# ON BOUNDARIES OF TEICHMÜLLER SPACES AND ON KLEINIAN GROUPS, III

BY

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The Teichmüller space of a marked Riemann surface $S$, of finite conformal type $(g, n)$ with signature or of a finitely generated Fuchsian group of the first kind representing $S$, was shown to have a complex structure by Ahlfors [3] (see also Rauch [23]). Bers [7] later showed that it could be embedded as a bounded domain in the space of bounded quadratic differentials on $S$; this space has dimension $3g-3+n$. The Bers embedding of the Teichmüller space lends itself naturally to questions about the boundary. In a two part, almost-joint paper, Bers [8] and Maskit [20] systematically examined that boundary. In particular, Bers showed that totally degenerate Kleinian groups are represented on the boundary. A totally degenerate Kleinian group has a region of discontinuity which is connected and simply connected. Using his deep construction techniques, Maskit gave an exhaustive list of those pathologies which might be expected to occur on the boundary and showed that they do indeed occur. We shall study those groups which are not pathological, in a sense to be defined in § 1, and are called regular. This class coincides with Bers' non-degenerate non-quasi-Fuchsian groups and Maskit's groups giving complete factorizations of $S$. Regularity is a planar concept although we will show in § 7 that regularity is equivalent to geometric finiteness.

Since we will need to refer on many occasions to the earlier papers in this series, Bers [8] will be denoted B-I and Maskit [20] will be denoted B-II.

The class of non-pathological boundary groups has been studied constructively by Maskit [20] and using the structure of their associated 3-manifolds, in the torsionfree case, by Marden [19] and Earle and Marden [12]. The techniques used in the present work are two-dimensional and non-constructive. They involve conformal and quasiconformal mappings and plane topology, although much of the work draws deeply, if not directly then in spirit, on the work of Maskit. Related studies in the space of Fuchsian groups have been conducted

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by Harvey [15]. Bers [10] has recently examined a compactification of the Riemann space using a concept of partition of a compact unmarked surface.

Theorems 1 and 5 show that the theory of regular $b$-groups is coextensive with the study of proper partitions of finite hyperbolic Riemann surfaces. Theorem 1 shows that any regular $b$-group induces a proper partition. Theorem 6 shows that any proper partition of the surface $S$ may be realized on the boundary of the Teichmüller space of $S$. This is exactly the existence of simultaneous uniformizations of properly partitioned Riemann surfaces or, in Bers' terminology [10], for Riemann surfaces with nodes. The uniqueness theorem for the uniformizations is corollary to Theorem 4.

Theorem 2 is a technical theorem on the limit sets of regular $b$-groups and it enables us to prove that the limit set is locally connected and of zero area.

In § 5 and § 6 we examine deformations on the boundary of Teichmüller space. In particular we show that there are partially degenerate groups on the boundary of the Teichmüller space $T(S)$ of $S$ if dim $T(S) > 1$.

Throughout the course of this work I have immeasureably profited from conversations with Lipman Bers. I should like to express my thanks to him and the other participants in the "Kleinian year" at the Mittag-Leffler Institute, where much of this work was initiated.

## § 1. Preliminaries

A *b-group* is a finitely generated Kleinian group $G$ with a simply connected component $\Delta = \Delta(G)$ of its set of discontinuity $\Omega(G)$. We denote by $\pi: \Delta \xrightarrow{G} S$ the projection onto $S = \Delta/G$. If there are two invariant components $G$ is quasi-Fuchsian. If $\Delta = \Omega(G)$, then $G$ is totally degenerate.

If $G$ is a Kleinian group an element $\gamma$ of $G$ is called *primary* if $\eta^n = \gamma$ has no solution in $G$ for $n \neq \pm 1$. If $\gamma$ is a primary parabolic element of $G$ generating a subgroup $G_1$, then a topological disc $B$ is called a *horocycle* for $G_1$ if $\eta(B) = B$ for $\eta \in G_1$ and $\eta(B) \cap B = \emptyset$ for $\eta \in G - G_1$. A $b$-group $G$ is called *regular* if it is neither quasi-Fuchsian nor totally degenerate and each primary parabolic transformation in $G$ has two disjoint horocycles. If the horocycles of $\gamma$ do not lie in $\Delta(G)$, then $\gamma$ is called an *accidental parabolic transformation.*

Let $S$ be a connected compact surface with the assignment at a finite number of points $z_i$ of integers $n_i \geq 2$ or the symbol $\infty$, for $1 \leq i \leq m$. $S$ may be given a complex structure so that it is punctured at every point carrying the symbol $\infty$ and has local parameter $t^{1/n_i}$ at points marked by $n_i$. The *hyperbolic* (or *Poincare*) *area* of $S$ is

$$A(S) = 2\pi\left[2(g-1) + \sum_{i=1}^{m} (1 - 1/n_i)\right], \tag{1.1}$$

where $g$ denotes the genus of $S$. The area of a finite union of disjoint connected Riemann surfaces is the sum of the areas of the individual surfaces.

A *marked surface* is a pair consisting of a surface together with a distinguished set of generators of the fundamental group of the surface. A *marked surface with signature* is a system $(S, \alpha_1, ..., \alpha_n)$ where $S$ is a surface with signature and $\alpha_i$, $1 \leqslant i \leqslant n$, is a system of curves (simple loops or slits) on $S$ determining a cover of $S$ by a simply connected domain which respects signature. If $S$ is not assigned integers at any point, this reduces to a marked surface. By abuse of language, we call the marking with signature an *extended marking*.

A conformal map $f$ of $\Delta$ onto the upper half-plane $U$ conjugates $G$ into a Fuchsian group $H$ of the first kind. $H$ is called the *Fuchsian equivalent* (or *model*) of $G$. If $\gamma$ is an accidental parabolic transformation, $\hat{\gamma} = f\gamma f^{-1}$ is hyperbolic. The non-Euclidean straight line connecting the fixed points of $\hat{\gamma}$ is called the *axis* $A_{\hat{\gamma}}$ of $\hat{\gamma}$ in $U$. $A_\gamma = f(A_{\hat{\gamma}})$ is called the *axis* of $\gamma$ in $\Delta$ and $\bar{A}_\gamma$ is called the *closed axis* of $\gamma$. $\bar{A}_\gamma$ is a Jordan curve intersecting $\Lambda(G)$ exactly in the fixed point $\gamma$. $\bar{A}_\gamma$ is a quasi-circle since $A_\gamma$ is analytic and is differentiable at the fixed point of $\gamma$. If $\pi$ is the natural projection $\pi\colon \Omega(G) \to \Omega(G)/G$ and $\gamma$ is an accidental parabolic transformation, then $\pi(A_\gamma)$ is a curve (not necessarily simple or a loop) on $\Delta/G$ called a *pinched curve*.

For any subset $D \subset \hat{C}$, we denote by $G_D$ the stability subgroup of $D$, the subgroup $\{\gamma \in G \mid \gamma(D) = D\}$. The stability subgroup of a parabolic fixed point is either infinite cyclic or a non-trivial finite extension of a cyclic group. In the latter case the stability group is called a *phantom group*. By abuse of language we also call phantom the Fuchsian equivalent of a phantom group.

The classification of elementary groups yields that phantom groups are all isomorphic. The elements of infinite order form a cyclic subgroup. The remaining elements of the group fall into two conjugacy classes, consisting of elements of order two.

## § 2. Regular $b$-groups and proper partitions

Let $S$ be a topologically finite orientable surface. Let $\pi\colon U \overset{G}{\to} S$ define a covering of $S$ by a simply connected domain $U$ which is ramified over each point $z_i$ of index $n_i$. A closed curve $\alpha$ determines a conjugacy class $[\gamma]$ of elements of $G$. $\alpha$ is said to be *admissible* if each $\gamma$ determined by $\alpha$ is primary in $G$ and each component $R$ of $\pi^{-1}(\alpha)$ is a crosscut of $S$ stabilized by the normal subgroup $N_G(\gamma)$ of $G$ generated by some $\gamma \in [\gamma]$. If each $\gamma$ in $G$ determined by $\alpha$ generated a subgroup $\langle\gamma\rangle$ which is self-normalized in $G$, then $\alpha$ is homotopically nontrivial simple loop on $S' = S - \{z_1, ..., z_l\}$. If $\eta \in N_G(\gamma) - \langle\gamma\rangle$, then $\eta$ is of order two and $\alpha$ contains $\pi(\zeta)$, where $\zeta$ is the fixed point of $\eta$. Since $G$ is discrete and is isomorphic to a Fuchsian group, $N_G(\gamma)$ is either $\langle\gamma\rangle$ or a phantom group. In the latter case the impression of $\alpha$ is a slit connecting two distinct points marked by the number two. $\alpha$ covers the slit twice.

If $\{\alpha_1, ..., \alpha_n\}$ is a collection of pairwise disjoint admissible curves on $S$, $S - \bigcup_i \alpha_i$ is a finite collection of finite surfaces, $S_1, ..., S_k$, with signature. $P = \{S_1, ..., S_k\}$ is called the *proper* (*topological*) *partition induced by* $\{\alpha_1, ..., \alpha_n\}$, if each $S_i$, as a surface with signature can be given the complex structure of a hyperbolic Riemann surface. The above condition on $S_i$ is equivalent to

$$2g_i + r_i + \sum_d{}'(1 - 1/n_d) \geqslant 2, \tag{2.1}$$

where $g_i$ is the genus of $S_i$, $r_i$ is the number of ideal boundary components of $S_i$, and $\Sigma'$ is the sum taken over those points contained in $S_i$.

We note several obvious facts for future reference. A surface $S$ of finite conformal type admits a proper nontrivial partition in precisely those cases where $S$ is hyperbolic and the Teichmüller space of $S$ has positive dimension. We assume $\{\alpha_1, ..., \alpha_n\}$ induces a proper partition of $S$, and $\tilde{\alpha}_j$ is an admissible curve on $S$, $\tilde{\alpha}_j \cap \alpha_i = \emptyset$ for $i \neq j$, which determines the same conjugacy class in $G$ as does $\alpha_j$. $\{\tilde{\alpha}_j, \alpha_1, ..., \hat{\alpha}_j, ..., \alpha_n\}$ then induces a proper partition of $S$, (as usual, the symbol $\hat{}$ denotes deletion of $\alpha_j$ from the list) and we call the induced partitions *equivalent*. This relation on partitions extends to an equivalence relation on partitions and we again call two related partitions *equivalent*. Any set of admissible curves inducing a proper partition of $S$ is equivalent to a set whose curves are piecewise analytic.

A topologically finite surface $S$ may be compactified by adjoining to $S$ one point for each ideal boundary component. We call the resulting surface $\hat{S}$.

LEMMA 1: *If $S$ is a Riemann surface of finite conformal type and $\{\alpha_1, ..., \alpha_n\}$ induces a proper partition of $S$, each curve of which is piecewise analytic then there exists a piecewise analytic closed curve $\beta$ on $\hat{S}$ with the following properties:*

(i) *$S - \beta$ is simply connected,*

(ii) *if $\alpha_i$ is a simple loop, $\alpha_i \cap (S - \beta)$ is a single open Jordan arc,*

(iii) *if $\alpha_i$ is a slit, then its impression is contained in $\beta$.*

*Proof:* We have noted above that $S$ is hyperbolic and of finite type, hence there exists a Fuchsian group $G$, finitely generated and of the first kind, so that $U/G$ is conformally equivalent to $S$, where $U$ is the upper half-plane. Let $\pi$: $U \to S$ be the natural projection. We denote by $D$ the Dirichlet region for the action of $G$ on $U$ and based at the point $i$. Let $\tilde{\beta}_1$ be the boundary of $D$ in $\bar{U}$. $\tilde{\beta}_1$ projects to a closed piecewise analytic curve on $\hat{S}$, hence on $S$. $\tilde{\beta}_1$ intersects the lift of any admissible curve in either a finite set of points, and/or in a finite set of common subarcs. By an arbitrarily small shift of basepoint, we may assume that the lift of a simple loop $\alpha_i$ intersects $\tilde{\beta}_1$ in a finite set of points, none of which is a vertex of $D$. Let $\{\alpha_{i_1}, \ldots, \alpha_{i_m}\}$ be a list of the components of the intersection of $D$ and the lift $\pi^{-1}(\alpha_i)$ of the simple loop $\alpha_i$. Each $\alpha_{i_j}$ is a cross-cut of $D$ going from one side $a$ of $D$ to side $b$. Side $a$ is paired by some element $\gamma$ in $G$ with $\gamma(a)$ which is also a side of $D$. Choose a piecewise analytic open Jordan arc $c$ from side $s$ to side $b$, not intersecting $\pi^{-1}(\bigcup \alpha_i)$ and separating $\alpha_{i_1}$ from that $\alpha_{i_j}$ which contains $\gamma(a)$. $D-c$ has two components $D_1$, containing $\alpha_{i_1}$, and $D_{2'}$, containing $\alpha_{i_j}$. $D' = \bar{D}_1 \bigcup \gamma(D_2)$ is a simply connected fundamental set for $G$ with piecewise analytic boundary. We thus reduce the number of components of $D-\pi^{-1}(\bigcup \alpha_i)$ by one. The process may be repeated until we find a simply connected fundamental set, again denoted $D$, whose boundary projects to a curve $\beta_2$ satisfying (i) and (ii).

We must now show how to modify the curve $\beta_2$, or alternately the fundamental set, the projection of whose boundary $\tilde{\beta}_2$ is $\beta_2$, to satisfy (iii). We now assume $\alpha_1$ is a slit. Any subarc of $\pi^{-1}(\alpha_i) \bigcap D$ may be used to split $D$ as above, and we may sequentially obtain a fundamental set, whose interior does not contain such subarcs. The boundary of this fundamental set projects to a curve satisfying the conditions of the lemma.

We will now prove a basic theorem about regular $b$-groups and explore some of it's consequences.

THEOREM 1: *Let $R$ be the set of axes of accidental parabolic transformations in a regular $b$-group $G$. Then there is a one-to-one correspondence $\psi$ between the components $\Delta_i$ of $\Delta^* = \Delta - \bigcup \{A_\gamma \in R\}$ and the components $\Omega_i$ of $\Omega(G) - \Delta$ so that $G_{\Delta_i} = G_{\Omega_i}$.*

*Proof:* Let $\Omega_1$ be a non-invariant component of $G$ and $G_1$ its stability subgroup. Since $G$ is not quasi-Fuchsian and $\Omega_1$ is a Jordan domain, $[G; G_1] = \infty$.

Let $H$ be the Fuchsian model of $G$ in $L$ and $H_1$ the subgroup of $H$ corresponding to $G_1$. Since both $G$ and $G_1$ are finitely generated and $U/H_1$ is of infinite area, the Dirichlet region $D$ of $H_1$ has at least one free side. That $D$ has a finite number of sides is a classical theorem of Nielsen (see Marden [17]). It follows that $U/H_1$ is the interior $\Sigma^0$ of the bordered Riemann surface $\Sigma = [U \cup (\Omega(H_1) \cap \hat{\mathbf{R}})]/H_1$. Let $\Sigma'$ be $\Sigma$ with its ramification points deleted. Further let $\alpha$ be a simple loop on $\Sigma' \cap \Sigma^0$ which is retractible in $\Sigma'$ into a border component. $\alpha$ determines a conjugacy class of hyperbolic elements of $H_1$ represented say by $\hat{\gamma}$. $\hat{\gamma}$ is also determined by the border curve, which implies that $H_1$ is discontinuous on $\hat{\mathbf{R}} - \{r_1, r_2\}$ where $r_1$ and $r_2$ are the fixed points of $\hat{\gamma}$. Under the given isomorphism of $H$ with $G$, we claim that the $\gamma \in G$ corresponding to $\hat{\gamma}$ is parabolic.

If it were loxodromic, then we would continuously extend the conformal map $f$: $L \to \Delta$, giving the Fuchsian model, to the fixed points of $\hat{\gamma}$. These points, together with the image under $f$ of the axis of $\hat{\gamma}$ and the axis of $\gamma$ in $\Omega_1$, form a Jordan curve, $J$. $J$ intersects $\partial\Omega_1$ in two points. We consider the arc $C_1$ of $\partial\Omega_1$ containing prime ends of $\Delta$ which are images under $f$ of $C$. $C_1$ contains fixed points of elements of $G_1$ since $\partial\Omega_1 = \Lambda(G_1)$ (Abikoff [1], § 1). In particular, in any neighborhood of a point $\lambda \in C_1$, we can find points equivalent to points in $f(D)$. When we pull these points back to $L$, we contradict the discontinuity of $H_1$ on $C$. We have shown that $\gamma$ is an accidental parabolic transformation.

We next show that if $\alpha = \pi_{H_1}(A_{\hat{\gamma}})$ where $\gamma$ is an accidental parabolic transformation in $G_1$, then $\alpha$ is retractible in $\Sigma' \cap \Sigma^0$ to a border curve of $\Sigma$. $\overline{A\gamma}$ does not separate $\Lambda(G_1)$ hence there is a subarc $C$ of $\hat{\mathbf{R}}$ minus the fixed points of $\gamma$ on which $H_1$ is discontinuous. $\alpha$ is retractible into $\pi_{H_1}(C)$.

Let $K_1$ be the convex hull of $\Lambda(H_1)$ in $U$. By the above reasoning together with the fact that axes of accidental parabolic transformations are pairwise disjoint (B-II Lemma 1), we have $K_1^0$ is a component of $\Delta^*$.

$K_1^0$ is $H_1$ invariant, hence $G_{f(K_1^0)} = G_1$. We have given an injection $\psi$ from the components of $\Omega(G) - \Delta$ to the components of $\Delta^*$.

We show that $\psi$ is a surjection. The $A_{\hat{\gamma}} \in R$ fall into finitely many equivalence classes and are determined by parabolic elements $\gamma_1, \ldots, \gamma_n$ of $G$ which are accidental with respect to $\Delta$. Let $A_1$ be the axis of $\hat{\gamma}_1$ which corresponds to $\gamma_1$. $\gamma_1$ has two horocycles, one on either side of $A_1$. By definition of $R$, one of the horocycles must be induced by a horocycle on a non-invariant component $\Omega_1$ of $G$. By definition of regularity, $\gamma_1$ must have another horocycle. Thus there is another component $\Omega_2$ of $\Omega(G)$ left invariant by $\gamma_1$. $\partial\psi(\Omega_1)$ and $\partial\psi(\Omega_2)$ both contain $A_1$.

Choose a point $z_0$ in $\Delta^*$, and suppose $C$ is the hyperbolic geodesic in $\Delta$ connecting $z_0$ and some point $z_1$ in $\psi(\Omega_1)$. $C$ crosses finitely many elements of $R$, and we may use the above argument inductively to show that each component of $\Delta^*$ crossed by $C$ lies in the image of $\psi$. The proof of the Theorem is complete.

Corollary 1: *If $G$ is a regular b-group with accidental parabolic conjugacy classes $[\gamma_1], \ldots, [\gamma_k]$, then $\{\pi(A_{\gamma_k}), \ldots, \pi(A_{\gamma_1})\}$ is a family of admissible curves defining a proper partition of $\Delta/G$.*

*Proof:* $R$ is a $G$-invariant union of crosscuts of $\Delta$. Each $A_\gamma \in R$, which does not pass through an elliptic fixed point, projects to a simple loop on $S = \Delta/G$. If $A_\gamma \in R$ passes through an elliptic fixed point then $\gamma$ is normalized by a phantom group and $\pi(A_\gamma)$ is a slit. Thus $\{\pi(A_{\gamma_1}), \ldots, \pi(A_{\gamma_n})\}$ is a family of disjoint admissible curves on $S$.

It remains to show that each component $S_i$ of $S - \bigcup \pi(A_\gamma)$ satisfies (2.1). $S_i = \Delta / G_{\Delta_i}$ which is homeomorphic to $\Omega_i / G_{\Omega_i}$. By Ahlfors Finiteness Theorem $\Omega_i / G_{\Omega_i}$ satisfies (2.1).

COROLLARY 2: *If $\Omega_1$ and $\Omega_2$ are noninvariant components of the b-group $G$ and $\lambda \in \partial\Omega_1 \cap \partial\Omega_2$, then $\lambda$ is the fixed point of an accidental parabolic transformation.*

*Proof:* Since $\lambda$ is an accumulation point of orbits under $G_{\Omega_1}$ and $G_{\Omega_2}$, $\Delta_1 = \psi^{-1}(\Omega_1)$ and $\Delta_2 = \psi^{-1}(\Omega_2)$ have $\lambda$ as a boundary point. $\Delta_2$ must lie in a single component of $C - \bar{\Delta}_1$. But any boundary curve of $\Delta_1$ is the closed axis of an accidental parabolic transformation $\gamma$. Hence $\bar{\Delta}_1 \cap \bar{\Delta}_2$ can consist at most of $A_\gamma$ and $\lambda$ must then be the fixed point of $\gamma$.

## § 3. The limit sets of regular *b*-groups

We give a classification of limit points of regular *b*-groups and use it to show that the limit set has zero area and is locally connected.

THEOREM 2: *If $G$ is a regular b-group with $\infty \in \Omega(G)$ and $\lambda \in \Lambda(G)$ then either:*

(i) *there exists a non-invariant component $\Omega_1$ of $\Omega(G)$ such that $\lambda \in \partial\Omega_1$, or*

(ii) *there exists an accidental parabolic transformation $\gamma \in G$ and a sequence $(\gamma_n) \subset G$ such that if $A_\gamma$ is the (closed) axis of $\gamma$, then $\gamma_n(A_\gamma)$ is a spherical nest about $\lambda$ and* diam $(\gamma_n(A_\gamma)) \to 0$.

*Proof of Theorem* 2: We use the Fuchsian model $H$ for $G$. Let $f: U \to \Delta$ be the conformal mapping conjugating $H$ onto $G$. We assume $\lambda$ does not lie in the boundary of any non-invariant component of $\Omega(G)$ since each such boundary is a quasi-circle (B-II). It follows that $\lambda$ lies in a prime end of $\Delta$ whose preimage in $U$ under $f$ is defined by a nested sequence of crosscuts corresponding to accidental parabolic transformations in $G$. It follows from Theorem 1 that there exist only finitely many non-conjugate accidental parabolic transformations in $G$, and therefore we may extract a subsequence of crosscuts $\hat{\alpha}_n$ which are images of a fixed crosscut $\hat{\alpha}_0$ under the maps $\hat{\gamma}_n \in H$. $f(\hat{\alpha}_0)$ together with the fixed point of $\gamma = f^{-1} \circ \hat{\gamma} \circ f$. the accidental parabolic transformation whose axis is $f(\hat{\alpha}_0)$ is the (closed) axis $A_\gamma$ of $\gamma$. It is trivial that $\gamma_n(A_\gamma)$ nest about $\lambda$ if $\gamma_n = f^{-1} \circ \hat{\gamma}_n \circ f$.

If diam $\gamma_n(A_\gamma)$ does not converge to zero, we can find a subsequence, again denoted $\gamma_n(A_\gamma)$, for which the diameters remain greater than some $\varepsilon > 0$. Since $\gamma_n(A_\gamma)$ accumulates only at limit points, we may conjugate $G$ so that $\infty$ lies in the invariant component without affecting the uniform positivity of the diam $\gamma_n(A_\gamma)$. Thus we may assume $\gamma_n^{-1}(\infty) \in \Delta$. Since $\gamma_n(A_\gamma) = \gamma_n \gamma^k(A_\gamma)$ for $k \in Z$, we may assume $\gamma_n^{-1}(\infty)$ lies in a fixed fundamental set for the cyclic group generated by $\gamma$. By definition of accidental transformation, horocycles for $\gamma$ lie in non-invariant components, hence the poles of the $\gamma_n$ are uniformly bounded away from the fixed point of $\gamma$ hence from $A_\gamma$. It follows from standard arguments that a subsequence of the $\gamma_n$ converge uniformly to a constant on $A_\gamma$, which contradicts the assumption that diam $\gamma_n(A_\gamma) > \varepsilon$.

The description of limit points of regular $b$-groups given in the previous theorem enables us to quickly show that regular $b$-groups have limit sets of zero area, using the Koebe-Maskit Theorem [21].

COROLLARY: *If $G$ is a regular b-group with $\infty \in \Omega(G)$, then the limit set of $G$ has zero area.*

*Proof:* The classification of limit points given in Theorem 2, decomposes $\Lambda(G)$ into two subsets. The first consists of those $\lambda \in \Lambda(G)$ which lie in the boundary of a non-invariant component. Each such boundary is a quasi-circle, hence of zero area. That there are countably many such quasi-circles in $\Lambda(G)$ is shown in Abikoff [1].

The limit points of $G$ not lying on the boundary of a non-invariant component form the relative residual limit set $\Lambda_R(G, \Delta)$. The accidental parabolic transformations $\gamma$, the images of whose axes nest about some $\lambda \in \Lambda_R(G, \Delta)$, may be chosen from a finite list since there are finitely many conjugacy classes of such $\gamma$. To show $\Lambda(G)$ has zero area, it therefore suffices to show that for one such $\gamma$,

$$\Lambda_\gamma = \{\lambda \in \bigcap_{n=1}^{\infty} \operatorname{Int} \gamma_n(A_\gamma): \gamma_n \in [\gamma_n] \in G/\langle\gamma\rangle\},$$

where $\langle\gamma\rangle$ is the group generated by $\gamma$. As noted in the proof of Theorem 2, we may assume that the poles of the $\gamma_n$ lie at a uniformly positive distance from $A_\gamma$. While this is not precisely the hypothesis of the Koebe-Maskit Theorem, the proof may be repeated verbatim to conclude that $\Lambda_\gamma$ has zero area.

We have shown that the metric structure of $\Lambda(G)$ is a rather simply consequence of Theorem 2 and known techniques. The next theorem shows that its topological structure follows in the same fashion.

THEOREM 3: *The limit set of a regular b-group is locally connected.*

*Proof:* Let $G$ be a regular b-group. We examine the two possible cases given by Theorem 2.

If $\{\lambda\} = \bigcap_{n=1}^{\infty} \operatorname{Int} \gamma_n(A_\gamma)$

then either

$$\gamma_{n_j}(\overline{\operatorname{Ext} A_\gamma}) \text{ or } \gamma_{n_j}(\overline{\operatorname{Int} A_\gamma}),$$

is a fundamental system of neighborhoods of $\lambda$ in $\mathbf{C}$. Since the situation is symmetric, we assume the latter. $\bar{A}_\gamma$ intersects $\Lambda(G)$ solely in the fixed point $\lambda_0$ of $\gamma$. If $\overline{\operatorname{Int} A_\gamma \cap \Lambda(G)} = A$ is connected, its images under $\gamma_{n_j}$ yield the neighborhood system required for local connectivity. So we must show $A$ is connected. If not, there exist at least two components $C_1$ and $C_2$ of $A$. $C_1$ and $C_2$ are closed in $A$ and, at least one, say $C_1$, does not contain $\lambda_0$. It follows from the plane separation theorem (Whyburn [24, p. 108]) that there exists a Jordan curve $J \subset \operatorname{Int} A_\gamma$, $J$ separating $C_1$ and $C_2$. $J$ is thus contained in $\Omega(G)$, hence in a single component of $\Omega(G)$, but every component of $\Omega(G)$ is simply connected. This contradiction proves that $\Lambda(G)$ is locally connected at $\lambda \in \Lambda_R(G, \Delta)$.

If $\lambda$ lies on the boundary of a non-invariant component of $\Omega(G)$ and $\Lambda(G)$ is not locally connected at $\lambda$, then $\lambda$ lies on a continuum of convergence of $\Lambda(G)$; i.e., there exists a nondegenerate subcontinuum $B$ of $\Lambda(G)$ at no point of which is $\Lambda(G)$ locally connected, (Whyburn [24, p. 19]). $B$ must lie entirely in $\Lambda(G) - \Lambda_R(G, \Delta)$. If $\Omega_i$ is a non-invariant component of $\Omega(G)$, then $\partial\Omega_i \cap B$ is closed in $B$. We also have that

$$B = \bigcup_{i=1}^{\infty} (B \cap \partial\Omega_i),$$

where $\Omega_i$ ranges over the non-invariant components of $\Omega(G)$. Since no continuum may be written as the union of a countable number (greater than one) of disjoint closed sets, (Whyburn [24, p. 16]) there exist two non-invariant components say $\Omega_1$ and $\Omega_2$, such that $B \cap \partial\Omega_1 \cap \partial\Omega_2 \neq \varnothing$. It follows from Corollary 2 to Theorem 1 that if $\lambda_1 \in B \cap \partial\Omega_1 \cap \partial\Omega_2$, then $\lambda_1$ is the fixed point of an accidental parabolic transformation. The following lemma shows that $\Lambda(G)$ is locally connected at such fixed points. It follows that $\Lambda(G)$ contains no continua of convergence and is therefore locally connected.

LEMMA 2: *If $\lambda$ is the fixed point of an accidental parabolic transformation, then $\Lambda(G)$ is locally connected at $\lambda$.*

*Proof:* Let $\partial\Omega_1$ and $\partial\Omega_2$ intersect exactly in the fixed point $\lambda_1$ of an accidental parabolic transformation $\gamma$. Since $\overline{\Omega}_i$ is a closed Jordan domain, i.e., topologically a closed disc, given any two sequences $(z_n^i)$, $\zeta(_n^i) \subset \partial\Omega_i$ with $\zeta_n^i \to \lambda_1$ and $z_n^i \to \lambda_1$, there exist open arcs $\beta_n^i$ in $\Omega_i$ such that $\overline{\beta}_n^i \cap \partial\Omega_i = \{\zeta_n^i, z_n^i\}$ and diam $\beta_n^i \to 0$. Let $\zeta_1^1$ (respectively $\zeta_1^2$) be a loxodromic fixed point on $\partial\Omega_1$ (respectively $\partial\Omega_2$) and draw a curve $\alpha$ connecting $\zeta_1^1$ and $\zeta_1^2$, which except for endpoints lies in $\Delta$. If

$$\zeta_n^i = \gamma^n(\zeta_i^1),$$
$$z_n^i = \gamma^{-n}(\zeta_1^i) \quad \text{for } n \in N,$$

and

$$\alpha_n = \gamma^n(\alpha_1) \quad \text{for } n \in Z - \{0\},$$

then as $n \to \pm\infty$, $\alpha_n \to \lambda$ uniformly. If necessary by conjugation and passing to a subsequence, we may assume $\lambda_1 \in \operatorname{Int} C_n$ for all $n \in Z$, where

$$C_n = \alpha_n \cup \alpha_{-n} \cup \beta_n^1 \cup \beta_n^2.$$

Since diam $C_n \to 0$, $B_n = \operatorname{Int} C_n$ is a fundamental system of neighborhoods of $\lambda_1$. To prove the lemma it suffices to show that the component of $B_n \cap \Lambda(G)$ containing $\lambda_1$ is a neighborhood of $\lambda_1$ in $\Lambda(G)$.

If $B_n \cap \Lambda(G)$ is not connected then we may assume $B_n \cap \Lambda(G)$ has infinitely many components, in particular more than four, since if not, we can forget those not containing $\lambda_n$ to obtain the required neighborhood. The proof now proceeds as for those $\lambda \in \Lambda_R(G, \Delta)$ using the plane separation theorem and we thus complete the proof of the lemma and of the theorem.

COROLLARY: *If $G$ is a regular $b$-group and $f$: $U \to \Delta(G)$, then $f$ has a continuous extension to $\overline{U}$.*

*Proof:* It is simply necessary to cite several classical results. It is known (Whyburn [24, p. 111–112]) that a plane region $\Delta$, whose boundary $\partial\Delta = \Lambda(G)$, is locally connected, has the property that every boundary point of $\Delta$ is accessible from $\Delta$. Since $\Delta$ is then a simply connected domain each of whose boundary points is accessible from $\Delta$, it is a consequence of the theory of prime ends (Goluzin [13, p. 45]) that any conformal map $f$ of the unit disc $U$ onto $\Delta$ has a continuous extension to $\overline{U}$. The proof of the corollary is now complete.

## § 4. Congruent regular *b*-groups

The previous section dealt with the plane topological and metric properties of the limit set of a regular $b$-group. We now proceed to examine quasiconformal deformations of regular $b$-groups. In Theorem 4, we will give precise conditions for two regular $b$-groups to be quasiconformally equivalent. For groups without torsion, the theorem has been proved by Marden [19].

In order to proceed, we must first formalize the concept of two regular $b$-groups that look alike. The relevant notion is that of congruence.

*Definition:* Let $G$ and $\hat{G}$ be regular $b$-groups with invariant components $\Delta$ and $\hat{\Delta}$ respectively. Let $\{\alpha_1, ..., \alpha_k\}$ (respectively $\{\hat{\alpha}_1, ..., \hat{\alpha}_k\}$) be the admissible curves on $S = \Delta/G$ (respectively $\hat{S} = \hat{\Delta}/\hat{G}$) defining the proper partition given by Theorem 1. $G$ and $\hat{G}$ are said to be *congruent* if there exists a homeomorphism $f: S \to \hat{S}$, preserving extended markings, such that $f(\alpha_i) = \hat{\alpha}_i$.

We note for future reference several immediate consequences of the definition. We not only obtain a correspondence between the non-invariant factors of $G$ and $\hat{G}$, but we also see the manner in which they are pieced together to form $S$ and $\hat{S}$. Thus $f$ lifts to a homeomorphism of $\Omega(G) \to \Omega(\hat{G})$, since factor congruence implies the topological equivalence of $S$ and $S'$. $f$ restricts to yield the topological equivalence of non-invariant factors, hence $G$ and $\hat{G}$ are isomorphic as abstract groups. If we assume that the $\alpha_i$ and $\hat{\alpha}_i$ are piecewise analytic, then $f$ may be taken quasi-conformal on $S$, hence we may lift to a quasiconformal homeomorphism $\tilde{f}$ of $\Omega(G)$ onto $\Omega(\hat{G})$. As corollary to Theorem 4, we will show that $\tilde{f}$ extends to a quasiconformal mapping of $\hat{\mathbf{C}}$.

We now introduce a normalization of quasiconformal mappings. Let $G$ and $\hat{G}$ be $b$-groups and $H$ (respectively $\hat{H}_1$) a non-invariant component subgroup of $G$ (respectively $\hat{G}$). If $w$ is a quasiconformal mapping of $\hat{\mathbf{C}}$ which conjugates $H_1$ into $\hat{H}_1$, then $w$ is called *axis-normalized* (*with respect to* $H_1$) if, for each parabolic $\gamma \in H_1$ which is accidental with respect to $\Omega(G)$, $w(A_\gamma) = A_{w\gamma w^{-1}}$. By projection to $\Omega(H_1)/H_1$, quasiconformal deformation of that surface and then lift to $\Omega(H_1)$, it is clear that every quasiconformal deformation of $H_1$ is equivalent to an axis-normalized quasiconformal deformation. Let $H_1$ and $H_2$ be non-invariant component subgroups of $G$ and assume there is a parabolic $\gamma \in H_1 \cap H_2$ which is accidental with respect to $\Omega(G)$. Let $\hat{G}$ be a $b$-group congruent to $G$ and $w_i$ an axis-normalized quasiconformal mapping of $\hat{\mathbf{C}}$ which conjugates $H_i$ into $\hat{H}_i$, for $i = 1, 2$. If $w_1 \gamma w_1^{-1} = w_2 \gamma w_2^{-1}$, then we may assume that $w_1 | A_\gamma = w_2 | A_\gamma$.

THEOREM 4: *If $G$ and $\hat{G}$ are congruent regular $b$-groups, then there exists a quasiconformal homeomorphism $\hat{w}$ of $\hat{\mathbf{C}}$ so that*

$$\hat{w} G \hat{w}^{-1} = \hat{G}.$$

*Proof:* The proof will have the following structure. A quasiconformal mapping $\hat{w}$: $\hat{\mathbf{C}} \to \hat{\mathbf{C}}$ conjugates $G$ into $\hat{G}$ if and only if $\hat{w}\eta_i\hat{w}^{-1} = \hat{\eta}_i^{-1}$, where $\eta_i$, $i = 1, \ldots, N$ is a system of generators of $G$ and $\{\hat{\eta}_i\}$ is a system of generators of $G$. Each $\eta_i$ (respectively $\hat{\eta}_i$) either stabilizes zero, one or two non-invariant components of $G$ (respectively $\hat{G}$). Let $F$ be a fundamental set for the action of $G$ on $\Delta$ given by a lift of $S - \beta$, $\beta$ as in Lemma 1, to a connected subset of $\Delta$. We assume $\infty \in F$. $F$ has a finite number of sides and intersects finitely many axes of accidental parabolic transformations. It is classical that the transformations $\eta_1, \ldots, \eta_N$ pairing the sides of $F$ generate $G$. Let $\Omega_i, \ldots, \Omega_k$ be the non-invariant components of $\Omega(G)$ corresponding to the components $F \cap \Delta^*$ as in Theorem 1. Let $G_i = \operatorname{Stab} \Omega_i$ for

$i \leqslant K$. A generator $n_j$ either lies in $G_i$ for some $i \leqslant K$ or maps some $\Omega_i$ into $\Omega_{i'}$, with $i, i' \leqslant K$. Thus to show $\hat{w}$ conjugates $G$ onto $\hat{G}$ is equivalent to showing

(1) $\hat{w}$ conjugates $G_i$ into $\hat{G}_i$ for $i \leqslant K$, and

(2) if $\eta_j(\Omega_i) = \Omega_{i'}$, for some $i, i' \leqslant K$ then $\hat{w}\eta_j\hat{w}^{-1} = \hat{\eta}_j \colon \hat{\Omega}_i \to \hat{\Omega}_{i'}$.

The proof consists of globally modifying a fixed mapping successively to satisfy (1) and (2). The modifications take place in a complementary component of a closed axis of an accidental parabolic transformation. The axis is a quasi-circle and is a removable singularity for quasiconformal mappings. The modifications must be performed infinitely many times but are defined on finitely many complementary components of closed axes, and then extended by elements of $G$ to the other complementary components. It then follows from normal convergence of uniformly quasiconformal mappings that the limit mapping is quasiconformal. The method is algorithmic.

*Modification* 1: We assume that we have an axis-normalized quasiconformal mapping $g_j$ so that $g_j H_j g_j^{-1} = \hat{H}_j$ where $H_j$ (respectively $\hat{H}_j$) is the group generated by $G_1, \ldots, G_j$ (respectively $\hat{G}_1, \ldots, \hat{G}_j$). We seek a quasiconformal mapping $G_{j+1}$ such that $g_{j+1} H_{j+1} g_{j+1}^{-1} = \hat{H}_{j+1}$. We may assume that $G_{j+1} \cap H_j$ is a cyclic parabolic group, generated by a transformation $\gamma$ which is accidental with respect to $\Delta(G)$. Let $w_{j+1}$ be an axis-normalized map conjugating $G_{j+1}$ into $\hat{G}_{j+1}$ such that $w_{j+1} | A_\gamma = g_j | A_\gamma$.

Let $D_1$ be the component of $\hat{\mathbf{C}} - A_\gamma$ containing $\Omega_{j+1}$ and $D_2 = \hat{\mathbf{C}} - \overline{D_1}$ and set

$$h_0(z) = \begin{cases} w_{j+1}(z) & \text{for } z \in \overline{D_1} \\ g_j(z) & \text{for } z \in \overline{D_2} \end{cases}$$

The dilatation $\mu_{h_0}$ satisfies $|\mu_{h_0}| \leqslant \max(|\mu_{w_{j+1}}|, |\mu_{g_j}|)$

We now order the $H_{j+1}$-images $A_{\theta_n}$ of $A_\gamma$ so that for $n_1 < n_2$, $A_{\theta_{n_1}}$ is separated from $A_\gamma$ by no more $H_{j+1}$-images of $A_\gamma$ than $A_{\theta_{n_1}}$ is. We now define a sequence of maps $h_n$. Let $D_n^1$ be the complementary disc of $A_{\theta_{n_1}}$ containing $\bigcup_{i=1}^{j+1} \Omega_i$ and $D_n^2 = \mathbf{C} - D_n^1$. Let

$$h_n(z) = \begin{cases} h_{n-1}(z) & \text{for } z \in D_n^1 \\ \hat{\theta}_n h_{n-1} \theta_n^{-1}(z) & \text{for } z \in D_n^2 \end{cases} \tag{4.1}$$

where $\hat{\theta}_n$ is the image of $\theta_n$ in $\hat{G}$ under the isomorphism established by the given congruence. By the assumption of axis-normalization, $h_n$ extends to a quasiconformal map of $\hat{\mathbf{C}}$. $(h_n)$ is a sequence of mappings with $|\mu_{h_n}| \leqslant |\mu_{h_0}|$. Thus $h_n \to h$ were $h$ is a quasiconformal map of $\hat{\mathbf{C}}$.

To show that $h$ is the required mapping $g_{j+1}$, we must only show that $h\eta h^{-1}(z) = \hat{\eta}(z)$

for each generator $\eta$ of $H_{j+1}$ and $z \in \Omega(G)$. For $z \in \Omega(G)$, $h(z) = h_n(z)$ for $n \geq n_0$ and by (4.1) we may assume

$$h_n(z) = \hat{\theta}_n h_{n-1} \theta_n(z).$$

A simple argument yields that $\eta = \theta_n$ and we have shown that we may find a quasiconformal mapping conjugating $H_k$ into $\hat{H}_k$.

The second modification again gives us an algorithm for redefining $g_k$ so that it commutes with the generators $\eta \in G - H_k$. We give the induction step.

*Modification* 2: Suppose $H_k < H_m < G$ where $H_m$ is obtained from the group generated by $H_k$ and some of the generators of $G$ in $G - H_k$. Let $\eta$ be a generator of $G$, $\eta \in G - H_m$. We denote by $H_{m+1}$ the group generated by $H_m$ and $\eta$. We further suppose $g$ is an axis-normalized quasiconformal map of $\hat{\mathbf{C}}$ conjugating $H_m$ into $H_m$. Under the correspondence established at the beginning of the proof, between non-invariant components of $G$ and components of $F \cap \Delta^*$ there are accidental parabolic transformations $\gamma_1, \gamma_2 \in H_m$ such that $\eta\,(\mathrm{Ext}\, A_{\gamma_1}) = \mathrm{Int}\, A_{\gamma_2}$. Since $g$ is axis-normalized, we may assume

$$\hat{\eta} \circ g \,|\, A_{\gamma_1} = g \,|\, A_{\gamma_2} = g \circ \eta \,|\, A_{\gamma_1} \tag{4.2}$$

Let $B$ be the doubly connected region bounded by $A_{\gamma_1}$ and $A_{\gamma_2}$. Define $h_1(z) = \hat{\eta}^k \circ g \circ \eta^{-1}(z)$ for $z \in \eta^k(B)$ and $k \in Z$. By (4.2) $h_1$ is continuous and injective. It has a continuous extension to the fixed points of $\eta$, hence is a homeomorphism of $\hat{\mathbf{C}}$. It is the limit of mappings $f_n$ with $|\mu_{f_n}| \leq |\mu_g|$ a.e., hence $h_1$ is quasiconformal. If $\mathfrak{A}_n$ denotes the set of axes, of accidental parabolic transformations $\gamma$ conjugate to $\gamma_1$ in $H_{m+1}$, which are separated from $A_{\gamma_1} \cup A_{\gamma_2}$ by $n-1$ $h_{m+1}$-images of $A_{\gamma_1}$ then $h_1$ satisfies

$$\hat{\theta} \circ g \,|\, A_\gamma = g \,|\, \theta(A_\gamma) = g \circ \theta \,|\, A_\gamma \quad \text{for } A_\gamma, \theta(A_\gamma) \in \mathfrak{A}_1. \tag{4.3}$$

A similar argument yields the following induction step. Let $A_{\sigma_1}, A_{\sigma_2} \in \mathfrak{A}_n$ and suppose $\tau_j \in H_{m+1}$, $\tau_j(A_{\sigma_1}) = A_{\sigma_2}$ and $h_{\tau_{j-1}}$ satisfies

$$\hat{\theta} h_{\tau_{j-1}} \,|\, A_\gamma = h_{\tau_{j-1}} \,|\, \theta(A_\gamma) = h_{\tau_{j-1}} \theta \,|\, A_\gamma \quad \text{for } A_\gamma, \theta(A_\beta) \in \mathcal{U}_n. \tag{4.4}$$

Define $h_{\tau_j}(z) = \hat{\tau}_j^k h_{\tau_{j-1}} \tau_j^{-k}(z)$ for $z \in \tau_j^k\,(\mathrm{Ext}\, A_{\sigma_1} \cap A_{\sigma_2})$ and $k \in Z$. By the normalization (4.4) and continuous extension to the fixed points of $\tau_j$, $h_{\tau_j}$ is continuous. The dilatation of $h_{\tau_j}$ is bounded by $|\mu_g|$ almost everywhere since $h_{\tau_j}$ is a limit of quasiconformal mappings with the given bound on their dilatations.

The induction step is first performed for all pairs of axes in $\mathfrak{A}_n$. The assumption (4.4) is then valid for $\mathfrak{A}_{n+1}$ and we may continue. In the limit we obtain a quasiconformal mapping $h$ with $|\mu_n| \leq |\mu_g|$ a.e. which conjugates $H_{m+1}$ into $\hat{H}_{m+1}$.

Since $H_m = G$ for some $m$, the process terminates with a quasiconformal map $\hat{w}$ satisfying $\hat{w}G\hat{w}^{-1} = \hat{G}$.

*Remarks:* The proof of Theorem 4 given above, depends most heavily on one property of regular $b$-groups. The stability subgroups of non-invariant components have in some sense, disjoint actions. More precisely, there is a canonically defined set of quasicircles, the axes of accidental parabolic transformations, which split the action of the group. We have used this property in the following fashion: given two non-invariant component subgroups with a common accidental parabolic transformation, we were able to simultaneously conjugate them, via a quasiconformal mapping, into two similarly situated subgroups.

COROLLARY: *If $G$ and $\hat{G}$ are factor congruent regular $b$-groups, then any quasiconformal mapping $w$: $\Omega(G) \to \Omega(\hat{G})$ conjugation $G$ onto $\hat{G}$, extends to a quasiconformal mapping of $\hat{\mathbf{C}}$ conjugating $G$ onto $\hat{G}$.*

*Proof:* If $\hat{w}$ is as in the Theorem, then $f = \hat{w}^{-1} \circ w$ is a quasiconformal mapping of $\Omega(G)$ inducing a trivial automorphism of $G$, hence has a quasiconformal extension by Maskit's Extension Theorem [22]. $w$ may then be extended to $\mathbf{C}$ by $\hat{w} \circ f$.

## §5. Convergence of quasi-Fuchsian groups to regular $b$-groups

We next give a constructive procedure for finding regular $b$-groups on the boundary of Teichmüller spaces. We first prove the following generalization of B-II, Theorem 12. If $G$ is a finitely generated Fuchsian group of the first kind and $P$ is a proper partition of $U/G$, we show that there is a regular $b$-group $H \in \partial T(G)$ which realizes the partition $P$ — in the sense of Theorem 1. We next show that, up to inner automorphism by elements of $\mathfrak{Möb}_{\mathbf{C}}$, all regular $b$-groups lie on the boundary of some Teichmüller space. For torison-free groups, the latter result has been obtained by Marden [18]. As corollary, we obtain a partial solution to a conjecture of Bers and the quasiconformal stability of regular $b$-groups. Stability is discussed in B-I.

We first prove the following lemma, which we call the *three point condition.*

LEMMA 3: *Let $G_0$ be a Kleinian group and $(w_n)$ a sequence of homeomorphisms such that $\chi_n(G_0) = G_n = w_n G_0 (w_n)^{-1} \subset \mathfrak{Möb}_{\mathbf{C}}$. We further assume:*

(i) *$G_n$ converges (in the sense of generators converging) to a group $H = \chi(G_0) \subset \mathfrak{Möb}_{\mathbf{C}}$ which is isomorphic to $G_0$,*

(ii) *there is an open set $D \subset \hat{\mathbf{C}}$, on which $w_n$ is uniformly quasiconformal,*

(iii) *for some* $z \in D$ *and* $\eta$ *and* $\gamma$ *non-commuting loxodromic elements of* $G_0$, $\gamma(z)$ *and* $\eta(z)$ *lie in* $D$,

*then* $H$ *is either discontinuous or non-discrete. Further, if* $H$ *is discontinuous* $H_{w(D)} = \chi(G_{0_D})$

*Proof of lemma:* Since the dilatation of $w_n|D$ is uniformly bounded, we may choose a subsequence converging normally to a mapping $w$ of $D$ into $\hat{\mathbf{C}}$. It follows from Lehto-Virtanen [16, p. 74] that $w$ is either $K$-quasiconformal, constant or a mapping of $D$ onto two points. We treat each of these cases separately.

If $w$ is quasiconformal, then for each $\zeta \in D \cap \Omega(G)$ there is a neighborhood $N$ of $w(\zeta)$ in $\Omega(G)$ such that $w_n(\hat{C} - D) \cap N = \emptyset$ for $n$ sufficiently large. $H$ is discontinuous in a neighborhood of $\zeta$ since

$$\text{card}\,\{\tau \in H\colon \tau(N) \cap N \neq \emptyset\} = \text{card}\,\{\tau \in G_0\colon \tau(w^{-1}(N) \cap w^{-1}(N) \neq \emptyset\} < \infty.$$

It also follows that $H_{w(D)} = \chi(G_{0_D})$.

If $w$ is the constant map $w(\zeta) = c$, then on the triple $\{z, \gamma(z), \eta(z)\}$ $w_n$ converges uniformly. If $\gamma_n = \chi_n(\gamma)$ and $\eta_n = \chi_n(\eta)$, then $\gamma_n \to \hat{\gamma}$ and $\eta_n \to \hat{\eta}$ where $\hat{\gamma}$, $\hat{\eta} \in H$. It follows easily that $\hat{\gamma}(c) = \hat{\eta}(c) = c$. Thus $\hat{\gamma}$ and $\hat{\eta}$ have a fixed point in common. They do not commute since $G$ and $H$ are isomorphic. They are also of infinite order. The previous three statements are incompatible for elements in a discrete group of Möbius transformations.

The last case may be trivially dealt with. If necessary, by removal of a finite number of points of $D - \{z, \gamma(z), \eta(z)\}$, we may assume that $D$ has at least two boundary points. It then follows from Lehto-Virtanen [16, p. 76] that $w(D)$ is connected and we then use the preceding arguments, to conclude that $H$ is either discontinuous or non-discrete.

If $S$ is a conformally finite surface with signature which is covered by the upper half-plane $U$ we have defined an admissible curve so that a connected component of its lift to $U$ is invariant under the normalizer of a cyclic hyperbolic subgroup of the cover group. We will now give a definition of an open set associated to a family of admissible curves on which we may support certain deformations, of the surface $S$, which take us to regular $b$-groups on the boundary of the Teichmüller space, $T(S)$. Let $\alpha$ be a piecewise analytic admissible curve on $S$. If $\alpha$ is a simple loop, a ***distinguished neighborhood*** of $\alpha$ is a tubular neighborhood $K$ of $\alpha$ whose boundary is a pair of disjoint piecewise analytic simple loops, each homotopic to $\alpha$ on $S' = S - \{z_1, ..., z_l\}$. If $\alpha$ is a slit with endpoints $z_i$ and $z_j$, a distinguished neighborhood of $\alpha$ is a disc on $S$ with piecewise analytic boundary $C$ and $C$ separates $z_i$ and $z_j$ from the other points at which $S$ is marked. In either case the lift to $U$ of a distinguished neighborhood of $\alpha$ has, as components, neighborhoods of the components of the lifts of $\alpha$.

Let $\{\alpha_1, ..., \alpha_n\}$ be a set of piecewise analytic admissible curves on $S$ defining a proper partition of $S$. Let $K_i$ be distinguished neighborhoods of the $\alpha_i$ having pairwise disjoint closures. $K = \bigcup_{i=1}^{n} K_i$ is called a *proper neighborhood* for the proper partition of $S$ defined by $\{\alpha_1, ..., \alpha_n\}$. This terminology is justified by the next theorem. Its proof is for the most part due to Bers [B-I, Theorem 11], and states that, up to homeomorphism, we may realize a proper partition of a marked surface $S = U/G_0$ in terms of the orbit space $\Omega(G)/G$ of a regular boundary group $G$ of the Teichmüller space $T(G_0)$.

THEOREM 5: *Let $G_0$ be a finitely generated Fuchsian group of the first kind and $\{\alpha_1, ..., \alpha_n\}$ a set of piecewise analytic admissible curves on $S = U/G_0$ defining a proper partition $\{S_1, ..., S_k\}$ of $S$. If $K$ is a proper neighborhood for this partition, then there exists a sequence $G_i$ of quasiconformal deformations of $G_0$, $G_1 = (w^{\mu_i})G_0(w^{\mu_i})^{-1}$ such that:*

(1) $\operatorname{supp} \mu_i \subset \pi^{-1}(K)$,

(2) $G_i \to G \in \partial T(G_0)$ *(in the sense, say, of generators converging) with $G$ a regular b-group,*

(3) $\Omega(G)/G = L/G_0 + \hat{S}_1 + ... + \hat{S}_K$ *where $\hat{S}_j$ is homeomorphic to $S_j$ and $L$ is the lower half-plane.*

*Proof:* Each distinguished neighborhood $K_i$ of one of the admissible curves, $\alpha_i$, i.e., component of $K$, is topologically an annulus. The annulus contains a simple piecewise analytic homotopically non-trivial loop $\alpha_i'$ in $K_i$. $\alpha_i'$ determines a hyperbolic element of $G_0$ Let $K_i'$ be a tubular neighborhood of $\alpha_i'$ contained in $K_i$. The elegant argument, due to Bers and cited above, shows that we may find a sequence of Beltrami differentials, so that

(a) $\operatorname{supp} \mu_i \subset K' = \bigcup K_i'$,

(b) $G_i = (w^{\mu_i})G_0(w^{\mu_i})^{-1} \to G \in \partial T(G_0)$ with an isomorphism $\chi$ induced by the conformal map $w = \lim w^{\mu_i} | L$ of $L$,

(c) if $\gamma_m \in G_0$ is a cover transformation determined by $\alpha_m$, then $\hat{\gamma}_m = \operatorname{Lim}_{i\to\infty} (w^{\mu_i})_m \times (w^{\mu_i})^{-1} \in G$ is an accidental parabolic transformation.

We now show that each proper part of $S$ appears—up to homeomorphism—in $\Omega(G)/G$. We consider $S_1 \cdot S_1$ is homeomorphic (as a marked surface with signature) to $S_1^* = (S_1 - K)^0$. Let $A$ be a component of $\pi^{-1}(S_1^*)$. On $A$, the maps $w^i$ are schlicht. We next show that $A$ contains points $z$, $\gamma(z)$, and $\eta(z)$ where $\gamma$ and $\eta$ are non-commuting loxodromic elements of $G_0$. $S_1^*$ may be given the conformal structure of a marked hyperbolic conformally finite surface. As such the stability group of $A$ is the cover group of a ramified cover of $S_1^*$ with the obvious ramifications. Thus $(G_0)_A$ is isomorphic to a finitely generated Fuchsian group of the first kind. In particular $(G_0)_A$ is a Fuchsian group which is not a finite extension of a cyclic group. In that case it is well-known that $(G_0)_A$ contains non-commuting loxodromic transformations.

The conditions of Lemma 3 are thus satisfied and $G$ is discrete since $G \in \partial T(G_0)$ and such groups are $b$-groups. Thus $(G_0)_A = G_{w(A)}$. Thus $(w \circ \pi^{-1}(S_1^*))/G$ appears in the quotient and is homeomorphic to $S_1$.

We next complete the proof of (3) by showing that $\Omega(G)/G$ has no other components save the base surface and those determined as above by proper parts. Suppose $\Omega_1$ is a component of $G$ which does not project into the proper parts defined above. $G_1 = G_{\Omega_1}$ is a finitely generated quasi-Fuchsian group of the first kind (see Abikoff [1]) and there is a loxodromic transformation $\hat{\gamma}$ with fixed points $\hat{\zeta}_1$ and $\hat{\zeta}_2$. $\hat{\gamma}$ has a pullback $\gamma$ which is hyperbolic. If the fixed points $\zeta_1$, $\zeta_2$ of $\gamma$ lie in the boundary of some component of $U = \pi^{-1}(K)$ then there exists another non-invariant component $\Omega_2$ of $\Omega(G)$ which is stabilized by $\gamma$. In this case we may find open Jordan arcs $J_1$ and $J_2$ lying in $\Omega_1$ and $\Omega_2$ respectively whose endpoints are $\hat{\zeta}_1$ and $\hat{\zeta}_2$. $\bar{J} = \bar{J}_1 \cup J_2$ is a Jordan curve in $\hat{\mathbf{C}} - \Delta(G)$ both of whose complementary components contain limit points. This is impossible for a function group. If the fixed points of $\gamma$ lie in the boundary of no component of $U - \pi^{-1}(K)$, $\zeta_1$ and $\zeta_2$ are separated, as boundary points of $L$, by the axis $A_\eta$ of the transformation $\eta \in G_0$, for which $\hat{\eta} = w\eta w^{-1}$ is parabolic. $\overline{w(A_\eta)}$ is a Jordan curve in $\overline{\Delta(G)}$ and $\overline{w(A_\eta)}$ meets $\Lambda(G)$ in exactly one point, the fixed point of $\hat{\eta}$. $\overline{w(A_\eta)}$ therefore cannot separate the fixed points $\hat{\zeta}_1$, $\hat{\zeta}_2$ of $\hat{\gamma}$ since $\hat{\zeta}_1$, $\hat{\zeta}_2$ lie on the Jordan curve $\Lambda(G_1) \subset \Lambda(G)$. But it is also true that $\overline{w(A_\eta)}$ must separate $\hat{\zeta}_1$ and $\hat{\zeta}_2$ since $w|L$ is schlicht. The proof of (3) is complete.

To complete the proof of the Theorem, it remains to show that $G$ is a regular $b$-group. That $G$ is a nonquasi-Fuchsian $b$-group follows from Bers' arguments. It is not totally degenerate since we have shown that it has non-invariant components. We must only show that each parabolic element $\hat{\gamma}$ of $G$ has two disjoint horocycles. If $\hat{\gamma}$ is non-accidental with respect to $\Delta(G)$, then $w^{-1}\hat{\gamma}w = \gamma$ is parabolic and has two disjoint horocycles, $B_1$ and $B_2$. The maps $w^{\mu_i}$ are schlicht on $B_1$ and $B_2$ as is the limit map $w$. We will need the following lemma.

LEMMA 4: *Under the conditions of Theorem* 5, *if $\hat{\gamma}$ is a primary accidental parabolic transformation with respect to $\Delta(G)$, then $\hat{\gamma}$ is determined on $\Delta(G)/G$ by an admissible curve $\alpha$ equivalent to one of the admissible curves defining the given partition of $S$.*

*Proof of Lemma* 4: We assume the statement is false and the curve $\alpha$ is a geodesic. We consider the intersection number of $\alpha$ and any admissible curve $\alpha_j$ defining the given partition. We may also assume that each $\alpha_j$ is a geodesic. We denote by $\alpha^*$ an arbitrary connected lift of $\alpha$ to $\Delta(G)$, and $\alpha_j^*$ a connected lift of $\alpha_j$. Up to sign the intersection number is then realized in the number of intersection points of $\alpha^*$ and $G\alpha_j^*$.

If $\alpha^* \cap \alpha_j^* = \emptyset$ for every lift of every $\alpha_j$, then $\alpha$ lies in some proper part $S_l$, and on $S_l$ is not homotopic to a power of a boundary curve. $\hat{\gamma}$ thus lies in the stability group of a non-invariant component $\Omega_1$ of $\Omega(G)$. $G_1 = G_{\Omega_1}$ is quasi-Fuchsian hence contains no parabolic transformations which are accidental with respect to either of its two components. Since $\alpha$ is not equivalent to a boundary curve of $S_l$, $\hat{\gamma}$ must be loxodromic, contrary to hypothesis.

$\alpha^*$ and $\alpha_j^*$ can intersect in at most one point.

We now assume $\alpha^* \cap \alpha_j^* = \{z_0\}$. The closures of $\alpha^*$ and $\alpha_j^*$ are Jordan curves $\overline{\alpha^*}$ and $\overline{\alpha_j^*}$ respectively. If $\alpha_j^*$ is the axis of $\hat{\gamma}_j \in G$, then $\overline{\alpha^*}$ and $\overline{\alpha_j^*}$ are the closed axes of $\hat{\gamma}$ and $\hat{\gamma}_j$ respectively, and are crosscuts of $\Delta(G)$. It follows that $\alpha^*$ and $\alpha_j^*$ intersect at two points, $z_0$ and the fixed point $z_1$ common to $\hat{\gamma}$ and $\hat{\gamma}_j$. Thus $\hat{\gamma}$ and $\hat{\gamma}_j$ lie in a cyclic subgroup of $G$, in which $\hat{\gamma}$ and $\hat{\gamma}_j$ are primary. The assumption that $\hat{\gamma}$ and $\hat{\gamma}_j$ are geodesic implies that they have the same impression, hence the partitions $\{\alpha_1, \ldots, \alpha_n\}$ and $\{\alpha, \alpha_1, \ldots, \hat{\alpha}_j, \ldots, \alpha_n\}$ are equivalent. The proof of the lemma is complete.

*Continuation of the proof of Theorem* 5: If $\hat{\gamma}$ is accidental with respect to $\Delta(G)$, then Lemma 4 implies that $\hat{\gamma}$ is determined by one of the admissible curves, say $\alpha_j$. It also follows that there are Jordan curves, $J_1$ and $J_2$, invariant under $\hat{\gamma}$, each contained except for the fixed point of $\hat{\gamma}$ in distinct non-invariant components of $\Omega(G)$. These curves contain disjoint horocyclic neighborhoods, and the proof of the Theorem is complete.

THEOREM 6: *Let $G$ be a regular b-group with Fuchsian equivalent $H$, then there exists $\eta \in \mathfrak{Möb}_{\mathbf{C}}$ so that*

$$\eta G \eta^{-1} \in \partial T(H).$$

*Proof:* It follows from Theorems 1 and 5, that $\partial T(H)$ contains a regular $b$-group $\hat{G}$ which is congruent to $G$ and such that $H$ is also the Fuchsian equivalent of $\hat{G}$. It then follows from the corollary to Theorem 4 that $G$ and $\hat{G}$ are quasiconformally equivalent via a map $w$ which is conformal on the invariant component $\Delta$ of $G$. By composing $w$ with a Möbius transformation $\eta$, we may assume $w(-i) = -i$ and $w'(-i) = 1$.

$\hat{G}$ has been defined as the limit of a sequence of groups $H_n = v_n H v_n^{-1}$. For notational purposes, we denote the induced isomorphism of groups defined by a mapping $f$ by $\chi_f$. In this notation $\hat{G} = \operatorname{Lim} \chi_{v_n}(H)$ the mappings $v_n$ of Theorem 5 converge to a conformal mapping only on $L \cup (\Omega - \pi^{-1}(K))$. In the present context this is not sufficient; we must, by additional assumptions control the behavior of the $v_n$ on a dense open subset of $\Omega(G)$.

We will alter the definition of $\hat{G}$ so that each $v_n$ is uniformly locally quasiconformal except along the lifts to $U$ of the curves $\alpha_1, \ldots, \alpha_n$ on $U/H$. Let $(K_i)$ be a sequence of proper neighborhoods of the admissible curves $\{\alpha_1, \ldots, \alpha_n\}$ so that $\bigcap K_i = \bigcup_{j=1}^{n} \alpha_j$. Let $\hat{K}_i = \pi^{-1}(K_i)$. If $\mu_n$ is the dilatation of $v_n$, we assume

$$\mu_n = \begin{cases} 0 & \text{for } z \in \mathbf{C} - \hat{K}_1 \\ \mu_i & \text{for } z \in K_i - K_n \quad \text{and} \quad i < n \end{cases}$$

and that $\mu_n$ is chosen on $K_n$ so that Bers' argument, as given in Theorem 5, remains valid. As before, $H_n = \chi_{v_n}(H)$ converges to a regular $b$-group, again denoted $\hat{G}$, which is factor congruent to $G$. But in this case $v_n$ converges uniformly on compact subsets of $\Omega^*(H) = \Omega(H) - \pi^{-1}(\cup \alpha_j)$ to a map $v$. We note that this rather complicated statement is in reality no more deep than the fact that one may map an annulus $A = \{0 < r_1 < |z| < 1\}$ onto a punctured disc via a homeomorphism which is quasiconformal off an arbitrarily small neighborhood of $|z| = r_1$.

At any point $z_0 \in \Omega(G)$, $v^{-1}(z_0)$ is well defined and $v_n \cdot v^{-1}$ has uniformly bounded dilatation at $z_0$. We now preceed to define a sequence of groups $H_n \in T(H)$, which converge to $G$, by locally defining a dilatation $\mu_n(z)$. Let

$$\mu_n(z) = \mu_{w^{-1}}(v \cdot v_n^{-1}(z))$$

on a fundamental set for $\Omega(H_n) - v_n(\pi^{-1}(\bigcup \alpha_j))$ and extend $\mu_n(z)$ to $\Omega(H_n) - v_n(\pi^{-1}(\bigcup \alpha_j))$ as a Beltrami differential for $H_n$, which defines $\mu_n$ almost everywhere. For all $n$, $|\mu_n| < \|\mu_w\|_\infty < 1$ hence there exist quasiconformal mappings $w_n$ whose dilation is exactly $\mu_n(z)$. We of course assume an interior normalization for $w_n$, i.e. $w_n(-i) = -i$, $w_n'(-i) = 1$. We claim that on some subsequence, again denoted $n$, $w_n \to w$. If so it is clear that $H_n = w_n H w_n^{-1}$ is a quasiconformal deformation of $H$ and $H_n \to G$. To prove the claim we note that $v \cdot v_n^{-1}$ converges pointwise to the identity on $\Omega(G)$ hence almost everywhere. Further,

$$\mu_n(\gamma_n(z)) = \mu_n(z) \frac{\overline{\gamma_n'(z)}}{\gamma_n'(z)} \quad \text{if } \gamma_n = \chi_{v_n}(\gamma)$$

but $\gamma_n \to \hat{\gamma} = \chi_{v_n}(\gamma)$ hence $\gamma_n' \to \hat{\gamma}'$, hence

$$\mu_n(\gamma_n(z)) \to \mu_{w^{-1}}(z) \frac{\overline{\hat{\gamma}'(z)}}{\hat{\gamma}'(z)}$$

and $\mu_n$ converges almost everywhere to $\mu_{w^{-1}}$. If follows easily from an approximation theorem of Bers which may be found in Lehto-Virtanen [16, p. 197] that $w_n \to w$, and the proof is complete.

We are now in a position to study deformations of regular $b$-groups. The next theorem proves a conjecture of Bers in the case of regular $b$-groups. Bers' conjecture as states in [9] is: for any group $G$ on the boundary of a Teichmüller space $T(H)$, one can find a complex manifold in $\partial T(H)$ of quasiconformal deformations of $G$ which is isomorphic to a product of Teichmüller spaces. To prove the conjecture in full generality, it would be necessary to prove this corollary for partially degenerate groups.

COROLLARY 1: *If $H$ is a finitely generated Fuchsian group of type $(g, n)$ and whose limit set is $\hat{\mathbf{R}} = \mathbf{R} \cup \{\infty\}$, let $S = L/H$, where $L$ the lower half-plane. Let $\{S_1, \ldots, S_k\}$ be a proper partition of $S$ and $G = \chi(H)$ be a regular $b$-group realizing this partition, i.e. $S_i$ is homeomorphic to $\Omega_i/\mathrm{Stab}\,\Omega_i$, where $\Omega_i$ is a non-invariant component of $\Omega(G)$. Then $\partial T(H)$ contains a complex manifold of quasiconformal deformations of $G$ isomorphic to $\prod_{j=1}^{k} T(\Omega_i/\mathrm{Stab}\,\Omega_i)$.*

*Proof:* If $H$ represents a triangle group, it is quasiconformally rigid, hence has no boundary groups. If $H$ is of type $(1, 1)$ or $(0, 4)$, the stability group of each non-invariant component of $G$ is a triangle group, the triviality of whose Teichmüller space implies the triviality of the assertion. If

$$\pi: \Omega(G) \to \Omega(G)/G = \Delta/G + S_i^* + \ldots + S_k^*$$

then any Beltrami differential $\mu_i$ on $S_i^*$ may be lifted to a Beltrami differential $\mu_i^*$ on $\pi^{-1}(S_i^*)$. We define

$$\mu(z) = \begin{cases} \mu_i^*(z) & z \in \pi^{-1}(S_i^*) \\ 0 & \text{elsewhere} \end{cases}$$

$\mu$ is a Beltrami differential for $G$. By solving the Beltrami equation with coefficient $\mu$ we obtain a quasiconformal mapping $w^{\mu}$ and a quasiconformal deformation $\chi_{\mu}(G) = G_1 = w_{\mu} G w_{\mu}^{-1}$. It is clear that $G_1$ is a regular $b$-group and $G_1 \in \partial T(H)$ by Theorem 6. But $\chi_{\mu}(G)$ depends holomorphically on $\mu$, hence on $\mu \,|\, (\pi^{-1}(S_i^*)/G) \in T(S_i^*)$. Thus $\partial T(G)$ contains a subset which has the complex analytic structure induced by the identification with $\prod_{i=1}^{k} T(S_i^*)$. Thus we have a canonically defined map

$$i: \prod_{i=1}^{k} T(S_i^*) \to \partial T(H).$$

To complete the proof of the theorem, it remains to show that $i$ is injective. Let $[\mu_i]$ (respectively $[\nu_i]$) be the equivalence class of $\mu_i$ (respectively $\nu_i$) in $T(S_i^*)$. If $i$ is not injective then $i([\mu_i], \ldots, [\mu_k]) = i([\nu_i], \ldots, [\nu_k])$ with $[\nu_i] \neq [\mu_i]$ for some $i$, i.e. $w_{\mu} G w_{\mu}^{-1} = w_{\nu} G w_{\nu}^{-1}$. It follows that $[\nu_i]$ and $[\mu_i]$ must be equivalent under the Teichmüller modular group $\Gamma(S_1^*)$ of $T(S_i^*)$. Since the modular group is discrete the mapping $i$ is an immersion. To prove that it is injective, it suffices to show that is it univalent on each $T(S_i^*)$ since this determines the conformal structure of $w_\mu \Omega_i / w_\mu(\text{Stab}\,\Omega_i) w^{\mu}_{-1}$. We may therefore assume $\mu(z) = \nu(z) = 0$ for $z \in \hat{\mathbf{C}} - \pi^{-1}(S_i^*)$. $w_\mu^{-1} w_\nu$ then gives a trivial deformation of $G$, hence, by Maskit [22], $[\mu_i] = [\nu_i]$. The proof is complete.

We note for future reference that the injection extends, again as an injection, to those boundary groups of $T(S_i^*)$ which are regular. These are $b$-groups with more loops pinched.

The corollary above states that there are many quasiconformal deformations of a regular $b$-group $G$ in a small neighborhood of $G$ in $\partial T(H)$. A simple dimension argument yields the next corollary. It may also be obtained using the constructibility of regular $b$-groups (B-II) and the quasiconformal stability of constructible groups (Abikoff [2]).

COROLLARY 2: *A regular b-group is quasiconformally stable.*

## §6. Limits of deformations on the boundary of Teichmüller space and irregular $b$-groups

We have in the previous section demonstrated the existence of injections of the Teichmüller spaces of non-invariant component subgroups (or proper parts) of a regular $b$-group $G$ into the boundary of the Teichmüller space of the Fuchsian equivalent $H$ of $G$. If the proper partition of $S = \Delta/G$ is induced by curves $\{\alpha_1, ..., \alpha_j\}$, we may perform further deformations of $G$ by taking admissible curves $\{\alpha_{j+1}, ..., \alpha_n\}$ so that $\{\alpha_1, ..., \alpha_n\}$ again induces a proper partition of $S$. In the same fashion as in Theorem 5, we may deform $G$ into a regular $b$-group $G'$ which is associated to the proper partition induced by $\{\alpha_1, ..., \alpha_n\}$. These are not the only limiting deformations which we may perform on regular $b$-groups. If $\Omega_1$ is a component of $G$, then $G_{\Omega_1} = G_1$ may be deformed in it's Teichmüller space into degenerate boundary point, i.e. there is a sequence $(\hat{G}) \subset \partial T(G_1)$ converging to a totally degenerate boundary point $G_1$. We next show that the injection $i$ preserves degeneracy. More specifically, we show that if $(\hat{G}_n) \subset T(G_1)$ and $\hat{G}_n \to \hat{G}$ a totally degenerate boundary group of $T(G_1)$, then every accumulation point in $\partial T(H)$ of $i(G_n)$ is either partially or totally degenerate. The latter occurs if $S_1$ is dense in $S = L/H$ and the former if it is not dense, i.e. $S$ has proper parts other than $S_1$ in some partition. The proof, as usual, relies on Bers' proof of the existence of totally degenerate groups.

THEOREM 7: *Let $G$ be a marked regular b-group with Fuchsian equivalent $H$ and invariant component $\Delta$ and let $\Omega_1$ be a non-invariant component of $G$. We further suppose $S = \Delta/G$ and $S_1 = \Omega_1/G_1$, $G_1 = \text{Stab}\,\Omega_1 < G$, and $T(H_1)$ is a Teichmüller space representing $T(S_1)$. If $H_n \in T(H_1)$ converge to a totally degenerate group $\hat{H}$ of $\partial T(H_1)$, then, under the injection $i$ of Corollary 1 to Theorem 6, a convergent subsequence of $i(H_n)$ converges to a b-group*

$\hat{G} \in \partial T(H)$. *If $S_1$ is the sole part of $S$ in the partition induced by $G$, then $\hat{G}$ is totally degenerate. Otherwise $\hat{G}$ is partially degenerate.*

*Proof:* We first note that under the imbedding $i$: $T(H_1) \to \partial T(H)$, $i(T(H_1))$ has a natural boundary. It is not clear that the boundaries of $T(H_1)$ and $i(T(H_1))$ are homeomorphic, but both are compact. We also know that every point on $\partial T(H)$ hence on $\partial i(T(H_1))$ represents a marked $b$-group, which is group theoretically isomorphic to $H$ and preserves markings. According to Bers, $\partial T(H_1)$ contains totally degenerate groups and we can choose a sequence $[\mu_n] \in T(H_1)$ converging to a point on $\partial T(H_1)$ which represents a totally degenerate group $H$. We assume that the $\mu_n$ are Teichmüller differentials, i.e. of minimal dilatation in their equivalence classes. The $\mu_n$ may be transported to $i(T(H_1))$ as in Corollary 1 to Theorem 6, and extended by 0 on $\hat{\mathbf{C}} - G\Omega_1$. We maintain the notation $\mu_n$ in $i(T(H_1))$ and choose a convergent subsequence, again denoted $(\mu_n)$. This convergence is to be taken in the following sense. If we identify $[\mu_n]$ and $H_n$ as elements of $T(H_1)$, then $i([\mu_n]) = \hat{H}_n \in \partial T(H)$ and $\hat{H}_n \to \hat{G}$ in the sense of generators converging. We denote the isomorphism of $\hat{H}_n$ and $\hat{G}$ by $\chi_n$ and $\chi$ respectively.

If $G$ has a non-invariant component $\Omega_2$, not equivalent to $\Omega_1$, then $\hat{G}_2 = \lim w_{\mu_n} \times (\operatorname{Stab} \Omega_2) w_{\mu_n}^{-1}$ is quasi-Fuchsian, since $w_{\mu_n} | \Omega_2$ is conformal and the argument of Lemma 3 may be carried over to show that $\lim w_{\mu_n} | \Omega_2$ is schlicht. In this case, $G$ is either regular or partially degenerate. To prove the Theorem we must show that $\hat{G}_1 = w_{\mu_n} (\operatorname{Stab} \Omega_2) w_{\mu_n}^{-1}$ is totally degenerate. To show that $\hat{G}_1$ is totally degenerate it suffices to show that $\hat{G}_1$ is quasiconformally equivalent to $\hat{H}$. Let $G_1^n = w_{\mu_n}(G_1) w_{\mu_n}^{-1}$. We will show that there exist global quasiconformal homeomorphisms $\omega_n$ conjugating $G_1^n$ into $H_n$ and whose dilatations are uniformally bounded away from 1. If so by passing to a subsequence we may assume $\omega_n$ converge to a global quasiconformal homeomorphism $\omega$ conjugating $\hat{G}_1$ into $\hat{H}$. Let $B_1$ be the component of $G_1$ containing $\Delta$, and $B_n = w_{\mu_n}(B_1)$. The surfaces $B_n/G_1^n$ are each quasiconformally equivalent to $B_1/G_1$, via maps $f_n$.

We claim that the mappings $f_n$ may be chosen to have dilatations $\tilde{\nu}_n$ uniformly bounded away from 1. If so, we may lift $\nu_n$ to $B_1$ and obtain a Beltrami differential $\nu_n$ for the group $G_1$ on $B_1$. If $D_1 = \hat{\mathbf{C}} - B_1$, the maps conjugating $G_1$ into $G_1^n$ and into $H_n$ have identical dilatation $\mu_n$ on $D_1$. We consider the following diagram on the page 233, where $\tilde{f}_n$ and $\tilde{g}_n$ are solutions of the Beltrami equations with the following dilatations:

$$\mu_{\tilde{f}_n} = \begin{cases} \nu_n \text{ on } B_1 \\ \mu_n \text{ on } D_1 \end{cases}$$

$$\mu_{\tilde{g}_n} = \begin{cases} 0 \text{ on } B_1 \\ \mu_n \text{ on } D_1 \end{cases}$$

$$\begin{tikzcd}
\hat{C} \arrow[d, "\tilde{f}_n"'] \arrow[dr, "\tilde{g}_n"] & \\
\hat{C} \arrow[r, "\omega_n"'] & \hat{C}
\end{tikzcd}$$

Since $\omega_n = \tilde{g}_n \cdot \tilde{f}_n^{-1}$, the usual formula for the dilatation of a composition (see Ahlfors [4], p. 10) we obtain

$$|\mu_{\omega_n}| \cdot (\tilde{f}_n(z)) = \begin{cases} 0 \text{ on } D_n = \tilde{f}_n(D_1) \\ \tilde{\nu}_n \text{ on } B_n \end{cases}$$

Hence the claim is sufficient to prove the Theorem. We first reduce the claim to a technical lemma.

We examine that component $\Delta_1$ of $\Delta^*$ which is stabilized by $G_1$. $\Delta/G_1$ is a finite bordered Riemann surface. $B_1/G_1$ may be obtained from $\Delta_1/G_1$ by adjoining a finite number of disjoint closed punctured discs. The mappings $w_{\mu_n}$ are conformal on $\Delta_1$, as are the mappings $\tilde{g}_n$. The surfaces $B_n/G_n$ and $\tilde{g}_n(B_1)/H_1$ are therefore quasiconformally equivalent via maps which are conformal except on a finite number of punctured discs with disjoint closures. Under these circumstances, the next lemma asserts that the surfaces $B_n/G_n$ are quasiconformally equivalent under mappings whose dilatations are uniformly bounded away from 1.

LEMMA 5: *Let $f$: $S_1 \to S_2$ be a quasiconformal homeomorphism of the marked finite Riemann surfaces $S_1$ and $S_2$ with signature which preserve the extended markings. For each puncture $z_i$, $1 \leqslant i \leqslant k$ on $S_1$ let $N_i$ be a punctured disc on $S_1$ whose boundary in $S_1$ is a quasicircle and such that the puncture corresponds to $z_i$. We assume that the neighborhoods $N_i$ have disjoint closures, and that $f | (S_1 - \bigcup_{i=1}^k N_i)$ is conformal. Then there is a quasiconformal mapping $g$ of $S_1$ onto $S_2$ again preserving markings such that:*

(1) *$g^{-1} \cdot f$ is homotopic to the identity on $S_1$, and,*

(2) sup $|\mu_g(z)| \leqslant C$ *where $C$ depends only on $S_1$ and $f | (S_1 - \bigcup_{i=1}^k N_1)$.*

*Proof:* We first reduce the problem to a problem on quasiconformal mappings of closed discs with given boundary correspondences.

For each $i$, let $\beta_i$ be the boundary curve of $N_i$ and $\beta_i'$ be a quasicircle in $S_1 - \bigcup_{i=1}^k N_i$ such that $\beta_i$ and $\beta_i'$ bound an annulus $R_i$ on $S_1$, which contains no ramification points. The

$R_i$ are taken with disjoint closures. It follows that $\bar{R}_i \cup N_i = \bar{N}'_i$ is also a punctured disc. Let $f_i$ denote $f|\bar{N}'_i$. $f_i$ is thus a quasiconformal mapping of a closed disc onto another closed disc, which is conformal in a fixed neighborhood of the boundary quasicircle and maps $z_i$ into a given point $f(z_i)$. Let $g_i$ be a quasiconformal mapping of $N'_i$ onto $f(N'_i)$ with the same boundary correspondence as $f_i$, $g(z_i) = f(z_i)$ and having minimal dilatation. We claim that $|\mu_{g_i}| \leqslant C_i$ where $C_i$ depends only on the values of $f$ along the curve $\beta'_i$. If so set

$$g(z) = \begin{cases} f(z) & \text{on } S_1 - \bigcup_{i=1}^{k} N'_i \\ g_i(z) & \text{on } N'_i \quad \text{for } 1 \leqslant i \leqslant k \end{cases}$$

$g$ then satisfies (1) and (2).

By composition with conformal mappings we may assume $N'_i$ and $f(N'_i)$ are both the upper half-plane. The claim is then equivalent to the following.

LEMMA 6: *Let $h$ be an Ahlfors-Beurling function* (*Ahlfors* [4], *p.* 63 *ff*) *$z_0$ a given point in $U$ and $K$ a compact set in $U$. Then, for each $z_1$ in $K$, there is a quasiconformal mapping $w$: $U \to U$ so that $w(z_0) = z_1$, $w|\mathbf{R} = h$ and $|\mu_w| \leqslant C < 1$ where $C$ depends only on $K$ and $h$.*

*Proof:* Let $K_1 \supset\supset K_2 \supset\supset K \cup \{z_0\}$ with $K_1$ and $K_2$ open rectangles and $\supset\supset$ denotes contains compactly. A recent theorem of Carleson [11] states that we may find a quasiconformal mapping of $U$ with the boundary values given by $h$ which is piecewise linear and is the identity on any compact subset of $U$. The dilatation of the Carleson mapping depends only on $h$ and the compact set. Let $w_1$ be a Carleson mapping with boundary values $h$ and which is the identity on $K_1$. Since both $z_0$ and $K$ lie compactly in $K_2$, for each $z_1 \in K$ there is a piecewise linear quasiconformal mapping $w_2$ of $K_1$ which is the identity on $\partial K_1$ and has dilatation bounded by a constant independent of the choice of $z_1$ in $K$. Set

$$w = \begin{cases} w_1 & \text{on } U - K_1 \\ w_2 & \text{on } K_1 \end{cases}$$

$w$ satisfies the requirements of the lemma and the proof of the theorem is complete.

## § 7. The geometric finiteness of regular $b$-groups

Various authors have proposed definitions of those non-quasi-Fuchsian $b$-groups which are in some sense non-singular. The original definition given by Bers [8] is that of non-degeneracy which is based on the hyperbolic area of the quotient surfaces. Maskit's notion of non-singularity, given in B-II, is that of a $b$-group yielding a complete factorization of the base (marked) surface. Marden [19] (also Earle and Marden [12]) uses the concept of geometric finiteness to distinguish the class of non-singular finitely generated torsion-free Kleinian groups. Various other analytic and topological properties of regular $b$-groups may be found in B-II. The equivalence of the concepts of regularity and non-degeneracy for non-quasi-Fuchsian $b$-groups follows easily from B-II. We will next show that geometric finiteness and regularity are equivalent for non-quasi-Fuchsian $b$-groups. We recall that a Kleinian group is called *geometrically finite* if it has a finite sided Dirichlet fundamental polyhedron (see Marden [19] and Beardon and Maskit [6]).

THEOREM 8: *A non-quasi-Fuchsian b-group G is regular if and only if it is geometrically finite.*

*Proof:* The proof given here is based on a characterization of geometrically finite Kleinian groups, given by Beardon and Maskit [6]. We recall several of the pertinent definitions and theorems from their paper. If $G$ is a Kleinian group, $\lambda \in \Lambda(G)$ is a *point of approximation* of $G$ if there exists some $z \in \Omega(G)$ and a sequence $(\gamma_n)$ in $G$ so that $|\gamma_n(z) - \gamma_n(\lambda)| \geq C > 0$. A parabolic fixed point $z$ is said to be cusped if:

(i) each primary parabolic transformation in $G$ fixing $z$ has two horocycles, or

(ii) the stabilizer of $z$ is not a finite extension of a cyclic group.

We note that case (ii) does not occur in $b$-groups. A Kleinian group $G$ is geometrically finite if and only if each limit point is either a point of approximation or a cusped parabolic fixed point. Parabolic fixed points are not points of approximation.

We assume $G$ is a geometrically finite non-quasi-Fuchsian $b$-group. Greenberg [14] (see also Marden [19]) has shown that $G$ is not totally degenerate. It then follows from (i) that $G$ is regular.

To prove the converse, we first note that by definition of regular $b$-group, condition (i) is satisfied by each parabolic fixed point. To discuss the other points of $\Lambda(G)$, we need the description given by Theorem 3. We assume $\infty \in \Omega(G)$. If $\lambda$ lies in the boundary of no non-invariant component of $\Omega(G)$, then there exists a closed axis $A_\gamma$ of an accidental parabolic transformation $\gamma \in G$ and a sequence $\gamma_n \in G$ so that $\gamma_n^{-1}(A_\gamma)$ and $\gamma_n(\lambda)$ lies in a fixed complementary component $B$ of $A_\gamma$. $|\gamma_n(\lambda) - \gamma_n(z)|$ has a positive lower bound unless both $\gamma_n(\lambda)$ and $\gamma_n(z)$ have subsequences converging to the fixed point of $\gamma$. It is clear that the set of maps of $A_\gamma$ into $\gamma_n^{-1}(A_\gamma)$ is exactly $B_n = \{\gamma_n^{-1}\gamma^k \mid k \in Z\}$ hence there exists some $\eta_n \in B_n$ so that $\eta_n(z)$ lies in the Ford fundamental region for the group generated by $\gamma$. Since $z \in \Delta$ so does $\eta_n(z)$, but the points in the Ford region for $\gamma$ near the fixed point of $\gamma$, lie in horocycles for $\gamma$. As $\gamma$ is accidental with respect to $\Delta$, these horocycles lie in non-invariant components. Thus $\eta_n(z)$ does not converge to the fixed point of $\gamma$ and $|\eta_n(\lambda)-\eta_n(z)|\geqslant C>0$ for some $C$.

To complete the proof of the theorem we must show that, if $\lambda\in\partial\Omega_i$, $\Omega_i$ a non-invariant component of $\Omega(G)$, is not a parabolic fixed point, then $\lambda$ is a point of approximation. By Ahlfors' Lemma [5], $\partial\Omega_i=\Lambda$ (Stab $\Omega_i$) and $G_i=$Stab $\Omega_i$ is quasi-Fuchsian of the first kind. The property to be proved being a topological invariant, and $G_i$ being the quasiconformal deformation of a Fuchsian group, it suffices to assume $G_i$ is Fuchsian. It is well known that for $G_i$ Fuchsian each $\lambda\in\Lambda(G_i)$ which is not a parabolic fixed point, is contained in a spherical nest of circles $\gamma_n^{-1}(B_\gamma)$, $\gamma_n\in G_i$ and $B_\gamma$ a circle passing through the fixed points of some $\gamma\in G_i$ and orthogonal to $\Lambda(G_i)$. Using the argument above we can find a sequence $\eta_n$ and a point $z\in\Omega(G)$ which define $\lambda$ to be a point of approximation and the theorem is proved.

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