant 1: |B_s^n| = 1\} \quad \text{and} \quad \tau_n = \inf\{t \geqslant 1: |\hat{B}_t^n| = 1\}.$$

Then $\{(\sigma_n, \tau_n), n=1, 2, ...\}$ are i.i.d. and thus it is easy to see that, with probability one, the family (σ_n, τ_n) is dense in $\{(s, t): s \geqslant 1, t \geqslant 1\}$. Define

$$M_{st} = \sum_{n=1}^{\infty} 2^{-n} E\{f(B_\sigma^n, \hat{B}_\tau^n) | \mathfrak{F}_{st}\}.$$

Then $\{M_{st}, s, t \geqslant 1\}$ is an L^1-bounded martingale with lim sup $= \infty$ and lim inf $= -\infty$ at each (σ_n, τ_n). These being dense, it follows that at each point $(s, t) \succ (1, 1)$,

$$\limsup_{(u,v)\uparrow(s,t)} M_{uv} = \limsup_{(u,v)\downarrow(s,t)} M_{uv} = \infty$$

and

$$\liminf_{(u,v)\uparrow(s,t)} M_{uv} = \liminf_{(u,v)\downarrow(s,t)} M_{uv} = -\infty.$$

§ 4. Line integrals

Let Γ be a curve in $\mathbf{R}_+^2$ given by the parametric representation:

$$\{z: z = \gamma(\sigma), 0 \leqslant \sigma \leqslant 1\}, \tag{4.1}$$

where $\gamma: [0, 1] \rightarrow \mathbf{R}_+^2$ is a continuous function. Let $M \in \mathcal{M}^2$ and suppose Γ is an increasing path, i.e. $\gamma(\sigma) \prec \gamma(\sigma')$ if $\sigma \leqslant \sigma'$. We can define line integrals along Γ with respect to M: just notice that $N_\sigma \overset{\text{def}}{=} M_{\gamma(\sigma)}$, $0 \leqslant \sigma \leqslant 1$, is a classical square integrable martingale and that therefore one can define $\int_\Gamma \phi \partial M = \int_0^1 \phi(\gamma(\sigma)) dN_\sigma$ as an Ito integral. But this works only for increasing paths and wouldn't allow us, for instance, to integrate around a circle. We will take another tack which will allow us to define line integrals for all reasonably smooth paths, including all increasing paths. We do this by first defining two integrals, denoted by $\int_\Gamma \phi \partial_1 M$ and $\int_\Gamma \phi \partial_2 M$. One might think of these as the integrals of the stochastic differential forms $\phi \partial_1 M$ and $\phi \partial_2 M$. (We will use the notation $\int \phi \partial M$ for line integrals to avoid confusion with the surface integral $\int \phi dM$.)

Let Γ be an oriented curve with the parametric representation (4.1). There is a curve $\hat{\Gamma}$ of the opposite orientation, which has the representation

$$\{z: z=\hat{\gamma}(\sigma)=\gamma(1-\sigma),\ 0\leqslant\sigma\leqslant 1\}.$$

DEFINITION. *Γ is of type I if it is an increasing path; of type II if $\sigma\leqslant\sigma'$ implies $\gamma(\sigma)\curlywedge\gamma(\sigma')$; and of type I′ (resp. II′) if $\hat{\Gamma}$ is of type I (resp. type II). We say Γ is of pure type if it is of type I, II, I′ or II′.*

Remarks. A type I curve is linearly ordered by "$\prec$", a type II curve by "$\curlywedge$". Horizontal (resp. vertical) lines are simultaneously of type I and II (resp. I and II′) but this will cause no confusion. If Γ is of types I or II, the fields $\mathcal{F}^1_{\gamma(\sigma)}$ increase with σ; if Γ is of type I or II′, $\mathcal{F}^2_{\gamma(\sigma)}$ increases with σ.

Given a curve Γ of pure type, we will define two processes on Γ, M_1^Γ and M_2^Γ, which may be thought of as coming from the horizontal and vertical increments, respectively, of M. A suggestive notation for this would be $dM_1^\Gamma=\partial_1 M$ and $dM_2^\Gamma=\partial_2 M$.

The easiest way to describe these processes is to introduce them first for stepped paths. A polygonal curve Γ is said to be a *stepped path* if its segments are either horizontal or vertical. Let Γ be an increasing stepped path with successive horizontal segments $h_1=[a_1,b_1], \dots, h_n=[a_n,b_n]$ and vertical segments $v_1=[c_1,d_1], \dots, v_m=[c_m,d_m]$, and with initial and final points z_0 and z_f respectively. Suppose, for the moment, that M is continuous and define

$$M_1^\Gamma(z_f)=\sum_{j=1}^{n}(M_{b_j}-M_{a_j})\quad\text{and}\quad M_2^\Gamma(z_f)=\sum_{k=1}^{m}(M_{d_k}-M_{c_k}).\tag{4.2}$$

One could proceed to define M_1^Γ and M_2^Γ for arbitrary type I curves by approximating them by stepped paths—and we shall do this later—but there is a more direct way. If $z\in\mathbf{R}^2_+$, let H_z (resp. V_z) be the horizontal (resp. vertical) line segment connecting z and the t-axis (resp. s-axis). If $z\in\Gamma$, denote by $\bar{D}^1_z$ (resp. $\bar{D}^2_z$) the closed area bounded by V_{z_0}, V_z (resp. H_{z_0}, H_z), Γ and the axis, and let $D^1_z=\bar{D}^1_z-V_{z_0}$ (resp. $D^2_z=\bar{D}^2_z-H_{z_0}$). Then, according to (4.2), we have

$$M_1^\Gamma(z_f)=M(D^1_{z_f})\quad\text{and}\quad M_2^\Gamma(z_f)=M(D^2_{z_f}).$$

Thus, suppose Γ is a curve of pure type with initial and final points z_0 and z_f respectively, and define D^1_z and D^2_z as above.

DEFINITION. *Let $M\in\mathcal{M}^2$. If Γ is of type I or II and $i=1$ (resp. of type I or II′ and $i=2$) put*

$$M_i^\Gamma(z_f) = M(D^i_{z_f}),$$

$$M_i^\Gamma(z) = E\{M(D^i_{z_f}) \mid \mathcal{F}^i_z\}, \quad z \in \Gamma. \tag{4.3}$$

Note that $D^i_{z_f} = D^i_z + (D^i_{z_f} - D^i_z)$ and, if $i=1$, for instance, that this last term is the union of $(D^1_{z_f} - D^1_z) \cap V_z$ and $(D^1_{z_f} - D^1_z) - V_z$. M being a martingale, $E\{M((D^1_{z_f} - D^1_z) - V_z) \mid \mathcal{F}^1_z\} = 0$. Thus, if $M((D^1_{z_f} - D^1_z) \cap V_z) = 0$ (resp. $M((D^2_{z_f} - D^2_z) \cap H_z) = 0$),

$$M_1^\Gamma(z) = M(D^1_z) \quad (\text{resp. } M_2^\Gamma(z) = M(D^2_z)). \tag{4.4}$$

This is the case if M does not charge vertical (resp. horizontal) lines, which happens for example if M is continuous, or if Γ contains no vertical (resp. horizontal) segments, or if M simply does not charge Γ.

PROPOSITION 4.1. *If Γ is of type I or II (resp. I or II′), then $\{M_1^\Gamma(z), \mathcal{F}^1_z, z \in \Gamma\}$ (resp. $\{M_2^\Gamma(z), \mathcal{F}^2_z, z \in \Gamma\}$) is a one-parameter square integrable right-continuous martingale, which is continuous if M is.*

This is immediate, except for the continuity. Before tackling that, we give some simple approximation properties.

PROPOSITION 4.2. *Let Γ and Γ' be curves of type I or II (resp. I or II′) both having initial point z_0 and final point z_f. Suppose Γ' lies above (resp. on the right of) Γ. If A is the open area enclosed by $\Gamma \cup \Gamma'$,*

$$E\{(M_i^{\Gamma'}(z_f) - M_i^\Gamma(z_f))^2\} = E\{\langle M\rangle(\bar{A} - \Gamma)\} \quad (i = 1, 2). \tag{4.5}$$

This is immediate since $M(\bar{A} - \Gamma) = M_i^{\Gamma'}(z_f) - M_i^\Gamma(z_f)$. Two direct consequences are:

COROLLARY 4.3. *Let Γ and Γ' be curves of pure type with the same initial point z_0 and final point z_f. If M does not charge $\Gamma \cup \Gamma'$ and if A is the area enclosed by $\Gamma \cup \Gamma'$,*

$$E\{(M_i^{\Gamma'}(z_f) - M_i^\Gamma(z_f))^2\} = E\{\langle M\rangle(A)\} \quad (i = 1, 2). \tag{4.6}$$

COROLLARY 4.4. *Let Γ be a curve of type I or II (resp. I or II′) with initial point z_0 and final point z_f. Let $\{\Gamma_n\}$ be a sequence of curves such that Γ_n lies above (resp. on the right of) Γ. If Γ_n converges to Γ, then $M_i^{\Gamma_n}(z_f)$ converges to $M_i^\Gamma(z_f)$ in L^2 for $i=1$ (resp. $i=2$).*

Note that a curve of pure type can be approximated from either above or below by a stepped path. For instance, if Γ is of type I, choose points $z_0 \prec z_1 \prec \dots \prec z_n = z_f$ on Γ. Then let Γ^+ and Γ^- be the upper and lower parts of the boundary of $\bigcup_{j=0}^{n-1} [z_j, z_{j+1}]$. The distance from any point of Γ^+ or Γ^- to Γ is less than $\sup_j |z_{j+1} - z_j|$. By taking finer and finer partitions of Γ, we obtain sequences $\{\Gamma_n^+\}$ (resp. $\{\Gamma_n^-\}$) of stepped paths decreasing (resp. increasing) to Γ.

Now suppose Γ is of pure type—say type I—and let $\{\Gamma_n\}$ be a sequence of stepped paths decreasing to Γ as per Corollary 4.4. We may suppose, by taking a subsequence if necessary, that $E\{(M_1^{\Gamma_n}(z_f) - M_1^{\Gamma}(z_f))^2\} < 2^{-n}$. If $z_0 = (s_0, t_0)$ and $z_f = (s_f, t_f)$, define for $s_0 \leqslant s \leqslant s_f$,

$$
\begin{aligned}
N_n(s) &= E\{M_1^{\Gamma_n}(z_f) \mid \mathfrak{F}_{s0}^1\}, \\
N(s) &= E\{M_1^{\Gamma}(z_f) \mid \mathfrak{F}_{s0}^1\}.
\end{aligned}
\tag{4.7}
$$

By the maximal inequality, $N_n(s)$ converges uniformly to $N(s)$. By (4.4)—see also (4.2)—if M is continuous, so is $N_n(s)$ and it follows that $N(s)$ is too. But if $(s, t) \in \Gamma$, then $M_1^{\Gamma}(s, t) = N(s)$, hence $M_1^{\Gamma}(z)$ is continuous and we have proved Proposition 4.1.

Note that it is only for type I curves that we have simultaneously defined M_1^{Γ} and M_2^{Γ}. Denote the restriction of M to Γ by M^{Γ}: $M_z^{\Gamma} = \{M_z, z \in \Gamma\}$. Then we have:

PROPOSITION 4.5. *Let* Γ *be an increasing path with initial point* z_0. *Let* $M, N \in \mathcal{M}^2$ *and suppose* M *does not charge* Γ. *Then*

(a) $M_1^{\Gamma} \perp N_2^{\Gamma}$, *i.e.* $\{M_1^{\Gamma}(z) N_2^{\Gamma}(z), \mathfrak{F}_z, z \in \Gamma\}$ *is a martingale;*
(b) $M_z^{\Gamma} - M_{z_0}^{\Gamma} = M_1^{\Gamma}(z) + M_2^{\Gamma}(z)$;
(c) $\langle M^{\Gamma} \rangle_z = \langle M_1^{\Gamma} \rangle_z + \langle M_2^{\Gamma} \rangle_z$.

Proof. Let $z \prec z'$ and set $A = D_{z'}^1 - D_z^1$, $B = D_{z'}^2 - D_z^2$, where D_z^i is defined as before. Notice that since Γ is increasing, M_1^{Γ} and N_2^{Γ} are adapted. Thus

$$E\{M_1^{\Gamma}(z) N(B) \mid \mathfrak{F}_z\} = E\{M_1^{\Gamma}(z) E\{N(B) \mid \mathfrak{F}_z^2\} \mid \mathfrak{F}_z\} = 0,$$

since N is a martingale. Similarly $E\{M(A) N_2^{\Gamma}(z) \mid \mathfrak{F}_z\} = 0$. It follows that, since $M_1^{\Gamma}(z') = M_1^{\Gamma}(z) + M(A)$ and $N_2^{\Gamma}(z') = N_2^{\Gamma}(z) + N(B)$,

$$E\{M_1^{\Gamma}(z') N_2^{\Gamma}(z') \mid \mathfrak{F}_z\} = M_1^{\Gamma}(z) N_2^{\Gamma}(z) + E\{M(A) N(B) \mid \mathfrak{F}_z\}.$$

We must show the last term vanishes. If $R \subset A$ is a rectangle with upper left-hand corner ξ,

$$E\{M(R) N(B) \mid \mathfrak{F}_z\} = E\{M(R) N(B \cap R_\xi) \mid \mathfrak{F}_z\} + E\{M(R) N(B - R_\xi) \mid \mathfrak{F}_z\}.$$

The first term vanishes, since $B \cap R_\xi \subset R_\xi$ and $E\{M(R) \mid \mathfrak{F}_\xi^1\} = 0$, while the last term vanishes because $E\{N(B - R_\xi) \mid \mathfrak{F}_\xi^2\} = 0$. As M does not charge Γ, we can write $M(A) = \lim_{n \to \infty} M(A_n)$, where A_n is a union of rectangles, so that

$$E\{M(A) N(B) \mid \mathfrak{F}_z\} = \lim_{n \to \infty} E\{M(A_n) N(B) \mid \mathfrak{F}_z\} = 0.$$

To see part (b), just note that if we take $z = z_0$ above,

$$M_{z'} - M_z = M(A \cup B) = M(A) + M(B) = M_1^{\Gamma}(z) + M_2^{\Gamma}(z),$$

for $M(A \cap B) = 0$, since $A \cap B \subset \Gamma$. Finally, (c) follows from (a) and (b). qed

If Γ is of type I or II and $i=1$, or of type I or II′ and $i=2$, then $M_i^\Gamma=\{M_i^\Gamma(z), \mathcal{F}_z^i, z\in\Gamma\}$ is a one-parameter square integrable martingale. Let $\langle M_i^\Gamma\rangle$ be the $\mathcal{F}_z^i$-predictable increasing process associated with M_i^Γ. If Γ is of type I or II and if $\phi=\{\phi_z, z\in\Gamma\}$ is $\mathcal{F}_z^1$-predictable and such that $\int_\Gamma \phi^2 d\langle M_1^\Gamma\rangle<\infty$ a.s., then one can define the Ito integral with respect to M_1^Γ in the usual way:

$$\phi\cdot M_1^\Gamma(z),\quad z\in\Gamma.$$

Since there is some danger of mistaking this for the integral $\phi\cdot M$, we will denote it, in general, by

$$\int_{\Gamma_z}\phi\partial_1 M,\quad z\in\Gamma, \tag{4.8}$$

or just $\int_\Gamma\phi\partial_1 M$ for the integral over all of Γ. Similarly, if Γ is of type I or II′ and $\phi=\{\phi_z, z\in\Gamma\}$ is $\mathcal{F}_z^2$-predictable and such that $\int_\Gamma\phi^2 d\langle M_2^\Gamma\rangle<\infty$ a.s., we can define $\phi\cdot M_2^\Gamma$ as an Ito integral, which we denote by

$$\int_{\Gamma_z}\phi\partial_2 M,\quad z\in\Gamma. \tag{4.9}$$

If Γ is of type I′ or II′ (resp. I′ or II), we define

$$\int_\Gamma\phi\partial_1 M=-\int_{\hat\Gamma}\phi\partial_1 M\quad\left(\text{resp. }\int_\Gamma\phi\partial_2 M=-\int_{\hat\Gamma}\phi\partial_2 M\right), \tag{4.10}$$

where $\hat\Gamma$ is defined in (4.2). Finally, if Γ is of pure type, we let

$$\int_\Gamma\phi\partial M=\int_\Gamma\phi\partial_1 M+\int_\Gamma\phi\partial_2 M. \tag{4.11}$$

Let us remark that the definition of $\int_\Gamma\phi\partial_i M$ can be immediately extended to compact curves which can be broken into a finite or countable number of curves of pure type. We will say that a curve is *piecewise-pure* if it consists of a finite number of curves of pure type.

If Γ is an increasing path, one can define $\int_\Gamma\phi\partial M$ directly, as discussed at the beginning of this section. If M does not charge Γ, the two definitions agree thanks to Proposition 4.5.

We close this section with a theorem which tells us when $\lim_{n\to\infty}\int_{\Gamma_n}\phi_n\partial_i M=\int_\Gamma\partial_i M$.

Let Γ and Γ_n, $n=0, 1, 2, \ldots$, be curves of the same type, either I or II, all having the same initial point $z_0=(s_0, t_0)$ and final point $z_1=(s_1, t_1)$ and such that Γ and Γ_n lie entirely below Γ_0. Denote the area enclosed by $\Gamma\cup\Gamma_n$ by A_n. For $s\geqslant 0$ and any curve Λ, let $v_\Lambda(s)$

be the point (s, τ), where $\tau = \inf\{t: (s, t) \in \Lambda\}$. If $M \in \mathcal{M}^2$, define martingales (relative to $\{\mathcal{F}^1_{s0}\}$) N and N_n by (4.7). Then

$$N(s) = M_1^{\Gamma}(v_{\Gamma}(s)), \quad s_0 \leqslant s \leqslant s_1,$$

$$N_n(s) = M_1^{\Gamma_n}(v_{\Gamma_n}(s)), \quad n = 0, 1, 2, \ldots, s_0 \leqslant s \leqslant s_1.$$

PROPOSITION 4.6. *Let $M \in \mathcal{M}^2_S$ and suppose that $\{\phi(z)\}$ and $\{\phi_n(z)\}$, $n = 1, 2, \ldots$, are $\mathcal{F}^1_z$-predictable processes defined for z in Γ and Γ_n respectively. Define functions ψ and ψ_n respectively by $\psi(s) = \phi(v_{\Gamma}(s))$ and $\psi_n(s) = \phi_n(v_{\Gamma_n}(s))$, $s_0 \leqslant s \leqslant s_1$. Suppose that*

(a) $\lim_{n\to\infty} E\{M^2(A_n)\} = 0;$

(b) $E\{\int_{s_0}^{s_1} \psi_n^2(s)\, d\langle N_0\rangle_s\} < \infty, E\{\int_{s_0}^{s_1} \psi^2(s)\, d\langle N_0\rangle_s\} < \infty;$

(c) $\lim_{n\to\infty} E\{\int_{s_0}^{s_1} (\psi_n(s) - \psi(s))^2\, d\langle N_0\rangle_s\} = 0.$

Then

$$\lim_{n\to\infty} \int_{\Gamma_n} \phi_n \partial_1 M = \int_{\Gamma} \phi \partial_1 M \quad \text{in } L^2.$$

Proof. Suppose for the moment that Γ_0 and Γ are stepped paths. Using the fact that Γ_0 lies above Γ and that M is a strong martingale, we see that N and $N_0 - N$ are orthogonal. This remains true in the general case, since then one can approximate Γ_0 and Γ by stepped paths and use Corollary 4.4 to pass to the limit. Similarly, N_n and $N_0 - N_n$ are orthogonal. Thus

$$\langle N_0\rangle = \langle N\rangle + \langle N_0 - N\rangle = \langle N_n\rangle + \langle N_0 - N_n\rangle.$$

It follows that $d\langle N\rangle \leqslant d\langle N_0\rangle$ and $d\langle N_n\rangle \leqslant d\langle N_0\rangle$. Since $\langle N - N_n\rangle \leqslant 2\langle N\rangle + 2\langle N_n\rangle$, we have $d\langle N - N_n\rangle \leqslant 4d\langle N_0\rangle$. Now

$$\int_{\Gamma} \phi\partial_1 M - \int_{\Gamma_n} \phi\partial_1 M = \int_{s_0}^{s_1} \psi\, dN - \int_{s_0}^{s_1} \psi_n\, dN_n$$

$$= \int_{s_0}^{s_1} (\psi - \psi_n)\, dN + \int_{s_0}^{s_1} (\psi_n - \psi)\, d(N - N_n) + \int_{s_0}^{s_1} \psi\, d(N - N_n).$$

Each of these integrals tends to zero in L^2, as $n \to \infty$. Indeed,

$$E\left\{\left(\int_{s_0}^{s_1} (\psi - \psi_n)\, dN\right)^2\right\} = E\left\{\int_{s_0}^{s_1} (\psi - \psi_n)^2\, d\langle N\rangle\right\} \leqslant E\left\{\int_{s_0}^{s_1} (\psi - \psi_n)^2\, d\langle N_0\rangle\right\},$$

which tends to zero by hypothesis. Similarly

$$E\left\{\left(\int_{s_0}^{s_1} (\psi_n - \psi)\, d(N - N_n)\right)^2\right\} = E\left\{\int_{s_0}^{s_1} (\psi_n - \psi)^2\, d\langle N - N_n\rangle\right\} \leqslant 4E\left\{\int_{s_0}^{s_1} (\psi_n - \psi)^2\, d\langle N_0\rangle\right\},$$

which also tends to zero. Finally,

$$E\left\{\left(\int_{s_0}^{s_1} \psi\, d(N-N_n)\right)^2\right\} = E\left\{\int_{s_0}^{s_1} \psi^2\, d\langle N-N_n\rangle\right\}.$$

Write $\psi^2 = \psi^2 I_{\{|\psi|\leqslant m\}} + \psi^2 I_{\{|\psi|>m\}} \leqslant m^2 + \psi^2 I_{\{|\psi|>m\}}$. Then the above expectation is

$$\leqslant m^2 E\left\{\int_{s_0}^{s_1} d\langle N-N_n\rangle\right\} + 4E\left\{\int_{s_0}^{s_1} \psi^2 I_{\{|\psi|>m\}}\, d\langle N_0\rangle\right\}$$

$$= m^2 E\{M^2(A_n)\} + 4E\left\{\int_{s_0}^{s_1} \psi^2 I_{\{|\psi|>m\}}\, d\langle N_0\rangle\right\}.$$

Let first n and then m tend to infinity. The first term goes to zero by hypothesis (a) and the second by the dominated convergence theorem. qed

Our main applications will be to the case where $M = W$. In this case, $d\langle N_0\rangle = t\,ds$, so the conditions become simpler.

COROLLARY 4.7. *Suppose $\phi = \{\phi_z, z \in \mathbf{R}_+^2\}$ is an $\mathcal{F}_z^1$-adapted measurable process such that*

(a) *$E\{\phi_z^2\}$ is bounded for z in compact sets;*

(b) *for all s, t, $\lim_{t'\to t} E\{(\phi_{st'} - \phi_{st})^2\} = 0$.*

If Γ and Γ_n, $n = 1, 2, \ldots$, are curves of type I or II, having the same initial and final points and such that the area enclosed by Γ and Γ_n tends to zero as $n \to \infty$, then we have

$$\lim_{n\to\infty} \int_{\Gamma_n} \phi\,\partial_1 W = \int_{\Gamma} \phi\,\partial_1 M \quad \textit{in } L^2.$$

Of course the symmetric versions of Proposition 4.6 and its corollary hold for integrals with respect to $\partial_2 M$ and $\partial_2 W$. In particular, if ϕ is $\mathcal{F}_z$-adapted and measurable, we can apply Corollary 4.7 to both $\partial_1 W$ and $\partial_2 W$ to get the following result:

COROLLARY 4.8. *Suppose $\phi = \{\phi_z, z \in \mathbf{R}_+^2\}$ is an adapted measurable process which is continuous in the L^2-mean. If Γ_n and Γ are curves of the same pure type, having the same initial and final points and such that the area bounded by $\Gamma \cup \Gamma_n$ goes to zero, then*

$$\lim_{n\to\infty} \int_{\Gamma_n} \phi\,\partial_i W = \int_{\Gamma} \phi\,\partial_i W \quad \textit{in } L^2 \quad (i = 1, 2).$$

§5. A mixed integral

Let $M \in \mathcal{M}^2(z_0)$. Let H_{st} be the horizontal line segment connecting (s, t) with the t-axis and consider the integral

$$I_{s_0 t} = \int_{H_{s_0 t}} \phi \partial_1 M. \tag{5.1}$$

Under suitable conditions, which we shall make precise shortly, we can integrate $I_{s_0 t}$ with respect to t to get an iterated integral

$$\int_0^{t_0} \int_0^{s_0} \phi \partial_1 M \, dt \overset{\text{def}}{=} \int_0^{t_0} I_{s_0 t} \, dt.$$

Note that it would make no sense to integrate over t first, then with respect to $\partial_1 M$, for $\partial_1 M$ depends on t.

Recall that the process $[M]^1$ is the unique process which is increasing and $\mathfrak{F}_{st}$-predictable in the first parameter and such that $M^2 - [M]^1$ is a 1-martingale. Suppose we have chosen $[M]^1_{st}$ measurably in the pairs (s, t), which we can certainly do if, for instance, M is a strong martingale, for $[M]^1$ is then right-continuous. If ϕ is a positive measurable process, then

$$\int_0^{t_0} \left(\int_0^{s_0} \phi_{st} \partial_s [M]^1_{st} \right) dt$$

makes sense.

Proposition 5.1. *Let* $M \in \mathcal{M}^2(z_0)$ *and suppose that* ϕ *is* $\mathfrak{F}^1_z$*-predictable and satisfies* $E\{\int_0^{t_0} \int_0^{s_0} \phi^2_{st} d_s [M]^1_{st} dt\} < \infty$. *Then there exists a measurable process* $\{I_z, z \prec z_0\}$ *such that*

(a) *for a.e. (Lebesgue) fixed* $t \leqslant t_0$,

$$P\left\{ I_{st} = \int_{H_{st}} \phi \partial_1 M, \text{ for each } s \leqslant s_0 \right\} = 1,$$

and consequently $\{I_{st}, \mathfrak{F}^1_{st}, s \leqslant s_0\}$ *is a one-parameter right-continuous martingale, continuous if* M *is;*

(b) $E\{\sup_{s \leqslant s_0} I^2_{st}\}$ *is a.e. finite and is integrable in* t;

(c) $E\{(\int_0^{t_0} I_{s_0 t} dt)^2\} \leqslant t_0 E\{\int_0^{t_0} \int_0^{s_0} \phi^2_{st} d_s [M]^1_{st} dt\}$.

Proof. Once we know $\{I_z\}$ is a mesurable process satisfying (a), (c) follows from the Schwarz inequality and the fact that, for a.e. $t \leqslant t_0$,

$$E\{I^2_{s_0 t}\} = E\left\{ \int_0^{s_0} \phi^2_{st} d_s [M]^1_{st} \right\}.$$

The proof of (a) is straightforward. If ϕ is a simple function, writing down the integrals explicitly makes it clear that it holds. If ϕ is $\mathcal{F}_z^1$-predictable, we can, by now-familiar arguments, find a sequence $\{\phi_n\}$ of $\mathcal{F}_z^1$-adapted simple functions such that

$$E\left\{\int_0^{t_0}\int_0^{s_0}(\phi_n-\phi)^2 d_s[M]_{st}^1\,dt\right\}\to 0.$$

By taking a subsequence, if necessary, we can suppose that for a.e. t,

$$E\left\{\int_0^{s_0}(\phi_n-\phi)^2 d_s[M]_{st}^1\right\}<2^{-n},$$

for large enough n. It follows that for a.e. t, $\int_{H_{st}}\phi_n\partial_1 M$ converges a.s. uniformly in s to $\int_{H_{st}}\phi\partial_1 M$. Thus, define

$$I_{st}=\begin{cases}\lim\limits_{n\to\infty}\displaystyle\int_{H_{st}}\phi_n\partial_1 M & \text{if the limit exists,}\\ 0 & \text{otherwise.}\end{cases}$$

Then (a) clearly holds. Furthermore, by the Doob inequality, for a.e. t,

$$E\{\sup_{s\leqslant s_0} I_{st}^2\}\leqslant 4E\{I_{s_0t}^2\}=4E\left\{\int_0^{s_0}\phi_{st}^2\,d_s[M]_{st}\right\},$$

which is t-integrable by hypothesis. qed

Define

$$\int_0^{s_0}\int_0^{t_0}\phi\partial_1 M\,dt=\int_0^{t_0}I_{st}\,dt,$$

where I_{st} is as in Proposition 5.1.

Remarks. By symmetry, one can also define $\int_0^{s_0}\int_0^{t_0}\phi\,\partial_2 M\,ds$ for $\mathcal{F}_z^2$-predictable ϕ. One can think of $\partial_1 M\,dt$ and $\partial_2 M\,ds$ as stochastic measures on $\mathbf{R}_+^2$. Accordingly, we will often use notation such as $\iint_A\phi\partial_1 M\,dt$, where $A\subset\mathbf{R}_+^2$.

COROLLARY 5.2. *With probability one, the process $\iint_{R_z}\phi\partial_1 M\,dt$ is right-continuous in z and is continuous if M is.*

Proof. We have $|I_{s_0t}|\leqslant\sup_{s\leqslant s_0}|I_{st}|\overset{\text{def}}{=}S_t$. Using (b) and Fubini, we see that $S_t(\omega)$ is integrable—even square integrable—for a.e. ω. Choose ω such that $S_t(\omega)$ is integrable and $I_{st}(\omega)$ right-continuous in s for a.e. t. By dominated convergence, if $s'\downarrow s$ and $t'\to t$, then

$$\int_0^{t'}I_{s'v}(\omega)\,dv\to\int_0^{t}I_{sv}(\omega)\,dv.$$

If M is continuous, so is $s\to I_{st}$, for a.e. t, and the same conclusion holds as $(s',t')\to(s,t)$.

§6. The measure J_M and Green's formula

If $M = \{M_z, z \in \mathbf{R}_+^2\}$ is a martingale, it induces a measure on $\mathbf{R}_+^2$ which is not, except in trivial cases, a product measure. Thus, in general, $dM \neq \partial_1 M \partial_2 M$. But there is a measure which does correspond to $\partial_1 M \partial_2 M$ and which we will call J_M. Let

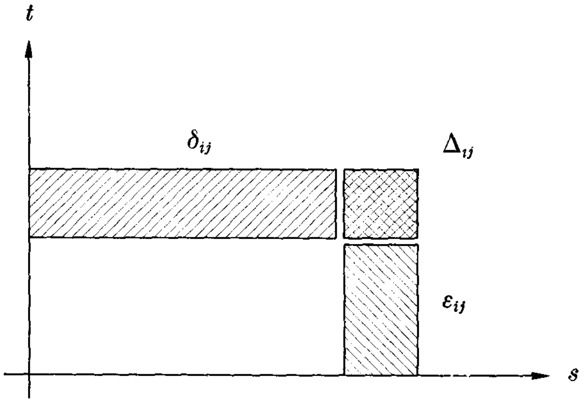
$$\psi(\zeta, \xi) = \begin{cases} 1 & \text{if } \zeta \mathrel{\substack{\wedge \\ \wedge}} \xi, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose $M \in \mathcal{M}_S^4(z_0)$ is continuous. Then $[M]^1 = [M]^2$ by Theorem 1.9. Denote the common value by $\langle M \rangle$; this is permissible by the remark following Proposition 1.8. Define

$$J_M(z) = \psi \cdot MM_z, \quad z \prec z_0.$$

It is not obvious from this formula that J_M induces $\partial_1 M \partial_2 M$. Let us look at it from a slightly different point of view. Divide R_{z_0} into squares with corners at the lattice points $z_{ij} = (2^{-n} i s_0, 2^{-n} j t_0)$, $i, j = 0, 1, \ldots, 2^n$. Let $\Delta_{ij} = (z_{ij}, z_{i+1, j+1}]$ and put

$$\delta_{ij} = (z_{0j}, z_{i, j+1}] \quad \text{and} \quad \varepsilon_{ij} = (z_{i0}, z_{i+1, j}].$$



Define

$$J_{ij}^n(z) = M(\delta_{ij} \cap R_z) M(\varepsilon_{ij} \cap R_z)$$

and

$$J_M^n(z) = \sum_{i, j=0}^{2^n - 1} J_{ij}^n(z).$$

This is an approximation to $\psi \cdot MM_z$ and in fact $J_M^n \to J_M$. Furthermore, it is clear that

$$J_M^n(\Delta_{ij}) = M(\varepsilon_{ij}) M(\delta_{ij}),$$

which gives the connection between J_M and $\partial_1 M \partial_2 M$, for $M(\varepsilon_{i\cdot})$ is the increment of M over $\overline{z_{ij} z_{i+1,j}}$ and $M(\delta_{ij})$ the increment of M over $\overline{z_{ij} z_{i,j+1}}$.

Let us calculate $\int_{R_{z_0}} M\,dM$. Approximate M by M^n defined by $M_z^n = M_{z_{ij}}$, if $z \in \Delta_{ij}$, and $M^n = 0$ on the axes. Let us write:

$$\int_{R_{z_0}} M^n dM = \sum_{i,j=0}^{2^n-1} M_{z_{ij}} M(\Delta_{ij}) = \sum_{i,j=0}^{2^n-1} M_{z_{ij}}(M(\varepsilon_{i,j+1}) - M(\varepsilon_{ij}))$$

$$= \sum_{i,j=0}^{2^n-1} (M_{z_{i,j+1}} M(\varepsilon_{i,j+1}) - M_{z_{ij}} M(\varepsilon_{ij})) + \sum_{i,j=0}^{2^n-1} (M_{z_{ij}} - M_{z_{i,j+1}}) M(\varepsilon_{i,j+1}).$$

The first sum on the right telescopes in j, while in the second, $M_{z_{ij}} - M_{z_{i,j+1}} = -M(\delta_{ij})$. Writing $M(\varepsilon_{i,j+1}) = M(\varepsilon_{ij}) + M(\Delta_{ij})$, the right-hand side becomes:

$$\sum_{i=0}^{2^n-1} M_{z_{i,2^n}} M(\varepsilon_{i,2^n}) - \sum_{i,j=0}^{2^n-1} M(\delta_{ii}) M(\varepsilon_{ij}) - \sum_{i,j=0}^{2^n-1} M(\delta_{ij}) M(\Delta_{ij}).$$

We can identify all three of these sums. Indeed, if H_{z_0} is the horizontal line segment joining z_0 to the t-axis and if we define

$$\bar{M}^n_{s,t_0} = \begin{cases} M_{2^{-n}is_0,t_0} & \text{if } s \in (2^{-n}is_0.\, 2^{-n}(i+1)s_0], \\ 0 & \text{if } s = 0, \end{cases}$$

$$\delta^n(z) = \begin{cases} M(\delta_{ii}) & \text{if } z \in \Delta_{ij}, \\ 0 & \text{on the axis,} \end{cases}$$

the above can be written in the form

$$\int_{H_{z_0}} \bar{M}^n \partial_1 M - J^n_M(z_0) - \int_{R_{z_0}} \delta^n dM.$$

Thus,

$$J^n_M(z_0) = \int_{H_{z_0}} \bar{M}^n \partial_1 M - \int_{R_{z_0}} M^n dM - \int_{R_{z_0}} \delta^n dM. \tag{6.1}$$

Now,

$$\sup_{n,\, z<z_0} (M^n_z - M_z)^2 \leqslant 2 \sup_{n,\, z<z_0} (M^n_z)^2 + 2 \sup_{z<z_0} M^2_z \leqslant 4 \sup_{z<z_0} M^2_z,$$

hence, by Theorem 1.2,

$$E\{\sup_{n,\, z<z_0} (M^n_z - M_z)^4\} \leqslant \text{const.}\, E\{M^4_{z_0}\} < \infty.$$

In view of the continuity of M, $\sup_{z<z_0} |M^n_z - M_z| \to 0$, a.s., so the above implies that

$E\{\sup_{z<z_0}(M_z^n - M_z)^4\} \to 0$. Furthermore

$$E\left\{\int_{R_{z_0}} (M^n - M)^2 d\langle M\rangle\right\} \leqslant E\{\sup_{z<z_0}(M_z^n - M_z)^2\langle M\rangle_{z_0}\}$$

$$\leqslant (E\{\sup_{z<z_0}(M_z^n - M_z)^4\}E\{(\langle M\rangle_{z_0})^2\})^{1/2}. \tag{6.2}$$

But as $\langle M\rangle = [M]^1$, $\{\langle M\rangle_z, z\in H_{z_0}\}$ is the increasing process associated with the ordinary martingale $\{M_z, z\in H_{z_0}\}$; hence by Burkholder's inequality (see [5], p. 276), $E\{(\langle M\rangle_{z_0})^2\} \leqslant$ const. $E\{M_{z_0}^4\}$, so the right-hand side of (6.2) tends to zero. Similarly

$$E\left\{\int_{H_{z_0}} (\bar{M}^n - M)^2 \partial_1\langle M\rangle\right\} \leqslant (E\{\sup_{z\in H_{z_0}}(\bar{M}_z^n - M_z)^4\}E\{(\langle M\rangle_{z_0})^2\})^{1/2},$$

which tends to zero. Turning to the last term of (6.1) and using the strength of M, we get

$$E\left\{\left(\int_{R_{z_0}} \delta^n\, dM\right)^2\right\} \leqslant E\{\sup_{i,j} M(\delta_{ij})^2\langle M\rangle_{z_0}\} \leqslant (E\{\sup_{i,j} M(\delta_{ij})^4\}E\{(\langle M\rangle_{z_0})^2\})^{1/2},$$

which tends to zero because of the continuity of M and the fact that

$$E\{\sup_{n,i,j} M(\delta_{ij})^4\} \leqslant \text{const.}\, E\{\sup_{z<z_0} M_z^4\} \leqslant \text{const.}\, E\{M_{z_0}^4\} < \infty.$$

We conclude from this that the right-hand side of (6.1) converges in L^2. The left-hand side converges in L^2 to $J_M(z_0)$, giving us

$$J_M(z_0) = \int_{H_{z_0}} M\partial_1 M - \int_{R_{z_0}} M\, dM. \tag{6.3}$$

But now since M is continuous and $\langle M\rangle = [M]^1$, the line integral in (6.3) is just $\frac{1}{2}(M_{z_0}^2 - \langle M\rangle_{z_0})$. Therefore,

$$J_M(z_0) = \tfrac{1}{2}M_{z_0}^2 - \int_{R_{z_0}} M\, dM - \tfrac{1}{2}\langle M\rangle_{z_0}. \tag{6.4}$$

Remarks.

1°. If $M = W$, we will write J instead of J_W.

2°. J_M is orthogonal to M in the sense that the product MJ_M is a weak martingale. In general, J_M is a martingale.

3°. For each $z \prec z_0 = (s_0, t_0)$, according to Theorem 2.5,

$$\langle J_M\rangle_z = \iint_{R_z\times R_z} I_{\{\zeta \curlywedge \xi\}}\, d\langle M\rangle_\zeta\, d\langle M\rangle_\xi.$$

In what follows, we will denote the element of measure $d\langle J_M\rangle_{st}$ by $d_s\langle M\rangle_{st} d_t\langle M\rangle_{st}$, so that if $\phi \in \mathcal{L}^2_{J_M}(z_0)$, we have

$$E\{(\phi \cdot J_M)^2_{z_0}\} = E\left\{\int_{R_{z_0}} \phi^2_{st}\, d_s\langle M\rangle_{st}\, d_t\langle M\rangle_{st}\right\}. \tag{6.5}$$

Note that

$$d_s\langle M\rangle_{st}\, d_t\langle M\rangle_{st} \leqslant d_s\langle M\rangle_{st_0} \times d_t\langle M\rangle_{s_0t}. \tag{6.6}$$

The classical version of Green's theorem requires the existence of partial derivatives. In our context, if $\Phi = \{\Phi_{st}\}$ is a process, the analogue of the existence of a partial derivative relative to t is the validity of the following equation:

$$\Phi_{st} = \Phi_{s0} + \int_{V_{st}} \phi\, \partial_2 M + \int_{V_{st}} \psi\, dv, \tag{6.7}$$

where V_{st} is the vertical line segment connecting the point (s, t) with the s-axis, and where ϕ and ψ are $\mathfrak{F}_z$-predictable processes such that

$$\int_0^t \phi^2_{sv}\, d_v\langle M\rangle_{sv} < \infty \quad \text{a.s.} \quad \text{and} \quad \int_0^t |\psi_{sv}|\, dv < \infty \quad \text{a.s.}$$

If (6.7) holds for a fixed s and each $t \leqslant t_0$, we say that Φ has *stochastic partial derivatives* (or, more simply, *stochastic partials*) ϕ *and* ψ *with respect to* (M, t) *along the line* V_{st_0}. If (6.7) holds for each $s \leqslant s_0$ and $t \leqslant t_0$, we say that Φ has *stochastic partials with respect to* (M, t) *in the region* R_{z_0}. The stochastic partials relative to (M, s) are similarly defined.

If $f(x; s, t)$ is twice continuously differentiable in x and continuously differentiable in s and t, then, by Ito's formula, the process $\{f(W_{st}; s, t),\ s, t \geqslant 0\}$ has stochastic partials with respect to both (W, s) and (W, t) everywhere.

One special case that deserves note is when Φ is a martingale. In that case, one can see that the function ψ in (6.7) vanishes and we say then that Φ has a stochastic partial ϕ.

We will make one further restriction: we suppose for the remainder of the section that the increasing process $\langle M\rangle$ is *deterministic*, i.e. independent of ω. This is true for $M = W$, for instance, but is in general extremely restrictive and will be in force for this section only.

THEOREM 6.1. (Green's formula for rectangles) *Let* $z_0 = (s_0, t_0)$ *and suppose that the processes* ϕ *and* ψ *are* $\mathfrak{F}_z$*-predictable and satisfy*

$$E\left\{\int_0^{s_0}\int_0^{t_0} \phi^2_{st}\, d_s\langle M\rangle_{st_0}\, d_t\langle M\rangle_{s_0t}\right\} < \infty$$

and

$$E\left\{\int_0^{s_0}\int_0^{t_0}\psi_{st}^2\, d_s\langle M\rangle_{st_0}\,dt\right\}<\infty.$$

Suppose in addition that the Φ *is an* $\mathfrak{F}_z$*-predictable process, having stochastic partials* ϕ *and* ψ *with respect to* (M, t) *along* V_{st_0}, *for* $d\langle M\rangle_{\cdot t_0}-$ *a.e.* $s\leqslant s_0$, *and such that* $E\{\int_0^{s_0}\Phi_{s0}^2\, d_s\langle M\rangle_{st_0}\}<\infty$. *Then if* $A\subset R_{z_0}$ *is a rectangle*

$$\int_{\partial A}\Phi\partial_1 M=\int_A\Phi\, dM+\int_A\phi\, dJ_M+\iint_A\psi\partial_1 M\, dt, \tag{6.8}$$

where the line integral is taken in the clockwise direction.

Proof. Let $A=(z_1, z_2]$, where $z_1=(s_1, t_1)\prec\prec z_2=(s_2, t_2)$. We can assume that $\Phi=0$ on the lower edge of A. Indeed, if we write $\Phi_{st}=\Phi_{st_1}+(\Phi_{st}-\Phi_{st_1})$, $(s, t)\in A$, then since Φ_{st_1} is independent of t,

$$\int_A\Phi_{st_1}\,dM_{st}=\int_{s_1}^{s_2}\Phi_{st_1}\,d_s(M_{st_2}-M_{st_1}),$$

which is just

$$\int_{\partial A}\Phi_{st_1}\partial_1 M.$$

Hence (6.8) holds iff it holds for $\{\Phi_{st}-\Phi_{st_1}\}$.

We first suppose ϕ and ψ are bounded simple functions. We can write A as a union of subrectangles A_i on which ϕ and ψ are constant. Notice that

$$\int_{\partial A}\Phi\partial_1 M=\sum_i\int_{\partial A_i}\Phi\partial_1 M,$$

since the line integrals over the interior portions of the boundaries of the A_i cancel out. Since the right side of (6.8) is the sum of the integrals over the A_i, it suffices to prove (6.8) for $A=A_i$, or equivalently, for the case where ϕ and ψ are constant on A. If these constant values are ϕ_1 and ψ_1 respectively, we can write

$$\Phi_{st}=\phi_1(M_{st}-M_{st_1})+\psi_1(t-t_1),\quad (s, t)\in A. \tag{6.9}$$

Note that $J_M(A)=J_M(R_{s_2t_2})-J_M(R_{s_1t_2})-J_M(R_{s_2t_1})+J_M(R_{s_1t_1})$, so that, by (6.3),

$$\int_A\phi\, dJ_M=\int_{\partial A}\phi_1 M\partial_1 M-\int_A\phi_1 M\, dM=\int_{\partial A}\phi_1(M_{st}-M_{st_1})\partial_1 M-\int_A\phi_1(M_{st}-M_{st_1})\,dM_{st}. \tag{6.10}$$

If $\{N_t\}$ is a continuous martingale with a one-dimensional parameter set, Ito's formula gives

$$d((t-t_1)N_t)=N_t\,dt+(t-t_1)dN_t.$$

Applying this to $N_t=M_{s_2t}-M_{s_1t}$ below, we get

$$\begin{aligned}\int_A \psi_1(t-t_1)\,dM_{st}&=\psi_1\int_{t_1}^{t_2}(t-t_1)\,d_t(M_{s_2t}-M_{s_1t})\\ &=\psi_1(t_2-t_1)\,(M_{s_2t_2}-M_{s_1t_2})-\psi_1\int_{t_1}^{t_2}(M_{s_2t}-M_{s_1t})\,dt\\ &=\int_{\partial A}\psi_1(t-t_1)\,\partial_1 M-\int_{t_1}^{t_2}\left(\int_{s_1}^{s_2}\psi_1\,d_s M_{st}\right)dt. \qquad (6.11)\end{aligned}$$

In view of (6.9), we need only add (6.10) and (6.11) and rearrange the terms to get (6.8). This proves the theorem for simple functions. Before completing the proof, we need a lemma.

LEMMA 6.2. *Suppose that ϕ and ψ satisfy the conditions of the theorem and that X and Y are $\mathfrak{F}_z$-predictable processes such that, for $d\langle M\rangle_{\cdot t_0}$*-a.e. $s\leqslant s_0$,

$$X_{st}=\int_{V_{st}}\phi\partial_2 M \quad\text{and}\quad Y_{st}=\int_{V_{st}}\psi\,dv,$$

for all $t\leqslant t_0$. Then we have

$$E\left\{\left(\int_{R_{z_0}}X\,dM\right)^2\right\}\leqslant E\left\{\left(\int_{H_{z_0}}X\partial_1 M\right)^2\right\}\leqslant\int_0^{s_0}\int_0^{t_0}E\{\phi_{st}^2\}\,d_s\langle M\rangle_{st_0}\,d_t\langle M\rangle_{s_0t}; \qquad (6.12)$$

$$E\left\{\left(\int_{R_{z_0}}Y\,dM\right)^2\right\}\leqslant t_0\int_0^{s_0}\int_0^{t_0}E\{\psi_{st}^2\}\,d_s\langle M\rangle_{st_0}\,dt; \qquad (6.13)$$

$$E\left\{\left(\int_{H_{z_0}}Y\partial_1 M\right)^2\right\}\leqslant t_0\int_0^{s_0}\int_0^{t_0}E\{\psi_{st}^2\}\,d_s\langle M\rangle_{st_0}\,dt. \qquad (6.14)$$

Proof. We have

$$E\{X_{st}^2\}=E\left\{\left(\int_{V_{st}}\phi\partial_2 M\right)^2\right\}=E\left\{\int_0^t\phi_{sv}^2\,d_v\langle M\rangle_{sv}\right\}\leqslant\int_0^{t_0}E\{\phi_{sv}^2\}\,d_v\langle M\rangle_{s_0v},$$

where we have used the fact that $\langle M\rangle$ is deterministic. Similarly, by the Schwarz inequality,

$$E\{Y_{st}^2\}=E\left\{\left(\int_0^t\psi_{sv}\,dv\right)^2\right\}\leqslant t_0\int_0^{t_0}E\{\psi_{sv}^2\}\,dv.$$

Then (6.12)–(6.14) follow from this and the fact that

$$E\left\{\left(\int_{R_{z_0}} X\,dM\right)^2\right\}=\int_{R_{z_0}} E\{X^2\}\,d\langle M\rangle$$

and

$$E\left\{\left(\int_{H_{z_0}} X\partial_1 M\right)^2\right\}=\int_0^{s_0} E\{X^2_{st_0}\}\,d_s\langle M\rangle_{st_0}. \qquad \text{qed}$$

It is now easy to finish the proof of Theorem 6.1. If ϕ and ψ satisfy the conditions of the theorem, we can find sequences $\{\phi_n\}$ and $\{\psi_n\}$ of bounded simple functions such that

$$\int_0^{s_0}\int_0^{t_0} E\{(\phi_n(s,t)-\phi(s,t))^2\}\,d_s\langle M\rangle_{st_0}\,d_t\langle M\rangle_{s_0t}\to 0$$

and

$$\int_0^{s_0}\int_0^{t_0} E\{\psi_n(s,t)-\psi(s,t))^2\}\,d_s\langle M\rangle_{st_0}\,dt\to 0.$$

Then (6.8) holds for

$$\Phi_n(s,t)\overset{\text{def}}{=}\int_{V_{st}}\phi_n\,\partial_2 M+\int_{V_{st}}\psi_n\,dv.$$

But by Lemma 6.2 and (6.6), we can pass to the limit, as $n\to\infty$, to see that (6.8) holds for Φ.

qed

THEOREM 6.3. *Let $D\subset R_{z_0}$ be a region whose boundary ∂D is piecewise-pure. Suppose that M does not charge ∂D and that Φ, ϕ and ψ satisfy the conditions of Theorem* 6.1.

Then

$$\int_{\partial D}\Phi\partial_1 M=\int_D \Phi\,dM+\int_D \phi\,dJ_M+\iint_D \psi\partial_1 M\,dt, \tag{6.15}$$

where the line integral is taken in the clockwise direction.

Proof. Let us break ∂D into a finite number of curves Γ_i, $i=1,\ldots,p$, each of which is of one of the types I, II, I′ or II′. Approximate each Γ_i by stepped paths Γ_i^n, of the same type as Γ_i and having the same initial and final points as Γ_i. We can do this in such a way that Γ_i^n and Γ_j^n intersect at most at their end points. Let D^n be the region bounded by $\cup_i\Gamma_i^n$. We can write D^n as a finite union of disjoint rectangles A_i and apply Theorem 6.1 to each of the A_i separately. But notice that

$$\int_{\partial D^n}\Phi\partial_1 M=\sum_i\int_{\partial A_i}\Phi\partial_1 M.$$

Thus, if we add over the A_i, we get (6.15) with D replaced by D^n.

Now, for each i, let the open region B_{in} enclosed by $\Gamma_i \cup \Gamma_i^n$ satisfy $\limsup_{n\to\infty} B_{in} = \emptyset$. By Proposition 4.6,

$$\lim_{n\to\infty} \int_{\Gamma_i^n} \Phi \partial_1 M = \int_{\Gamma_i} \Phi \partial_1 M \quad \text{in } L^2,$$

hence

$$\lim_{n\to\infty} \int_{\partial D^n} \Phi \partial_1 M = \int_{\partial D} \Phi \partial_1 M \quad \text{in } L^2.$$

But now the surface integrals over D^n on the right-hand side of (6.15) clearly converge, so we can pass to the limit, as $n \to \infty$, and the proof is complete. qed

The symmetric equation to (6.15) is

$$-\int_{\partial D} \Phi \partial_2 M = \int_D \Phi \, dM + \int_D \hat{\phi} \, dJ_M + \iint_D \hat{\psi} \partial_2 M \, ds, \tag{6.16}$$

where here we suppose that Φ is $\mathcal{F}_2$-predictable, has stochastic partials $\hat{\phi}$ and $\hat{\psi}$ with respect to (M, s) and that the hypotheses analogous to those of Theorem 6.1 are satisfied. Subtracting (6.16) from (6.15) gives

$$\int_{\partial D} \Phi \partial M = \int_D (\phi - \hat{\phi}) \, dJ_M + \iint_D \psi \partial_1 M \, dt - \iint_D \hat{\psi} \partial_2 M \, ds. \tag{6.17}$$

If Φ is known to be a martingale, then both ψ and $\hat{\psi}$ must vanish and (6.17) simplifies considerably to

$$\int_{\partial D} \Phi \partial M = \int_D (\phi - \hat{\phi}) \, dJ_M. \tag{6.18}$$

If $M = W$, then ϕ and $\hat{\phi}$ must be equal by Theorem 9.12. This may be true in general.

One application of this theorem is to get a "two time-dimensional version" of Ito's formula. We consider only the simplest case. Suppose f is four times continuously differentiable on $\mathbf{R}$ and $f''(W), f'''(W) \in \mathcal{L}_W^2$. By Ito's formula along the line t = constant,

$$f(W_{st}) = f(0) + \int_0^s f'(W_{ut}) \, d_u W_{ut} + \frac{t}{2} \int_0^s f''(W_{ut}) \, du. \tag{6.19}$$

Applying Green's formula (6.15) to the stochastic integral, the right-hand side of (6.19) becomes:

$$f(0) + \int_{R_{st}} f'(W) \, dW + \int_{R_{st}} f''(W) \, dJ + \int_0^t \left[\int_0^s \frac{u}{2} f'''(W_{uv}) \, d_u W_{uv} \right] dv + \frac{t}{2} \int_0^s f''(W_{ut}) \, du. \tag{6.20}$$

Now, by Ito's formula, we can write

$$\frac{1}{2}sf''(W_{sv}) = \int_0^s \frac{u}{2} f'''(W_{uv})\, d_u W_{uv} + \frac{v}{4}\int_0^s u f^{\mathrm{IV}}(W_{uv})\, du + \frac{1}{2}\int_0^s f''(W_{uv})\, du. \tag{6.21}$$

If we solve this for the integral involving f''' and substitute the resulting expression for the term in brackets in (6.20), we get

$$\begin{aligned} f(W_{st}) = f(0) &+ \int_{R_{st}} f'(W)\, dW + \int_{R_{st}} f''(W)\, dJ \\ &- \frac{1}{2}\int_{R_{st}} \left[f''(W) + \frac{uv}{2} f^{\mathrm{IV}}(W) \right] du\, dv - \frac{1}{2}\int_{\partial R_{st}} f''(W)\,(u dv - v du), \end{aligned} \tag{6.22}$$

which is the formula we advertised. We consider that it is less useful than Green's formula and Ito's formula used separately, but it has some applications. Here is one.

THEOREM 6.4. *There exists a process* $\{\phi(x, s, t) : x \in \mathbf{R},\ (s, t) \in \mathbf{R}_+^2\}$ *which is a.s. jointly continuous in* x, s *and* t *and such that, for a.e.* ω,

$$\int_0^s \int_0^t uv f(W_{uv}(\omega))\, du\, dv = \int_{\mathbf{R}} \phi(x, s, t; \omega) f(x)\, dx, \tag{6.23}$$

for each bounded Borel function f *on* $\mathbf{R}$ *and each* $(s, t) \in \mathbf{R}_+^2$.

Proof. Let $g_{\varepsilon x} \in C^4(\mathbf{R})$ be of compact support and such that $g_{\varepsilon x}^{\mathrm{IV}}(\cdot) \approx \frac{1}{2}\varepsilon^{-1} I_{[x-\varepsilon,\, x+\varepsilon]}(\cdot)$. Then $g_{\varepsilon x}$ satisfies the conditions which allow to apply (6.22). Solve this equation for the integral of $g_{\varepsilon x}^{\mathrm{IV}}$:

$$\begin{aligned} \int_{R_{st}} uv\, g_{\varepsilon x}^{\mathrm{IV}}(W_{uv})\, du\, dv = 4g_{\varepsilon x}(0) - 4g_{\varepsilon x}(W_{st}) + 4\int_{R_{st}} g'_{\varepsilon x}(W)\, dW + 4\int_{R_{st}} g''_{\varepsilon x}(W)\, dJ \\ - 2\int_{\partial R_{st}} g''_{\varepsilon x}(W)\,(u dv - v du) - 2\int_{R_{st}} g''_{\varepsilon x}(W)\, du\, dv. \end{aligned} \tag{6.24}$$

It is easily seen that we can actually let $g_{\varepsilon x}^{\mathrm{IV}}(y) = \frac{1}{2}\varepsilon^{-1} I_{[x-\varepsilon,\, x+\varepsilon]}(y)$ without affecting the validity of (6.24). Now let $\varepsilon \to 0$ and note that $\lim_{\varepsilon\to 0} g_{\varepsilon x}(y) = \frac{1}{6}[(y-x)^+]^3$, $\lim_{\varepsilon\to 0} g'_{\varepsilon x}(y) = \frac{1}{2}[(y-x)^+]^2$ and $\lim_{\varepsilon\to 0} g''_{\varepsilon x}(y) = (y-x)^+$. It can be verified without difficulty that each of the integrals on the right-hand side of (6.24) converges and that the limit and the integral can be interchanged. It follows, that the left-hand side converges as well and we have

$$\begin{aligned} \phi(x, s, t) &\overset{\text{def}}{=} \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon}\int_{R_{st}} uv\, I_{\{x-\varepsilon \leqslant W_{uv} \leqslant x+\varepsilon\}}\, du\, dv \\ &= \frac{2}{3}[(-x)^+]^3 - \frac{2}{3}[(W_{st}-x)^+]^3 + 2\int_{R_{st}} [(W-x)^+]^2 dW + 4\int_{R_{st}} (W-x)^+ dJ \\ &\quad - 2\int_{\partial R_{st}} (W-x)^+ (u dv - v du) - 2\int_{R_{st}} (W-x)^+ du\, dv. \end{aligned} \tag{6.25}$$

Now $\phi(x, s, t)$ is clearly continuous in (s, t), by Theorem 2.2. In fact it is continous in the triple (x, s, t). This is clear for all terms except possibly the two stochastic integrals. If $x, y \in [-x_0, x_0]$ and $z_0 \in \mathbf{R}_+^2$, then, since $|(W-x)^+ - (W-y)^+| \leqslant |x-y|$,

$$E\{\sup_{z<z_0}([(W-x)^+]^2 \cdot W_z - [(W-y)^+]^2 \cdot W_z)^2\} \leqslant 16\, E\{([(W-x)^+]^2 \cdot W_{z_0} - [(W-y)^+]^2 \cdot W_{z_0})^2\}$$

$$= 16 \int_{R_{z_0}} E\{([(W-x)_z^+]^2 - [(W-y)_z^+]^2)^2\}\, dz \leqslant \text{const.}\ (x-y)^2,$$

where the constant depends on x_0 and z_0. By a theorem of Kolmogorov, for each z, $x \to [(W-x)^+]^2 \cdot W_z$ has a continuous version, and in fact this version will be equicontinuous as z varies in a bounded set. Thus since we already know that $z \to [(W-x)^+]^2 \cdot W_z$ is continuous, it follows that $(x, z) \to [(W-x)^+]^2 \cdot W_z$ is continuous. Exactly the same reasoning holds for $(W-x)^+ \cdot J_z$, which establishes the continuity.

Now let us verify (6.23). Let $h_{\varepsilon x}$ be in $C^4(\mathbf{R})$, of compact support and such that $h_{\varepsilon x}^{\mathrm{IV}}(y) = \frac{1}{2}\varepsilon^{-1} \int_0^x I_{[x'-\varepsilon,\, x'+\varepsilon]}(y)\, dx'$. Replace $g_{\varepsilon x}$ by $h_{\varepsilon x}$ in (6.24), let $\varepsilon \to 0$ and note that $h_{\varepsilon x}^{\mathrm{IV}}(y)$ converges to $I_{(0,\, x)}(y)$, while $h_{\varepsilon x}$, $h'_{\varepsilon x}$ and $h''_{\varepsilon x}$ converge to their limits denoted by h_x, h'_x and h''_x, respectively. Since the limits and the integrals can be interchanged, (6.24) becomes

$$\int_{R_{st}} uv I_{(0,\, x)}(W_{uv})\, du\, dv = 4h_x(0) - 4h_x(W_{st}) + 4\int_{R_{st}} h'_x(W)\, dW$$

$$+ 4\int_{R_{st}} h''_x(W)\, dJ - 2\int_{\partial R_{st}} h''_x(W)\,(u dv - v du) - 2\int_{R_{st}} h''_x(W)\, du\, dv. \tag{6.26}$$

But $h_x(y) = \int_0^y g_{x'}(y)\, dx'$, where $g_x(y) = \frac{1}{6}[(y-x)^+]^3$, and it is easily seen that we can change the order of integration of each of the integrals on the right of (6.26), e.g.

$$\int_{R_{st}} h'_x(W)\, dW = \int_{R_{st}} \left(\int_0^x g'_{x'}(W)\, dx' \right) dW = \int_0^x \left(\int_{R_{st}} g'_{x'}(W)\, dW \right) dx'.$$

Do this to each term and compare with (6.25) to see that

$$\int_{R_{st}} uv I_{(0,\, x)}(W_{uv})\, du\, dv = \int_{\mathbf{R}} I_{(0,\, x)}(y)\, \phi(y, s, t)\, dy. \tag{6.27}$$

This verifies (6.23) in case $f(y) = I_{(0,\, x)}(y)$. It follows that, for a.e. ω, (6.23) is true simultaneously for all f of the form $I_{(x_1\, x_2]}$, where x_1 and x_2 are rationals. Since both sides are linear in f, a monotone class argument shows that (6.23) holds simultaneously for all bounded Borel measurable functions. qed

Now ϕ differs from the local time by the factor uv appearing on the left-hand side of (6.23). This may or may not seem awkward. However, we can define the local time at x up till time (s, t) by

$$L(x, s, t) = \int_{R_{st}} \frac{1}{uv} d_{u,v}\, \phi(x, u, v).$$

Remark. The existence of a local time can also be proved starting with the local time $L_t(x, s)$ at x for the Brownian motion $\{W_{st}, s \in \mathbf{R}_+\}$ and setting

$$L(x, s, t) = \int_0^t L_v(x, s)\, dv.$$

One can show that $L(x, s, t)$ so defined is jointly continuous in x, s and t.

§ 7. Increasing processes associated with line integrals

In this section and for the remainder of the paper, we suppose that $\mathcal{F}_z = \sigma(W_\xi, \xi \prec z)$. Let $X = \{X_z, z \in \mathbf{R}^2_+\}$ be a square integrable martingale which vanishes on the axes. It has a continuous version and we know, by the Wong-Zakai theorem (Theorem 3.1), that there exist $\phi \in \mathcal{L}^2_W$ and $\psi \in \mathcal{L}^2_{WW}$ such that

$$X = \phi \cdot W + \psi \cdot WW. \tag{7.1}$$

As we have seen, the increasing process $\langle X \rangle$ associated with X is given by

$$\langle X \rangle_z = \int_{R_z} \phi^2(\xi)\, d\xi + \iint_{R_z \times R_z} \psi^2(\zeta, \xi)\, d\zeta\, d\xi. \tag{7.2}$$

$\langle X \rangle$ is absolutely continuous, so X does not charge sets of Lebesgue measure zero in $\mathbf{R}^2_+$. In particular, it does not charge rectifiable curves. Hence, the theory of line integrals developed in § 4 is valid for X.

If Γ is a curve of type I or II (resp. I′ or II′) the process $\{X_1^\Gamma(z), \mathcal{F}^1_z, z \in \Gamma\}$ (resp. $\{X_2^\Gamma(z), \mathcal{F}^2_z, z \in \Gamma\}$) defined in § 4 will be a continuous square integrable martingale with a one dimensional parameter set. Thus, there is a unique continuous increasing process, which we will denote by $\langle X_1^\Gamma \rangle$ (resp. $\langle X_2^\Gamma \rangle$) such that

$$\{(X_1^\Gamma(z))^2 - \langle X_1^\Gamma \rangle_z, \mathcal{F}^1_z, z \in \Gamma\} \text{ (resp. } \{(X_2^\Gamma(z))^2 - \langle X_2^\Gamma \rangle_z, \mathcal{F}^2_z, z \in \Gamma\})$$

is a martingale. As usual, one defines the covariation of X_i^Γ and Y_i^Γ by

$$\langle X_i^\Gamma, Y_i^\Gamma \rangle = \tfrac{1}{2}\{\langle (X+Y)_i^\Gamma \rangle - \langle X_i^\Gamma \rangle - \langle Y_i^\Gamma \rangle\} \quad (i = 1, 2).$$

We will calculate these increasing processes explicitly. We begin with the case where Γ is a horizontal or vertical line segment. By (7.1) it is enough to calculate $\langle (\phi \cdot W)_i^\Gamma \rangle$, $\langle (\psi \cdot WW)_i^\Gamma \rangle$ and $\langle (\phi \cdot W)_i^\Gamma, (\psi \cdot WW)_i^\Gamma \rangle$.

The case of $\phi \cdot W$ is easily handled: it is a strong martingale and the increasing process along a horizontal or vertical line segment is the same as the two-parameter increasing process (Theorem 1.9).

PROPOSITION 7.1. *Let* $\phi, \hat{\phi} \in \mathcal{L}^2_W$, *and put* $M = \phi \cdot W$ *and* $\hat{M} = \hat{\phi} \cdot W$. *If* H *is a horizontal line,*

$$\langle M_1^H \rangle_z = \langle M \rangle_z = \int_{R_z} \phi_\zeta^2 \, d\zeta, \quad z \in H; \tag{7.3}$$

$$\langle M_1^H, \hat{M}_1^H \rangle_z = \int_{R_z} \phi_\zeta \hat{\phi}_\zeta \, d\zeta, \quad z \in H. \tag{7.4}$$

Similarly, if V *is a vertical line,*

$$\langle M_2^V \rangle_z = \langle M \rangle_z = \int_{R_z} \phi_\zeta^2 \, d\zeta, \quad z \in V; \tag{7.5}$$

$$\langle M_2^V, \hat{M}_2^V \rangle_z = \int_{R_z} \phi_\zeta \hat{\phi}_\zeta \, d\zeta, \quad z \in V. \tag{7.6}$$

Finally,

$$\langle M_2^H \rangle_z = \langle M_1^V \rangle_z \equiv 0. \tag{7.7}$$

The case of the martingale $\phi \cdot WW$ is not so simple. The one-parameter increasing process is no longer the same as the two-parameter increasing process.

We need to say a few words about measurability of functions defined by integrals. If $f(z, z', \omega)$ is jointly measurable in z, z' and ω, and if for each z', $f(\zeta, z', \cdot)$ is $\mathcal{F}^1_\zeta$-adapted and $E\{\int_{R_z} f^2(\zeta, z') d\zeta\} < \infty$, then $\int_{R_z} f(\zeta, z') dW_\zeta$ makes sense for each z'. Using an argument of C. Doleans-Dade [3], one can define this integral simultaneously for each z' and jointly measurably in z' and ω. Under our hypotheses we can only define it for a.e. z', for $E\{\int_{R_z} f^2(\zeta, z') d\zeta\}$ is only finite for a.e. z'. However, this suffices for our purposes and a simpler argument of the type given in § 5 provides the joint measurability.

PROPOSITION 7.2. *Let* $\phi \in \mathcal{L}^2_W$ *and let* $\psi, \hat{\psi} \in \mathcal{L}^2_{WW}$. *Define* $M = \phi \cdot W$, $N = \psi \cdot WW$ *and* $\hat{N} = \hat{\psi} \cdot WW$. *Let* H *be a horizontal line and let* A *be the area under* H. *Then*

$$\langle N_1^H \rangle_z = \int_{R_z} \left(\int_A \psi(\zeta, \xi) \, dW_\zeta \right)^2 d\xi, \quad z \in H; \tag{7.8}$$

$$\langle N_1^H, \hat{N}_1^H \rangle_z = \int_{R_z} \left(\int_A \psi(\zeta, \xi) \, dW_\zeta \right) \left(\int_A \hat{\psi}(\zeta', \xi) \, dW_{\zeta'} \right) d\xi, \quad z \in H. \tag{7.9}$$

If V *is a vertical line and* B *is the area to the left of* V,

$$\langle N_2^V \rangle_z = \int_{R_z} \left(\int_B \psi(\zeta, \xi) \, dW_\xi \right)^2 d\zeta, \quad z \in V; \tag{7.10}$$

$$\langle N_2^V, \hat{N}_2^V \rangle_z = \int_{R_z} \left(\int_B \psi(\zeta, \xi)\, dW_\xi \right) \left(\int_B \hat{\psi}(\zeta, \xi')\, dW_{\xi'} \right) d\zeta, \quad z \in V. \tag{7.11}$$

Finally,

$$\langle N_1^V \rangle_z = \langle N_2^H \rangle_z \equiv 0; \tag{7.12}$$

$$\langle M_1^H, N_1^H \rangle_z = \int_{R_z} \phi(\xi) \left(\int_A \psi(\zeta, \xi)\, dW_\zeta \right) d\xi, \quad z \in H; \tag{7.13}$$

$$\langle M_2^V, N_2^V \rangle_z = \int_{R_z} \phi(\zeta) \left(\int_B \psi(\zeta, \xi)\, dW_\xi \right) d\zeta, \quad z \in V. \tag{7.14}$$

Proof. (7.9) and (7.11) are direct consequences of (7.8) and (7.10), while the pairs of equations (7.8) and (7.10), (7.13) and (7.14) are symmetric. Since (7.12) is clear, it is enough to prove only (7.8) and (7.13).

Let us first remark that if $z \in H$ and $\xi \prec z$, then $\psi I_A(\zeta, \xi) = 0$ unless ζ is also dominated by z, since $\psi(\zeta, \xi) = 0$ unless $\zeta \curlywedge \xi$. Thus the stochastic integrals over A in (7.8) and (7.13) are really integrals over R_z.

Let us consider (7.8) in the case where ψ is a simple function. Let $z \prec z' \in H$, $z \neq z'$, and partition $R_{z'}$ into a finite number of half-open rectangles Δ_i such that $\psi(\zeta, \xi)$ is constant on $\Delta_i \times \Delta_j$. We can assume that every Δ_i lies either entirely in R_z or in $R_{z'} - R_z$. Let β_{ij} be the value of ψ on $\Delta_i \times \Delta_j$ and write

$$N_{z'} - N_z = \sum_{\substack{i, j:\\ \Delta_j \subset R_{z'} - R_z}} \beta_{ij}\, W(\Delta_i)\, W(\Delta_j).$$

Then

$$E\{N_{z'}^2 - N_z^2 \mid \mathfrak{F}_z^1\} = E\{(N_{z'} - N_z)^2 \mid \mathfrak{F}_z^1\} = \sum_{\substack{i, j, k, l:\\ \Delta_j, \Delta_l \subset R_{z'} - R_z}} E\{\beta_{ij} \beta_{kl}\, W(\Delta_i)\, W(\Delta_j)\, W(\Delta_k)\, W(\Delta_l) \mid \mathfrak{F}_z^1\}.$$

If $j \neq l$ the conditional expectation vanishes, so this equals

$$\sum_{\substack{i, j, k:\\ \Delta_j \subset R_{z'} - R_z}} E\{\beta_{ij} \beta_{kj}\, W(\Delta_i)\, W(\Delta_k)\, W^2(\Delta_j) \mid \mathfrak{F}_z^1\} = E\Big\{ \sum_{\substack{j:\\ \Delta_j \subset R_{z'} - R_z}} W^2(\Delta_j) \sum_{i, k} \beta_{ij} \beta_{kj}\, W(\Delta_i)\, W(\Delta_k) \mid \mathfrak{F}_z^1\Big\}. \tag{7.15}$$

We can identify the sum over i and k, for

$$\sum_i \beta_{ij}\, W(\Delta_j) = \int_{R_z} \psi(\zeta, z_j +)\, dW_\zeta,$$

where z_j is the lower left-hand corner of Δ_j and $\psi(\zeta, z_j+) = \lim_{\substack{\xi \to z_j\\ z_j \ll \xi}} \psi(\zeta, \xi)$. Thus the right-hand side of (7.15) is equal to

$$E\left\{ \sum_{\substack{j:\\ \Delta_j \subset R_{z'} - R_z}} W^2(\Delta_j) \left(\int_{R_{z'}} \psi(\zeta, z_j +)\, dW_\zeta \right)^2 \mid \mathfrak{F}_z^1 \right\}. \tag{7.16}$$

Now $\psi(\zeta, z_j+)$ vanishes if $\zeta \in A - R_{z'}$, so we can replace $R_{z'}$ by A in $\int_{R_{z'}} \psi(\zeta, z_j+)\,dW_\zeta$ without changing its value. Since

$$E\left\{W^2(\Delta_j)\left(\int_A \psi(\zeta, z_j+)\,dW_\zeta\right)^2 \mid \mathcal{F}^1_z\right\} = E\left\{m(\Delta_j)\left(\int_A \psi(\zeta, z_j+)\,dW_\zeta\right)^2 \mid \mathcal{F}^1_z\right\},$$

where m is Lebesgue measure, (7.16) equals

$$E\left\{\int_{R_{z'}-R_z}\left(\int_A \psi(\zeta, \xi)\,dW_\zeta\right)^2 d\xi \mid \mathcal{F}^1_z\right\}. \tag{7.17}$$

Set

$$\mathrm{A}_z = \int_{R_z}\left(\int_A \psi(\zeta, \xi)\,dW_\zeta\right)^2 d\xi = \int_{R_z}\left(\int_{R_z} \psi(\zeta, \xi)\,dW_\zeta\right)^2 d\xi.$$

Then $\mathrm{A} = \{\mathrm{A}_z\}$ is $\mathcal{F}_z$-adapated and continuous, hence $\mathcal{F}_z$-predictable. We have just seen that $\{N^2_z - \mathrm{A}_z, z \in H\}$ is a martingale relative to $\{\mathcal{F}^1_z\}$, hence, by uniqueness, $\mathrm{A} = \langle N^H_1 \rangle$.

This proves (7.8) for simple functions. In the general case, if $\psi \in \mathcal{L}^2_{WW}$, there exists a sequence $\{\psi_n\} \subset \mathcal{L}^2_{WW}$ of simple functions such that for all z,

$$\iint_{R_z \times R_z} E\{(\psi_n(\zeta, \xi) - \psi(\zeta, \xi))^2\}\,d\zeta\,d\xi \to 0.$$

Since $(\psi_n \cdot WW_z)^2 \to (\psi \cdot WW_z)^2$ in L^1, the theorem will be proved if we can show that

$$\int_{R_z}\left(\int_A \psi_n(\zeta, \xi)\,dW_\zeta\right)^2 d\xi \to \int_{R_z}\left(\int_A \psi(\zeta, \xi)\,dW_\zeta\right)^2 d\xi \quad \text{in } L^1.$$

Applying the Schwarz inequality:

$$\begin{aligned}
&E\left\{\left|\int_{R_z}\left(\left(\int_A \psi_n(\zeta, \xi)\,dW_\zeta\right)^2 - \left(\int_A \psi(\zeta, \xi)\,dW_\zeta\right)^2\right)d\xi\right|\right\} \\
&\leqslant \int_{R_z} E\left\{\left|\left(\int_A (\psi_n - \psi)\,dW_\zeta\right)\left(\int_A (\psi_n + \psi)\,dW_\zeta\right)\right|\right\}d\xi \\
&\leqslant \left(\int_{R_z} E\left\{\left(\int_A (\psi_n - \psi)\,dW_\zeta\right)^2\right\}d\xi\right)^{1/2}\left(\int_{R_z} E\left\{\left(\int_A (\psi_n + \psi)\,dW_\zeta\right)^2\right\}d\xi\right)^{1/2} \\
&\leqslant \left(\iint_{R_z \times R_z} E\{(\psi_n - \psi)^2\}\,d\zeta\,d\xi\right)^{1/2}\left(\iint_{R_z \times R_z} E\{(\psi_n + \psi)^2\}\,d\zeta\,d\xi\right)^{1/2},
\end{aligned}$$

which goes to zero as $n \to \infty$, since the second term is bounded, while the first term goes to zero.

The proof of (7.13) is similar, so we will give fewer details. Keeping the same notation,

$$E\{M_{z'}N_{z'} - M_zN_z \mid \mathcal{F}^1_z\} = E\{(M_{z'} - M_z)(N_{z'} - N_z) \mid \mathcal{F}^1_z\}.$$

If ϕ and ψ are bounded and simple and if α_i is the value of ϕ on Δ_i, the right-hand side becomes equal

$$E\{\sum_{\substack{i,j,k:\\ \Delta_i,\Delta_k\subset R_{z'}-R_z}} \alpha_i\beta_{jk}W(\Delta_i)W(\Delta_j)W(\Delta_k)\,|\,\mathcal{F}_z^1\}.$$

The conditional expectation vanishes unless $i=k$, so the last term equals

$$E\{\sum_{\substack{i:\\ \Delta_i\subset R_{z'}-R_z}} \alpha_i W(\Delta_i)^2\sum_j\beta_{ji}W(\Delta_j)\,|\,\mathcal{F}_z^1\}$$

$$=E\left\{\sum_i \phi(z_i+)\,W(\Delta_i)^2\int_{R_{z'}}\psi(\zeta,z_i+)\,dW_\zeta\,|\,\mathcal{F}_z^1\right\}$$

$$=E\left\{\int_{R_{z'}-R_z}\phi(\xi)\left(\int_A\psi(\zeta,\xi)\,dW_\zeta\right)d\xi\,|\,\mathcal{F}_z^1\right\}.$$

Thus, if

$$\mathbf{B}_z=\int_{R_z}\phi(\xi)\left(\int_A\psi(\zeta,\xi)\,dW_\zeta\right)d\xi,$$

we have seen that

$$E\{M_{z'}N_{z'}-M_zN_z\,|\,\mathcal{F}_z^1\}=E\{\mathbf{B}_{z'}-\mathbf{B}_z\,|\,\mathcal{F}_z^1\}.$$

Since $\mathbf{B}=\{\mathbf{B}_z\}$ is adapted, of bounded variation and continuous, this identifies $\mathbf{B}$ with $\langle M_1^H, N_1^H\rangle$. The passage to general ϕ and ψ being similar to the previous calculation, we leave it to the reader. qed

The next two theorems extend Propositions 7.1 and 7.2 to more general curves.

Let Γ be a curve of type I or II (resp. I or II'). We denote by D_Γ^- (resp. D_Γ^+) the region bounded by Γ, the s-axis (resp. t-axis) and the lines parallel to the t-axis (resp. s-axis) which pass through the initial and final points of Γ. If Γ has the parametric representation $\{z: z=\gamma(\sigma),\ 0\leqslant\sigma\leqslant 1\}$ and if $z=\gamma(\tau)\in\Gamma$, Γ_z will denote the curve $\{z: z=\gamma(\sigma),\ 0\leqslant\sigma\leqslant\tau\}$.

THEOREM 7.3. *Let $\phi\in\mathcal{L}_W^2$. Then if $M=\phi\cdot W$,*

$$\langle M_1^\Gamma\rangle_z=\int_{D_{\Gamma_z}^-}\phi_\zeta^2\,d\zeta,\quad z\in\Gamma\quad (\textit{of type I or II});\tag{7.18}$$

$$\langle M_2^\Gamma\rangle_z=\int_{D_{\Gamma_z}^+}\phi_\zeta^2\,d\zeta,\quad z\in\Gamma\quad (\textit{of type I or II}').\tag{7.19}$$

Proof. If Γ is a stepped path, $\langle M_1^\Gamma\rangle$ will be constant on the vertical segments, while on the horizontal segments we can use (7.3) to compute $d\langle M_1^\Gamma\rangle$. The result is (7.18). In general, if Γ is of type I (resp. II), let $z\prec z'\in\Gamma$ (resp. $z\curlywedge z'\in\Gamma$) and let $\{\Gamma_n\}$ be a sequence of stepped paths decreasing to Γ and such that $\Gamma \cap \Gamma_n$ includes z and z' as well as the initial and final points z_0 and z_f, respectively, of Γ. By Corollary 4.4, $M_1^{\Gamma_n}(z_f)$ converges in L^2 to $M_1^\Gamma(z_f)$. Now, for $\xi \in \Gamma$, $M_1^\Gamma(\xi) = E\{M_1^\Gamma(z_f) \mid \mathcal{F}_\xi^1\}$ and the equation also holds with Γ replaced by Γ_n. Since z and z' are in $\Gamma \cap \Gamma_n$, it follows that

$$E\{M_1^{\Gamma_n}(z')^2 - M_1^{\Gamma_n}(z)^2 \mid \mathcal{F}_z^1\} \to E\{M_1^\Gamma(z')^2 - M_1^\Gamma(z)^2 \mid \mathcal{F}_z^1\},$$

the convergence being in L^1. On the other hand, $D_{(\Gamma_n)_z}^-$ decreases to $D_{\Gamma_z}^-$, hence $\int_{D_{(\Gamma_n)_z}^-} \phi_\zeta^2 \, d\zeta$ decreases to $\int_{D_{\Gamma_z}^-} \phi_\zeta^2 \, d\zeta$. It follows that, if we let $\mathbf{A}_z = \int_{D_{\Gamma_z}^-} \phi_\zeta^2 \, d\zeta$,

$$E\{M_1^\Gamma(z')^2 - M_1^\Gamma(z)^2 \mid \mathcal{F}_z^1\} = E\{\mathbf{A}_{z'} - \mathbf{A}_z \mid \mathcal{F}_z^1\}.$$

Since $\mathbf{A} = \{\mathbf{A}_z\}$ is adapted, continuous and increasing, we can conclude, by the uniqueness of the increasing process, that $\mathbf{A} = \langle M_1^\Gamma \rangle$. This proves (7.18) and, by symmetry, (7.19). qed

We need some notation. Let Γ be a curve and let $z = (s, t)$. We denote by $A_z(\Gamma)$ the region $\{(u, v): v \leqslant \inf\{\tau: (s, \tau) \in \Gamma\}\}$, and by $B_z(\Gamma)$ the region $\{(u, v): u \leqslant \inf\{\sigma: (\sigma, t) \in \Gamma\}\}$, where $\inf \varnothing = 0$.

THEOREM 7.4. *Let $\phi \in \mathcal{L}_W^2$, $\psi \in \mathcal{L}_{WW}^2$ and set $M = \phi \cdot W$ and $N = \psi \cdot WW$. Then, if Γ is of type I or II,*

$$\langle N_1^\Gamma \rangle_z = \int_{D_{\Gamma_z}^-} \left(\int_{A_\xi(\Gamma)} \psi(\zeta, \xi) \, dW_\zeta \right)^2 d\xi, \quad z \in \Gamma; \tag{7.20}$$

$$\langle M_1^\Gamma, N_1^\Gamma \rangle_z = \int_{D_{\Gamma_z}^-} \phi(\xi) \left(\int_{A_\xi(\Gamma)} \psi(\zeta, \xi) \, dW_\zeta \right) d\xi, \quad z \in \Gamma. \tag{7.21}$$

If Γ is of type I or II′,

$$\langle N_2^\Gamma \rangle_z = \int_{D_{\Gamma_z}^+} \left(\int_{B_\zeta(\Gamma)} \psi(\zeta, \xi) \, dW_\xi \right)^2 d\zeta, \quad z \in \Gamma; \tag{7.22}$$

$$\langle M_2^\Gamma, N_2^\Gamma \rangle_z = \int_{D_{\Gamma_z}^+} \phi(\zeta) \left(\int_{B_\zeta(\Gamma)} \psi(\zeta, \xi) \, dW_\xi \right) d\zeta, \quad z \in \Gamma. \tag{7.23}$$

The proof of Theorem 7.4 is entirely similar to that of Theorem 7.3, so we leave it. Notice that in Theorems 7.3 and 7.4, if Γ is increasing, the increasing processes are adapted and are thus the processes associated with the martingales considered relative to the fields $\mathcal{F}_z$ as well as $\mathcal{F}_z^i$.

The formulas in Theorems 7.3 and 7.4 can be given more simply if we write them in terms of differentials: if $M = \phi \cdot W$ and $N = \psi \cdot WW$, we have

$$\partial_1\langle M\rangle = d\langle M_1^\Gamma\rangle = \left(\int_0^t \phi^2(s,v)\,dv\right) ds,$$

$$\partial_2\langle M\rangle = d\langle M_2^\Gamma\rangle = \left(\int_0^s \phi^2(u,t)\,du\right) dt,$$

$$\partial_1\langle N\rangle = d\langle N_1^\Gamma\rangle = \left(\int_0^t dv\left(\int_{R_{st}} \psi(\zeta;s,v)\,dW_\zeta\right)^2\right) ds,$$

$$\partial_2\langle N\rangle = d\langle N_2^\Gamma\rangle = \left(\int_0^s du\left(\int_{R_{st}} \psi(u,t;\xi)\,dW_\xi\right)^2\right) dt,$$

$$\partial_1\langle M,N\rangle = d\langle M_1^\Gamma, N_1^\Gamma\rangle = \left(\int_0^t dv\phi(s,v)\int_{R_{st}} \psi(\zeta;s,v)\,dW_\zeta\right) ds,$$

$$\partial_2\langle M,N\rangle = d\langle M_2^\Gamma, N_2^\Gamma\rangle = \left(\int_0^s du\phi(u,t)\int_{R_{st}} \psi(u,t;\xi)\,dW_\xi\right) dt.$$

By Proposition 4.5, if X is a square integrable martingale and Γ an increasing path,

$$\langle X^\Gamma\rangle = \langle X_1^\Gamma\rangle + \langle X_2^\Gamma\rangle.$$

This gives us a way to compute the increasing process associated to X along any increasing path. In terms of differentials, we can write

$$\partial\langle X\rangle = \partial_1\langle X\rangle + \partial_2\langle X\rangle.$$

One particular case is $X = W$:

$$\partial_1\langle W\rangle = tds, \quad \partial_2\langle W\rangle = sdt,$$

and

$$\partial W = tds + sdt.$$

§ 8. Strong martingales and path-independent variation

We begin this section with a characterization of the strong martingales. Again, the fields $\mathfrak{F}_z$ will be those generated by W.

THEOREM 8.1. *$X \in \mathfrak{M}^2$ is a strong martingale iff there exists $\phi \in \mathcal{L}_W^2$ such that $X = \phi \cdot W$.*

Proof. Suppose that $X = \phi \cdot W$, where $\phi \in \mathcal{L}_W^2$. Then X is a strong martingale, by Theorem 2.2 (a). Conversely, suppose that $X \in \mathfrak{M}^2$ is a strong martingale. By the Wong-Zakai theorem (Theorem 3.1), there exist $\phi \in \mathcal{L}_W^2$ and $\psi \in \mathcal{L}_{WW}^2$ such that

$$X = \phi \cdot W + \psi \cdot WW.$$

Since $\phi\cdot W$ is a strong martingale, it follows that $\psi\cdot WW$ is also a strong martingale. We will prove that $\psi\cdot WW\equiv 0$. For that purpose, consider a rectangle $A=(z,z'], z\prec\prec z'$, and divide the rectangle $(0,z']$ into four disjoint subrectangles $(0,z]$, A, B and C (B to the left of A). We have

$$\psi\cdot WW(A)=\iint_{B\times C}\psi\, dW\, dW+\iint_{B\times A}\psi\, dW\, dW+\iint_{A\times C}\psi\, dW\, dW+\iint_{A\times A}\psi\, dW\, dW.$$

Now, the conditional expectation of the last three terms on the right-hand side, given $\mathcal{F}_z^1\vee\mathcal{F}_z^2$, is zero, while that of the first term equals

$$\iint_{B\times C}E\{\psi(\zeta,\xi)\mid\mathcal{F}_z^1\vee\mathcal{F}_z^2\}\,dW_\zeta\, dW_\xi.$$

(This can be easily seen by considering first simple functions and then passing to the limit.) Hence

$$E\{\psi\cdot WW(A)\mid\mathcal{F}_z^1\vee\mathcal{F}_z^2\}=\iint_{B\times C}E\{\psi(\zeta,\xi)\mid\mathcal{F}_z^1\vee\mathcal{F}_z^2\}\,dW_\zeta\, dW_\xi, \tag{8.1}$$

and since $\psi\cdot WW$ is a strong martingale, both sides of (8.1) vanish. Thus

$$\iint_{B\times C}(E\{\psi(\zeta,\xi)\mid\mathcal{F}_z^1\vee\mathcal{F}_z^2\})^2\,d\zeta\, d\xi=0,$$

which implies that $E\{\psi(\zeta,\xi)\mid\mathcal{F}_z^1\vee\mathcal{F}_z^2\}=0$, and hence that $E\{\psi(\zeta,\xi)\mid\mathcal{F}_z\}=0$, for a.e. pair $(\zeta,\xi)\in B\times C$. This being true for each z, z', for a.e. pair (ζ,ξ), we have

$$E\{\psi(\zeta,\xi)\mid\mathcal{F}_z\}=0, \tag{8.2}$$

for a.e. $z\in R_{\zeta\vee\xi}$, by Fubini's theorem. Take such a pair (ζ,ξ) and choose a sequence $z_n\in R_{\zeta\vee\xi}$ such that (8.2) holds and $z_n\nearrow\zeta\vee\xi$. Since $\mathcal{F}_{\zeta\vee\xi}=\lim_{n\to\infty}\mathcal{F}_{z_n}$, it follows that

$$\psi(\zeta,\xi)=E\{\psi(\zeta,\xi)\mid\mathcal{F}_{\zeta\vee\xi}\}=\lim_{n\to\infty}E\{\psi(\zeta,\xi)\mid\mathcal{F}_{z_n}\}=0.$$

qed

Let $X\in\mathcal{M}^2$. We say that the variation of X is *path-independent* if for any two increasing paths Γ and Λ with initial point 0 and the same final point z,

$$\langle X^\Gamma\rangle_z=\langle X^\Lambda\rangle_z.$$

The idea of path-independent variation was introduced by Wong and Zakai [18] and turns out to be connected with the concept of strong martingale. Indeed, a strong martingale has path-independent variation, for, if $X\in\mathcal{M}_S^2$ and if H_z and V_z denote respectively the horizontal and vertical line segments connecting the point z with the axes, then $\langle X^{H_z}\rangle_z=$

$\langle X^{V_z}\rangle_z = \langle X\rangle_z$, by Theorem 1.9, and consequently, if Γ is an increasing path, $z, z' \in \Gamma$, $z \prec z'$, and z'' denotes the intersection of $H_{z'}$ with the vertical line through z, we have

$$\begin{aligned}
&E\{(X^{\Gamma}_{z'})^2 - (X^{\Gamma}_z)^2 \mid \mathfrak{F}_z\} \\
&\quad = E\{X^2_{z'} - X^2_{z''} \mid \mathfrak{F}_z\} + E\{X^2_{z''} - X^2_z \mid \mathfrak{F}_z\} \\
&\quad = E\{\langle X^{H_{z'}}\rangle_{z'} - \langle X^{H_{z''}}\rangle_{z''} \mid \mathfrak{F}_z\} + E\{\langle X^{V_{z''}}\rangle_{z''} - \langle X^{V_z}\rangle_z \mid \mathfrak{F}_z\} \\
&\quad = E\{\langle X\rangle_{z'} - \langle X\rangle_z \mid \mathfrak{F}_z\}.
\end{aligned}$$

We have not succeeded in proving that, in general, the converse is also true, i.e. that each martingale with path-independent variation is a strong martingale. However, several indications let us believe that path-independence is a second characterization of the strong martingales.

We will prove here the converse for a particular class of martingales.

THEOREM 8.2. *Suppose that $X \in \mathfrak{M}^2$ has the representation*

$$X = \phi \cdot W + \chi \cdot J,$$

where $\phi \in \mathcal{L}^2_W$ and $\chi \in \mathcal{L}^2_J$. If the variation of X is path-independent, then $\chi \cdot J \equiv 0$.

Proof. By hypothesis,

$$\langle X^{H_{st}}\rangle_{st} = \langle X^{V_{st}}\rangle_{st}.$$

If we vary t, keeping s fixed, $t \to \langle X^{V_{st}}\rangle_{st}$ increases. Thus $t \to \langle X^{H_{st}}\rangle_{st}$ also increases. (This is the only place we use the fact that the variation is path-independent.) Let us calculate $\langle X^{H_{st}}\rangle$. By Propositions 7.1 and 7.2, setting

$$\psi(s, \tau; \sigma, t) = \begin{cases} \chi(\sigma, \tau) & \text{if } s < \sigma \text{ and } t < \tau, \\ 0 & \text{otherwise,} \end{cases}$$

in (7.8) and (7.13), we have

$$\begin{aligned}
\langle X^{H_{st}}\rangle_{st} = {} & \int_{R_{st}} \phi^2(\sigma, \tau)\, d\sigma\, d\tau + 2\int_{R_{st}} \phi(\sigma, \tau) \left(\int_\tau^t \chi(\sigma, v)\, d_v W_{\sigma v}\right) d\sigma\, d\tau \\
& + \int_{R_{st}} \left(\int_\tau^t \chi(\sigma, v)\, d_v W_{\sigma v}\right)^2 d\sigma\, d\tau. \qquad (8.3)
\end{aligned}$$

If $\{M_t\}$ is a continuous square integrable one-parameter martingale such that $M_0 = 0$, then, by Ito's formula,

$$M_t^2 = 2\int_0^t M_v\, dM_v + \langle M\rangle_t.$$

If we apply this (for a.e. σ) to the martingale (relative to $\{\mathcal{F}^2_{0t}\}$)

$$M^\sigma_t = \int_\tau^{\tau\vee t} \chi(\sigma, v)\, d_v W_{\sigma v}, \quad t \geqslant 0,$$

we get

$$(M^\sigma_t)^2 = 2\int_\tau^{\tau\vee t}\left(\int_\tau^{\tau\vee v} \chi(\sigma, v')\, d_{v'} W_{\sigma v'}\right) dM^\sigma_v + \sigma\int_\tau^{\tau\vee t} \chi^2(\sigma, v)\, dv.$$

Putting the above in (8.3) yields

$$\langle X^{H_{st}}\rangle_{st} = \int_{R_{st}} \phi^2(\sigma, \tau)\, d\sigma\, d\tau + \int_{R_{st}} \chi^2(\sigma, \tau)\, \sigma\tau\, d\sigma\, d\tau$$
$$+ 2\int_{R_{st}} \phi(\sigma, \tau)\left(\int_\tau^t \chi(\sigma, v)\, d_v W_{\sigma v}\right) d\sigma\, d\tau + 2\int_{R_{st}} \left[\int_\tau^t\left(\int_\tau^v \chi(\sigma, v')\, d_{v'} W_{\sigma v'}\right) dM^\sigma_v\right] d\sigma\, d\tau.$$

The first two terms on the right-hand side increase in t, as does $\langle X^{H_{st}}\rangle_{st}$. It follows that the sum S_t of the last two terms is of bounded variation in t. On the other hand, the first of the last two terms is clearly a continuous martingale, while the second, being of the form $\int_{R_{st}}(\int_0^t M^\sigma_v dM^\sigma_v)\, d\sigma d\tau$, is a continuous local martingale. We conclude that $\{S_t\}$ is a continuous local martingale of bounded variation. Hence $S_t \equiv 0$, and so $\langle S\rangle_t \equiv 0$, where $\langle S\rangle_t$ is the associated increasing process. Now, $\langle S\rangle_t$ is easily calculated:

$$\langle S\rangle_t = 4\int_{R_{st}}\left[\int_{R_{sv} - R_{uv}} \chi(\sigma, v)\left(\phi(\sigma, \tau) + \int_\tau^v \chi(\sigma, v')\, d_{v'} W_{\sigma v'}\right) d\sigma\, d\tau\right]^2 du\, dv.$$

It follows that, for a.e. $(u, v) \in R_{st}$,

$$\int_{R_{sv} - R_{uv}} \chi(\sigma, v)\left(\phi(\sigma, \tau) + \int_\tau^v \chi(\sigma, v')\, d_{v'} W_{\sigma v'}\right) d\sigma\, d\tau = 0.$$

But (s, t) is arbitrary, hence, for a.e. (s, v),

$$\chi(s, v)\int_0^v \left(\phi(s, \tau) + \int_\tau^v \chi(s, v')\, d_{v'} W_{sv'}\right) d\tau = 0. \tag{8.4}$$

We can eliminate the exceptional set of measure zero for which (8.4) fails by modifying χ slightly: simply set $\chi(s, v; \omega) = 0$, whenever (8.3) does not hold, and leave it unchanged otherwise. This changes χ only on a $(s, v; \omega)$-set of measure zero, so that $\chi\cdot J$ remains unchanged, and likewise, the stochastic integral in (8.4) is unchanged except for a set of s of measure zero. With this modification, we have that (8.4) holds, for a.e. s, identically in v.

Let $t>0$. Fix an s for which $\int_0^t|\phi(s,\tau)|\,d\tau<\infty$ a.s., $\int_0^t E\{\chi^2(s,\tau)\}\,d\tau<\infty$ and (8.4) holds for all $v\leq t$. Notice that

$$M_v \overset{\text{def}}{=} \int_0^v \left(\int_\tau^v \chi(s,v')\,d_{v'}W_{sv'}\right)d\tau, \quad v\leq t,$$

is a continuous square integrable martingale (relative to $\{\mathcal{F}^2_{0v}\}$). We know that, with probability one, $v\to M_v$ is constant on an interval iff $v\to\langle M\rangle_v$ is constant on the same interval ($\langle M\rangle$ is the associated increasing process). Now, it is easy to see that

$$\langle M\rangle_v = s\int_0^v \tau^2\chi^2(s,\tau)\,d\tau, \quad v\leq t.$$

It follows that, with probability one, $v\to M_v$ is constant on an interval iff $\chi(s,v)=0$ a.e. on the same interval; hence, with probability one, the total variations of $v\to M_v$ over $[0,t]$ and over the closure of $\{v\colon \chi(s,v)\neq 0\}$ coincide. But from (8.4), for each v in the closure of this set, we have

$$M_v = -\int_0^v \phi(s,\tau)\,d\tau.$$

It follows that, with probability one, $v\to M_v$ is of bounded variation over $[0,t]$, hence constant on this interval, which implies that $\chi(s,v)=0$ for a.e. $v\in[0,t]$. We conclude that $\chi(s,t)=0$ for a.e. (s,t) and hence that $\chi\cdot J\equiv 0$.

§ 9. Holomorphic processes

We say that a process $\Phi=\{\Phi_z, z\in\mathbf{R}^2_+\}$ is *holomorphic in* $\mathbf{R}^2_+$, or, more simply, *holomorphic* if there exists an adapated measurable process $\phi=\{\phi_z, z\in\mathbf{R}^2_+\}$ such that $E\{\phi_z^2\}$ is bounded for z in compact sets and such that for all $z\in\mathbf{R}^2_+$ and any increasing path $\Gamma\subset\mathbf{R}^2_+$ with initial point 0 and final point z,

$$\Phi_z=\Phi_0+\int_\Gamma \phi\,\partial W, \tag{9.1}$$

where Φ_0 is a constant. We call ϕ a *derivative* of Φ. In terms of stochastic differentials, (9.1) is

$$\partial\Phi=\phi\,\partial W.$$

In spite of the fact that we are working with purely real-valued processes, we think there is some justification for the adjective holomorphic. Several of the classical theorems about holomorphic functions have their analogues here, notably the theorems that a holomorphic process has a holomorphic derivative, that the integral of a holomorphic process is holomorphic, and the theorem of the existence of power series expansions.

Let us begin with some remarks. If Φ is holomorphic, we can write, for all z,

$$\Phi_z = \Phi_0 + \int_{H_z} \phi \partial W = \Phi_0 + \int_{V_z} \phi \partial W. \tag{9.2}$$

We will say that a process $\Phi = \{\Phi_z, z \in \mathbf{R}_+^2\}$ is *weakly holomorphic* if there exists an adapted measurable process $\phi = \{\phi_z, z \in \mathbf{R}_+^2\}$ such that, for each z, $\int_{H_z} E\{\phi^2\} ds$ and $\int_{V_z} E\{\phi^2\} dt$ are finite and (9.2) holds for a constant Φ_0. In this case, we will call ϕ a *weak derivative* of Φ.

We have just seen that a holomorphic process is weakly holomorphic. Conversely, if (9.2) holds for all z, it is easily seen that (9.1) holds for stepped paths Γ. If ϕ were continuous in the mean, we could approximate a given piecewise-pure curve by stepped paths and use Corollary 4.4 to pass to the limit. In this case, weakly holomorphic would imply the validity of (9.1) for any piecewise-pure curve $\Gamma \subset \mathbf{R}_+^2$ with initial point 0 and final point z, hence, in particular, it would imply holomorphic. Since we are making no such assumption on ϕ, the class of weakly holomorphic processes is—apparently—larger than the class of holomorphic processes. We will see later that both notions are the same and imply the existence of a continuous derivative ϕ, so that to say Φ is holomorphic will be equivalent to saying that there exists ϕ satisfying (9.1) for an increasing path and such that

$$\int_\Gamma \phi \partial W = 0,$$

for any closed piecewise-pure curve $\Gamma \subset \mathbf{R}_+^2$.

PROPOSITION 9.1. *Suppose Φ is weakly holomorphic. Then Φ is a square integrable martingale.*

Proof. Note first that

$$E\{\Phi_{st}^2\} = \Phi_0^2 + E\left\{\left(\int_{V_{st}} \Phi \partial M\right)^2\right\} = \Phi_0^2 + s\int_0^t E\{\phi_{sv}^2\}\, dv,$$

which is finite. Further, let $z = (s, t)$ and $z' = (s', t')$. If $z \prec z'$,

$$\Phi_{z'} = \Phi_z + \int_s^{s'} \phi_{ut}\, d_u W_{ut} + \int_t^{t'} \phi_{s'v}\, d_v W_{s'v},$$

and the conditional expectations, given $\mathcal{F}_z$, of both stochastic line integrals vanish, so Φ is a martingale. qed

Thanks to Corollary 3.4, the preceeding proposition implies that a weakly holomorphic process always has a continuous version. We will thus assume that all weakly holomorphic processes are continuous.

The class of holomorphic processes is clearly a vector space over the rea's. It contains constants and it contains W itself, which has the derivative 1. There are many more holomorphic processes. Here, for instance, is a description of a large class of them.

Consider a real-valued function $f(x; s, t)$ on $\mathbf{R} \times \mathbf{R}_+ \times \mathbf{R}_+$ which has continuous partial derivatives of the second order in x and of the first order in s and t and which vanishes if $x=0$. Let us look at the process

$$X_{st} = f(W_{st}; s, t), \quad (s, t) \in \mathbf{R}_+^2.$$

Write Ito's formula along the lines $t=\text{const.}$ and $s=\text{const.}$:

$$X_{st} = \int_0^s \frac{\partial f}{\partial x}(W_{ut}; u, t)\, d_u W_{ut} + \int_0^s \left[\frac{t}{2}\frac{\partial^2 f}{\partial x^2}(W_{ut}; u, t) + \frac{\partial f}{\partial u}(W_{ut}; u, t)\right] du; \tag{9.3a}$$

$$X_{st} = \int_0^t \frac{\partial f}{\partial x}(W_{sv}; s, v)\, d_v W_{s,v} + \int_0^t \left[\frac{s}{2}\frac{\partial^2 f}{\partial x^2}(W_{sv}; s, v) + \frac{\partial f}{\partial v}(W_{sv}; s, v)\right] dv. \tag{9.3b}$$

In order that X be a martingale, both terms in square brackets must vanish:

$$\frac{t}{2}\frac{\partial^2 f}{\partial x^2} + \frac{\partial f}{\partial s} = 0 \quad \text{and} \quad \frac{s}{2}\frac{\partial^2 f}{\partial x^2} + \frac{\partial f}{\partial t} = 0. \tag{9.4}$$

Consequently, $s(\partial f/\partial s) = t(\partial f/\partial t)$, which implies that f depends on s and t only through their product. If $g(x, st) \overset{\text{def}}{=} f(x; s, t)$, we find from (9.4) that g satisfies

$$\frac{1}{2}\frac{\partial^2 g}{\partial x^2} + \frac{\partial g}{\partial t} = 0, \tag{9.5}$$

which is the backward heat equation. Conversely, if $g(x, t)$ has continuous partial derivatives of the second order in x and of the first order in t and satisfies the backward heat equation, then (9.3a) and (9.3b) imply

$$g(W_{st}, st) = g(0, 0) + \int_0^s \frac{\partial g}{\partial x}(W_{ut}, ut)\, d_u W_{ut}$$

$$= g(0, 0) + \int_0^t \frac{\partial g}{\partial x}(W_{sv}, sv)\, d_v W_{sv}.$$

Thus if $E\{(\partial g/\partial x(W_{st}, st))^2\}$ is bounded for (s, t) in bounded sets, $\{g(W_{st}, st)\}$ is holomorphic with derivative $\{\partial g/\partial x(W_{st}, st)\}$.

There is a special class of solutions of the backward heat equation which will be particularly interesting. These are the Hermite polynomials. Denote by $H_n(x, t)$ the n^{th}

Hermite polynomial. H_n is a polynomial in both x and t. It can be defined by the formula

$$H_n(x,t)=\frac{(-t)^n}{n!}e^{x^2/2t}\frac{\partial^n}{\partial x^n}e^{-x^2/2t}. \tag{9.6}$$

One can see that the first few H_n are given by $H_0(x,t)\equiv 1$, $H_1(x,t)=x$, $H_2(x,t)=\frac{1}{2}x^2-\frac{1}{2}t$. If we fix $t>0$, then $\{H_n(\cdot,t)\}_{n=0}^{\infty}$ is a complete orthogonal set relative to the weight function $(2\pi t)^{-\frac{1}{2}}e^{-x^2/2t}$, so that for $s,t>0$ we have

$$E\{H_m(W_{st},st)H_n(W_{st},st)\}=\begin{cases}0 & \text{if } m\neq n,\\ \dfrac{(st)^n}{n!} & \text{if } m=n.\end{cases} \tag{9.7}$$

We will not need many of the detailed properties of the Hermite polynomials, but the following well-known facts will be useful. The generating function expansion is given by

$$e^{xy-\frac{1}{2}ty^2}=\sum_{n=0}^{\infty}H_n(x,t)y^n,\quad x,y\in\mathbf{R},\quad t\in\mathbf{R}_+. \tag{9.8}$$

Differentiating with respect to x and t and equating the coefficients leads to the equations:

$$\frac{\partial}{\partial x}H_n=H_{n-1};\quad \frac{\partial}{\partial t}H_n=-\tfrac{1}{2}H_{n-2}, \tag{9.9}$$

from which it follows that H_n satisfies the backward heat equation.

By our remarks above, we have

PROPOSITION 9.2. *$\{H_n(W_{st},st),\ (s,t)\in\mathbf{R}_+^2\}$ is a holomorphic process. Its derivative is $\{H_{n-1}(W_{st},st),\ (s,t)\in\mathbf{R}_+^2\}$.*

It follows that finite sums of Hermite polynomials are holomorphic. More generally:

PROPOSITION 9.3. *Suppose $\{a_n\}_{n=0}^{\infty}$ is a sequence of real numbers such that $\sum_0^{\infty}a_n^2(t^n/n!)<\infty$ for all $t>0$. Then the process Φ defined by*

$$\Phi_{st}=\sum_{n=0}^{\infty}a_nH_n(W_{st},st) \tag{9.10}$$

is holomorphic with derivative ϕ given by

$$\phi_{st}=\sum_{n=1}^{\infty}a_nH_{n-1}(W_{st},st), \tag{9.11}$$

the convergence taking place in L^2.

Proof. By (9.7),

$$E\left\{\left(\sum_0^m a_nH_n(W_{st},st)\right)^2\right\}=\sum_0^m a_n^2\frac{(st)^n}{n!}.$$

This is bounded by

$$\sum_0^\infty a_n^2 \frac{(st)^n}{n!} < \infty .$$

It follows that the series in (9.10) converges in L^2 and the same is true for the series in (9.11). Consider now

$$\phi_{st}^{(m)} = \sum_1^m a_n H_{n-1}(W_{st}, st),$$

and let

$$\Phi_{st}^{(m)} = a_0 + \int_{V_{st}} \phi^{(m)} \partial W = a_0 + \sum_1^m a_n H_n(W_{st}, st).$$

Then $\lim_{m\to\infty} \Phi_{st}^{(m)} = \Phi_{st}$ in L^2. To finish the proof, we need only check that

$$\lim_{m\to\infty} \int_{V_{st}} \phi^{(m)} \partial W = \int_{V_{st}} \lim_{m\to\infty} \phi^{(m)} \partial W.$$

By (9.7),

$$E\{(\phi_{st} - \phi_{st}^{(m)})^2\} = \sum_{m+1}^\infty a_n^2 \frac{(st)^{n-1}}{(n-1)!},$$

so that

$$E\left\{\left(\int_{V_s} (\phi - \phi^{(m)}) \partial W\right)^2\right\} = s\int_0^t \sum_{m+1}^\infty a_n^2 \frac{(sv)^{n-1}}{(n-1)!} dv = \sum_{m+1}^\infty a_n \frac{(st)^n}{n!}.$$

Thus,

$$\lim_{m\to\infty} \int_{V_{st}} \phi^{(m)} \partial W = \int_{V_{st}} \phi \partial W. \qquad \text{qed}$$

PROPOSITION 9.4. *Suppose that $f(x, t)$ has continuous partial derivatives of the second order in x and of the first order in t and that $\{f(W_{st}, st)\}$ is a holomorphic process. Then, for each $(s, t) \in \mathbf{R}_+^2$,*

$$f(W_{st}, st) = \sum_{n=0}^\infty a_n H_n(W_{st}, st), \tag{9.12}$$

where the convergence takes place in L^2 and where, for $s, t > 0$,

$$a_n = \frac{n!}{(st)^n} E\{f(W_{st}, st) H_n(W_{st}, st)\}. \tag{9.13}$$

This is a special case of Theorem 9.15 below, so we won't give a detailed proof now, but we just indicate how to get the coefficients. If we fix $s, t > 0$, we can use the fact that the H_n form a complete orthogonal set to see that

$$f(x, st) = \sum_0^\infty a_n H_n(x, st),$$

where

$$a_n = \frac{n!}{(st)^n} \frac{1}{\sqrt{2\pi st}} \int_{-\infty}^{\infty} f(x, st) \exp\left(-\frac{x^2}{2st}\right) dx.$$

(This is just another way of writing (9.13).) We leave it to the reader to show that the definition is independent of s and t.

This gives some picture of holomorphic processes of the form $\Phi_{st} = f(W_{st}; s, t)$, but it is very restrictive to suppose that Φ is of this form. A priori we know only that Φ_{st} is $\mathcal{F}_{st}$-measurable, but not that it is a function of W_{st} itself. We want to investigate the general case.

First, if Φ is weakly holomorphic with a weak derivative ϕ, we can apply Green's formula to the rectangle R_z. Since $\int_{\partial R_z} \Phi \partial_1 W = \int_{H_z} \Phi \partial W$, we have

$$\int_{H_z} \Phi \partial W = \int_{R_z} \Phi \, dW + \int_{R_z} \phi \, dJ. \tag{9.14}$$

Similarly,

$$\int_{V_z} \Phi \partial W = \int_{R_z} \Phi \, dW + \int_{R_z} \phi \, dJ. \tag{9.15}$$

THEOREM 9.5. *Let Φ be weakly holomorphic and define Ψ by taking a continuous version of $\Psi_z = \int_{H_z} \Phi \partial W$. Then Ψ is holomorphic.*

Proof. By (9.14) and (9.15)

$$\Psi_z = \int_{H_z} \Phi \partial W = \int_{V_z} \Phi \partial W.$$

Hence Ψ is weakly holomorphic. Since Φ is continuous in L^2, it follows that Ψ is holomorphic. qed

To go further we must find out what it means for a process to have stochastic partial derivatives. Classically, the existence of partials is a smoothness condition, but here the derivatives are with respect to the "measure" W (see (6.7)) which—whatever else it may be—is not a product measure, so the interpretation is more complicated.

Lemma 9.6. *Let $z_0 = (s_0, t_0) \in \mathbf{R}^2_+$ and let $\phi = \{\phi_z, z \in H_{z_0}\}$ be adapted, measurable and satisfying*

$$\int_0^{s_0} E\{\phi^2_{ut_0}\}\,du < \infty.$$

Then for any $z \prec z_0$,

$$E\left\{\int_{H'_{z_0}} \phi\partial_1 W \mid \mathcal{F}_z\right\} = \int_{H_z} \psi\partial_1 W, \tag{9.16}$$

where $\psi(u, v)$ is any measurable version of the conditional expectation $E\{\phi_{ut_0} \mid \mathcal{F}_{uv}\}$, $(u, v) \prec z_0$.

Proof. Suppose first that $z = (s_0, t)$ for some $t < t_0$ and that ϕ of the form

$$\phi_{ut_0} = \alpha I_{(s_1, s_2]}(u),$$

where $s_1 < s_2 \leqslant s_0$ and α is bounded and $\mathcal{F}_{s_1 t_0}$-measurable. Then if $u > s_1$, (F4) tells us that

$$E\{\alpha \mid \mathcal{F}_{ut}\} = E\{\alpha \mid \mathcal{F}_{s_1 t}\}.$$

Call this random variable β. It follows that, for each $u \geqslant 0$,

$$E\{\phi_{ut_0} \mid \mathcal{F}_{ut}\} = \beta I_{(s_1, s_2]}(u). \tag{9.17}$$

Now, the left-hand side of (9.16) equals

$$E\{\alpha(W_{s_2 t_0} - W_{s_1 t_0}) \mid \mathcal{F}_{s_0 t}\} = E\{\alpha E\{W_{s_2 t_0} - W_{s_1 t_0} \mid \mathcal{F}^1_{s_1 t} \vee \mathcal{F}^2_{s_1 t}\} \mid \mathcal{F}_{s_0 t}\}.$$

But W is a strong martingale; hence

$$E\{(W_{s_2 t_0} - W_{s_1 t_0}) - (W_{s_2 t} - W_{s_1 t}) \mid \mathcal{F}^1_{s_1 t} \vee \mathcal{F}^2_{s_1 t}\} = 0,$$

so the above equals

$$E\{\alpha \mid \mathcal{F}_{s_0 t}\}\,(W_{s_2 t} - W_{s_1 t}) = \beta(W_{s_2 t} - W_{s_1 t}),$$

which, upon comparison with (9.17), is seen to be equal

$$\int_0^{s_0} E\{\phi_{ut_0} \mid \mathcal{F}_{ut}\}\,d_u W_{ut}.$$

Thus (9.16) holds for simple ϕ in the case $z = (s_0, t)$. We then approximate ϕ by simple functions ϕ_n in such a way that

$$\int_0^{s_0} E\{(\phi_n(u, t_0) - \phi(u, t_0))^2\}\,du \to 0.$$

Let $\psi_n(u, t) = E\{\phi_n(u, t_0) \mid \mathcal{F}_{ut}\}$ and $\psi(u, t) = E\{\phi(u, t_0) \mid \mathcal{F}_{ut}\}$. Since $E\{(\psi_n - \psi)^2\} \leqslant E\{(\phi_n - \phi)^2\}$,

$$\int_0^{s_0} E\{(\psi_n - \psi)^2\}\,du \to 0.$$

Thus

$$\int_{H_z} \psi_n \partial_1 W \to \int_{H_z} \psi \partial_1 W \quad \text{in } L^2.$$

Since (9.16) holds for each ϕ_n and since

$$\int_{H_{z_0}} \phi_n \partial_1 W \to \int_{H_{z_0}} \phi \partial_1 W \quad \text{in } L^2,$$

it follows that

$$\int_{H_z} \psi \partial_1 W = E\left\{\int_{H_{z_0}} \phi \partial_1 W \mid \mathcal{F}_z\right\}.$$

This proves (9.16) for the case $z=(s_0, t)$. But now, if $z=(s,t)$, $s<s_0$, by conditioning both sides of (9.16) on $\mathcal{F}_{st_0}$, we see that

$$\int_{H_{st}} \psi \partial_1 W = E\left\{\int_{H_{s_0 t}} \psi \partial_1 W \mid \mathcal{F}_{st_0}\right\} = E\left\{E\left\{\int_{H_{s_0 t_0}} \phi \partial_1 W \mid \mathcal{F}_{s_0 t}\right\} \mid \mathcal{F}_{st_0}\right\} = E\left\{\int_{H_{s_0 t_0}} \phi \partial_1 W \mid \mathcal{F}_{st}\right\}.$$

qed

To say that a martingale M has a stochastic partial with respect to (W, s) along a given line H_{z_0} is a stronger condition than it might appear, for it implies that M has a stochastic partial with respect to (W, s) in all of R_{z_0}. Indeed,

Proposition 9.7. *Let $M \in \mathcal{M}^2$ and let $z_0 = (s_0, t_0) \in \mathbf{R}^2_+$. Suppose that M has a stochastic partial derivative ϕ with respect to (W, s) along H_{z_0}. Then M has a stochastic partial derivative ψ with respect to (W, s) in R_{z_0}, where ψ is the adapted 2-martingale given by*

$$\psi_{st} = E\{\phi_{st_0} \mid \mathcal{F}_{st}\}.$$

Proof. If $z \prec z_0$, Lemma 9.6, implies that

$$M_z = E\{M_{z_0} \mid \mathcal{F}_z\} = E\left\{\int_{H_{z_0}} \phi \partial_1 W \mid \mathcal{F}_z\right\} = \int_{H_z} \psi \partial_1 W. \qquad \text{qed.}$$

Recall that if $M \in \mathcal{M}^2$, by the Wong-Zakai theorem (Theorem 3.1) there exist $\phi \in \mathcal{L}^2_W$ and $\psi \in \mathcal{L}^2_{WW}$ such that

$$M = \phi \cdot W + \psi \cdot WW. \tag{9.18}$$

We say a real-valued function $f(t)$ is *essentially constant* if there is a real number α such that $f(t) = \alpha$ for a.e. (Lebesgue) t.

THEORFM 9.8. *Let* $M \in \mathcal{M}^2$. *If* ϕ *and* ψ *are the functions in the representation* (9.18), *then a necessary and sufficient condition that* M *have a stochastic partial derivative with respect to* (W, s) *along the line segment* $H_{s_0 t_0}$ *is that for* (s, τ) *and* (s, τ') *outside of a negligible set* F *and such that* $s \leqslant s_0$ *and* $\tau < \tau' \leqslant t_0$,

$$\phi_{s\tau} - \phi_{s\tau'} = \int_{R_{\infty t_0}} [\psi(z; s, \tau') - \psi(z; s, \tau)]\, dW_z. \tag{9.19}$$

In this case, for a.e. $z = (u, v) \in R_{s_0 t_0}$ *and a.e.* $s \leqslant s_0$, $\psi(z; s, \tau)$ *is a.s. an essentially constant function of* τ, *for* $\tau \leqslant v$, *and the partial derivative* ϱ *satisfies, for a.e.* $s \leqslant s_0$,

$$\varrho_{st_0} = \phi_{s\tau} + \int_{R_{\infty t_0}} \psi(z; s, \tau)\, dW_z, \quad \textit{for a.e. } \tau \leqslant t_0, \tag{9.20}$$

where $R_{\infty t_0}$ *is the area under the horizontal line* $t = t_0$.

Proof. Suppose that M has a stochastic partial α with respect to (W, s) along the line $H_{s_0 t_0}$. Then noting that M vanishes on the axes, we have

$$M_{st_0} = \int_{H_{st_0}} \alpha \partial_1 W, \quad s \leqslant s_0.$$

Thus, if we write $H = H_{s_0 t_0}$,

$$\langle M_1^H, W_1^H \rangle_{st_0} = t_0 \int_0^s \alpha_{ut_0}\, du,$$

so that

$$\alpha_{st_0} = \frac{1}{t_0} \frac{\partial}{\partial s} \langle M_1^H, W_1^H \rangle_{st_0}, \quad \text{for a.e. } s \leqslant s_0.$$

Thus

$$\langle M_1^H \rangle_{st_0} = \frac{1}{t_0} \int_0^s \left(\frac{\partial}{\partial u} \langle M_1^H, W_1^H \rangle_{ut_0} \right)^2 du. \tag{9.21}$$

We have calculated both $\langle M_1^H \rangle$ and $\langle M_1^H, W_1^H \rangle$ in terms of ϕ and ψ in §7. From (7.4) and (7.13) we see that

$$\langle M_1^H, W_1^H \rangle_{st_0} = \int_0^s \int_0^{t_0} \left(\phi_{uv} + \int_{R_{\infty t_0}} \psi(z; u, v)\, dW_z\right) du\, dv.$$

Thus, for a.e. $s \leqslant s_0$, we have that

$$\frac{\partial}{\partial s} \langle M_1^H, W_1^H \rangle_{st_0} = \int_0^{t_0} \left(\phi_{sv} + \int_{R_{\infty t_0}} \psi(z; s, v)\, dW_z \right) dv. \tag{9.22}$$

Similarly, from (7.3), (7.8) and (7.13),

$$\langle M_1^H \rangle_{st_0} = \int_0^s \int_0^{t_0} \left(\phi_{uv} + \int_{R_{\infty t_0}} \psi(z; u, v)\, dW_z \right)^2 du\, dv. \tag{9.23}$$

Putting (9.22) and (9.23) into (9.21), we get

$$\int_0^s \int_0^{t_0} \left(\phi_{uv} + \int_{R_{\infty t_0}} \psi(z; u, v)\, dW_z \right)^2 dv\, du = \frac{1}{t_0} \int_0^s \left(\int_0^{t_0} \left(\phi_{uv} + \int_{R_{\infty t_0}} \psi(z; u, v)\, dW_z \right) dv \right)^2 du. \tag{9.24}$$

This holds with probability one for all $s \leqslant s_0$, so we must have for a.e. $s \leqslant s_0$ that

$$t_0 \int_0^{t_0} \left(\phi_{sv} + \int_{R_{\infty t_0}} \psi(z; s, v)\, dW_z \right)^2 dv = \left(\int_0^{t_0} \left(\phi_{sv} + \int_{R_{\infty t_0}} \psi(z; s, v)\, dW_z \right) dv \right)^2. \tag{9.25}$$

But the integral equation

$$t \int_0^t f^2(v)\, dv = \left(\int_0^t f(v)\, dv \right)^2$$

has the unique solution $f = \text{constant}$, since by the Schwarz inequality

$$\left(\int_0^t 1 \cdot f(v)\, dv \right)^2 \leqslant \left(\int_0^t 1^2 dv \right) \left(\int_0^t f^2(v)\, dv \right) = t \int_0^t f^2(v)\, dv,$$

with equality iff 1 and f are linearly dependent, i.e. iff f is equal a.e. to a constant.

Applying this to (9.25) we get that for a.e. $s \leqslant s_0$, there exists, a r.v. $\varrho(s)$ such that

$$\phi_{sv} + \int_{R_{\infty t_0}} \psi(z; s, v)\, dW_z = \varrho(s), \quad \text{for a.e. } v \leqslant t_0. \tag{9.26}$$

To get (9.19), we have only to set $v = \tau$ and then $v = \tau'$ and subtract. (Notice that we can choose $\varrho(s)$ measurably: indeed, we could take

$$\varrho(s) = \frac{1}{t_0} \int_0^{t_0} \left(\phi_{sv} + \int_{R_{\infty t_0}} \psi(z; s, v)\, dW_z \right) dv.)$$

Now let us show that $\psi(z; s, \tau)$ must be an essentially constant function of τ. Fix an $s \leqslant s_0$ for which (9.19) holds outside of a negligible set of (τ, τ'). Notice that the left-hand side of (9.19) is $\mathfrak{F}^2_{s\tau'}$-measurable while the right-hand side can be written

$$\int_{R_{\infty \tau'}} (\psi(z; s, \tau') - \psi(z; s, \tau))\, dW_z + \int_{R_{\infty t_0} - R_{\infty \tau'}} (\psi(z; s, \tau') - \psi(z; s, \tau))\, dW_z.$$

The first term is $\mathfrak{F}^2_{s\tau'}$-measurable while the second is orthogonal to $\mathfrak{F}^2_{s\tau'}$. The only way the sum can be $\mathcal{F}_{s\tau'}$-measurable is that the second term vanishes. This can happen only if

$$\psi(z;\, s, \tau') = \psi(z;\, s, \tau), \quad \text{for a.e. } z \in R_{\infty t_0} - R_{\infty \tau'}.$$

Apply this to all pairs $\tau < \tau' < t_0$ and use Fubini. We see there must exist a function $\chi(z, s)$ such that, for a.e. $z = (u, v)$, $\psi(z;\, s, \tau) = \chi(z, s)$ for a.e. $\tau \leqslant v$, which proves the penultimate statement of the theorem.

Finally, let us show that (9.19) is sufficient. Write

$$M_{st_0} = \int_{R_{st_0}} \phi(\xi)\, dW_\xi + \iint_{R_{st_0} \times R_{st_0}} \psi(\zeta, \xi)\, dW_\zeta\, dW_\xi,$$

and apply the "stochastic Fubini's theorem" (Theorem 2.6) to the second integral. If $\xi = (u, v)$, we have

$$M_{st_0} = \int_{R_{st_0}} \left(\phi_{uv} + \int_{R_{st_0}} \psi(z;\, u, v)\, dW_z \right) dW_{uv}.$$

If (9.19) holds, so does (9.26), hence

$$M_{st_0} = \int_{R_{st_0}} \varrho(u)\, dW_{uv}.$$

Since the integrand is independent of v, this last is equal to

$$\int_{H_{st_0}} \varrho \partial_1 W,$$

i.e. $\{M_{st_0}\}$ has a stochastic partial ϱ with respect to (W, s) and (9.20) is satisfied. qed

Theorem 9.9. *Let Φ be a weakly holomorphic process. Then Φ is holomorphic and admits a derivative Φ' (necessarily unique) which is itself a holomorphic process.*

Proof. Let Φ be weakly holomorphic. Then there exist $\phi \in \mathcal{L}^2_W$ and $\psi \in \mathcal{L}^2_{WW}$ such that

$$\Phi = \Phi_0 + \phi \cdot W + \psi \cdot WW.$$

Being weakly holomorphic, Φ has stochastic partials with respect to both (W, s) and (W, t). Applying Theorem 9.8 we see that there is a negligible set $G \subset \mathbf{R}^2_+$ such that if $(s, t) \notin G$,

$$\tau \to \psi(\sigma, t;\, s, \tau) \quad \text{is a.s. essentially constant in } [0, t] \text{ for a.e. } \sigma,$$

$$\sigma \to \psi(\sigma, t;\, s, \tau) \quad \text{is a.s. essentially constant in } [0, s] \text{ for a.e. } \tau.$$

By replacing ψ if necessary by

$$\hat{\psi}(\sigma, t; s, \tau) = \begin{cases} \dfrac{1}{st} \displaystyle\int_0^s \int_0^t \psi(\sigma', t; s, \tau')\, d\sigma' d\tau' & \text{if } (s, t) \notin G \quad \text{and } \sigma \leqslant s, \tau \leqslant t, \\ 0 & \text{otherwise,} \end{cases}$$

we can suppose that for each (s, t), $\psi(\sigma, t; s, \tau)$ is a.s. constant in $\sigma \leqslant s$ and $\tau \leqslant t$. (Note that $\hat{\psi}(z, z') = \psi(z, z')$ for a.e. pair (z, z'), so that $\hat{\psi} \cdot WW = \psi \cdot WW$.) Let χ be defined by $\chi(s, t) = \psi(0, t; s, 0)$, and note that

$$\iint_{R_z \times R_z} \psi(\zeta, \xi)\, dW_\zeta dW_\xi = \int_{R_z} \chi\, dJ,$$

so that

$$\Phi_z = \Phi_0 + \int_{R_z} \phi\, dW + \int_{R_z} \chi\, dJ.$$

Let us apply Theorem 9.8 again. If (s, t) and (s, t') are not in some negligible set and if $t \leqslant t'$, then

$$\phi_{st'} - \phi_{st} = \int_{R_{\infty t'}} [\psi(z; s, t) - \psi(z; s, t')]\, dW_z.$$

But $\psi(u, v; s, t) = 0$ if $v < t$ or $s < u$, and equals $\chi(s, v)$ if $v \geqslant t$ and $s \geqslant u$, so that this integral becomes

$$\int_{R_{st'} - R_{st}} \psi(u, v; s, t)\, dW_{uv} = \int_{V_{st'}} \chi \partial_2 W - \int_{V_{st}} \chi \partial_2 W.$$

By the symmetric argument, we conclude that if (s, t), (s, t') and (s', t) are not in some negligible set F and $s \leqslant s'$, $t \leqslant t'$, then

$$\begin{aligned} \phi_{st'} - \phi_{st} &= \int_{V_{st'}} \chi \partial_2 W - \int_{V_{st}} \chi \partial_2 W, \\ \phi_{s't} - \phi_{st} &= \int_{H_{s't}} \chi \partial_1 W - \int_{H_{st}} \chi \partial_1 W. \end{aligned} \tag{9.27}$$

Let B be the set of (s, t) such that $(s, t) \notin F$, and for a.e. s' and t', $(s', t) \notin F$ and $(s, t') \notin F$. Since F is negligible, $\mathbf{R}_+^2 - B$ is negligible. Moreover, it follows from (9.27) that $\{\phi_z, \mathfrak{F}_z, z \in B\}$ is a square integrable martingale. Thus, let Φ' be a continuous version of the square integrable martingale defined, for each $z \in \mathbf{R}_+^2$, by

$$\Phi'_z = E\{\phi_{z'} \mid \mathfrak{F}_z\}, \quad z' \in B, z \prec z'.$$

Then $\Phi'_z = \phi_z$ for a.e. z, so that $\Phi' \cdot W = \phi \cdot W$. Furthermore Φ' has stochastic partial χ with respect to both (W, s) and (W, t), for a.e. line H_z and V_z. By Proposition 9.7, we can conclude three things: first that Φ' has stochastic partials χ^1 and χ^2, in $\mathbf{R}_+^2$, with respect to (W, s) and (W, t), respectively; secondly that χ^1 (resp. χ^2) is an adapted square integrable 2-martingale (resp. 1-martingale); and finally that, if z is not in some negligible set,

$$\chi_z^1 = \chi_z^2 = \chi_z.$$

Now let D be the set associated to this negligible set in the manner in which B was associated to F above. Then it is easily seen that $\{\chi_z, \mathcal{F}_z, z \in D\}$ is a square integrable martingale. Thus, let $\hat{\chi}$ be a continuous version of the square integrable martingale defined, for each $z \in \mathbf{R}_+^2$, by

$$\hat{\chi}_z = E\{\chi_{z'} \mid \mathcal{F}_z\}, \quad z' \in D, z \prec z'.$$

Then $\hat{\chi}_z = \chi_z^1$ (resp. $\hat{\chi}_z = \chi_z^2$), except perhaps for a negligible set of z of the form $N \times \mathbf{R}_+$ (resp. $\mathbf{R}_+ \times N$). Hence $\hat{\chi}$ is a stochastic partial of Φ' with respect to both (W, s) and (W, t) in $\mathbf{R}_+^2$. But now we are almost done. Indeed, we have shown that Φ' is weakly holomorphic with weak derivative $\hat{\chi}$ and thus holomorphic, since $\hat{\chi}$ is continuous. Furthermore, $\hat{\chi}_z = \chi_z$ for a.e. z, so that $\hat{\chi} \cdot J = \chi \cdot J$ and, by the Green's formula,

$$\int_{H_z} \Phi' \partial_1 W = \int_{R_z} \Phi' dW + \int_{R_z} \hat{\chi} dJ = \int_{R_z} \phi \, dW + \int_{R_z} \chi \, dJ = \Phi_z - \Phi_0.$$

Similarly,

$$\int_{V_z} \Phi' \partial_2 W = \Phi_z - \Phi_0.$$

We conclude that Φ' is a weak derivative of Φ, and since Φ' is continuous, that Φ is holomorphic with derivative Φ'. qed

Remarks. 1°. In the sequel, by derivative of a holomorphic process we will always mean the *holomorphic derivative.*

2°. If Φ is holomorphic, it has derivatives of all orders. Denoting by Φ' and Φ'' respectively the first and second derivatives of Φ, we have, by Green's formula,

$$\Phi_z = \Phi_0 + \int_{R_z} \Phi' dW + \int_{R_z} \Phi'' dJ. \tag{9.28}$$

There is a slight variation of Theorem 9.9 which is of interest, not so much for itself as for its curious similarity to the elementary but basic theorem in the theory of functions of several complex variables which states that a function of $n \geq 2$ complex variables which is holomorphic in some neighborhood of the boundary of a bounded domain $D \subset \mathbf{C}^n$ can be extended to be holomorphic in all of D. Our result could be phrased: a process which is holomorphic on the boundary of a rectangle R_z can be extended to be holomorphic in all of $\bar{R}_z$.

THEOREM 9.10. *Let* $z_0 \in \mathbf{R}^2_+$ *and* $\phi = \{\phi_z, z \in H_{z_0} \cup V_{z_0}\}$ *be an adapted measurable process such that* $\int_{H_{z_0}} E\{\phi^2\}du$ *and* $\int_{V_{z_0}} E\{\phi^2\}dv$ *are finite. Suppose that*

$$\int_{H_{z_0}} \phi \partial_1 W = \int_{V_{z_0}} \phi \partial_2 W$$

and let Φ *be a continuous version of* $\{\Phi_z, z \in H_{z_0} \cup V_{z_0}\}$ *defined by*

$$\Phi_z = \int_{H_z} \phi \partial_1 W \quad \text{if } z \in H_{z_0}, \quad \text{and} \quad \Phi_z = \int_{V_z} \phi \partial_2 W \quad \text{if } z \in V_{z_0}.$$

Then there exists a process $\hat{\Phi}$ *which is holomorphic in the closure of* R_{z_0} *and which equals* Φ *on* $H_{z_0} \cup V_{z_0}$.

Proof. Define $\hat{\Phi}$ by taking a continuous version of $\hat{\Phi}_z = E\{\Phi_{z_0} | \mathcal{F}_z\}$, $z \in R_{z_0}$. Then $\hat{\Phi} = \Phi$ on $H_{z_0} \cup V_{z_0}$. Thus $\hat{\Phi}$ has stochastic partials with respect to (W, s) and (W, t) along H_{z_0} and V_{z_0} respectively. By Proposition 9.7, $\hat{\Phi}$ has stochastic partials in all of R_{z_0}. But this is all we really used in the proof of Theorem 9.9, so it follows as above that $\hat{\Phi}$ is holomorphic in the closure of R_{z_0}. qed

Now we will turn our attention to a different aspect of our subject: series expansions.

If Φ is holomorphic, it has a derivative which we denote by Φ'. Likewise Φ' has a derivative Φ'' and so on. We denote the n^{th} derivative of Φ by $\Phi^{(n)}$. Now suppose Φ and Ψ are holomorphic processes. Fix a $t > 0$. By Ito's formula (or by direct calculation),

$$\Phi_{st}\Psi_{st} = \Phi_0\Psi_0 + \int_{H_{st}} \Phi \partial \Psi + \int_{H_{st}} \Psi \partial \Phi + t \int_{H_{st}} \Phi' \Psi' du,$$

so that

$$E\{\Phi_{st}\Psi_{st}\} = \Phi_0\Psi_0 + t\int_0^s E\{\Phi'_{ut}\Psi'_{ut}\}du. \tag{9.29}$$

If we use (9.29) to expand $E\{\Phi'_{ut}\Psi'_{ut}\}$ and substitute this in the last term of (9.29) we see that

$$E\{\Phi_{st}\Psi_{st}\} = \Phi_0\Psi_0 + st\Phi'_0\Psi'_0 + t^2 \int_0^s \int_0^{s_1} E\{\Phi''_{ut}\Psi''_{ut}\}du\, ds_1.$$

By induction we have

PROPOSITION 9.11. *Suppose* Ψ *and* Φ *are holomorphic. Then*

$$E\{\Phi_{st}\Psi'_{st}\} = \sum_{j=0}^{n} \Phi_0^{(j)}\Psi_0'^{(j)}\frac{(st)^j}{j!} + t^{n+1}\int_0^s\int_0^{s_n}\dots\int_0^{s_1} E\{\Phi_{ut}^{(n+1)}\Psi_{ut}'^{(n+1)}\}\,du\,ds_1\dots ds_n; \tag{9.30}$$

$$E\{\Phi_{st}^2\} = \sum_{j=0}^{n} (\Phi_0^{(j)})^2\frac{(st)^j}{j!} + t^{n+1}\int_0^s\int_0^{s_n}\dots\int_0^{s_1} E\{(\Phi_{ut}^{(n+1)})^2\}\,du\,ds_1\dots ds_n. \tag{9.31}$$

Note that since $\Phi^{(n+1)}$ is a martingale, $E\{(\Phi_{ut}^{(n+1)})^2\}$ increases in u, so the last term in (9.31) is bounded above by

$$\frac{(st)^{n+1}}{(n+1)!}E\{(\Phi_{st}^{(n+1)})^2\}.$$

LEMMA 9.12. *Let Φ be holomorphic. Then, for each $(s,t)\in\mathbf{R}_+^2$,*

$$\lim_{n\to\infty}\frac{(\mathrm{st})^n}{n!}E\{(\Phi_{st}^{(n)})^2\} = 0.$$

Proof. Define $g(s,t)=E\{\Phi_{st}^2\}$ and $g_n(s,t)=E\{(\Phi_{st}^{(n)})^2\}$. By (9.29),

$$g(s,t) = \Phi_0^2 + t\int_0^s g_1(u,t)\,du.$$

From this and the symmetric equation with s and t interchanged, we see that

$$\frac{\partial g}{\partial s} = tg_1 \quad\text{and}\quad \frac{\partial g}{\partial t} = sg_1, \tag{9.32}$$

which implies that $s(\partial g/\partial s)=t(\partial g/\partial t)$. Since g has continuous partial derivatives, we conclude from this that g depends on s and t only through their product. The same being true for g_n, we define $f(x)$ and $f_n(x)$ by

$$f(st)=g(s,t),\quad f_n(st)=g_n(s,t).$$

From (9.32), $f'=f_1$. Similarly, $f'_n=f_{n+1}$. Thus

$$f_n(x)=f^{(n)}(x). \tag{9.33}$$

Both f and f_n are positive and infinitely differentiable, so we can apply Taylor's theorem:

$$f(x)=f(x_0)+\sum_0^N f^{(n)}(x_0)\frac{(x-x_0)^n}{n!}+f^{(N+1)}(\theta)\frac{(x-x_0)^{N+1}}{(N+1)!}.$$

But if $x>x_0$, this is

$$\geqslant \sum_0^N f^{(n)}(x_0)\frac{(x-x_0)^n}{n!},$$

for all the coefficients are positive. Thus the last series converges. Take $x = 2x_0$ to see that

$$\lim_{n\to\infty} f^{(n)}(x_0)\frac{x_0^n}{n!} = \lim_{n\to\infty} f_n(x_0)\frac{x_0^n}{n!} = 0. \qquad \text{qed}$$

From (9.31) and the Lemma 9.12, we have

COROLLARY 9.13. *If* Φ *is holomorphic,*

$$E\{\Phi_{st}^2\} = \sum_{n=0}^{\infty} (\Phi_0^{(n)})^2 \frac{(st)^n}{n!}. \tag{9.34}$$

Let us now apply (9.30) with $\{\Psi_{st}\} = \{H_n(W_{st}, st)\} = H_n$. Since $H'_n = H_{n-1}$ (Proposition 9.2), it follows that $H_n^{(n)} = H_0 \equiv 1$ and $H_n^{(n+1)} \equiv 0$. Furthermore, $H_n(0,0) = 0$ if $n \geqslant 1$. Thus the only non-zero term in (9.30) is the n^{th}, so:

$$E\{\Phi_{st} H_n(W_{st}, st)\} = \frac{(st)^n}{n!}\Phi_0^{(n)}. \tag{9.35}$$

COROLLARY 9.14. *If* Φ *is holomorphic and there is some* $s > 0$, $t > 0$ *for which* $E\{\Phi_{st} H_n(W_{st}, st)\} = 0$ *for all* n, *then* $\Phi \equiv 0$.

Proof. By (9.35), if $E\{\Phi_{st} H_n(W_{st}, st)\} = 0$, $\Phi_0^{(n)}$ must vanish. By Corollary 9.13, $E\{\Phi_{st}^2\} = 0$ for *all* $s \geqslant 0$, $t \geqslant 0$; hence $\Phi \equiv 0$. qed

This brings us to the main theorem.

THEOREM 9.15. *If* Φ *is holomorphic, then, for each* $(s,t) \in \mathbf{R}_+^2$,

$$\Phi_{st} = \sum_{n=0}^{\infty} \Phi_0^{(n)} H_n(W_{st}, st), \tag{9.36}$$

where the convergence is taken in L^2.

Proof. Define Ψ by $\Psi_{st} = \Sigma_0^\infty \Phi_0^{(n)} H_n(W_{st}, st)$. Since $\Sigma_0^\infty (\Phi_0^{(n)})^2((st)^n/n!) < \infty$ by Corollary 9.13, the series converges in L^2 and Ψ is holomorphic (by Proposition 9.3).

Since the H_n form an orthogonal set

$$E\{\Psi_{st} H_n(W_{st}, st)\} = E\{\Phi_0^{(n)} H_n^2(W_{st}, st)\} = \Phi_0^{(n)}\frac{(st)^n}{n!}.$$

But by (9.35)

$$E\{\Phi_{st} H_n(W_{st}, st)\} = \frac{(st)^n}{n!}\Phi_0^{(n)}.$$

Thus $E\{(\Phi_{st} - \Psi_{st}) H_n(W_{st}, st)\} = 0$ for all n. By Corollary 9.14, $\Phi - \Psi \equiv 0$. qed

For a quick application of the preceeding:

Proposition 9.16. *Suppose* Φ *is holomorphic and that there exists* $s>0$ *and* $t>0$ *such that* $P\{\Phi_{st}=0\}=1$. *Then* $\Phi\equiv 0$.

Proof. By (9.34), $0=E\{\Phi_{st}^2\}=\sum_0^\infty(\Phi_0^{(n)})^2((st)^n/n!)$. Thus $\Phi_0^{(n)}=0$ for all n. By Theorem 9.15, $\Phi\equiv 0$. qed

Corollary 9.17. *If* Φ *and* Ψ *are holomorphic processes such that for some* $s>0$ *and* $t>0$, $\Phi_{st}=\Psi_{st}$, *then* $\Phi\equiv\Psi$.

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Received March 18, 1974