A GENERALIZATION TO OVERDETERMINED SYSTEMS OF THE NOTION OF DIAGONAL OPERATORS

II. *Hyperbolic operators*

BY

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Chapter I: Introduction

Section 0. Introduction

One of the problems in the theory of overdetermined systems of linear partial differential equations is to prove the existence of local solutions of the inhomogeneous equation $Du=v$. In general, if D is overdetermined, v must satisfy a compatibility condition $D'v=0$ for some operator D'. The best possible result is that the compatibility condition is not only necessary, but sufficient for the existence of local solutions; that is, if $\underline{E}^0$, $\underline{E}^1$,

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and $\underline{E}^2$, are sheaves of germs of differentiable sections of vector bundles, then the complex of sheaves

$$0 \longrightarrow \Theta \longrightarrow \underline{E}^0 \xrightarrow{D} \underline{E}^1 \xrightarrow{D'} \underline{E}^2 \tag{0.1}$$

is exact, where Θ is the solution sheaf of the homogeneous equation.

D. C. Spencer [12] has shown that, granted certain reasonable assumptions about D, there exists a complex

$$0 \longrightarrow \Theta \longrightarrow \underline{C}^0 \xrightarrow{D^0} \underline{C}^1 \xrightarrow{D^1} \cdots \xrightarrow{D^{n-1}} \underline{C}^n \longrightarrow 0 \tag{0.2}$$

of sheaves and first order linear operators such that the cohomology of (0.2) at $\underline{C}^1$ is the same as that of (0.1) at $\underline{E}^1$. Thus it suffices to look at complexes of first order operators satisfying the properties of Spencer sequences.

In [9], the author showed that if D is elliptic and satisfies a condition called the δ-estimate, then (0.2) is exact. The proof consists of showing that the Neumann boundary value problem is solvable on sufficiently convex small domains and that the harmonic space is zero on these domains. In this paper, hyperbolic complexes satisfying the δ-estimate are considered, and it is shown that for such complexes, under suitable conditions, the Cauchy boundary value problem is solvable. This does not show that the complex (0.2) is exact, but rather gives an isomorphism of the global cohomology corresponding to (0.2) over certain domains with the cohomology of a related complex on the lower dimensional Cauchy surface.

The final theorem of the paper proves local existence and uniqueness of solutions for the Cauchy problem for complexes of first order linear differential operators which are symbol surjective (such as Spencer sequences), which satisfy the δ-estimate and a hyperbolicity condition, and some conditions on the regularity of the characteristic variety.

Of these conditions, the δ-estimate is the least familiar. It is a homological condition on the δ-complex of an operator, and initially looks strange, but it has several nice properties, which are listed here with an indication of where the proofs can be found.

(1) If D satisfies the δ-estimate, then D is involutive. [9, Theorem II.1.7].

(2) If D satisfies the δ-estimate, then every prolongation of D satisfies the δ-estimate [9, Theorem II.1.4 and II.3.1].

(3) If the first operator of a symbol surjective complex of linear first order operators satisfies the δ-estimate, then all operators in the complex do. [This paper, Theorem 6.2.]

(4) If D satisfies the δ-estimate, then every operator in the Spencer complex for D does. [9, Theorem II.2.1.]

(5) If S is a non-characteristic submanifold and D_0 is the tangential complex associated with S and D, and if D satisfies the δ-estimate, then D_0 does. [This paper, Theorem 7.8.]

(6) If the constant $(1/2)\,(k+1)^2$ which appears in the δ-estimate (6.1) is replaced by $(1/2)\,(k+1)^2+\varepsilon$ for any $\varepsilon>0$, the resulting condition is so strong that it is satisfied only by those operators which have no characteristics, real or complex, and are of finite type. Such operators are essentially the exterior derivative. [This paper, Corollary 8.3.]

(7) On the other hand, it is conjectured that if $(1/2)\,k^2$ is replaced by $(1/2)\,k^2-\varepsilon$ for any $\varepsilon>0$, the resulting condition is so weak that it is satisfied by any involutive operator.

(8) If D is an elliptic operator which satisfies the δ-estimate, then the analytic methods which prove the $\bar{\partial}$-Neumann problem is solvable also prove the Neumann problem for the Spencer complex for D is solvable [9].

The goal of this paper is to justify the following statement:

(9) If D is hyperbolic and satisfies the δ-estimate, then the analytic methods which prove the Cauchy problem for a symmetric hyperbolic system is solvable also work to show that the Cauchy problem for the Spencer sequence of D is solvable.

Sections 1–3 state the Cauchy problem and what is meant by "existence and uniqueness" of solutions. Sections 4–6 define the δ-estimate and collect the results we need in this paper. Section 7 is an exposition of the Guillemin decomposition of a complex relative to a foliation by non-characteristic surfaces. These results are due to Guillemin, but since they are unpublished, we reproduce them here. Section 8 contains a main result of the paper, that the symbols of certain operators in the Guillemin decomposition of a δ-estimate complex restrict to the "cohomology" or "harmonic space" of complexes of symbols of other operators and that these restrictions are normal (commute with their adjoints). Since normal matrices can be diagonalized this theorem is the justification for considering δ-estimate operators generalizations of diagonal operators. Section 9 shows that if the complex is hyperbolic, these normal matrices are in fact symmetric. Section 10 applies this result to prove existence and uniqueness of solutions to the Cauchy problem by reducing it to solving symmetric hyperbolic Cauchy problems. Section 11 gives some examples.

Section 1. Complexes of first order linear differential operators

In this paper the phrase "differentiable of class C^∞" as it applies to vector bundles, sections, germs, etc., has been suppressed; it is assumed in all instances where it makes sense. N denotes the non-negative integers, $N=\{0, 1, 2, \dots\}$.

Let Ω be an open submanifold of a manifold X of dimension n, and let $\{E^i: i \in N\}$ be a sequence of finite dimensional vector bundles over Ω. For simplicity, we assume the scalar field for the vector bundles is $\mathbf{C}$, the complex numbers, but in many cases it suffices to consider real vector bundles. The sheaf of germs of sections of the bundle E^i is denoted by $\underline{E}^i$, and the space of global sections over Ω is denoted by $\Gamma(\Omega, \underline{E}^i)$, by $\mathcal{E}^i(\Omega)$, or by $\mathcal{E}^i$. For each x in Ω, let E^i_x (resp. $\underline{E}^i_x$) denote the fiber (resp. stalk) of the bundle E^i (resp. sheaf $\underline{E}^i$) at x. $T^*(\Omega)$, or simply T^*, is the complexified cotangent bundle of Ω.

We consider linear partial differential operators D: $\mathcal{E}^i \to \mathcal{E}^j$ where i and j are in N. D may be expressed in some neighborhood of x_0 in terms of a local coordinate chart of Ω and local trivializations of E^i and E^j as

$$D = \sum_{0 \leqslant |\alpha| \leqslant k} A_\alpha(x)\,(\partial/\partial x)^\alpha$$

For each cotangent vector $(x, \xi) \in T^*(\Omega)$ (here $x \in \Omega$ and $\xi \in T^*_x$), define a vector space morphism $\sigma_\xi(D)$: $E^i_x \to E^j_x$, called the symbol morphism of D at (x, ξ), as follows: At x, choose a germ f of a function such that $f(x) = 0$ and $df(x) = \xi$. For any germ $u \in \underline{E}^i_x$, set

$$\sigma_\xi(D)\,(u(x)) = \frac{1}{m!}\,D(f^m u)\,(x)$$

where m is the order of D at x. It is easy, and left to the reader, to check that this is well-defined. If D is represented in terms of local coordinates and trivializations as above, and if $\xi = \Sigma \xi_i dx^i$, then

$$\sigma_\xi(D) = \sum_{|\alpha| = m} A_\alpha(x)\,\xi^\alpha.$$

This shows that if $\xi \in \Gamma(\Omega, \underline{T^*(\Omega)})$ is a smooth cotangent vector field on Ω, the vector space morphisms

$$\sigma_{\xi(x)}(D) : E^i_x \to E^j_x$$

piece together to give a vector bundle morphism $\sigma_\xi(D)$: $E^i \to E^j$, that the map $\sigma(D)$: $T^* \to \operatorname{Hom}(E^i, E^j)$ is smooth, and that $\sigma_{\xi(x)}(D)$: $T^*_x \to \operatorname{Hom}(E^i_x, E^j_x)$ is a homogeneous polynomial function of order m in ξ. These show that there is a (linear) vector bundle morphism $\sigma(D)$: $S^m T^* \otimes E^i \to E^j$, where $S^m T^*$ is the bundle of m-fold symmetric tensor products on T^*.

In this paper we shall consider complexes of first order linear differential operators

$$\mathcal{E}^0 \xrightarrow{D^0} \mathcal{E}^1 \xrightarrow{D^1} \cdots$$

It is convenient to consider the graded vector bundle E, which in degree i is E^i, and the graded operator D which in degree i is D^i: $\mathcal{E}^i \to \mathcal{E}^{i+1}$ and is therefore an operator of degree one. Since $D^{i+1} \circ D^i = 0$, we have $D \circ D = 0$. Let Hom (E, E) be the graded vector bundle of graded homomorphisms. In degree i, it is $\operatorname{Hom}^i(E, E)$, the bundle of graded homomorphisms of degree i.

Fix $x \in \Omega$. For each $\xi \in T_x^*$, $\sigma_\xi(D)$: $E_x^i \to E_x^{i+1}$ so the assignment of $\sigma_\xi(D)$ to ξ gives a map $\sigma(D)$: $T_x^* \to \operatorname{Hom}_x^1(E, E)$, which is linear since D is first order. We extend this to a vector space morphism

$$\sigma(D) : \otimes^k T_x^* \to \operatorname{Hom}_x^k(E, E)$$

by setting $\sigma(D)(\xi_1 \otimes \ldots \otimes \xi_k) = \sigma_{\xi 1}(D) \circ \ldots \sigma_{\xi k}(D)$ and extending linearly. Since $D \circ D = 0$ implies $\sigma_\xi(D) \circ \sigma_\xi(D) = 0$ for every ξ, this morphism factors through a well-defined, graded of degree zero, vector space morphism

$$\sigma(D) : \Lambda T_x^* \to \operatorname{Hom}_x(E, E)$$

where ΛT_x^* is the exterior algebra of T_x^*.

We have shown that the symbol morphism of D induces on E_x the structure of a ΛT_x^*-module. Clearly, $\mathcal{E}(\Omega)$ is then a $\Gamma(\Omega, \underline{\Lambda T^*(\Omega)})$-module, and for each x, $\underline{E}_x$ is a $\underline{\Lambda T_x^*}$-module. We say, loosely, that E is a ΛT^*-module.

Denote the multiplication of $u \in E_x$ by $\omega \in T_x^*$ by $\omega \wedge u$, so that if $\xi \in T_x^*$, $\xi \wedge u = \sigma_\xi(D) u$.

The interaction between the operator D and the module structure on E given by $\sigma(D)$ is given by the following proposition due to V. W. Guillemin, which can be stated either in terms of germs of sections, as we have done here, or in terms of global sections.

LEMMA 1.1: *For each* $\omega \in \underline{\Lambda^i T}_x$ *and each* $u \in \underline{E}_x$,

$$D(\omega \wedge u) = d\omega \wedge u + (-1)^i \omega \wedge Du.$$

Proof: If ω is a 0-form, the lemma is an immediate consequence of the definition of the symbol map. The proof for forms of higher degree follows from a simple induction argument.

Remark: The cohomology of the complex D: $\mathcal{E} \to \mathcal{E}$ is the graded vector space which is

$$H^i(\mathcal{E}(\Omega), D) = \frac{\ker D^i : \mathcal{E}^i \to \mathcal{E}^{i+1}}{\operatorname{im} D^{i-1} : \mathcal{E}^{i-1} \to \mathcal{E}^i}.$$

V. W. Guillemin has observed that this lemma implies that the cohomology of the complex D: $\mathcal{E} \to \mathcal{E}$ is a module over the de Rham cohomology ring.

Section 2. Examples

We pause to give several examples of complexes of first order differential operators and the corresponding module structures. The two best known examples of complexes of linear first order differential operators are the de Rham and Dolbeault complexes, Examples (i) and (ii) below. Example (iii) gives a method by which a complex is derived from a given complex and a submanifold of codimension at least one. The resulting complex, the tangential complex, is used in the statement of the Cauchy problem.

(i) *The de Rham complex.* In this case, $E^i = \Lambda^i T^*(\Omega)$, and D_i: $\underline{\Lambda^i T^*} \to \underline{\Lambda^{i+1} T^*}$ is the exterior differential, usually denoted by d. If ξ is in T_x^*, the symbol morphism $\sigma_\xi(d)$: $\Lambda T_x^* \to \Lambda T_x^*$ is exterior multiplication by ξ, and the ΛT_x^*-module E_x is ΛT_x^*.

(ii) *The Dolbeault complex.* In this case, X is a complex analytic manifold, $E^i = \Lambda^{p,i} T^*$ the bundle of differential forms of type (p, i), and D_i: $\underline{\Lambda^{p,i} T^*} \to \underline{\Lambda^{p,i+1} T^*}$ is the Cauchy-Riemann operator in several complex variables, usually denoted by $\bar\partial$. The symbol morphism for ξ in T_x^*, $\sigma_\xi(\bar\partial)$: $\Lambda^{p,i} T_x^* \to \Lambda^{p,i+1} T_x^*$ is exterior multiplication by $\xi^{(0,1)}$, where $\xi^{(0,1)}$ is the projection of ξ onto $\Lambda^{0,1} T_x^*$ in the direct sum decomposition $T_x^* = \Lambda^{1,0} T_x^* \oplus \Lambda^{0,1} T_x^*$. Thus the ΛT_x^*-module E is isomorphic to $\Lambda^{0,*} T_x^* \otimes \Lambda^{p,0} T_x^*$, where $\Lambda^{0,*} T_x^*$ is the graded algebra which in degree i is $\Lambda^{0,i} T_x^*$.

(iii) *Tangential complexes.* For each $x \in \Omega$, the bundle ΛT_x^* is an algebra over the real or complex numbers, and is, *a fortiori*, a ring. Similarly $\Gamma(\Omega, \underline{\Lambda T^*})$ and $\underline{\Lambda T^*}$ are rings. A differential ideal $\mathcal{J}$ in $\Gamma(\Omega, \underline{\Lambda T^*})$ is a subset of $\Gamma(\Omega, \underline{\Lambda T^*})$ which is an ideal which is d-closed (if $\alpha \in \mathcal{J}$, then $d\alpha \in \mathcal{J}$). Similarly we may define a differential ideal $\mathcal{J}$ in $\underline{\Lambda T^*}$. Given a differential ideal $\mathcal{J}$, define a submodule $\mathcal{J}\mathcal{E}$ of $\mathcal{E}$ to be the submodule consisting of all finite sums of the form $\Sigma \omega_i \wedge e_i$ where each $\omega_i \in \mathcal{J}$ and each $e_i \in \mathcal{E}(\Omega)$. Since $\mathcal{J}$ is d-closed, it follows from Lemma 1.1 that the submodule $\mathcal{J}\mathcal{E}$ is D-closed.

We obtain the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & (\mathcal{J}\mathcal{E})^{i-1} \arrow[r, "D"] \arrow[d, "i"] & (\mathcal{J}\mathcal{E})^{i} \arrow[r, "D"] \arrow[d, "i"] & (\mathcal{J}\mathcal{E})^{i+1} \arrow[r] \arrow[d, "i"] & \cdots \\
\cdots \arrow[r] & \mathcal{E}^{i-1} \arrow[r, "D"] \arrow[d, "\pi"] & \mathcal{E}^{i} \arrow[r, "D"] \arrow[d, "\pi"] & \mathcal{E}^{i+1} \arrow[r] \arrow[d, "\pi"] & \cdots \\
\cdots \arrow[r] & \mathcal{E}^{i-1}/(\mathcal{J}\mathcal{E})^{i-1} \arrow[r, "D_\mathcal{J}"] \arrow[d] & \mathcal{E}^{i}/(\mathcal{J}\mathcal{E})^{i} \arrow[r, "D_\mathcal{J}"] \arrow[d] & \mathcal{E}^{i+1}/(\mathcal{J}\mathcal{E})^{i+1} \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where $(\mathcal{J}\mathcal{E})^j$ is by definition $\mathcal{J}\mathcal{E} \cap \mathcal{E}^j$, the maps i and π are inclusion and projection onto the quotient, and the maps $D_\mathcal{J}$ are defined to make the diagram commute.

We examine several instances of this. Let S be a submanifold of Ω with codimension q. There is a smallest differential ideal $\mathcal{J}$ which contains all functions which vanish on S.

If S is given locally by the equations $\varrho_1 = \varrho_2 = \dots = \varrho_q = 0$, then locally $\mathfrak{J}$ is the ideal spanned by $\{\varrho_1, \dots, \varrho_q, d\varrho_1, \dots, d\varrho_q\}$ over the functions.

The corresponding complex

$$D_{\mathfrak{J}}: \mathcal{E}/\mathfrak{J}\mathcal{E} \to \mathcal{E}/\mathfrak{J}\mathcal{E}$$

is called the tangential complex for $D: \mathcal{E} \to \mathcal{E}$ tangential to S.

Since $\mathfrak{J}$ is d-closed, $\mathfrak{J}\mathcal{E}$ is D-closed. It is easy to show that $\mathfrak{J}\mathcal{E}$ is the least D-closed submodule of $\mathcal{E}$ which contains all sections of $\mathcal{E}$ which vanish on S. We also have:

LEMMA 2.1.: *If $u \in \mathfrak{J}\mathcal{E}$, then there exist sections u' and u'' such that $u = u' + Du''$, $u'|_S = 0$, and $u''|_S = 0$.*

Proof: Let $u = \Sigma \omega_i \wedge e_i$ be in $(\mathfrak{J}\mathcal{E})$ where each $\omega_i \in \mathfrak{J}$. Since $\mathfrak{J}$ is generated by $\{\varrho_1, \dots, \varrho_q, d\varrho_1, \dots, d\varrho_q\}$ over the functions, we can write $\omega_i = \Sigma \alpha_{ij} \wedge \varrho_j + \beta_{ij} \wedge d\varrho_j$ where α_{ij} and β_{ij} are differential forms. We can choose signs ε^{ij} such that $D(\Sigma \varepsilon^{ij} \beta_{ij} \wedge \varrho_j \wedge e_i) = \Sigma \beta_{ij} \wedge d\varrho_j \wedge e_i + u'''$ where u''' vanishes on S. Thus we let $u'' = \Sigma \varepsilon^{ij} \beta_{ij} \wedge \varrho_j \wedge e_i$ and let $u' = \Sigma \alpha_{ij} \wedge \varrho_j \wedge e_i - u'''$.

Let $\mathfrak{J}_x$ be the ideal in ΛT^*_x consisting of the values at x of forms in $\mathfrak{J}$. Clearly $\mathfrak{J}_x$ is generated by $\varrho_1(x), \dots, \varrho_q(x), d\varrho_1(x), \dots, d\varrho_q(x)$ over the constants. If $x \notin S$, then $1 \in \mathfrak{J}_x$ so $\mathfrak{J}_x = \Lambda T^*_x$ and $(\mathfrak{J}\mathcal{E})_x = E_x$ where $(\mathfrak{J}\mathcal{E})_x$ is the submodule of E_x consisting of the values at x of sections in $\mathfrak{J}\mathcal{E}$. Thus the space $\mathcal{E}/\mathfrak{J}\mathcal{E}$ is concentrated on S. It is not necessarily the space of sections of a vector bundle on S since the dimension of $(\mathcal{E}/\mathfrak{J}\mathcal{E})_x = E_x/(\mathfrak{J}\mathcal{E})_x$ may jump, but in the case where S is non-characteristic as defined in section 7, dim $(\mathcal{E}/\mathfrak{J}\mathcal{E})_x$ will be constant and $\mathcal{E}/\mathfrak{J}\mathcal{E}$ can be considered to be the space of sections of a vector bundle over S.

An explicit construction of the tangential complex for the Dolbeault complex appears in [1] and [2]. In the latter paper it is pointed out that the tangential de Rham complex on S is just the de Rham complex on S.

A similar construction is obtained by taking $\mathfrak{J}$ to be the ideal of an integrable codistribution. That is, $\mathfrak{J}$ is a differential ideal generated by forms of degree one. By the Frobenius theorem, the ideal determines locally a foliation of Ω; that is, it is possible to find locally functions ϱ_l such that $\{d\varrho_1, \dots, d\varrho_q\}$ generates $\mathfrak{J}$. The surfaces given by $\varrho_1 = \text{const}, \dots, \varrho_q = \text{const.}$ give the corresponding local foliation of Ω.

In this case it is no longer true that $(\mathcal{E}/\mathfrak{J}\mathcal{E})_x$ vanishes except for x on a given surface.

We have that $D_{\mathfrak{J}}$ differentiates only in directions tangential to the surfaces of the foliation: $\varrho_1 = \text{const}, \dots, \varrho_q = \text{const}$. More precisely, if $\varphi = \varphi(\varrho_1, \dots, \varrho_q)$ is a function which is constant on each surface, and if e is in $\mathcal{E}$, then $D(\varphi e) = (d\varphi \wedge e + \varphi \wedge De)$ by Lemma 1.1. Since φ is a function of $\varrho_1, \dots, \varrho_q$, $d\varphi$ is a combination of $d\varrho_1, \dots, d\varrho_q$, and hence $d\varphi \wedge e \in \mathfrak{J}\mathcal{E}$. Thus

$$D_{\mathfrak{J}}(\varphi e) = \varphi \wedge D_{\mathfrak{J}} e$$

and $D_{\mathcal{J}}$ commutes with functions constant along the sheets of the foliation. Thus $D_{\mathcal{J}}$ does not differentiate except in directions tangential to the foliation.

(iv) *Maxwell's equations in an isotropic homogeneous medium.* Let E be the electric field strength, H the magnetic field strength, D the dielectric displacement, B the magnetic induction, J the electric current density, ϱ the charge density, and σ, ε, and μ the conductivity, permittivity, and permeability. Assume the last three are constant.

Maxwell's equations are

$$\begin{aligned} \operatorname{curl} H &= J + \partial/\partial t D \\ \operatorname{curl} E &= -\partial/\partial t B \\ \operatorname{div} B &= 0 \\ \operatorname{div} D &= \varrho \end{aligned}$$

These are supplemented by the constitutive equations

$$\begin{aligned} D &= \varepsilon E \\ B &= \mu H \\ J &= \sigma E. \end{aligned}$$

Let $\mathbf{E} = E + \sqrt{\mu/\varepsilon}\, iH$. Then Maxwell's and the constitutive equations reduce to:

$$\begin{aligned} \operatorname{curl} \mathbf{E} - i\sqrt{\mu\varepsilon}\ \partial/\partial t \mathbf{E} - i\sigma\sqrt{\mu/\varepsilon}\operatorname{Re}\mathbf{E} &= 0 \\ \operatorname{div}\mathbf{E} &= \varrho/\varepsilon \end{aligned}$$

Let $X = \mathbf{R}^4$, let E^0, E^1, and E^2 be the bundles over X with fibers $\mathbf{C}^3$, $\mathbf{C}^3 \times \mathbf{C}^1$, and $\mathbf{C}^1$, respectively.

Let D^0: $\mathcal{E}^0 \to \mathcal{E}^1$ be the Maxwell operator defined by

$$D^0 u = (\operatorname{curl} u - i\sqrt{\mu\varepsilon}\,\partial/\partial t\, u - i\sigma\sqrt{\mu\varepsilon}\operatorname{Re} u, \operatorname{div} u)$$

and let D^1: $\mathcal{E}^1 \to \mathcal{E}^2$ be defined by

$$D^1(v_0, v_1) = \operatorname{div} v_0 + i\sqrt{\mu\varepsilon}\,\partial/\partial t\, v_1 + i\sigma\sqrt{\mu\varepsilon}\operatorname{Re} v_1$$

Then $D^1 \cdot D^0 = 0$, and

$$\mathcal{E}^0 \xrightarrow{D^0} \mathcal{E}^1 \xrightarrow{D^1} \mathcal{E}^2$$

is a complex.

Let S be the surface given by $t = 0$. Then $\mathcal{J}\mathcal{E}^0$ is isomorphic to the space of sections of $\mathbf{C}^3$ over S, $\mathcal{J}\mathcal{E}^1$ is isomorphic to sections of $\mathbf{C}^1$ over S, and $D^0_{\mathcal{J}}$ corresponds to div under these isomorphisms. $\mathcal{J}\mathcal{E}^2$ and $D^1_{\mathcal{J}}$ are zero.

Section 3. The Cauchy problem

Let S be a submanifold of Ω of codimension q. Let v be in $\mathcal{E}^{i+1}$ and let u_0 be a smooth section of E^i over S; i.e., $u_0 \in \Gamma(S, \underline{E}^i|_s)$. The Cauchy problem is:

$$\text{Find } u \in \mathcal{E}^i \text{ such that } Du=v \text{ and } u|_S=0.$$

We want to know when there exist solutions and to what extent they are unique. In this section we show that the best possible existence and uniqueness statements are equivalent to the exactness of the complex D: $\mathcal{J}\mathcal{E} \to \mathcal{J}\mathcal{E}$ defined in the previous section, where $\mathcal{J}$ is the smallest differential ideal containing all functions which vanish on S.

There are clearly some necessary conditions for the solvability of the Cauchy problem. Since $D \circ D = 0$, necessarily $Dv=0$. If $\tilde{u}_0$ is any section in $\mathcal{E}^i$ which extends u_0, then $u-\tilde{u}_0$ vanishes on S and so is in $\mathcal{J}\mathcal{E}$. Thus $D(u-\tilde{u}_0)=v-D\tilde{u}_0$ must be in $\mathcal{J}\mathcal{E}$. Clearly, whether $v-D\tilde{u}_0$ is in $\mathcal{J}\mathcal{E}$ depends only on v and u_0, and not on the extension $\tilde{u}_0$. The Cauchy problem can now be stated:

$$\text{Find } u \in \mathcal{E}^i \text{ such that } Du = v \text{ and } u|_S = u_0 \text{ where } Dv = 0 \text{ and } v - D\tilde{u}_0 \in \mathcal{J}\mathcal{E} \text{ for every extension } \tilde{u}_0 \text{ of } u_0. \tag{3.1}$$

We claim that this is equivalent to the following problem:

$$\text{Find } u' \in \mathcal{E}^i \text{ such that } Du' = v' \text{ and } u'|_S = 0 \text{ where } v' \in \mathcal{J}\mathcal{E} \text{ and } Dv' = 0. \tag{3.2}$$

Clearly, (3.2) is a special case of (3.1). Conversely, if (3.2) can be solved for $v'=v-D\tilde{u}_0$, then $u=u'+\tilde{u}_0$ solves (3.1.)

Now we claim that (3.2) is equivalent to the following problem:

$$\text{Find } u \in \mathcal{E}^i \text{ such that } Du = v \text{ and } u \in \mathcal{J}\mathcal{E} \text{ where } v \in \mathcal{J}\mathcal{E} \text{ and } Dv = 0. \tag{3.3}$$

If we can solve (3.2) we can certainly solve (3.3) since if $u|_S=0$ then $u \in \mathcal{J}\mathcal{E}$. If we can solve (3.3), we have by Lemma 2.1 that there is a section $u'' \in \mathcal{J}\mathcal{E}$ such that $u=u'+Du''$ where $u'|_S=0$, and u' solves (3.2).

The final form (3.3) is the form of the Cauchy problem we shall consider. The solvability of this problem is equivalent to the exactness of the complex

$$\dots \longrightarrow (\mathcal{J}\mathcal{E})^{i-1} \xrightarrow{D} (\mathcal{J}\mathcal{E})^i \xrightarrow{D} (\mathcal{J}\mathcal{E})^{i+1} \longrightarrow \dots \tag{3.4}$$

If $v \in (\mathcal{J}\mathcal{E})^{i+1}$ and $Dv=0$, we want to find $u \in (\mathcal{J}\mathcal{E})^i$ with $Du=v$. This can be done if and only if the cohomology class of v in the cohomology of (3.4) is zero. Solutions to the Cauchy problem are not in general unique if $i \geq 1$ since if $Du=v$ and $w \in (\mathcal{J}\mathcal{E})^{i-1}$, then $D(u+Dw)=v$, so $u+Dw$ is also a solution. We say that the solution of the Cauchy problem is unique in the cohomology sense if the only multiplicity of solutions is of this kind; i.e., if u and u' satisfy $Du=Du'=v$, then $u-u'$ represents the zero cohomology class in the cohomology of (3.4). Let $H^i(\mathcal{J}\mathcal{E})$ represent the cohomology of (3.4) at $(\mathcal{J}\mathcal{E})^i$. We have shown:

THEOREM 3.1: *The Cauchy problem* (3.3) *has a solution if and only if the cohomology class of v in $H^{i+1}(\mathcal{J}\mathcal{E})$ is zero. The solution, if it exists, is unique in the cohomology sense if and only if $H^i(\mathcal{J}\mathcal{E})=0$. The Cauchy problem can be solved for every $v \in (\mathcal{J}\mathcal{E})^{i+1}$ with $Dv=0$ if and only if $H^{i+1}(\mathcal{J}\mathcal{E})=0$.*

We conclude this section by proving that the solvability of the Cauchy problem for all admissible data is equivalent to the existence of a canonical isomorphism of $H(\mathcal{E})$, the cohomology of the complex

$$\dots \longrightarrow \mathcal{E}^{i-1} \xrightarrow{D} \mathcal{E}^i \xrightarrow{D} \mathcal{E}^{i+1} \longrightarrow \dots$$

and $H(\mathcal{E}/\mathcal{J}\mathcal{E})$, the cohomology of the complex

$$\dots \longrightarrow (\mathcal{E}/\mathcal{J}\mathcal{E})^{i-1} \xrightarrow{D_J} (\mathcal{E}/\mathcal{J}\mathcal{E})^i \xrightarrow{D_J} (\mathcal{E}/\mathcal{J}\mathcal{E})^{i+1} \longrightarrow \dots$$

The isomorphism is given by considering the long exact sequence of the short exact sequence of chain maps (2.1):

$$\dots \longrightarrow H^i(\mathcal{J}\mathcal{E}) \xrightarrow{i^*} H^i(\mathcal{E}) \xrightarrow{\pi^*} H^i(\mathcal{E}/\mathcal{J}\mathcal{E}) \longrightarrow H^{i+1}(\mathcal{J}\mathcal{E}) \xrightarrow{i^*} \dots$$

We have $\pi^*: H(\mathcal{E}) \to H(\mathcal{E}/\mathcal{J}\mathcal{E})$ is an isomorphism if and only if $H(\mathcal{J}\mathcal{E})=0$, which we have shown is equivalent to the existence and uniqueness (in the cohomology sense) of solutions to the Cauchy problem for D.

Chapter II: The δ-estimate

Section 4. The δ-complex

We have seen in section 1 that if D: $\mathcal{E}^i \to \mathcal{E}^j$ is a differential operator of order k, it determines a smooth vector bundle morphism $\sigma(D): S^kT^* \otimes E^i \to E^j$ where S^kT^* is the space of symmetric k-fold tensor products on T^* and $\sigma(D)(\xi^k \otimes e) = \sigma_\xi(D)(e)$. For each $x \in \Omega$, define the vector space $(g_k^i)_x$ to be the kernel of $\sigma(D)$; i.e.,

$$0 \longrightarrow (g_k^i)_x \longrightarrow (S^kT^* \otimes E^i)_x \xrightarrow{\sigma(D)} E_x^j$$

is exact. The mth prolongation of $\sigma(D)$ is $\sigma_m(D)$: $(S^{m+k}T^* \otimes E^i)_x \to (S^mT^* \otimes E^j)_x$, and is defined to be the restriction to $(S^{m+k}T^* \otimes E^i)_x$ of $I \otimes \sigma(D)$: $(S^mT^* \otimes S^kT^* \otimes E^i)_x \to (S^mT^* \otimes E^j)_x$. The mth prolongation of $(g_k^i)_x$ is $(g_{m+k}^i)_x$, the kernel of $\sigma_m(D)$. It is easy to check that $(g_{m+k}^i)_x = (S^mT^* \otimes g_k^i)_x \cap (S^{m+k}T^* \otimes E^i)_x$.

If $\sigma(D)$ has constant rank, then $g_k^i = \bigcup_{x \in \Omega} (g_k^i)_x$ is a vector bundle, and g_{m+k}^i is a vector bundle for all $m \geqslant 0$, provided in addition g_k^i is involutive, which will be the case in this paper, since by [9, Theorem II.1.7], operators which satisfy the δ-estimate are involutive.

We shall assume that $\sigma(D)$ has constant rank, although this assumption is not essential for much of the algebra in this paper.

Let δ: $S^mT^* \to T^* \otimes S^{m-1}T^*$ be the unique bundle morphism such that $\delta(\xi^1 \cdot \xi^2 \cdot \ldots \cdot \xi^m) = \sum_{i=1}^m \xi_i \otimes (\xi^1 \cdot \ldots \cdot \hat{\xi}^i \cdot \ldots \cdot \xi^m)$ for all $\xi^1, \ldots, \xi^m$ in T^*. Extend δ to a linear map δ: $\Lambda^l T^* \otimes S^mT^* \otimes E \to \Lambda^{l+1}T^* \otimes S^{m-1}T^* \otimes E$ by setting $\delta(u \otimes v \otimes e) = u \wedge \delta v \otimes e$ if $u \in \Lambda^l T^*$, $v \in S^mT^*$, and $e \in E$. Clearly $\delta^2 = 0$. It is possible to think of δ as formal exterior differentiation of polynomials which are elements of ST^*.

The following square commutes:

$$\begin{array}{ccc} \Lambda^l T^* \otimes S^{k+m+1}T^* \otimes E^i & \xrightarrow{\sigma_{m+1}(D)} & \Lambda^l T^* \otimes S^{m+1}T^* \otimes E^j \\ \downarrow \delta & & \downarrow \delta \\ \Lambda^{l+1}T^* \otimes S^{k+m}T^* \otimes E^i & \xrightarrow{\sigma_m(D)} & \Lambda^{l+1}T^* \otimes S^mT^* \otimes E^j \end{array}$$

Therefore $\delta(\Lambda^l T^* \otimes g^i_{k+m+1}) \to \Lambda^{l+1}T^* \otimes g^i_{k+m}$, and we obtain the δ-complex for each $m \geq k$:

$$0 \longrightarrow g^i_m \xrightarrow{\delta} T^* \otimes g^i_{m-1} \xrightarrow{\delta} \ldots \xrightarrow{\delta} \Lambda^{m-k}T^* \otimes g^i_k \xrightarrow{\delta} \Lambda^{m-k+1}T^* \otimes S^{k-1}T^* \otimes E^i \tag{4.1$_m$}$$

Definition 4.1: The *δ-cohomology* of g^i_k is the cohomology of the sequences (4.1$_m$) where $m \geq k$. The bundle g^i_k is *involutive* if the δ-cohomology is zero. The operator D: $\mathcal{E}^i \to \mathcal{E}^j$ is *involutive* if the corresponding g^i_k is involutive.

For each $m \geq k$, the sequence

$$0 \longrightarrow g^i_{m+1} \longrightarrow T^* \otimes g^i_m \longrightarrow \Lambda^2 T^* \otimes g^i_{m-1}$$

is always exact; the first non-trivial cohomology occurs at $\Lambda^2 T^* \otimes g^i_{m-1}$.

Section 5. Metrics and extensions of metrics

Assume that smooth inner products are given on the fibers of the bundles T^* and E^i. We extend these as follows. If $V_1, \ldots, V_m$ are vector bundles with smooth hermitian inner products, there is a unique inner product on $V_1 \otimes \ldots \otimes V_m$ such that $\langle v_1 \otimes \ldots \otimes v_m, w_1 \otimes \ldots \otimes w_m \rangle_x = \langle v_1, w_1 \rangle_x \ldots \langle v_m, w_m \rangle_x$. Thus we have an inner product on $\otimes^m T^*$. Define an inner product on S^mT^* by setting $\langle x, y \rangle = \langle \alpha x, \alpha y \rangle$ for x and $y \in S^mT^*$, where α is the injection of S^mT^* into $\otimes^m T^*$ generated by

$$\alpha(\xi^1 \cdot \ldots \cdot \xi^m) = \frac{1}{m!} \sum \xi^{\pi(1)} \otimes \ldots \otimes \xi^{\pi(m)}, \quad \pi \in S(m)$$

where $S(m)$ is the permutation group on $\{1, \ldots, m\}$. Similarly, define an inner product on

$\Lambda^l T^*$ by using the injection β: $\Lambda^l T^* \to \otimes^l T^*$ generated by

$$\beta(\xi^1 \wedge \ldots \wedge \xi^l) = \frac{1}{l!}\sum(-1)^\pi \xi^{\pi(1)} \otimes \ldots \otimes \xi^{\pi(l)}.$$

We now have inner products on each bundle $\Lambda^l T^* \otimes S^m T^* \otimes E$, and hence on all sub-bundles of these, so we have inner products on all bundles in the δ-complexes.

If D: $\mathcal{E} \to \mathcal{E}$ is a complex of linear first order differential operators, we want to have some relationship among the inner products on the bundles E^0, E^1, ... For this reason, we define:

Definition 5.1.: A complex D: $\mathcal{E} \to \mathcal{E}$ of linear first order differential operators is called *symbol surjective* if for every $i \geqslant 0$, $\sigma(D^i)$: $T^* \otimes E^i \to E^{i+1}$ is surjective.

It is clear that a complex is symbol surjective if and only if the ΛT^*-module E is generated by elements in E^0.

If a complex D: $\mathcal{E} \to \mathcal{E}$ is symbol surjective and inner products on T^* and E^0 are given, we define inner products on E^i, $i > 0$, inductively by identifying E^{i+1} with the orthocomplement of g^i, in $T^* \otimes E^i$. Under this identification E^{i+1} acquires the inner product of a sub-bundle of $T^* \otimes E^i$. Note that with this inner product, $\sigma(D^i)^*$: $E^{i+1} \to T^* \otimes E^i$ is an isometry.

The assumption of symbol surjectivity is a reasonable one. It is satisfied by the de Rham and Dolbeault sequences. Further, D. C. Spencer [12] has shown that corresponding to an operator D: $\mathcal{E}^i \to \mathcal{E}^j$ satisfying certain reasonable hypotheses there is a complex

$$0 \longrightarrow \Theta \longrightarrow \underline{C}^0 \xrightarrow{D^0} \underline{C}^1 \xrightarrow{D^1} \ldots \longrightarrow \underline{C}^n \longrightarrow 0$$

of first order linear operators which formally resolves the sheaf Θ of germs of solutions of the homogeneous equation $Du = 0$. This complex, the Spencer sequence of D, is symbol surjective. In Section 7 below we shall show that the tangential complex corresponding to a symbol surjective complex and a non-characteristic Cauchy surface is symbol surjective.

Section 6. The δ-estimate

Definition 6.1: A linear differential operator of order k, D: $\mathcal{E}^i \to \mathcal{E}^j$, *satisfies the δ-estimate* if and only if there exist inner products on T^* and E^i such that in the sequence

$$0 \longrightarrow g^i_{k+1} \xrightarrow{\delta} T^* \otimes g^i_k \xrightarrow{\delta} \Lambda^2 T^* \otimes S^{k-1} T^* \otimes E^i$$

$$\text{if } x \in T^* \otimes g^i_k \cap \ker \delta^*, \text{ then } \|\delta x\|^2 \geqslant (\tfrac{1}{2})\, k^2 \|x\|^2. \tag{6.1}$$

Henceforth, when we assume that D satisfies the δ-estimate, we shall assume that the inner products given on T^* and E^i are the ones which give the estimate (6.1).

We collect here some of the results of [9] which we shall need.

THEOREM 6.2: *If* D: $\mathcal{E} \to \mathcal{E}$ *is a symbol surjective complex of first order linear operators, and if* D^0: $\mathcal{E}^0 \to \mathcal{E}^1$ *satisfies the* δ*-estimate, then for each* $i \geqslant 0$, D^i: $\mathcal{E}^i \to \mathcal{E}^{i+1}$ *satisfies the* δ*-estimate.*

Proof: This is essentially Theorem II.2.1 of [9] ,which was proved for D: $\mathcal{E} \to \mathcal{E}$ a Spencer sequence of some operator. However, the only properties of D that were used hold for any symbol surjective complex.

THEOREM 6.3: *Consider the complex*

$$\Lambda^{l-1}T^* \otimes S^{m+1}T^* \otimes E \xrightarrow{\delta} \Lambda^l T^* \otimes S^m T^* \otimes E \xrightarrow{\delta} \Lambda^{l+1}T^* \otimes S^{m-1}T^* \otimes E.$$

The symmetric map $\delta^*\delta + \delta\delta^* \in \operatorname{Aut}(\Lambda^l T^* \otimes S^m T^* \otimes E)$ *has two eigenspaces:* $\ker \delta$, *on which the eigenvalue is* $((m+1)(m+l))/l$, *and* $\ker \delta^*$, *on which the eigenvalue is* $(m(m+l))/(l+1)$.

Proof: [9, Theorem I.7.1].

THEOREM 6.4.: *Consider the complex*

$$0 \longrightarrow S^2T^* \otimes E \xrightarrow{\delta} T^* \otimes T^* \otimes E \xrightarrow{\delta} \Lambda^2 T^* \otimes E \longrightarrow 0.$$

The symmetric map $S = \frac{1}{4}\delta\delta^* - \delta^*\delta \in \operatorname{Aut}(T^* \otimes T^* \otimes E)$ *is the linear map for which* $S(\xi^1 \otimes \xi^2 \otimes e) = \xi^2 \otimes \xi^1 \otimes e$; *it is called the switching map.*

Proof: [9, p. 108].

THEOREM 6.5: *Consider the complex*

$$0 \longrightarrow g_2 \xrightarrow{\delta} T^* \otimes g_1 \xrightarrow{\delta} \Lambda^2 T^* \otimes E.$$

On g_2 *the map* $\delta^*\delta$ *is* 4 *times the identity.*

Proof: [9, Lemma II.1.1].

Section 7. The Guillemin decomposition

The results of this section up through Theorem 7.7 are all due to V. W. Guillemin [5]; since they do not appear in any published work, we reproduce them here.

Let D: $\mathcal{E} \to \mathcal{E}$ be a symbol surjective complex of linear first order differential operators. The following theorem follows from the work of V. W. Guillemin and D. G. Quillen.

THEOREM 7.1: *The following properties of a vector subspace* U_x *of* T_x^* *are equivalent:*

(1) $0 \longrightarrow E_x^0 \xrightarrow{\sigma_\xi(D^0)} E_x^1$ *is exact for every* $\xi \in U_x$;

(2) $0 \longrightarrow E_x^0 \xrightarrow{\sigma_\xi(D^0)} E_x^1 \longrightarrow \dots \xrightarrow{\sigma_\xi(D^{n-1})} E_x^n \longrightarrow 0$

is exact for every $\xi \in U_x$;

(3) $0 \longrightarrow U_x \otimes E_x^0 \xrightarrow{\sigma(D_0)} E_x^1$ *is exact;*

(4) *For each* $m > 0$,

$$0 \longrightarrow S^m U_x \otimes E_x^0 \xrightarrow{\sigma_{m-1}(D^0)} S^{m-1} U_x \otimes E_x^1 \longrightarrow \dots \xrightarrow{\sigma(D^{m-1})} E^m \longrightarrow 0$$

is exact; and

(5) *The* ΛT_x^**-module* E_x *is free as a* ΛU_x*-module.*

Remark: It is important that T_x^* be the complexified cotangent space; the theorem is false if U_x and T_x^* are real vector spaces.

Proof: Quillen's theorem [10, 3] says that for each $\xi \in T_x^*$, the exactness of the sequences in (1) and (2) are equivalent. The proof was originally given for Spencer sequences only, but it applies also to any symbol surjective complex. In [4], V. W. Guillemin proves (1) and (3) are equivalent and (2) and (4) are equivalent. In [5], he shows (2) and (5) are equivalent.

Definition 7.2: A subspace U_x of T_x^* satisfying the above conditions is called *non-characteristic*. A sub-bundle U of $T^*(\Omega)$ (a co-distribution in the sense of differential geometry) is *non-characteristic* if U_x is non-characteristic for each $x \in \Omega$. A submanifold $S \subset \Omega$ is *non-characteristic at* x if U_x, the annihilator of the tangent space of S at x or the normal bundle of S at x, is non-characteristic; and S is *non-characteristic* if it is non-characteristic at each $x \in S$.

We assume always that a non-characteristic sub-bundle or subspace is the compexification of a real sub-bundle or subspace of T^*. A sub-bundle is called integrable if it is integrable as a co-distribution.

Let U be a non-characteristic sub-bundle of $T^*(\Omega)$ with fiber dimension q. Denote by $\sigma: \Lambda^i U \otimes E \to E$ the morphism given by $\sigma(\omega \otimes e) = \omega \wedge e$, where $\wedge$ is the multiplication in the ΛT^*-module E. We obtain a filtration

$$E = E_0 \supset E_1 \supset \dots \supset E_q \supset E_{q+1} = 0 \tag{7.1}$$

of E, where $E_i = \sigma(\Lambda^i U \otimes E)$. Let $E_j^k = E^k \cap E_j$ and let $E^{i,j} = E_j^{i+j} / E_{j+1}^{i+j}$. Since we have inner products on all the bundles, we may take $E^{i,j}$ to be the orthocomplement of E_{j+1}^{i+j} in E_j^{i+j}.

We obtain

$$E^i = \bigoplus_{j+k=i} E^{j,k} \tag{7.2}$$

where the direct sum is an orthogonal direct sum, and $E^{j,k} = \sigma(\Lambda^k U \otimes E^{j,0})$. Since E is free as a ΛU-module, $E^{j,k} \cong \Lambda^k U \otimes E^{j,0}$.

We claim that each $E^{i,j}$ is a vector bundle over Ω. All that is necessary is to show that the fiber dimension of $E^{i,j}$ is constant. Since $\Lambda^j U \otimes E^{i,0} \cong E^{i,j}$, it suffices to show that each $E^{i,0}$ is a vector bundle. We prove this by induction. Since $E^{0,0} = E^0$, it is true for $i=0$. Assume it is true for all $i<k$. Then by (7.2), $\dim E^k_x = \Sigma_{i+j=k} \dim E^{i,j}_x$. By the inductive assumption all summands on the right except $\dim E^{k,0}_x$ are constant; hence $\dim E^{k,0}_x$ is constant, and $E^{k,0}$ is a bundle.

Corresponding to (7.2) there is a decomposition of D into a sum $D = D_0 + D_1 + \ldots + D_q$, with $D_r: \mathcal{E}^{i,j} \to \mathcal{E}^{i-r+1,j+r}$ being the component of D with bidegree $(-r+1, r)$. The equation $D \circ D = 0$ gives

$$\sum_{0 \leqslant j \leqslant i} D_{i-j} \circ D_j = 0 \tag{7.3}$$

for each $i \geqslant 0$. In particular, $D_0 \circ D_0 = 0$ and

$$D_1 \circ D_0 + D_0 \circ D_1 = 0. \tag{7.4}$$

These facts may be summarized by observing that

$$\begin{tikzcd}
\mathcal{E}^{2,0} \arrow[r] & \ldots & & \\
\mathcal{E}^{1,0} \arrow[u, "D_1"] \arrow[r, "D_0"] & \mathcal{E}^{1,0} \arrow[u] \arrow[r] & \ldots & \\
\mathcal{E}^{0,0} \arrow[u, "D_1"] \arrow[r, "D_0"] & \mathcal{E}^{1,0} \arrow[u, "D_1"] \arrow[r, "D_0"] & \mathcal{E}^{2,0} \arrow[r] & \ldots
\end{tikzcd} \tag{7.5}$$

is a spectral sequence decomposition of the complex D: $\mathcal{E} \to \mathcal{E}$.

This decomposition certainly depends on the decomposition (7.2), which depends on the inner products on E. The operator D_0, however, has some canonical significance. If $u \in \mathcal{E}_i$, then we may write $u = \Sigma \omega_\alpha \wedge u_\alpha$ where $\omega_\alpha \in \Gamma(\Omega, \underline{\Lambda^i T^*(\Omega)})$, so by Lemma 1.1, $Du = \Sigma d\omega_\alpha \wedge u_\alpha + (-1)^i \omega_\alpha \wedge Du_\alpha$, which is in $\mathcal{E}_i$. Thus D respects the filtration, and we obtain a quotient operator $\tilde{D}$: $\mathcal{E}_j/\mathcal{E}_{j+1} \to \mathcal{E}_j/\mathcal{E}_{j+1}$. Since $E_j/E_{j+1} = \oplus_i E^{i,j}$, we may identify $\mathcal{E}_j/\mathcal{E}_{j+1}$ with $\oplus_i \mathcal{E}^{i,j}$. Under this correspondence, $\tilde{D}$ corresponds to D_0.

The following property of the operators D_r is important.

THEOREM 7.3: *Let ξ be a section of U. Then $\sigma_\xi(D_r) = 0$ except when $r=1$, and $\sigma_\xi(D_1)$: $E^{i,j} \to E^{i,j+1}$ is multiplication by ξ in the module E.*

Proof: $\sigma_\xi(D)$: $E \to E$ is multiplication by ξ, and according to the decomposition (7.2),

$\sigma_\xi(D)$ decomposes into $\sigma_\xi(D_0)+\sigma_\xi(D_1)+\ldots+\sigma_\xi(D_q)$ where $\sigma_\xi(D_r)$ has bidegree $(-r+1, r)$. If $\xi \in U$, then $\sigma_\xi(D)$: $E_j \to E_{j+1}$ and hence has bidegree (0, 1). The theorem follows.

The significance of this theorem is in the following theorem.

THEOREM 7.4: *If $\{S_\alpha\}$ is a local foliation of Ω by submanifolds of codimension q, and if U is the normal bundle to the foliation (i.e., U_x is the annihilator of the tangent space of S_α at x, where S_α is the submanifold containing x), then all the operators D_r, except D_1, are intrinsically defined as operators on the sheets of the foliation. That is, they do not involve any differentiation in the normal directions.*

Proof: $\sigma_\xi(D_r)$ for $\xi \in U$ is the coefficient of differentiation in a normal direction.

Let S be a non-characteristic submanifold of Ω with codimension q. Then locally there are q smooth functions $\{\varrho_1, \ldots, \varrho_q\}$ such that for each x in S, $\{d\varrho_1, d\varrho_2, \ldots, d\varrho_q\}$ is a linearly independent set which spans a non-characteristic subspace U_x of T_x^*. By a continuity argument, and by shrinking Ω if necessary, the forms $\{d\varrho_1, \ldots, d\varrho_q\}$ will span an integrable non-characteristic sub-bundle U of $T^*(\Omega)$. The surfaces $\{\varrho_i = \text{constant}, 1 \leqslant i \leqslant q\}$ will then be the leaves of a foliation of Ω by non-characteristic submanifolds, of which S is one leaf. Let $\mathfrak{J}$ be the least differential ideal containing all functions which vanish on S; it is generated by $\{\varrho_1, \ldots, \varrho_q, d\varrho_1, \ldots, d\varrho_q\}$ over the functions. If $x \notin S$, then some $\varrho_i(x) \neq 0$, and $(\mathfrak{J})_x = T_x^*$, so $(\mathfrak{J}\mathcal{E})_x = E_x$. If $x \in S$, then $\varrho_1(x) = \ldots = \varrho_q(x) = 0$, so $(\mathfrak{J}\mathcal{E})_x = \sigma(U \otimes E) = E_1$ and $D_{\mathfrak{J}}$: $\mathcal{E}/\mathfrak{J}\mathcal{E} \to \mathcal{E}/\mathfrak{J}\mathcal{E}$ is simply the restriction to S of the tangential operator $\tilde{D}$: $\mathcal{E}_0/\mathcal{E}_1 \to \mathcal{E}_0/\mathcal{E}_1$, which we have shown to be D_0 when quotients are identified with orthocomplements. We repeat this as a theorem.

THEOREM 7.5: *Let S be a non-characteristic submanifold of Ω; let U be an integrable non-characteristic sub-bundle extending the normal bundle of S; and let $\mathcal{E}^{*,0} = \oplus_i \mathcal{E}^{i,0}$. Then the complex $D_{\mathfrak{J}}$: $\mathcal{E}/\mathfrak{J}\mathcal{E} \to \mathcal{E}/\mathfrak{J}\mathcal{E}$ is isomorphic to the complex D_0: $\mathcal{E}^{*,0} \to \mathcal{E}^{*,0}$ restricted to S. The isomorphism is the one obtained by identifying quotient spaces with orthocomplements with respect to the given inner products.*

For each i we have a complex D_0: $\mathcal{E}^{*,i} \to \mathcal{E}^{*,i}$. We claim that this complex is essentially the complex D_0: $\mathcal{E}^{*,0} \to \mathcal{E}^{*,0}$. More precisely, we have:

THEOREM 7.6: *The following diagram commutes:*

$$\begin{array}{ccc}
\Gamma(\Omega, \underline{\Lambda^i U \otimes E^{*,0}}) & \xrightarrow{(-1)^i (I \otimes D_0)} & \Gamma(\Omega, \underline{\Lambda^i U \otimes E^{*,0}}) \\
\downarrow \sigma & & \downarrow \sigma \\
\mathcal{E}^{*,i}(\Omega) & \xrightarrow{\quad D_0 \quad} & \mathcal{E}^{*,i}(\Omega)
\end{array}$$

Proof: An arbitrary section of $\Gamma(\Omega, \underline{\Lambda^i \otimes E^{*,0}})$ can be written $\Sigma\omega_\alpha \otimes e_\alpha$ with $d\omega_\alpha = 0$ since U is integrable, and therefore has a basis of exact sections. Then $D(\sigma(\Sigma\omega_\alpha \otimes e_\alpha)) = D(\Sigma\omega_\alpha \wedge e_\alpha) = \Sigma(d\omega_\alpha \wedge e_\alpha + (-1)^i \omega_\alpha \wedge De_\alpha) = (-1)^i \Sigma\omega_\alpha \wedge De_\alpha$. Since D_0 is the part of D with bidegree $(1, 0)$, it is clear that $D_0(\sigma(\Sigma\omega_\alpha \otimes e_\alpha)) = (-1)^i \Sigma\omega_\alpha \wedge D_0 e_\alpha = \sigma((-1)^i \Sigma\omega_\alpha \otimes D_0 e_\alpha)$.

THEOREM 7.7: *If D: $\mathcal{E} \to \mathcal{E}$ is a symbol surjective complex of first order linear differential operators, and U is a non-characteristic sub-bundle of $T^*(\Omega)$, then D_0: $\mathcal{E}^{*,0} \to \mathcal{E}^{*,0}$ is a symbol surjective complex of first order differential operators.*

Proof: It suffices to show $\sigma(D_0^i)$: $T^* \otimes E^{i,0} \to E^{i+1,0}$ is surjective for each $i \geq 0$. By definition, $E^{i+1,0}$ is the orthocomplement of $\sigma(U \otimes E^i)$ in E^{i+1}, so π: $E^{i+1} \to E^{i+1,0}$ is surjective, where π is orthogonal projection. By the decomposition (7.2) and the definition of D_0, the following diagram commutes and is exact.

$$\begin{tikzcd}
T^* \otimes E^i \arrow[r, "\sigma(D^i)"] \arrow[d, "I \otimes \pi"] & E^{i+1} \arrow[r] \arrow[d, "\pi"] & 0 \\
T^* \otimes E^{i,0} \arrow[r, "\sigma(D_0^i)"] \arrow[d] & E^{i+1,0} \arrow[d] & \\
0 & 0 &
\end{tikzcd}$$

An elementary diagram chase shows that $\sigma(D_0^i)$ is surjective. In fact, $\sigma(D_0^i)$: $W \otimes E^{i,0} \to E^{i+1,0}$ is surjective, where W is the orthocomplement of U, since $\sigma(D_0^i)$ is zero on $U \otimes E^{i,0}$.

THEOREM 7.8: *If D: $\mathcal{E} \to \mathcal{E}$ is a symbol surjective complex of linear first order differential operators, and U is a non-characteristic sub-bundle of $T^*(\Omega)$, and if D^0: $\mathcal{E}^0 \to \mathcal{E}^1$ satisfies the δ-estimate, then for each $i \geq 0$, D_0^i: $\mathcal{E}^{i,0} \to \mathcal{E}^{i+1,0}$ satisfies the δ-estimate. More precisely, if W is the orthocomplement of U in T^*, if k_1^i is the kernel of $\sigma(D_0^i)$: $W \otimes E^{i,0} \to E^{i+1,0}$, and if $k_2^i = W \otimes k_1^i \cap S^2 W \otimes E^{i,0}$, then the δ-estimate is satisfied in the complex*

$$0 \longrightarrow k_2^i \xrightarrow{\delta} W \otimes k_1^i \xrightarrow{\delta} \Lambda^2 W \otimes E^{i,0};$$

that is, if $x \in W \otimes k_1^i \cap \ker \delta^*$, then $\|\delta x\|^2 \geq \frac{1}{2}\|x\|^2$.

Remark: We shall solve the codimension q Cauchy problem by solving a succession of codimension one Cauchy problems. This theorem says that one of our main hypotheses is preserved after passing to the tangential complex. If $\{S_\alpha\}$ is a foliation of Ω by non-characteristic surfaces having $U|_{S_\alpha}$ as normal bundles, there is no canonical inclusion of $T^*(S_\alpha)$ into $T^*(\Omega)$, but an inner product on $T^*(\Omega)$ induces an isomorphism of $\bigcup_\alpha T^*(S_\alpha)$ with W. If we consider D_0^i to be an operator restricted to some S_α, in order to consider the δ-estimate for D_0^i, we need an inner product on $T^*(S_\alpha)$, which obviously should be the one it obtains from its identification with $W|_{S_\alpha}$. The theorem then says that with this choice of inner product, D_0^i satisfies the δ-estimate.

Proof of Theorem 7.8: It suffices to prove that D_0^0: $\mathcal{E}^{0,0} \to \mathcal{E}^{1,0}$ satisfies the δ-estimate, for the theorem then follows from Theorems 6.2 and 7.7. To say that $\|\delta x\|^2 \geqslant \frac{1}{2}\|x\|^2$ for $x \in W \otimes k_1^i \cap \ker \delta^*$ is equivalent to saying that all the eigenvalues of $\delta^*\delta$: $\ker \delta^* \to \ker \delta^*$ are bounded below by $\frac{1}{2}$. We shall show that there is a dim ker δ^* dimensional subspace V of $W \otimes k_1^i$ on which $(\delta^*\delta - \frac{1}{2}I) \geqslant 0$ (where I is the identity). This implies what we want, for if $x \in V$, we may write $x = x_0 + x_1$ where $x_1 \in \ker \delta$ and $x_0 \in \ker \delta^*$, so $\langle x_0, x_1 \rangle = 0$. The map $x \to x_0$ is clearly an isomorphism by dimension considerations since any x in the kernel would be a 0-eigenvector for $\delta^*\delta$. By assumption, $\langle \delta^*\delta x, x \rangle \geqslant \frac{1}{2}\|x\|^2$, so $\langle \delta^*\delta x_0, x_0 \rangle \geqslant \frac{1}{2}(\|x_0\|^2 + \|x_1\|^2) \geqslant \frac{1}{2}\|x_0\|^2$. Thus all the eigenvalues of $\delta^*\delta$ on $\ker \delta^*$ are bounded below by $\frac{1}{2}$.

Consider the exact commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & & U \otimes E^0 \arrow[r, "I"] \arrow[d, "i \otimes I"] & U \otimes E^0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & g_1^0 \arrow[r, "i"] \arrow[d] & T^* \otimes E^0 \arrow[r, "\sigma"] \arrow[d, "\pi \otimes I"] & E^1 \arrow[r] \arrow[d, "\pi"] & 0 \\
0 \arrow[r] & k_1^0 \arrow[r, "i"] \arrow[d] & W \otimes E^{0,0} \arrow[r] \arrow[d] & E^{1,0} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where i represents various inclusion maps, π represents orthogonal projections, and σ represents the symbol of D^0, or alternatively, the multiplication in the ΛT^*-module E. The map from $W \otimes E^{0,0} \to E^{1,0}$ is $\sigma(D_0^0)$ restricted to $W \otimes E^{0,0}$, or alternatively, the multiplication in the ΛW-module $E^{*,0}$. The map $\pi \otimes I$: $T^* \otimes E^0 \to W \otimes E^{0,0}$ (recall $E^{0,0} = E^0$) induces an isomorphism π: $g_1^0 \to k_1^0$, as can be checked by an easy diagram chase.

Now consider the exact commutative diagram:

$$\begin{tikzcd}
 & & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & 0 \arrow[r] \arrow[d] & S^2U \otimes E^0 \arrow[r, "I"] \arrow[d] & S^2U \otimes E^0 \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] \arrow[d, dashed] & U \otimes g_1^0 \arrow[r, "\delta"] \arrow[d] & U \otimes T^* \otimes E^0 \arrow[r, "1 \otimes \sigma"] \arrow[d] & U \otimes E^1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & g_2^0 \arrow[r, "\delta"] \arrow[d, dashed, "\pi"] & T^* \otimes g_1^0 \arrow[r, "\delta"] \arrow[d, "\pi"] & \Lambda^2 T^* \otimes E^0 \arrow[r, "\delta"] \arrow[d, "\pi"] & E^2 \arrow[r] \arrow[d, "\pi"] & 0 \\
0 \arrow[r] & k_2^0 \arrow[r, "\delta"] \arrow[d, dashed] & W \otimes k_1^0 \arrow[r, "\delta"] \arrow[d] & \Lambda^2 W \otimes E^{0,0} \arrow[r] \arrow[d] & E^{2,0} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 & 0 &
\end{tikzcd}$$

It is not hard to verify that this diagram is exact and commutative. The maps in the third column are δ maps composed with inclusions, except for π: $\Lambda^2 T^* \otimes E^0 \to \Lambda^2 W \otimes E^{0,0}$ which is orthogonal projection. The map π: $T^* \otimes g_1^0 \to W \otimes k_1^0$ is the tensor product of orthogonal projection of T^* onto W with π: $g_1^0 \to k_1^0$ defined by the previous diagram. A simple diagram chase shows that the dashed arrows may be added so that the diagram remains exact and commutative.

We may now commence the proof. Consider $x \in W \otimes k_1^0$ with $\delta^* x = 0$. Then $\pi^* x \in T^* \otimes g_1^0$ and $\delta^* \pi^* x = \pi^* \delta^* x = 0$. By the assumption that D^0 satisfies the δ-estimate, $\|\delta \pi^* x\|^2 \geqslant \frac{1}{2}\|\pi^* x\|^2$. We may just as well consider $\pi^* x$ to be in $T^* \otimes T^* \otimes E^0$, since δ on $T^* \otimes g_1^0$ is the restriction of δ on $T^* \otimes T^* \otimes E^0$, so we have $\pi^* x \in T^* \otimes T^* \otimes E^0$ with $\langle (\delta^*\delta - \frac{1}{2}I)\pi^* x, \pi^* x\rangle \geqslant 0$. By Theorem 6.3, if we consider the trivial δ-complex

$$0 \longrightarrow S^2 T^* \otimes E^0 \xrightarrow{\delta} T^* \otimes T^* \otimes E^0 \xrightarrow{\delta} \Lambda^2 T^* \otimes E^0 \longrightarrow 0,$$

the identity on $T^* \otimes T^* \otimes E^0$ is $\frac{1}{4}\,\delta\delta^* + \delta^*\delta$. Thus $\delta^*\delta - \frac{1}{2}I = \frac{1}{2}(\delta^*\delta - \frac{1}{4}\delta\delta^*) = \frac{1}{2}S$, by Theorem 6.4, where S: $T^* \otimes T^* \otimes E^0 \to T^* \otimes T^* \otimes E^0$ is the linear map for which $S(\xi^1 \otimes \xi^2 \otimes e) = \xi^2 \otimes \xi^1 \otimes e$. Therefore,

$$\langle S\pi^* x, \pi^* x\rangle \geqslant 0.$$

We noted that π: $T^* \otimes g_1^0 \to W \otimes k_1^0$ is the tensor product of orthogonal projection and π: $g_1^0 \to k_1^0$, so π^* is the tensor product of the inclusion $W \to T^*$ and π^*: $k_1^0 \to g_1^0$, and $\pi^* x \in W \otimes g_1^0$. If we write $\pi^* x = x_0 + x_1$ where $x_0 \in W \otimes W \otimes E^0$ and $x_1 \in W \otimes U \otimes E^0$, then

$$\langle Sx_0, x_0\rangle = \langle Sx_0 + Sx_1, x_0 + x_1\rangle = \langle S\pi^* x, \pi^* x\rangle \geqslant 0,$$

since x_0, $Sx_0 \in W \otimes W \otimes E^0$, $x_1 \in W \otimes U \otimes E^0$, $Sx_1 \in U \otimes W \otimes E^0$, and these three subspaces are mutually orthogonal. Since the diagram

$$\begin{array}{ccc} T^* \otimes g_1^0 & \xrightarrow{i} & T^* \otimes T^* \otimes E^0 \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ W \otimes k_1^0 & \xrightarrow{i} & W \otimes W \otimes E^0 \end{array}$$

commutes, it follows that $\pi x_0 = x_0$ and $\pi x_1 = 0$. Thus $x_0 = \pi\pi^* x$. We have shown that if $\delta^* x = 0$, then $\langle S\pi\pi^* x, \pi\pi^* x\rangle \geqslant 0$. Clearly $\pi\pi^*$ is an isomorphism of $W \otimes k_1^0$, so there is a dim ker δ^* dimensional space, namely $\pi\pi^*$ ker δ^*, on which $(\delta^*\delta - \frac{1}{2}I) = S \geqslant 0$. We remarked at the beginning of the proof that this is sufficient to prove the theorem.

Section 8. The δ-estimate and normality

The Cauchy problem for a surface of codimension q can be solved by solving a succession of codimension one problems, so we shall make the simplifying assumption that

S is a non-characteristic surface of codimension one, and that U is an integrable non-characteristic line bundle.

Assume that S is given by the equation $\varrho=0$ where $d\varrho \neq 0$ on S. By shrinking Ω if necessary, we may assume that $d\varrho$ is the basis of a non-characteristic sub-bundle U in T^*. Let $\zeta = d\varrho/|d\varrho|$ be a unit basis for U. The Guillemin decomposition in this case is particularly simple, since $\Lambda^j U=0$ if $j>1$, and hence $E^{0,i}=0$ if $i>1$. The complex $D\colon \mathcal{E}\to\mathcal{E}$ becomes

$$\begin{tikzcd} \mathcal{E}^{0,1} \arrow[r, "D_0"] & \mathcal{E}^{1,1} \arrow[r, "D_0"] & \cdots \\ \mathcal{E}^{0,0} \arrow[u, "D_1"] \arrow[r, "D_0"] & \mathcal{E}^{1,0} \arrow[u, "D_1"] \arrow[r, "D_0"] & \cdots \end{tikzcd} \tag{8.1}$$

where $\mathcal{E}^i = \mathcal{E}^{i,0}\oplus\mathcal{E}^{i-1,1}$. By the isomorphism $\sigma\colon U\otimes E^{i,0}\cong E^{i,1}$, for each i the operator $D_1\colon E^{i,0}\to E^{i,1}$ transforms to an operator which we continue to denote by $D_1^i\colon E^{i,0}\to E^{i,0}$. It is easy to check that $\sigma_\zeta(D_1^i)$ is the identity on $E^{i,0}$.

By Theorem 7.6, the diagram (8.1) is isomorphic (via σ^{-1}) to

$$\begin{tikzcd} \mathcal{E}^{0,0} \arrow[r, "-D_0^0"] & \mathcal{E}^{1,0} \arrow[r, "-D_0^1"] & \mathcal{E}^{2,0} \arrow[r] & \cdots \\ \mathcal{E}^{0,0} \arrow[u, "D_1^0"] \arrow[r, "D_0^0"] & \mathcal{E}^{1,0} \arrow[u, "D_1^1"] \arrow[r, "D_0^1"] & \mathcal{E}^{2,0} \arrow[u, "D_1^2"] \arrow[r] & \cdots \end{tikzcd} \tag{8.2}$$

where D_0^i denotes the restriction of D_0 to $\mathcal{E}^{i,0}$. Equation (7.4) transforms under this isomorphism into

$$D_0^i \circ D_1^i = D_1^{i+1}\circ D_0^i \tag{8.3}$$

for each $i\geqslant 0$. In particular, for each cotangent vector field $\xi\in T^*(\Omega)$,

$$\sigma_\xi(D_0^i)\,\sigma_\xi(D_1^i) = \sigma_\xi(D_1^{i+1})\,\sigma_\xi(D_0^i).$$

If K_ξ^i denotes the kernel of $\sigma_\xi(D_0^i)\colon E^{i,0}\to E^{i+1,0}$, this implies

$$\sigma_\xi(D_1^i)\colon K_\xi^i \to K_\xi^i. \tag{8.4}$$

The cohomology of the complex of vector bundle morphisms

$$\cdots \longrightarrow E^{i-1,0} \xrightarrow{\sigma_\xi(D_0^{i-1})} E^{i,0} \xrightarrow{\sigma_\xi(D_0^i)} E^{i+1,0} \longrightarrow \cdots$$

is isomorphic to $H_\xi^i = K_\xi^i \cap \ker \sigma_\xi(D_0^{i-1})^*$. If $k=0$, $H_\xi^i = K_\xi^i$. In spite of (8.4), we do not know *a priori* that $\sigma_\xi(D_1^i)\colon H_\xi^i\to H_\xi^i$. However, this is part of the conclusion of the following theorem, which is the crux of this paper.

THEOREM 8.1: *If D: $\mathcal{E}\to\mathcal{E}$ is a symbol surjective complex of first order linear differential operators which satisfies the δ-estimate, and if U is an integrable one dimensional non-characteristic sub-bundle of T^*, then for each ξ in T^*_x*

$$\sigma_\xi(D_1^i): H_\xi^i \to H_\xi^i,$$

and the restriction of $\sigma_\xi(D_1^i)$ to H_ξ^i is a normal linear map; that is, if

$$\sigma_\xi(D_1^i)': H_\xi^i \to H_\xi^i$$

is the adjoint of $\sigma_\xi(D_1^i)|_{H_\xi^i}$ then

$$[\sigma_\xi(D_1^i), \sigma_\xi(D_1^i)'] = \sigma_\xi(D_1^i)\,\sigma_\xi(D_1^i)' - \sigma_\xi(D_1^i)'\,\sigma_\xi(D_1^i) = 0.$$

Proof: The proof of this theorem is quite long and hard, and except for Corollary 8.3, none of the results in this paper depend on the proof. The reader, if he is willing to take the theorem on faith, may skip directly to section 9 for the consequences of the theorem.

Remark: If $\sigma_\xi(D_1^i)$: $H_\xi^i \to H_\xi^i$ is normal and $a \neq 0$, then $\sigma_{\xi'}(D_1^i)$: $H_\xi^i \to H_\xi^i$ is normal, where $\xi' = a\xi + b\zeta$, since because of the identities $\sigma_\zeta(D_0) = 0$, and $\sigma_\zeta(D_1^i) = 1$, $\sigma_{\xi'}(D_1^i) = a\sigma_\xi(D_1^i) + bI$, and $H_{\xi'}^i = H_\xi^i$. Thus it suffices to prove the theorem assuming $\|\xi\| = 1$ and $\langle\zeta, \xi\rangle = 0$.

We first prove the following lemma due to V. W. Guillemin.

LEMMA 8.2: *Let H be a subspace of $T^*(\Omega)$, and define $h_1^i = H\otimes E^i \cap g_1^i$, and $h_2^i = H\otimes h_1^i \cap S^2H\otimes E^i$. If D^i: $\mathcal{E}^i \to \mathcal{E}^{i+1}$ satisfies the δ-estimate, then in the complex*

$$0 \longrightarrow h_2^i \xrightarrow{\ \delta\ } H\otimes h_1^i \xrightarrow{\ \delta\ } \Lambda^2 H \otimes E^i$$

$$\text{if } x \in H\otimes h_1^i \cap \ker \delta^*, \quad \text{then } \|\delta x\|^2 \geq \tfrac{1}{2}\|x\|^2. \tag{8.5}$$

Proof: We proceed by induction on the codimension of H in T^*, so it is sufficient to assume it is one. Let $\eta \in T^*$ be orthogonal to H so that T^* is the span of H and η.

Assume the lemma is false. Then there is an $x \in H\otimes h_1^i$ such that $x \perp \delta h_2^i$ and $\|\delta x\|^2 < \frac{1}{2}\|x\|^2$. We shall show shortly that there is a $y \in \operatorname{span}(\eta)\otimes g_1^i$ such that $(x+y) \perp \delta g_2^i$. Since $\eta \perp H$, we have that $x \perp y$ and $\delta x \perp \delta y$. Furthermore, $\|\delta y\|^2 \leq \frac{1}{2}\|y\|^2$ since $\eta\wedge\eta = 0$ and $\|\eta\wedge\zeta\|^2 = \frac{1}{2}\|\eta\otimes\zeta\|^2$ if $\eta\perp\zeta$. Therefore, $\|\delta x + \delta y\|^2 = \|\delta x\|^2 + \|\delta y\|^2 < \frac{1}{2}(\|x\|^2 + \|y\|^2) = \frac{1}{2}\|x+y\|^2$. This contradicts the assumption that D^i: $\mathcal{E}^i \to \mathcal{E}^{i+1}$ satisfies the δ-estimate, so the lemma is true.

We prove now that if $x \in H\otimes h_1^i \cap \ker\delta^*$, then there is a $y \in \operatorname{span}(\eta)\otimes g_1^i$ such that $(x+y) \perp \delta g_2^i$. Clearly $\delta g_2^i \supset \delta h_2^i$ and we can let $\{u_1, \ldots, u_s\}$ be a basis of the orthocomplement of δh_2^i in δg_2^i. Since $\delta g_2^i \subset T^*\otimes g_1^i = H\otimes g_1^i \otimes \operatorname{span}(\eta)\otimes g_1^i$, we can write $u_r = v_r + \eta\otimes w_r$ where $v_r \in H\otimes g_1^i$ and $w_r \in g_1^i$. We claim that we can find a w, a linear combination of the w_r's such that $(x+\eta\otimes w)\perp\delta g_2^i$. Since both x and $\eta\otimes w$ are orthogonal to δh_2^i, it suffices to find w such that $\langle\eta\otimes w, \eta\otimes w_r\rangle = -\langle x, u_r\rangle$, which can be done if the only solution to $\langle\eta\otimes w, \eta\otimes w_r\rangle = 0$, $1\leqslant r\leqslant s$, is $\eta\otimes w=0$. But this is clearly so since w is a linear combination of the w_r's. Let $y=\eta\otimes w$. This completes the proof of the lemma.

We apply the lemma by letting H be the space spanned by ξ and ζ. We have for any $x\in H\otimes h_1^i$ with $x\perp\delta h_2^i$, $\|\delta x\|^2\geqslant\frac{1}{2}\|x\|^2$. By Theorem 6.5, if $x\in\delta h_2^i$, then $\|\delta^* x\|^2=4\|x\|^2$, so the estimate (8.5) is equivalent to

$$8\|\delta x\|^2+\|\delta^* x\|^2\geqslant 4\|x\|^2 \text{ for all } x\in H\otimes h_1^i. \tag{8.6}$$

By the Guillemin decomposition, $E^i\cong E^{i-1,0}\otimes E^{i,0}$. We shall represent elements of E^i by pairs (y, z) with $y\in E^{i-1,0}$ and $z\in E^{i,0}$. If $u\in\mathcal{E}^{i-1,0}$ and $v\in\mathcal{E}^{i,0}$, then by diagram (8.2),

$$D^i(u, v) = (D_1^i v - D_0^{i-1}u,\ D_0^i v). \tag{8.7}$$

An arbitrary element $x\in H\otimes E^i$ is

$$x=(\xi\otimes y_1+\zeta\otimes y_2,\ \xi\otimes z_1+\zeta\otimes z_2),$$

where $y_j\in E^{i-1,0}$ and $z_j\in E^{i,0}$ for $j=1, 2$. Then $x\in h_1^i$ if and only if $\sigma(D^i)x=0$, or by (8.7),

$$(\sigma(D_1^i)(\xi\otimes z_1+\zeta\otimes z_2)-\sigma(D_0^{i-1})(\xi\otimes y_1+\zeta\otimes y_2),\ \sigma(D_0^i)(\xi\otimes z_1+\zeta\otimes z_2))=0.$$

Since $\sigma_\zeta(D_0)=0$ and $\sigma_\zeta(D_1^i)=I$, this becomes

$$(\sigma_\xi(D_1^i)z_1+z_2-\sigma_\xi(D_0^{i-1})y_1,\ \sigma_\xi(D_0^i)z_1)=0.$$

Thus,

$$h_1^i=\{(\xi\otimes y_1+\zeta\otimes y_2,\ \xi\otimes z_1+\zeta\otimes(\sigma_\xi(D_0^{i-1})y_1-\sigma_\xi(D_1^i)z_1)) : y_1, y_2\in E^{i-1,0} \text{ and } z_1\in K_\xi^i\}. \tag{8.8}$$

Thus an arbitrary element in $H\otimes h_1^i$ can be written

$$(\xi\otimes\xi\otimes y_{11}+\xi\otimes\zeta\otimes y_{12}+\zeta\otimes\xi\otimes y_{21}+\zeta\otimes\zeta\otimes y_{22},\ \xi\otimes\xi\otimes z_{11}+\xi\otimes\zeta\otimes(\sigma_\xi(D_0^{i-1})y_{11}$$
$$-\sigma_\xi(D_1^i)z_{11})+\zeta\otimes\xi\otimes z_{12}+\zeta\otimes\zeta\otimes(\sigma_\xi(D_0^{i-1})y_{12}-\sigma_\xi(D_1^i)z_{12})) \text{ where } y_{jk}\in E^{i-1,0} \text{ and } z_{1k}\in K_\xi^i$$

By definition, h_2^i is the intersection $H\otimes h_1^i\cap S^2H\otimes E^i$, and δ: $h_2^i\to H\otimes h_1^i$ is a constant times inclusion, so the above element is in δh_2^i if and only if it is symmetric in ξ and ζ. This means $y_{12}=y_{21}$ and $z_{12}=\sigma_\xi(D_0^{i-1})y_{11}-\sigma_\xi(D_1^i)z_{11}$. Thus,

$$\delta h_2^i=\{(\xi\otimes\xi\otimes y_{11}+(\xi\otimes\zeta+\zeta\otimes\xi)y_{12}+\zeta\otimes\zeta\otimes y_{22},\ \xi\otimes\xi\otimes z_{11}+(\xi\otimes\zeta+\zeta\otimes\xi)$$
$$\otimes(\sigma_\xi(D_0^{i-1})y_{11}-\sigma_\xi(D_1^i)z_{11})+\zeta\otimes\zeta\otimes(\sigma_\xi(D_0^{i-1})y_{12}-\sigma_\xi(D_1^i)(\sigma_\xi(D_0^{i-1})y_{11}$$
$$-\sigma_\xi(D_1^i)z_{11})) : y_{11}, y_{12}, y_{22}\in E^{i-1,0} \text{ and } z_{11}\in K_\xi^i\}. \tag{8.9}$$

Let π_ξ be orthogonal projection of K_ξ^i onto H_ξ^i. For simplicity denote $\sigma_\xi(D_0^{i-1})$ by σ_0, $\sigma_\xi(D_1^i)$ by σ_1, and $\pi_\xi\sigma_1$ by $\bar\sigma_1$. Then $\bar\sigma_1$: $H_\xi^i\to H_\xi^i$ and we may consider the adjoint of

$\bar{\sigma}_1|_{H^i_\xi}$, denoted by $\bar{\sigma}'_1: H^i_\xi \to H^i_\xi$. We have $\pi_\xi \sigma_0 = 0$ since the image of σ_0 is orthogonal to H^i_ξ, and the kernel of π_ξ is the image of σ_0. The map $A: H^i_\xi \to H^i_\xi$ defined by $A = I + \bar{\sigma}'_1 \bar{\sigma}_1$ is positive definite, and hence an automorphism. Let w_2 be an arbitrary element of H^i_ξ, and define $w_1 = A^{-1}\bar{\sigma}'_1 A w_2$. Since $\sigma_1 w_1 - \bar{\sigma}_1 w_1 \in \ker \pi_\xi = \text{image}\, \sigma_0$, there is a z_1 in $E^{i-1,0}$ such that $\sigma_0 z_1 + \sigma_1 w_1 - \bar{\sigma}_1 w_1 = 0$, and similarly there is a z_2 such that

$$\sigma_0 z_2 + \sigma_1 w_2 - \bar{\sigma}_1 w_2 = 0. \tag{8.10}$$

It is easy to check that $x \in H \otimes h^i_1$ where

$$x = (\xi \otimes \xi \otimes z_1 + (\xi \otimes \zeta + \zeta \otimes \xi) \otimes z_2,\ -\xi \otimes \xi \otimes w_1 + \xi \otimes \zeta \otimes \bar{\sigma}_1 w_1 - \zeta \otimes \xi \otimes w_2 + \zeta \otimes \zeta \otimes \bar{\sigma}_1 w_2). \tag{8.11}$$

Then $\delta x = (0, \xi \wedge \zeta \otimes (\bar{\sigma}_1 w_1 + w_2))$, so

$$\|\delta x\|^2 = \tfrac{1}{2}\|\bar{\sigma}_1 w_1 + w_2\|^2.$$

Also,

$$\|x\|^2 = \|z_1\|^2 + 2\|z_2\|^2 + \|w_1\|^2 + \|\bar{\sigma}_1 w_1\|^2 + \|w_2\|^2 + \|\bar{\sigma}_1 w_2\|^2.$$

To apply (8.6), we need to calculate $\|\delta^* x\|^2$. We have

$$\|\delta^* x\|^2 = \sup_{y \in h^i_2} \left(\frac{\langle \delta^* x, y\rangle}{\|y\|}\right)^2 = \sup_{y \in h^i_2} \left(\frac{\langle x, \delta y\rangle}{\|y\|}\right)^2.$$

Since by Theorem 6.5,

$$\|\delta y\|^2 = \langle \delta^* \delta y, y\rangle = 4\|y\|^2, \tag{8.12}$$

we have,

$$\|\delta^* x\| = 4 \sup_{y \in h^i_2} \left(\frac{\langle x, \delta y\rangle}{\|\delta y\|}\right)^2.$$

Let δy have the expansion given in (8.9). Then clearly,

$$\|\delta y\|^2 \geqslant \|y_{11}\|^2 + 2\|y_{12}\|^2,$$

and

$$\langle x, \delta y\rangle = \langle z_1, y_{11}\rangle + 2\langle z_2, y_{12}\rangle - \langle w_1, z_{11}\rangle + \langle \bar{\sigma}_1 w_1, \sigma_0 y_{11} - \sigma_1 z_{11}\rangle$$
$$- \langle w_2, \sigma_0 y_{11} - \sigma_1 z_{11}\rangle + \langle \bar{\sigma}_1 w_2, \sigma_0 y_{12} + \sigma_1 \sigma_1 z_{11} - \sigma_1 \sigma_0 y_{11}\rangle.$$

Now observe:

(1) w_1, $\bar{\sigma}_1 w_1$, w_2, and $\bar{\sigma}_1 w_2$ are in H^i_ξ and so are orthogonal to the image of σ_0.

(2) By (8.3), σ_1: image $\sigma_0 \to$ image σ_0.

(3) $z_{11} = \pi_\xi z_{11} + \sigma_0 z'$ for some z' since image $\sigma_0 = \ker \pi_\xi$,
$\sigma_1 z_{11} = \pi_\xi \sigma_1 \pi_\xi z_{11} + \sigma_0 z''$, and
$\sigma_1 \sigma_1 z_{11} = \pi_\xi \sigma_1 \pi_\xi \sigma_1 \pi_\xi z_{11} + \sigma_0 z'''$.

Thus,

$$\langle x, \delta y\rangle = \langle z_1, y_{11}\rangle + 2\langle z_2, y_{12}\rangle - \langle w_1, \pi_\xi z_{11}\rangle - \langle \bar\sigma_1 w_1, \bar\sigma_1 \pi_\xi z_{11}\rangle + \\ + \langle w_2, \bar\sigma_1 \pi_\xi z_{11}\rangle + \langle \bar\sigma_1 w_2, \bar\sigma_1\bar\sigma_1 \pi_\xi z_{11}\rangle.$$

Taking adjoints and using the definition of w_1, we obtain

$$\langle x, \delta y\rangle = \langle z_1, y_{11}\rangle + 2\langle z_2, y_{12}\rangle. \tag{8.13}$$

Therefore

$$\|\delta^* x\|^2 = 4 \sup_{y\in h_2^i} \left(\frac{\langle x, \delta y\rangle}{\|\delta y\|}\right)^2 \leqslant 4 \sup_{y_{11}, y_{12}} \frac{(\langle z_1, y_{11}\rangle + 2\langle z_2, y_{12}\rangle)^2}{\|y_{11}\|^2 + 2\|y_{12}\|^2}.$$

By applying the Cauchy-Schwartz inequality to the inner product on $E^{i-1,0} \oplus E^{i-1,0}$ defined by $\langle (z_1, z_2), (y_{11}, y_{12})\rangle = \langle z_1, y_{11}\rangle + 2\langle z_2, y_{12}\rangle$, we obtain

$$\|\delta^* x\|^2 \leqslant 4(\|z_1\|^2 + 2\|z_2\|^2).$$

We now have what we need to apply (8.6) to obtain, after some cancellation,

$$\|\bar\sigma_1 w_1 + w_2\|^2 \geqslant \|w_1\|^2 + \|\bar\sigma_1 w_1\|^2 + \|w_2\|^2 + \|\bar\sigma_1 w_2\|^2,$$

or

$$0 \geqslant \|w_1 - \bar\sigma_1' w_2\|^2 + \|\bar\sigma_1 w_2\|^2 - \|\bar\sigma_1' w_2\|^2.$$

Since $\|w_1 - \bar\sigma_1' w_2\|^2 \geqslant 0$, we have

$$\|\bar\sigma_1' w_2\|^2 \geqslant \|\bar\sigma_1 w_2\|^2. \tag{8.14}$$

Since w_2 is an arbitrary element of H_ξ^i, (8.14) shows that $\bar\sigma_1 \bar\sigma_1' - \bar\sigma_1' \bar\sigma_1$ is positive semi-definite on H_ξ^i. Since the map is symmetric, it can be diagonalized; since it is positive semi-definite, all the eigenvalues are $\geqslant 0$; and since it is a commutator, the trace, or the sum of the eigenvalues, is 0. Thus,

$$\bar\sigma_1 \bar\sigma_1' = \bar\sigma_1' \bar\sigma_1. \tag{8.15}$$

The fact that what was originally an inequality is in fact an equality has some strong consequences. If we had a strict inequality

$$\|\delta^* x\|^2 < 4(\|z_1\|^2 + 2\|z_2\|^2),$$

then (8.14) would be a strict inequality, which is impossible. Therefore,

$$\|\delta^* x\|^2 = 4(\|z_1\|^2 + 2\|z_2\|^2). \tag{8.16}$$

The choice of $y = \delta^* x$ gives equality in the Cauchy-Schwartz inequality $\|\delta^* x\|^2 = (\langle \delta^* x, y\rangle^2 / \|y\|)$. Let $y = \delta^* x$ have the expansion given in (8.9). We must have $\|\delta y\|^2 = \|y_{11}\|^2 + 2\|y_{12}\|^2$, for if $\|\delta y\|^2 > \|y_{11}\|^2 + 2\|y_{12}\|^2$, we would have $\|\delta^* x\|^2 < 4(\|z_1\|^2 + 2\|z_2\|^2)$, a contradiction. Now

$$4(\|z_1\|^2+2\|z_2\|^2)=\|\delta^*x\|^2=4\,\frac{\langle x,\delta y\rangle}{\|\delta y\|}=4\,\frac{(\langle z_1,y_{11}\rangle+2\langle z_2,y_{22}\rangle)^2}{\|y_{11}\|^2+2\|y_{12}\|^2}$$

or
$$\langle z_1,y_{11}\rangle+2\langle z_2,y_{22}\rangle=(\|z_1\|^2+2\|z_2\|^2)^{\frac{1}{2}}(\|y_{11}\|^2+2\|y_{12}\|^2)^{\frac{1}{2}},$$

which is an instance of equality in the Cauchy-Schwartz inequality. Therefore, (z_1, z_2) is proportional to (y_{11}, y_{12}), and from (8.12) we conclude $y_{11}=4z_1$ and $y_{12}=4z_2$. Therefore $\delta y=\delta\delta^*x=(\xi\otimes\xi\otimes 4z_1+(\xi\otimes\zeta+\zeta\otimes\xi)\otimes 4z_2+\zeta\otimes\zeta\otimes y_{22},\ \xi\otimes\xi\otimes z_{11}+(\xi\otimes\zeta+\zeta\otimes\xi)\otimes(4\sigma_0 z_1-\sigma_1 z_{11})+\zeta\otimes\zeta\otimes(4\sigma_0 z_2+\sigma_1\sigma_1 z_{11}-4\sigma_1\sigma_0 z_1))$, where y_{22} and z_{11} are yet to be determined. We know $\|\delta y\|^2=\|\delta\delta^*x\|^2=4\|\delta^*x\|^2=16(\|z_1\|^2+2\|z_2\|^2)$. If we calculate $\|\delta y\|^2$ directly from the above expression and compare it with this, we may conclude $y_{22}=0$, $z_{11}=0$, $\sigma_0 z_1=0$ and $\sigma_0 z_2=0$. The last equation with (8.10) implies that $\sigma_1 w_2=\bar\sigma_1 w_2$. Therefore $\sigma_1\colon H^i_\xi\to H^i_\xi$, and $[\sigma_1, \sigma_1']=0$ on H^i_ξ. This completes the proof of the theorem.

COROLLARY 8.3: *If in the complex*

$$0\longrightarrow g_2^0\xrightarrow{\ \delta\ }T^*\otimes g_1^0\xrightarrow{\ \delta\ }\Lambda^2T^*\otimes E^0,$$

it is true that for every $x\in T^*\otimes g_1^0$ *satisfying* $\delta^*x=0$,

$$\|\delta x\|^2>\tfrac{1}{2}\|x\|^2,$$

then $g_1^0=0$.

Proof: We check easily that under this hypothesis, if $x\in H\otimes h_1^0\cap\ker\delta^*$ in the complex

$$0\longrightarrow h_2^0\xrightarrow{\ \delta\ }H\otimes h_1^0\xrightarrow{\ \delta\ }\Lambda^2H\otimes E^0$$

then $\|\delta x\|^2>\frac{1}{2}\|x\|^2$. If the space $K_\xi^0=H_\xi^0$ is non-zero for some ξ, the above proof constructs an $x\in H\otimes h_1^0\cap\ker\delta^*$, namely (8.11) where z_1 and z_2 can be taken to be zero, for which $\|\delta^*x\|^2=\frac{1}{2}\|x\|^2$, which contradicts the hypothesis. Thus, for every ξ, $K_\xi^0=0$. Since every ξ, real or complex, is non-characteristic, T^* is a non-characteristic sub-bundle of T^*, or by Theorem 7.1, the complex $0\to T^*\otimes E^0\to E^1$ is exact. Therefore, $g_1^0=0$.

D. C. Spencer, in [12], has shown that a complex $D\colon\mathcal{E}\to\mathcal{E}$ which is completely integrable, or flat, for which $g_1^0=0$ is essentially a direct sum of copies of the de Rham complex.

Suppose U_q is an integrable q-dimensional non-characteristic sub-bundle of T^*, locally spanned by $\{d\varrho_1, \dots, d\varrho_q\}$. Let $\zeta_\alpha=d\varrho_\alpha/\|d\varrho_\alpha\|$. By the isomorphism $\sigma\colon U\otimes\mathcal{E}^{i,0}\cong E^{i,1}$ the operator $D_1\colon\mathcal{E}^{i,0}\to\mathcal{E}^{i,1}$ corresponds to $\Sigma_{\alpha=1}^q\zeta_\alpha\otimes D_{1,\alpha}$, where $D_{1,\alpha}\colon\mathcal{E}^{i,0}\to\mathcal{E}^{i,0}$. Then

$$\sigma_\xi(D_0^i)\,\sigma_\xi(D_{1,\alpha}^i)=\sigma_\xi(D_{1,\alpha}^{i+1})\,\sigma_\xi(D_0^i)$$

and
$$[\sigma_\xi(D_{1,\alpha}^i),\sigma_\xi(D_{1,\beta}^i)]\,\big|_{\ker\sigma_\xi(D_0^i)}=0$$

by 7.3. Let $H^i_{q,\xi}=\ker\sigma_\xi(D_0^i)\cap\ker\sigma_\xi(D_0^{i-1})^*$.

The following generalizes Theorem 8.1:

THEOREM 8.4: *If D: $\mathcal{E}\to\mathcal{E}$ is a symbol surjective complex of first order linear differential operators which satisfies the δ-estimate, and if U_q is an integrable q-dimensional non-characteristic sub-bundle of T^*, then for each $\xi\in T_x^*$ and each $\alpha=1, ..., q$.*

$$\sigma_\xi(D^i_{1,\alpha}):H^i_{q,\xi}\to H^i_{q,\xi}$$

and the restriction of $\sigma_\xi(D^i_{1,\alpha})$ to $H^i_{q,\xi}$ is normal.

Proof: We may suppose $\alpha=q$. Let U_{q-1} be the span of $\{d\varrho_1, ..., d\varrho_{q-1}\}$. Then U_{q-1} is non-characteristic, and there is a complex corresponding to U_{q-1}, call it $(D_0)_{q-1}$, which is tangential to the surfaces defined by the equations $\varrho_\beta=$constant, $\beta=1, ..., q-1$. The subspaces of the cotangent spaces of these surfaces determined by ϱ_q are non-characteristic for $(D_0)_{q-1}$, and so there is a corresponding decomposition of $(D_0)_{q-1}$. It is clear that the tangential complex of this decomposition is D_0 and the D_1 of the decomposition is $D_{1,q}$. Since by Theorem 7.8 the δ-estimate holds for $(D_0)_{q-1}$, we may apply Theorem 8.1 to prove the theorem.

Chapter III: The Cauchy problem

Section 9. Characteristics and hyperbolicity

In this section we discuss some of the consequences of Theorem 8.1 for the structure of the characteristic variety of D: $\mathcal{E}\to\mathcal{E}$; then we shall define symmetric hyperbolic complexes and show that for such complexes, the conclusion of Theorem 8.1 may be replaced by "$\sigma_\xi(D^i_1)|_{H^i_\xi}$ is symmetric."

Recall that since D: $\mathcal{E}\to\mathcal{E}$ is a symbol surjective complex, a cotangent vector $\xi\in T_x^*$ is characteristic if the sequence

$$0\longrightarrow E^0_x\xrightarrow{\sigma_\xi(D)}E^1_x\longrightarrow\cdots \tag{9.1}$$

fails to be exact. The set of all characteristic vectors for a complex is $\mathfrak{V}$, a projective variety in $T^*(\Omega)$, called the characteristic variety of the complex.

Let U_q be an integrable non-characteristic sub-bundle of T^* locally spanned by $\{d\varrho_1, ..., d\varrho_q\}$, and let $\zeta_\alpha=d\varrho_\alpha/\|d\varrho_\alpha\|$. If W is the orthocomplement of U_q, and if $K^0_{q,\xi}$ is the kernel of $\sigma_\xi(D^0_0)_q$: $E^{0,0}_x\to E^{1,0}_x$, where $\xi\in W_q$ and $(D_0)_q$: $\mathcal{E}^{*,0}\to\mathcal{E}^{*,0}$ is the tangential complex corresponding to U_q, then $\eta=(\xi-\sum^q_{\alpha=1}\lambda_\alpha\zeta_\alpha)\in T_x^*$ is characteristic for D: $\mathcal{E}\to\mathcal{E}$ if and only if there is an eigenvector $e\in K^0_{q,\xi,x}$ such that for each α, $\sigma_\xi(D^0_{1,\alpha})e=\lambda_\alpha e$. This is clear, since η is characteristic iff there is an $e\in E^0_x$ such that $\sigma_\eta(D^0)e=0$. But $\sigma_\eta(D^0)e=0$ iff

$$\sigma_\eta(D^0_0)e=0 \tag{9.2}$$

and

$$\sigma_\eta(D^0_1)e=\sigma(\textstyle\sum^q_{\alpha=1}\zeta_\alpha\otimes\sigma_\eta(D^0_{1,\alpha})e)=0. \tag{9.3}$$

Since for each α, $\sigma_{\zeta}(D_0^0)_q=0$, (9.2) means $e \in \ker \sigma_\xi(D_0^0)=K_{q,\xi}^0$. Since σ is an isomorphism and $\sigma_{\zeta_\alpha}(D_{1,\beta}^0)=\delta_\beta^\alpha I$, (9.3) means $\sigma_\xi(D_{1,\alpha}^0)e-\lambda_\alpha e=0$ which establishes the claim.

Thus if π: $T^* \to W_q$ is orthogonal projection, we have that $\pi^{-1}(\xi)$ intersects $\mathfrak{V}$ at the points $\xi-\Sigma\lambda_\alpha\zeta_\alpha$, where $(\lambda_1, ..., \lambda_q)$ ranges over the sets of eigenvalues of $\sigma_\xi(D_{1,1}^0)$, ..., $\sigma_\xi(D_{1,q}^0)$. Since the maps $\sigma_\xi(D_{1,\alpha}^0)$ commute on K_ξ^0, and, if D: $\mathcal{E} \to \mathcal{E}$ satisfies the δ-estimate, are normal on $K_{q,\xi}^0$, they can be simultaneously diagonalized there. Thus

$$\bigoplus_{\eta \in \pi^{-1}(\xi)} \ker \sigma_\eta(D^0) \cong \ker \sigma_\xi(D_0^0)_q = K_{q,\xi}^0 = H_{q,\xi}^0.$$

In general

$$\bigoplus_{\eta \in \pi^{-1}(\xi)} (\ker \sigma_\eta(D^i) \cap \ker \sigma_\eta(D^{i-1})^*) \cong \ker \sigma_\xi(D_0^i)_q \cap \ker \sigma_\xi(D^{i-1})_q^* = H_{q,\xi}^i,$$

as can be seen readily by considering the symbol spectral sequence corresponding to (7.5), since the limit cohomology in this case is isomorphic to the kernel of

$$H_{q,\xi}^i \xrightarrow{\Sigma\zeta_\alpha \otimes \sigma_\eta(D_{1,\alpha}^i)} U \otimes H_{q,\xi}^i$$

If U_q can be chosen to be maximally non-characteristic (i.e., there is no non-characteristic U' properly containing U_q), then every $\xi \in W_q$ is characteristic for D_0, and the characteristic variety consists of at most dim $K_{q,\xi}^0$ sheets, each of dimension $n-q$, lying over ξ.

Let S be a submanifold of Ω of codimension q and let U be an integrable non-characteristic sub-bundle of $T^*(\Omega)$ extending the normal bundle of S.

Definition 9.1: The pair U and D: $\mathcal{E} \to \mathcal{E}$ is called *hyperbolic* if for every $x \in \Omega$, and real cotangent vector ξ *in* T_x^*, $U_x+\xi$ contains only real characteristic vectors.

Remark: There is no restriction on the multiplicity of the characteristics of D. The situation is analogous to that of determined hyperbolic systems. Strong hyperbolicity assumes no symmetry, but does assume simplicity of characteristics; symmetric hyperbolicity assumes symmetry, but puts no restrictions on the multiplicity of characteristics. Our case corresponds to symmetric hyperbolicity with the δ-estimate taking the place of symmetry.

THEOREM 9.2: *Under the hypotheses of Theorem* 8.1, *if* (U, D) *is hyperbolic, then for each real* ξ,

$$\sigma_\xi(D_1^i): H_\xi^i \to H_\xi^i$$

is symmetric.

Proof: By Theorem 8.1, we know $\sigma_\xi(D_1^i)$ is normal, and therefore can be diagonalized by an orthonormal basis of H_ξ^i. It will be symmetric if and only if the eigenvalues on the diagonal are real. The case $i=0$ is already done, for the characteristics in $U+\xi$ are $\xi-\lambda\zeta$

where λ ranges over all eigenvalues of $\sigma_\xi(D_1^0)$. Since all of these characteristics are real, all the eigenvalues are real.

The case $i>1$ is nearly as easy. Let λ be any eigenvalue of $\sigma_\xi(D_1^i)$ and let e be a corresponding eigenvector. Then $(\xi-\lambda\zeta)\otimes e \in T^*\otimes E^{i,0}\subset T^*\otimes E^i$ (where ζ is a unit basis vector of U), and

$$\sigma(D^i)((\xi-\lambda\zeta)\otimes e) = (\sigma(D_1^i)((\xi-\lambda\zeta)\otimes e),\ \sigma(D_0^i)((\xi-\lambda\zeta)\otimes e)).$$

Since $\sigma_\zeta(D_0^i)=0$ and $\sigma_\zeta(D_1^i)=I$,

$$\sigma(D^i)((\xi-\lambda\zeta)\otimes e) = (\sigma_\xi(D_1^i)e-\lambda e,\ \sigma_\xi(D_0^i)e) = 0.$$

We claim further that $e\perp\sigma_{\xi-\lambda\zeta}(D^{i-1})(E^{i-1})$, for the projection of $\sigma_{\xi-\lambda\zeta}(D^{i-1})(E^{i-1})$ onto $E^{i,0}$ is $\sigma_{\xi-\lambda\zeta}(D_0^{i-1})(E^{i-1,0})$, and for any $f\in E^{i-1,0}$,

$$\langle\sigma_{\xi-\lambda\zeta}(D_0^{i-1})f,e\rangle = \langle\sigma_\xi(D_0^{i-1})f,e\rangle = \langle f,\sigma_\xi(D_0^{i-1})^*e\rangle = 0$$

since $e\in H_\xi^i$.

Thus $e\in\ker\sigma_{\xi-\lambda\zeta}(D^i)\cap\ker\sigma_{\xi-\lambda\zeta}(D^{i-1})^*$, so the complex of vector bundle morphisms

$$\cdots\longrightarrow E^{i-1}\xrightarrow{\sigma_{\xi-\lambda\zeta}(D^{i-1})}E^i\xrightarrow{\sigma_{\xi-\lambda\zeta}(D^i)}E^{i+1}\longrightarrow\cdots$$

cannot be exact. Therefore $\xi-\lambda\zeta$ is a characteristic vector in $U+\xi$, and λ must be real. Since λ was an arbitrary eigenvalue of $\sigma_\xi(D_1^i)$, we have that $\sigma_\xi(D_1^i)$ is symmetric.

Section 10. The solution of the Cauchy problem

In this section the result of Theorem 9.2 is used to construct pseudodifferential operators A^i: $\mathcal{E}^{i+1,0}\to\mathcal{E}^{i,0}$, of order zero and tangential to a foliation by non-characteristic surfaces, such that

$$L^i = D_1^i + D_0^{i-1}A^{i-1} + A^iD_0^i\colon\ \mathcal{E}^{i,0}\to\mathcal{E}^{i,0}$$

is a symmetric hyperbolic pseudodifferential operator. This in turn will be used to obtain the local existence and uniqueness theorem for solutions of the Cauchy problem.

In order to guarantee the smoothness of the symbols of the operators A^i, we must make some assumptions about the regularity of the characteristics of the complex D: $\mathcal{E}\to\mathcal{E}$. Recall that a cotangent vector $(x,\zeta)\in T^*(\Omega)$ is in $\mathcal{V}$, the characteristic variety of D: $\mathcal{E}\to\mathcal{E}$, if the complex $\sigma_\zeta(D)$: $\mathcal{E}_x\to\mathcal{E}_x$ fails to be exact. Let $\mathcal{V}_x=\mathcal{V}\cap T_x^*(\Omega)$. Following Guillemin [6], we have

Definition 10.1: A characteristic (x,ζ) in $\mathcal{V}$ is *generic* if

(a) $\mathcal{V}_x$ is non-singular at ζ; and

(b) The dimension of the cohomology of $\sigma_\zeta(D)$: $E_x\to E_x$ is at a local minimum.

Generic characteristics are Cohen-Macaulay points of the characteristic variety, so that

$$E_x^{i-1} \xrightarrow{\sigma_\xi(D)} E_x^i \xrightarrow{\sigma_\xi(D)} E_x^{i+1}$$

is exact if i is greater than the codimension of $\mathcal{V}_x$ in $T_x^*(\Omega)$ [5].

Let U_1 be the normal bundle of a foliation by non-characteristic sub-manifolds of codimension one. Our regularity hypothesis is:

Hypothesis A:

(a) *There is an integrable maximal non-characteristic sub-bundle* $U = U_q \subset T^*(\Omega)$ *(with fiber dimension* q*) which contains* U_1*; and*

(b) *Every characteristic in* $\mathcal{V}_q$ *is generic, where* $\mathcal{V}_q \subset W_q = (U_q)^\perp$ *is the characteristic variety of* $(D_0)_q$: $\mathcal{E}_q^{*,0} \to \mathcal{E}_q^{*,0}$*, the tangential complex corresponding to* U_q.

THEOREM 10.2: *If* D: $\mathcal{E} \to \mathcal{E}$ *is a symbol surjective complex of first order differential operators satisfying the* δ*-estimate, if* U_1 *is an integrable non-characteristic sub-bundle with fiber dimension one, and if* (U_1, D) *is hyperbolic, then for every* $(x, \xi) \in W_1$ *there exist maps* $a^i(x, \xi) \in \operatorname{Hom}(E_x^{i+1,0}, E_x^{i,0})$*, homogeneous of degree zero in* ξ*, such that for each* i*,* x*, and* ξ*,*

$$\sigma_\xi(D_1^i) + \sigma_\xi(D_0^{i-1})\, a^{i-1}(x, \xi) + a^i(x, \xi)\, \sigma_\xi(D_0^i) : E_x^{i,0} \to E_x^{i,0}$$

is symmetric.

Further, if D_0: $\mathcal{E}^{*,0} \to \mathcal{E}^{*,0}$ *satisfies Hypothesis A, the maps* $a^i(x, \xi)$ *may be chosen to be smooth in* (x, ξ) *except where* $\xi = 0$.

Proof: It clearly suffices to find $a^i(x, \xi)$ for $(x, \xi) \in \Sigma(W_1)$, the unit sphere bundle in W_1, and to extend a^i to be homogeneous of degree zero in ξ. Let p: $\Sigma(W_1) \to \Omega$ be the canonical projection, and $p^*E^{i,0}$ the pull-back of $E^{i,0}$ to a bundle over $\Sigma(W_1)$. Let $\sigma(D_1^i)$ denote the smooth section of $\operatorname{Hom}(p^*E^{i,0}, p^*E^{i,0})$ given by

$$\sigma(D_1^i)\,(x, \xi) = \sigma_\xi(D_1^i) : E_x^{i,0} \to E_x^{i,0},$$

and similarly for $\sigma(D_0^i)$.

Fix a point $(x, \xi) \in \Sigma(W_1)$. Let b^i: $E_x^{i,0} \to E_x^{i,0}$ be $\sigma(D_1^i)$ on H_ξ^i and zero on the orthocomplement. By Theorem 9.2, b is symmetric, and clearly $b\sigma_\xi(D_0) = \sigma_\xi(D_0) b = 0$, so $b - \sigma(D_1)$. is a cochain map from the complex $\sigma_\xi(D_0)$: $E_x^{*,0} \to E_x^{*,0}$ to itself. It induces a map on the cohomology which must be zero since its restriction to H_ξ is zero. A simple exercise [cf. 11, p. 205] shows that there is a cochain homotopy a such that $b - \sigma(D_1) = a\sigma(D_0) + \sigma(D_0)a$, which proves the first part of the theorem.

Now we show that Hypothesis A allows us to choose the a's smoothly. The problem is to extend $\sigma(D_1)|_H$, which is a symmetric map on a family of vector spaces, to a bundle map b which commutes with $\sigma(D_0)$, is symmetric, and agrees with $\sigma(D_1)$ on a bundle which includes H. Then $\sigma(D_1)-b$ vanishes on H and so induces the zero map on cohomology. Thus $\sigma(D_1)-b=a\sigma(D_0)+\sigma(D_0)a$. Since the a's may be chosen to be zero on a bundle including H, they may be constructed explicitly by Cramer's rule using the exactness of the $\sigma(D_0)$ complex, and are hence smooth.

By Hypothesis Aa, U is maximally non-characteristic, so every vector $\eta\in W_q$ is characteristic for $(D_0)_q$: $\mathcal{E}_q^{*,0}\to\mathcal{E}_q^{*,0}$, Thus $\mathcal{V}_q=W_q$, and by Hypothesis Ab and the observation that all vectors in $\mathcal{V}_q$ are Cohen-Macaulay, the cohomology of

$$0\to E^{0,0}_{q,x}\xrightarrow{\sigma_\eta(D_0^0)q}E^{1,0}_{q,x}\xrightarrow{\sigma_\eta(D_0^1)q}\cdots\longrightarrow E^{n-q,0}_{q,x}\longrightarrow 0$$

is concentrated in the first position, and the dimension of the cohomology is independent of $(x,\eta)\in W_q$. Thus it is a vector bundle H_q^0 over W_q and pulls back to a bundle $\pi^*H_q^0$ over W_1, where π: $W_1\to W_q$ is orthogonal projection.

We can choose functions $\{\varrho_1, ..., \varrho_q\}$ such that $\{d\varrho_1\}$ is locally a basis of U_1 and $\{d\varrho_1, ..., d\varrho_q\}$ is locally a basis of U_q. The sub-bundle $U'\subset W_1=U_1^\perp$ determined by the restrictions of the functions $\{\varrho_2, ..., \varrho_q\}$ to the leaves of the foliation given by $\varrho_1=$constant is non-characteristic for the tangential complex D_0: $\mathcal{E}^{*,0}\to\mathcal{E}^{*,0}$ corresponding to U_1. There is a spectral sequence decomposition of the symbol complex $\sigma(D_0)$: $p^*E^{*,0}\to p^*E^{*,0}$ corresponding to (7.5), namely

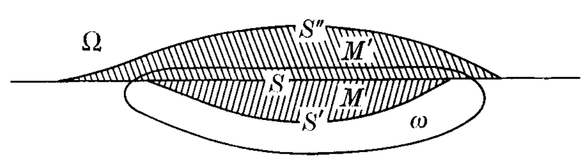
$$\begin{array}{ccccc}
\cdots & & & & \\
\uparrow & & & & \\
\Lambda^2U'\otimes p^*E_q^{0,0} & \longrightarrow & \cdots & & \\
\uparrow & & \uparrow & & \\
U'\otimes p^*E_q^{0,0} & \longrightarrow & U'\otimes p^*E_q^{1,0} & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \\
p^*E_q^{0,0} & \longrightarrow & p^*E_q^{1,0} & \longrightarrow & \cdots
\end{array}$$

Let $D_{1,1}$: $\mathcal{E}_q^{i,0}\to\mathcal{E}_q^{i,0}$ be defined as in § 8. By Theorem 8.4, $\sigma(D_{1,1})|_{\pi^*H_q^0}$ is normal, and hence, by hyperbolicity, symmetric. Thus $I\otimes\sigma(D_{1,1})|_{\Lambda U'\otimes\pi^*H_q^0}$ is symmetric. On $\Lambda U'\otimes\pi^*H_q^0$, $\sigma(D_0)$ is $\sigma(D_0^0)_q\oplus\Sigma_{\alpha=2}^q\zeta^\alpha\wedge\sigma(D_{1,\alpha})$, where $\zeta^\alpha=d\varrho^\alpha/\|d\varrho^\alpha\|$. As remarked in § 8, $\sigma(D_{1,1})$ commutes with $\sigma(D_{1,\alpha})$ on $\pi^*H_q^0$, and $\sigma(D_0^0)_q$ is zero on $\pi^*H_q^0$, so $\sigma(D_{1,1})$ commutes with $\sigma(D_0)$ on $\Lambda U'\otimes\pi^*H_q^0$.

Now $\sigma(D)$ restricted to $\Lambda U'\otimes p^*E_q^{0,0}$ is $I\otimes\sigma(D_0)_q\oplus\Sigma_{\alpha=1}^q\zeta^\alpha\wedge(I\otimes\sigma(D_{1,\alpha}))$, so $\sigma(D_1)$ restricted to $\Lambda U'\otimes\pi^*H_q^0$ is $I\otimes\sigma(D_{1,1})$. Let b: $p^*E^{*,0}\to p^*E^{*,0}$ be a symmetric bundle map commuting with $\sigma(D_0)$ which extends $I\otimes\sigma(D_{1,1})|_{\Lambda U'\otimes\pi^* H_q^0}$. Then $\sigma(D_1)-b$ is zero on $\Lambda U'\otimes\pi^* H_q^0$, hence it is zero on H, and we can find smooth maps a such that

$$\sigma(D_1)-b=a\sigma(D_0)+\sigma(D_0)a.$$

In order to consider local existence and uniqueness of solutions to the Cauchy problem, we must consider lens-shaped domains. Let Ω be an open submanifold of X contained in a coordinate neighborhood, and let S be a non-characteristic submanifold defined by the equation $\varrho=0$. Let Φ' and Φ'' be non-zero smooth functions with compact support on S with supp $\Phi'\subset$ interior supp $\Phi''\subset\subset S\cap\Omega$, and $\Phi'\leqslant 0\leqslant\Phi''$. Let S' and S'' be the submanifolds defined by the equations $\varrho=\Phi'$ and $\varrho=\Phi''$. We assume that S' and S'' are non-characteristic, which is clearly the case if Φ' and Φ'' are sufficiently small in the C^1 sense. Let M be the closure of $\{x\in\Omega\colon \Phi'(x)<\varrho(x)<0\}$.



There exists an open set $\omega\subset\Omega$ such that $\omega\cap S''=\phi$ and $M\subset\subset\omega$. We shall call the compact set M a lens-shaped domain. The definition is more restrictive than necessary, but it helps to avoid technical complications in the following proof.

THEOREM 10.2: *Let Ω, S, S', S'', and M be as above and let U and U' be integrable non-characteristic sub-bundles extending the normal bundles of S and S'. Let $\mathfrak{J}$ be the smallest differental ideal containing all functions which vanish on S. If $D\colon \mathcal{E}\to\mathcal{E}$ is a symbol surjective complex of first order differential operators satisfying the δ-estimate and Hypothesis A, and if (U, D) and (U', D) are hyperbolic, then*

$$0\longrightarrow \mathfrak{J}\mathcal{E}^0(M)\xrightarrow{D^0}\mathfrak{J}\mathcal{E}^1(M)\xrightarrow{D^1}\dots\longrightarrow\mathfrak{J}\mathcal{E}^n(M)\longrightarrow 0$$

is exact, where all sections of bundles over M are taken to be differentiable up to the boundary. That is, the Cauchy problem for $D\colon \mathcal{E}\to\mathcal{E}$ with initial data on S has solutions over M and they are unique in the sense of cohomology.

Proof: Since the question is local in a coordinate neighborhood, we may assume $\Omega\subset R^n$ and that a trivialization has been chosen for the graded vector bundle E. Choose coordinates $(t, x_1, \dots, x_{n-1})$ such that $t=0$ defines S' and $\{dt\}$ is a basis for U'. Let Ω^+ be the subset of Ω for which $t\geqslant 0$, and $\omega^+=\omega\cap\Omega^+$.

Let $H_{(m,s)}(R^n)$ be the Sobolev space with norm

$$\|u\|^2_{(m,s)} = (2\pi)^{-n} \int (1+|\xi|^2)^m (1+|\xi'|^2)^s |\hat{u}(\xi)|^2 d\xi$$

where $\xi = (\tau, \xi_1, \ldots, \xi_{n-1})$ is dual to $(t, x_1, \ldots, x_{n-1})$, and $\xi' = (0, \xi_1, \ldots, \xi_{n-1})$. The trivialization of E enables us to extend this to define $\|u\|^2_{(m,s)}$ for $u \in \mathcal{E}(\omega)$ if supp $u \subset \subset \omega$. Denote by $\mathring{H}_{(m,s)}(\omega, E)$ the completion of the space of compactly supported sections in $\mathcal{E}(\omega)$ with respect to the norm $\|u\|_{(m,s)}$. Denote by ω^- the intersection $\omega \cap \{(t,x): t \leq 0\}$, and by $\mathring{H}(\omega^-, E)$ the corresponding Sobolev space. Denote by $H_{(m,s)}(\omega^+, E)$ the quotient space $\mathring{H}_{(m,s)}(\omega, E)/\mathring{H}_{(m,s)}(\omega^-, E)$; that is, a distribution $u \in (\mathcal{E}(\omega^+))'$ is in $H_{(m,s)}(\omega^+, E)$ iff there exists a distribution $U \in \mathring{H}_{(m,s)}(\omega, E)$ with $u = U$ in ω^+. The norm of u is defined by

$$\|u\|_{(m,s)} = \inf \|U\|_{(m,s)},$$

the infimum being taken over all such U.

There is a nested sequence of compact sets $M = K_0 \subset K_1 \subset \ldots \subset \omega$ such that $K_\nu \subset K_{\nu+1}$ and $\cup K = \omega$. Any pseudodifferential operator P can be modified as in [8] by adding an operator of order $-\infty$ so that if supp $u \subset K_\nu$ then supp $Pu \subset K_{\nu+2}$. By Theorem 10.1 and Hypothesis A, there is a smooth map $a^i(x, \xi)$ which is the map of Theorem 10.1 smoothed in a neighborhood of the zero section of $(U^1)^\perp$. Let A^i be a pseudodifferential operator defined on each surface $\{t = \text{constant}\}$, modified as above so that if supp $u \in K_\nu$, then supp $A^i u \in K_{\nu+2}$, with asymptotic symbol $a^i(x, \xi)$. For each i, if $L^i = D^i_1 + A^i D^i_0 + D^{i-1}_0 A^{i-1}$, then $L^i + L^{i*}$ is a pseudodifferential of order zero defined on each surface $\{t = \text{constant}\}$.

We shall first prove that the short sequence

$$0 \to \mathcal{J}\mathcal{E}^0(M) \xrightarrow{D^0} \mathcal{J}\mathcal{E}^1(M) \tag{10.1}$$

is exact, beginning with the following lemma:

LEMMA 10.4: *Let u be a smooth section in $\mathcal{E}^0(\omega)$ with compact support. Consider the Guillemin decomposition with respect to U'. Under the hypotheses of Theorem* 10.2,

$$|\langle u, (D^0_1 + D^{0*}_1)u\rangle| \leq \text{const}\,\{\|D^0_0 u\| + \|u\|\}\|u\|,$$

where all inner products and norms are in

$$H_{(0,0)}(\omega^+, E) = L_2(\omega^+, E).$$

Proof: As above, if $B = L^0 + L^{0*}$, where $L^0 = D^0_1 + A^0 D^0_0$, B is a tangential pseudodifferential operator of order zero. Thus

$$|\langle u, (D^0_1 + D^{0*}_1)u\rangle| \leq |\langle u, Bu\rangle| + |\langle u, A^0 D^0_0 u\rangle| + \langle u, D^{0*}_0 A^{0*} u\rangle|$$

Since the operators on the right are tangential to the surfaces $\{t=\text{constant}\}$ and u has compact support in the x-directions, integration by parts incurs no boundary terms and yields

$$|\langle u, (D_1^0+D_1^{0*})u\rangle| \leqslant |\langle u, Bu\rangle| + 2|\langle A^{0*}u, D_0^0 u\rangle| \leqslant \text{const}\{\|D_0^0 u\| + \|u\|\}\|u\|,$$

since the zero order operators A and B are bounded in the L_2 sense.

By taking limits, we obtain

COROLLARY 10.5: *If $u \in H_{(1,0)}(\omega^+, E^0)$, then under the hypotheses of Theorem* 10.2,

$$|\langle u, (D_1^0+D_1^{0*})u\rangle| \leqslant \text{const}\{\|D_0^0 u\| + \|u\|\}\|u\|.$$

To prove

$$0 \longrightarrow \mathcal{J}\mathcal{E}^0(M) \xrightarrow{D^0} \mathcal{J}\mathcal{E}^1(M)$$

is exact, let $u \in \mathcal{J}\mathcal{E}^0(M)$ with $D^0 u=0$, so that $u|_S=0$, $D_1^0 u=0$, and $D_0^0 u=0$. Let $\tilde{u}$ be the extension of u to ω^+ given by $\tilde{u}=0$ outside M. Since $u|_S=0$, $\tilde{u}$ is continuous and has L_2 first order derivatives, so $\tilde{u} \in H_{(1,0)}(\omega^+, E^0)$, and $D_1^0\tilde{u}=0$ and $D_0^0\tilde{u}=0$. Now let $\tilde{v}=e^{-Nt}\tilde{u}$. Since $D_1^0-\partial/\partial t$ is an operator tangential to the surfaces $t=\text{constant}$, $D_1^0\tilde{v}+N\tilde{v}=0$. Thus

$$0=\int_{\omega^+}\langle D_1^0\tilde{v}+N\tilde{v}, \tilde{v}\rangle = \int_{\omega^+}\langle \tilde{v}, D_1^{0*}\tilde{v}+N\tilde{v}\rangle + \int_{S'}\|\tilde{u}\|^2,$$

since the boundary terms incurred by integrating by parts are all zero except where $t=0$. Since $D_1^{0*}+N=(D_1^0+D_1^{0*})-(D_1^0+N)+2N$,

$$0=\int_{\omega^+}\langle \tilde{v}, (D_1^0+D_1^{0*})\tilde{v}\rangle + 2N\int_{\omega^+}\|\tilde{v}\|^2 + \int_{S'}\|\tilde{u}\|^2,$$

and by Corollary 10.5.

$$\left|2N\int_{\omega^+}\|\tilde{v}\|^2\right| \leqslant \text{const.}\ \{\|D_0^0\tilde{v}\| + \|\tilde{v}\|\}\|\tilde{v}\|,$$

where the constant on the right is independent of N. Since $\sigma_{dt}(D_0^0)=0$, $D_0^0\tilde{v}=e^{-Nt}D_0^0\tilde{u}=0$, so $|2N\int_{\omega^+}\|\tilde{v}\|^2| \leqslant \text{const}\int_{\omega^+}\|\tilde{v}\|^2$. By choosing N sufficiently large we obtain a contradiction unless $\tilde{v}=\tilde{u}=0$. This proves that (10.1) is exact.

To prove that

$$\mathcal{J}\mathcal{E}^{i-1}(M) \xrightarrow{D^{i-1}} \mathcal{J}\mathcal{E}^i(M) \xrightarrow{D^i} \mathcal{J}\mathcal{E}^{i+1}(M)$$

is exact for $i \geqslant 1$, we begin with the following lemma concerning the existence and uniqueness of solutions of the Cauchy problem for the determined symmetric hyperbolic pseudodifferential operator L^i.

LEMMA 10.6: *If* $u \in \mathcal{E}^{i,0}(\Omega^+)$ *and supp* $u \subset \bar{\omega}^+$, *there exists a unique* $v \in \mathcal{E}^{i,0}(\Omega^+)$ *with supp* $v \subset \bar{\omega}^+$ *such that* $L^i v = u$ *in* Ω^+.

The proof is a rather straightforward copy of the proof of the existence and uniqueness of solutions of the Cauchy problem for a determined symmetric hyperbolic operator. The section v is constructed in $\bigcap_s H_{(0,s)}(\omega^+, E^{i,0})$ which means supp $v \subset \bar{\omega}^+$ and v is infinitely differentiable in the tangential directions. That v is infinitely differentiable in the normal direction follows by solving $L^i v = u$ for the normal derivatives.

Consider now a section $u \in \mathcal{J}\mathcal{E}^i(M)$ with $D^i u = 0$. This section is not in any of the previously mentioned Sobolev spaces, but this is remedied by the following

LEMMA 10.7: *Let* $u \in \mathcal{J}\mathcal{E}^i(M)$ *satisfy* $D^i u = 0$. *There is a section* $v \in \mathcal{J}\mathcal{E}^{i-1}(M)$ *such that* $D^{i-1}v - u$ *vanishes to infinite order on* S. *In fact* v *can be chosen in* $\mathcal{E}^{i-1,0}(M)$ *so that* $v|_S = 0$.

This is a consequence of the solvability of the formal Cauchy problem on non-characteristic surfaces; see [2, Theorem 7.4]. The section v is constructed by determining what its normal derivatives must be on S and extending by the Whitney extension theorem.

Let $\tilde{u}$ be $D^{i-1}v - u$ in M and $\tilde{u} = 0$ on $\omega^+ - M$. Since $D^{i-1}v - u$ vanishes to infinite order on S, $\tilde{u}$ is smooth and supp $u \subset \bar{\omega}^+$. If we can solve $D^{i-1}w = \tilde{u}$, then in M, $D^{i-1}w = D^{i-1}v - u$ so $u = D^{i-1}(v - w)$. Thus as far as solvability of the equation $D^{i-1}v = u$ is concerned, we may assume without loss of generality that u vanishes identically on $\omega^+ - M$.

Write $u = u_0 + u_1$ where $u_0 \in \mathcal{E}^{i,0}(\omega^+)$ and $u_1 \in \mathcal{E}^{i-1,0}(\omega^+)$, $D_0^i u_0 = 0$ and $D_1^i u_0 = D_0^{i-1} u_1$. Solve the Cauchy problem $L^{i-1}v = u_1 + A^{i-1}u_0$, supp $v \subset \bar{\omega}^+$. We claim that

$$D^{i-1}(-A^{i-2}v, v) = (u_1, u_0) = u;$$

that is,

$$D_0^{i-1}v = u_0 \quad \text{and} \quad D_1^{i-1}v + D_0^{i-2}A^{i-2}v = u_1.$$

First we show $D_0^{i-1}v = u_0$. Since $D_0^i D_0^{i-1} = 0$ and $D_1^i D_0^{i-1} = D_0^{i-1} D_1^{i-1}$, we have

$$L^i D_0^{i-1} v = D_0^{i-1} L^{i-1} v = D_0^{i-1}(u_1 + A^{i-1}u_0)$$

Thus $$L^i(D_0^{i-1}v - u_0) = D_0^{i-1}(u_1 + A^{i-1}u_0) - (D_1^i + A^i D_0^i + D_0^{i-1}A^{i-1})u_0 = 0$$

since $D_0^i u_0 = 0$ and $D_0^{i-1}u_1 = D_1^i u_0$. Since supp $v \subset \bar{\omega}^+$, supp $D_0^{i-1}v \subset \bar{\omega}^+$ and since also supp $u_0 \subset \bar{\omega}^+$ we have $D_0^{i-1}v - u_0$ is a solution of the determined Cauchy problem $L^i w = 0$, supp $w \subset \bar{\omega}^+$. By Lemma 10.6, $D_0^{i-1}v = u_0$.

Now we show $D_1^{i-1}v + D_0^{i-2}A^{i-2}v = u_1$. We have

$$D_1^{i-1}v + D_0^{i-2}A^{i-2}v = L^{i-1}v - A^{i-1}D_0^{i-1}v = u_1 + A^{i-1}u^0 - A^{i-1}D_0^{i-1}v = u_1$$

Thus there is a section $w = (-A^{i-2}v, v)$ with support in $\bar{\omega}^+$ such that $D^{i-1}w = u$. It is not clear, and in general not true, that $w \in \mathcal{J}\mathcal{E}^{i-1}(M)$. If $i = 1$, however, w vanishes to infinite order on S and is thus in $\mathcal{J}\mathcal{E}^{i-1}(M)$, for in this case we can consider the lens-shaped domain M' between S and S''. On M', $u=0$, and so by the exactness of (10.1) on M', $w=0$ on M' and since w is smooth, w vanishes to infinite order on S.

If $i \geq 2$, we can solve the equation $Dw' = w$ on M' and extend w' arbitrarily but smoothly to M. Then $D(w-Dw') = u$ on M and $w-Dw'$ vanishes on M' and hence to infinite order on S, and so $w-Dw' \in \mathcal{J}\mathcal{E}^{i-1}(M)$. This completes the proof of Theorem 10.3.

Section 11. Examples

(i) *The de Rham complex.* This was defined in section 2. It is usually treated as an elliptic complex but since it has no characteristics, real or complex, it is also hyperbolic. The δ-estimate is satisfied vacuously since $g_1^0 = 0$, and Hypothesis A is also satisfied. The construction of the pseudodifferential operators A^i is not necessary in this case, since if S is given by the equation $\varrho = 0$, D_1 is $\partial/\partial\varrho$.

The solvability of the Cauchy problem in this case is an analytical expression of the fact that de Rham cohomology is invariant under deformation retracts.

(ii) *The coercive Neumann problem.* A slight generalization of the above situation was considered by W. J. Sweeney in [14]. He showed that the Neumann boundary value problem for an elliptic complex

$$\mathcal{E}^0 \xrightarrow{D^0} \mathcal{E}^1 \xrightarrow{D^1} \mathcal{E}^2$$

is coercive, or elliptic, provided that the normal bundle of the boundary of a compact manifold-with-boundary is hyperbolic. That is, if the boundary is given by $\varrho = 0$, then Sweeney's condition is that there be no characteristics of the form $\xi + \lambda d\varrho$, where ξ is a real covector and $\lambda \in \mathbf{C}$.

The results of this paper apply to show that the cohomology of the boundary complex is isomorphic to the cohomology of the full complex on a neighborhood of the boundary. The hypotheses of the δ-estimate and Hypothesis A are unnecessary in this case, since $H_\xi^i = 0$ for every real ξ; i.e., the boundary complex is elliptic (otherwise, there would be characteristics of the full complex of the form $\xi + \lambda d\varrho$, where λ is an eigenvalue of $\sigma_\xi(D_1^i)$ on H_ξ^i). For details, see [7].

(iii) *Maxwell's equations in an isotropic homogeneous medium.* It is easy to verify that the bundle spanned by $\{dt\}$ is maximal non-characteristic and hyperbolic, and that all small deformations of it are hyperbolic. Thus there are lens-shaped domains on which all our hypotheses are satisfied, except possibly for the δ estimate. In fact, the δ-estimate is also satisfied, as can be verified by computation of the eigenvalues of $\delta\delta^* + \delta^*\delta$ on $T^* \otimes g_1^0$.

The tangential complex is

$$\mathbf{C}^3 \xrightarrow{\text{div}} \mathbf{C}^1 \to 0$$

and the normal operators $\qquad D_1^0 : \mathbf{C}^3 \to \mathbf{C}^3$

and $\qquad D_1^1 : \mathbf{C}^1 \to \mathbf{C}^1$

are given by

$$D_1^0 u = \frac{\partial u}{\partial t} + \frac{i}{\sqrt{\mu\varepsilon}} \text{ curl } u + \frac{\sigma}{\varepsilon} \operatorname{Re} u$$

and

$$D_1^1 v = \frac{\partial v}{\partial t} + \frac{\sigma}{\varepsilon} \operatorname{Re} v.$$

Since these are already symmetric hyperbolic, the construction of the operators A^i is not necessary when the Maxwell equations are given in their usual form.

(iv) *The wave equation.* If the second order equation $\partial^2 u/\partial x^2 - \partial^2 u/\partial y^2 = 0$ is reduced to a firstorder overdetermined system by the introduction of new variables, the result satisfies all our hypotheses.

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