EMBEDDING-OBSTRUCTION FOR SINGULAR ALGEBRAIC VARIETIES IN $\mathbf{P}^N$

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0. Introduction

An old and classical problem in algebraic geometry is the following: Try to classify —in some sense—all the subvarieties of $\mathbf{P}_k^N$, say with $k=\bar{k}$.

If the degree of the variety is small, such a classification can be carried out, see Swinnerton-Dyer [21] and the anonymous note [23]. But as the degree gets larger, the problem rapidly becomes much more difficult.

Recently there has been a renewed interest in this area. We will not attempt to give a complete account of this, but only mention some of the papers which are closest to our own interests. For further literature, the reader may for instance consult the references in the papers listed here: R. Hartshorne [4] and [5], E. Lluis [13] and [14], Peters and Simonis [16], J. Roberts [17] and [18], L. Szpiro [21], as well as our papers [8], [9] and [10].

In [10] we have proved the following result: Let k be an algebraically closed field (it suffices to assume that k is infinite), and let X be a non-singular, projective k-variety, embedded in $\mathbf{P}_k^N$. Let $c(X)=c((\Omega^1_{X/k})^*)$ be the total Chern-class of X, and let

$$s(X)=\frac{1}{c(X)}=1+s_1(X)+\ldots+s_n(X)$$

($n=\dim(X)$) be the (total) Segre-class of X (this name was introduced in [11]). With respect to the given projective embedding of X it then makes sense to talk about the degree of $s_l(X)$, put

$$d_l=\deg(s_l(X)),\quad l=0,1,\ldots,n.$$

In particular $d_0=\deg(X)=d$. Now let

$$\beta_j=\sum_{l=0}^{j-n}\binom{j+1}{j-n-l}d_l-d_0^2,$$

for $n\leqslant j\leqslant 2n$, and $\beta_j=0$ for $j>2n$. Then we show that ([9], [10] Theorem 4.2):

THEOREM. *Let $m\geqslant n$. Then the non-singular embedded variety X can be embedded into $\mathbf{P}_k^m$ via a projection from $\mathbf{P}_k^N$ if and only if $\beta_j=0$ for all $j\geqslant m$.*

This result was obtained independently by Peters and Simonis in [16]. See also [8].

Throughout this work we assume $k=\bar{k}$.

The purpose of the present paper is to *generalize this theorem to the case when X has singularities.*

One should note that this is necessary in order to treat the affine case of the situation described in the above theorem: In fact, let Y be a non-singular closed subscheme of $\mathbf{A}_k^N$. *What is the necessary and sufficient condition that Y may be embedded into $\mathbf{A}_k^m$ by a projection?* In section 7 we give the answer to this question for projections from a center which does not contain any points at infinity of Y. Here the main point is that even when Y is non-singular, it may have singularities at infinity, and this gives a contribution to the embedding obstruction which is not covered by the theory in [10].

In trying to formulate a generalization of the above theorem, one is confronted with the obstacle that no theory of Chern-classes for singular varieties is available in the litterature, except recently for complex algebraic varieties by the work of MacPherson [15]. Moreover, Baum, Fulton and MacPherson have recently made some remarkable advances in this direction with their Riemann-Roch theorem for singular varieties, [2].

The techniques developed in their paper include a theory of Chern-classes for locally free sheaves on a singular variety. However, this does not suffice as $\Omega^1_{X/k}$ is not locally free for a singular variety X.

On the other hand the full theory of Chern-classes is not really needed for our purpose. In fact, what is needed here is a generalization of the degrees $d_1, \ldots, d_n$ of the Segre-classes $s_1(X), \ldots, s_n(X)$ for an embedded (singular) variety $X \hookrightarrow \mathbf{P}_k^N$.

It turns out that in the singular case there are two natural generalizations of the projective invariant d_l, namely the invariants p_l and $q_{\alpha,l}$ introduced in section 1. In the non-singular case we get $p_l = q_{\alpha,l} = d_l$, see section 2.

Both of these sets of invariants are used for the embedding-obstruction in the singular case. In fact, let $i: X \hookrightarrow \mathbf{P}_k^N$ be a closed embedding of the variety X into $\mathbf{P}_k^N$. Let

$$\boldsymbol{\gamma}_l = (\gamma_l, \bar{\gamma}_l),$$

where

$$\gamma_l = d^2 - \sum_{j=0}^{l-n} \binom{l+1}{l-n-j} p_j,$$

for $l = n, \ldots, 2n$, and

$$\gamma_l = 0$$

for $l \geqslant 2n+1$. Moreover,

$$\bar{\gamma}_{\alpha,l} = \sum_{j=0}^{l+n-\varrho_\alpha} \binom{l+1}{l+n-\varrho_\alpha-j} q_{\alpha,j},$$

for $l = \varrho_\alpha - n, \ldots, \varrho_\alpha$. For $n \leqslant l < \varrho_\alpha - n$ we put $\bar{\gamma}_{\alpha,l} = 1$, and for $l > \varrho_\alpha$, we let $\bar{\gamma}_{\alpha,l} = 0$. Here $1 \leqslant \alpha \leqslant r = r(X, i)$, where $\mathbf{P}(\Omega^1_{X/k}) = P_1 \cup \ldots \cup P_r$ is the decomposition of $\mathbf{P}(\Omega^1_{X/k})$ into irreducible components (as a set) and $\varrho_\alpha = \dim(P_\alpha)$. Se section 6. Finally, we put

$$\bar{\gamma}_l = (\bar{\gamma}_{1,l}, \ldots, \bar{\gamma}_{r,l})$$

and

$$\boldsymbol{\gamma}_l = (\gamma_l, \bar{\gamma}_l).$$

We then prove (Theorem 6.3) that X may be embedded into $\mathbf{P}_k^m$ via a projection from $\mathbf{P}_k^N$ if and only if the entries of

$$\boldsymbol{\gamma}_m, \boldsymbol{\gamma}_{m+1}, \ldots$$

are all zero.

In section 7 we prove an affine analogue of Theorem 6.3, namely Theorem 7.2. For an affine variety Y one may also obtain embeddings by projecting from a center which contains points at infinity of Y. We hope to return to this later.

We hope that the result of this paper indicate that the invariants p_l and $q_{\alpha,l}$ are worth a closer study. It would, for instance, be interesting to try to relate them to the invariants studied by Horonaka in [6] and [7], or for the complex case, to look for a connection with MacPhersons theory of Chern-classes in [15]. We hope to return to this later.

Chapter I. Projective invariants for projective varieties with singularities

1. Definition of the invariants

Let $i: X \hookrightarrow \mathbf{P}_k^N$ be a projective variety X, embedded in $\mathbf{P}_k^N$ by the embedding i. We assume that X is reduced and irreducible.

Then

$$\begin{tikzcd} \mathbf{P}(\Omega^1_{X/k}) \arrow[r, hook, "\alpha"] \arrow[d, "g_X"] & \mathbf{P}(\Omega^1_{\mathbf{P}^N/k}) = T \arrow[d, "f"] \\ X \arrow[r, hook, "i"] & \mathbf{P}_k^N \end{tikzcd} \tag{1.1}$$

where α is a closed embedding and

$$\mathbf{P}(\Omega^1_{X/k}) = \text{Proj}\ (S_{O_X}(\Omega^1_{X/k})).$$

Moreover, δ_X: $X \hookrightarrow X \times_k X = Y$ denotes the diagonal embedding, π_X: $\widetilde{X \times_k X} \to X \times_k X$ the blowing-up with center in the diagonal and $\mathbf{T}(X) = \pi_X^{-1}(\Delta_X)$ the exceptional divisor. Then

$$\mathbf{T}(X) = \text{Proj}\ (Gr_I(O_Y)),$$

where I is the Ideal on Y which defines Δ_X. Thus as

$$Gr_I(O_Y) = O_Y/I \oplus I/I^2 \oplus I^2/I^3 \oplus \ldots$$

is a quotient of $S_{O_X}(\Omega^1_{X/k})$, we get

$$\begin{tikzcd} \mathbf{T}(X) \arrow[r, hook, "\beta"] \arrow[dr, "h"'] & \mathbf{P}(\Omega^1_{X/k}) \arrow[d, "g"] \\ & X \end{tikzcd} \tag{1.2}$$

$$A(T) = \mathbf{Z}[t, \xi],$$

where

$$\xi^N + (N+1)t\xi^{N-1} + \ldots + \binom{N+1}{i} t^i \xi^{N-i} + \ldots + (N+1)t^N = 0.$$

f_*: $A(T) \to A(\mathbf{P}_k^N) = \mathbf{Z}[t]$ is of degree $-N+1$, and $\mathbf{T}(X)$ is of codimension $2N-1-(2n-1) = 2N-2n$ in T, so for all r

$$f_*(\text{cl}_T(\mathbf{T}(X))\,\xi^r) \in A^{2N-2n-N+1+r}(\mathbf{P}_k^N),$$

i.e.

$$f_*(\text{cl}_T(\mathbf{T}(X))\,\xi^r) = e_{N-2n+1+r}\, t^{N-2n+1+r}.$$

If $r \geq 2n$, let

$$e_{N-2n+1+r} = 0.$$

One now has the following

LEMMA 1.3. $\mathbf{T}(X)$ *is irreducible.*

Proof. Let U be the open set of all regular points in X. Then $h^{-1}(U)$ is an open subset of $\mathbf{T}(X)$, put $Y=\overline{h^{-1}(U)}$. $h^{-1}(U)$ is a $\mathbf{P}^{n-1}$-bundle over U, so $h^{-1}(U)$ and hence Y are of dimension $2n-1$.

Further, for all points $x \in X$,

$$\mathbf{T}(X)_x = \operatorname{Proj}\,(k \oplus m_{X,x}/m_{X,x}^2 \oplus m_{X,x}^2/m_{X,x}^3 \oplus \dots)$$

is the projectivized tangent cone of X at x. Hence $\dim\,(\mathbf{T}(X)_x)=n-1$, from which it follows that $\dim\,(\mathbf{T}(X))=2n-1=\dim\,(Y)$, cf. EGA IV Corollàire (5.6.6).

Suppose now that Z is another irreducible component of $\mathbf{T}(X)$. Since $h(Y)=X$, we must have $h(Z)\neq X$, and since for all $x\in h(Z)$ $\dim Z_x \leqslant n-1$, we get as above that $\dim\,(Z)<2n-1$.

At the beginning of section 4 it is noted that $\mathbf{T}(X)$ may be embedded into a variety **Bl** as the intersection of two subvarieties, one of codimension 1 and the other of dimension $2n$. Hence $\dim\,(Z)\geqslant 2n-1$, a contradiction.

We have (cf. [10], section 9)

PROPOSITION 1.4. $e_{N-2n+1+r}=0$ *unless* $n-1\leqslant r\leqslant 2n-1$.

Proof. It suffices to show that $e_{N-2n+1+r}=0$ for $r\leqslant n-2$. Let $\Delta_r=\Sigma_i n_i \Delta_{r,i}$ be a cycle such that

$$\operatorname{cl}_T(\Delta_r)=\operatorname{cl}_T(\mathbf{T}(X))\xi^r.$$

Then

$$\dim \Delta_{r,i}=2n-1-r.$$

Now $f(\Delta_{r,i})\subseteq X$, so $f_*(\operatorname{cl}_T(\Delta_r))=0$ if $2n-1-r\geqslant n+1$, i.e. if $r\leqslant 2n-1-n-1=n-2$.

By Proposition 1.4 it is reasonable to make the following definition:

Definition 1.5. $p_s(X,i)=e_{N-n+s}$ for $s=0, \dots, n$.

Let $P_1, \dots, P_r$, $r=r(X,i)$ be the irreducible components of $\mathbf{P}(\Omega^1_{X/k})$, and let $\varrho_1, \dots, \varrho_r$ be their dimensions. We have

$$f_*(\operatorname{cl}_T(P_\alpha)\,\xi^r)=\tilde{e}_{\alpha,N-\varrho_\alpha+r}\,t^{N-\varrho_\alpha+r}$$

where $\tilde{e}_{\alpha,j}\in\mathbf{Z}$. Put $\tilde{e}_{\alpha,j}=0$ for $j\geqslant N+1$.

PROPOSITION 1.6. $\tilde{e}_{\alpha,N-\varrho_\alpha+r}=0$ *unless* $n\geqslant\varrho_\alpha-r\geqslant 0$, *i.e. unless* $\varrho_\alpha\geqslant r\geqslant\varrho_\alpha-n$.

Proof. For simplicity, we delete the subscript α. By definition $\bar{e}_{N-\varrho+r}=0$ if $N-\varrho+r\geqslant N+1$, i.e. if $\varrho-r\leqslant -1$. Moreover, as in the proof of Proposition 1.4, $f(P)\subseteq X$ implies that $\bar{e}_{N-\varrho+r}=0$ if $\varrho-r\geqslant n+1$.

We now make the following

Definition 1.7. $q_{\alpha,s}(X,i)=\bar{e}_{\alpha,N-n+s}$ for $\alpha=1,\ldots,r(X,i)$ and $s=0,\ldots,n$.

If $r=1$, we put $q_{1,s}=q_s$.

2. The non-singular case. Relation to the Chern- and the Segre-classes

The purpose of this section is to give an interpretation of the invariants p, q introduced in the previous section in the case that X is smooth over k.

If $i\colon X\hookrightarrow \mathbf{P}_k^N$ is an embedding of the n-dimensional smooth variety X into $\mathbf{P}_k^N$, and if $\alpha\in A^j(X)$, recall that deg (α) is given by

$$i_*(\alpha)=\deg(\alpha)\,t^{j+N-n}$$

where t is the class of a hyperplane in $\mathbf{P}_k^N$. Let

$$c(X)=1+c_1(X)+\ldots+c_n(X),$$

be the total Chern-class of X, and let

$$s(X)=\frac{1}{c(X)}=1+s_1(X)+\ldots+s_n(X)$$

be the total Segre-class of X. We put $s_0(X)=c_0(X)=1$. Then we have the

PROPOSITION 2.1. *If X is a smooth projective variety, embedded in $\mathbf{P}_k^N$ by the embedding $i\colon X\hookrightarrow\mathbf{P}_k^N$, then*

$$p_l(X,i)=q_l(X,i)=\deg(s_l(X))$$

for $l=0,\ldots,n$.

Proof. We have

$$\mathbf{T}(X)=\mathbf{P}(\Omega^1_{X/k}),$$

and thus $p_l(X,i)=q_l(X,i)$ for $l=0,\ldots,n$. With notation as in (1.1) we now have

$$\alpha^*(\xi)=\xi_X,$$

where $\xi_X=\xi_{\Omega^1_{X/k}}$, see [10], (3.3). Thus

$$\mathrm{cl}_T(\mathbf{P}(\Omega^1_{X/k}))\,\xi^r = \alpha_*(\xi^r_X).$$

and hence

$$f_*(\mathrm{cl}_T(\mathbf{P}(\Omega^1_{X/k}))\,\xi^r) = f_*\,\alpha_*(\xi^\alpha_X) = i_*\,g_*(\xi^\alpha_X).$$

By Lemma 11.1.3 in [10] (where the Segre-classes are referred to as the inverse Chern-classes) one now obtains

$$g_*(\xi^r_X) = \begin{cases} 0 & \text{for} \quad r \leqslant n-2 \\ s_{r-(n-1)}(X) & \text{for} \quad r \geqslant n-1. \end{cases}$$

This gives

$$i_*\,g_*(\xi^r_X) = \begin{cases} 0 & \text{for} \quad r \leqslant n-2 \\ \deg\,(s_{r-(n-1)}(X))\,t^{r-(n-1)+N-n} & \text{for} \quad r \geqslant n-1. \end{cases}$$

Hence

$$\deg\,(s_{r-(n-1)}(X)) = e_{N-2n+1+r} = p_{-n+1+r}(X,\,i),$$

and the claim follows.

In particular we thus obtain

$$\begin{aligned} s_1 &= -c_1(X) \\ s_2 &= c_1(X)^2 - c_2(X) \\ s_3 &= -c_1(X)^3 + 2c_2(X)\,c_1(X) - c_3(X) \\ &\ldots\ldots\ldots \end{aligned}$$

As in [10] we let

$$d_i = \deg\,(s_i).$$

3. Elementary properties

The invariants p_s and $q_{\alpha,s}$ defined in section 1 depend on the scheme X as well as on the embedding i. In what follows we shall need some very simple observations on how these invariants change when the embedding i is changed in a certain manner.

Let

$$l : \mathbf{P}^N_k \hookrightarrow \mathbf{P}^M_k$$

be a projective embedding. l may for instance identify $\mathbf{P}^N_k$ with a linear subspace of $\mathbf{P}^M_k$ in which case we say that l is linear. Or l may represent some twisted embedding of $\mathbf{P}^N_k$ into $\mathbf{P}^M_k$, like the Veronese-embedding of $\mathbf{P}^2_k$ into $\mathbf{P}^5_k$.

One has the group-homomorphism

$$l_* : A(\mathbf{P}^N_k) = \mathbf{Z}[t] \to A(\mathbf{P}^M_k) = \mathbf{Z}[T],$$

where $t^{N+1} = 0$, $T^{M+1} = 0$. Put

$$l_*(t^\alpha) = \delta_\alpha(l)\,T^{\alpha+M-N}.$$

In particular $\delta_\alpha(l)=1$ when l is linear.

Now let i: $X \hookrightarrow \mathbf{P}_k^N$ be a projective embedding of the scheme X. We then have the following

PROPOSITION 3.1. *If $n=\dim(X)$, then with notations as above*

$$p_s(X, l \circ i) = \delta_{N-n+s}(l)p_s(X, i), \tag{3.1.1}$$

$$q_{\alpha,s}(X, l \circ i) = \delta_{N-n+s}(l)q_{\alpha,s}(X, i). \tag{3.1.2}$$

Proof. We get the diagram

$$\begin{tikzcd}
\mathbf{T}(X) \arrow[r, hook] \arrow[ddr, "h"] & \mathbf{P}(\Omega^1_{X/k}) \arrow[r, hook, "j"] \arrow[dd, "g"] & \mathbf{P}(\Omega^1_{\mathbf{P}_k^N/k}) \arrow[r, hook, "m"] \arrow[d, equal] & \mathbf{P}(\Omega^1_{\mathbf{P}_k^M/k}) \arrow[d, equal] \\
 & & T \arrow[d, "f"] & U \arrow[d, "F"] \\
 & X \arrow[r, hook, "i"] & \mathbf{P}_k^N \arrow[r, hook, "l"] & \mathbf{P}_k^M
\end{tikzcd}$$

Since now (cf. [10], (3.3))

$$\xi_T^r = m^*(\xi_U^r),$$

we obtain the following for any cycle Z on T

$$m_*(\mathrm{cl}_T(Z)\,\xi_T^r) = m_*(\mathrm{cl}_T(Z)\,m^*(\xi_U^r)) = m_*(\mathrm{cl}_T(Z))\,\xi_U^r = \mathrm{cl}_U(Z)\,\xi_U^r.$$

by means of the Projection Formula. This gives

$$F_*(\mathrm{cl}_U(\mathbf{T}(X))\,\xi_U^r) = F_*(m_*(\mathrm{cl}_T\,\mathbf{T}(X))\,(\xi_T^r)) = l_*(f_*(\mathrm{cl}_T\,\mathbf{T}(X))\,\xi_T^r)).$$

Hence with the notation introduced in section 1 we get the following (e corresponds to $\mathbf{P}_k^N$, while E corresponds to $\mathbf{P}_k^M$):

$$E_{M-2n+1+r}T^{M-2n+1+r} = l_*(e_{N-2n+1+r}t^{N-2n+1+r}) = e_{N-2n+1+r}\,\delta_{N-2n+1+r}(l)\,T^{M-2n+1+r}.$$

(3.1.1) is immediate from this.

(3.1.2) is shown similarly.

The following corollary is immediate

COROLLARY 3.2. *If l is a linear embedding, then p_s and $q_{\alpha,s}$ are unchanged.*

Chapter II. Secant bundle and secant schemes for singular varieties

4. Secant bundle and the secant variety

First, recall the following from [10](1): Let $\pi\colon \mathbf{Bl} \to \mathbf{P}_k^N \times_k \mathbf{P}_k^N$ denote the blowing-up of $\mathbf{P}_k^N \times_k \mathbf{P}_k^N$ with center in the diagonal Δ. As before, let

$$f: T = \mathbf{P}(\Omega^1_{\mathbf{P}_k^N/k}) \to \mathbf{P}_k^N$$

be the projectivized cotangent-bundle of $\mathbf{P}_k^N$. Then $\pi^{-1}(\Delta) = T$, and we have the commutative diagram

$$\begin{tikzcd} T \arrow[r, hook] \arrow[d, "f"] & \mathbf{Bl} \arrow[d, "\pi"] \\ \mathbf{P}_k^N \arrow[r, hook, "\delta"] & \mathbf{P}_k^N \times_k \mathbf{P}_k^N \end{tikzcd}$$

where δ is the diagonal embedding. Note that $\mathbf{T}(X) = T \cap \widetilde{X \times_k X}$. Moreover, we have (see [10], Proposition 8.6):

PROPOSITION 4.1. *There is a projective morphism $\lambda = \mathbf{Bl} \to T$, which is a $\mathbf{P}^1$-bundle, such that the following diagram is commutative:*

$$\begin{tikzcd} \mathbf{Bl} \arrow[r, "\lambda"] \arrow[d, "\pi"] & T \arrow[d, "f"] \\ \mathbf{P}_k^N \times_k \mathbf{P}_k^N \arrow[r, "pr_2"] & \mathbf{P}_k^N \end{tikzcd} \tag{4.1.1}$$

Moreover, λ induces the identity on $T = \pi^{-1}(\Delta)$, and if x is a k-point of $\mathbf{P}_k^N$, then the fiber over x

$$\begin{tikzcd} \mathbf{Bl}_x \arrow[d, "\pi_x"] \arrow[dr, "\lambda_x"] & \\ \mathbf{P}_k^N & T_x = \mathbf{P}_k^{N-1} \end{tikzcd} \tag{4.1.2}$$

is the blowing-up of $\mathbf{P}_k^N$ with center x and the corresponding $\mathbf{P}^1$-bundle λ_x where

$$y \mapsto \pi_x(\lambda_x^{-1}(y))$$

establishes a $1-1$ correspondence between k-points y in $\mathbf{P}_k^{N-1}$ and lines in $\mathbf{P}_k^N$ passing through x.

As in [10] (Definition 8.10) we now put

$$\mathbf{Sb}(X, i) = \lambda(\widetilde{X \times_k X}),$$

(1) Similar techniques may be found in [1] and [12].

where $\widetilde{X \times_k X}$ denotes the strict transform of $X \times_k X$ under the blowing-up π. The induced morphism f_X: $\mathbf{Sb}(X, i) \to X$ is referred to as the *secant bundle* of the embedded variety i: $X \hookrightarrow \mathbf{P}_k^N$.(¹) As alway i is deleted when no confusion is possible. In [10] the following is proved (Proposition 8.9):

PROPOSITION 4.2. $\mathrm{pr}_1\,(\pi(\lambda^{-1}(\mathbf{Sb}(X))))$ *is the closure of the union of all lines in* $\mathbf{P}_k^N$ *with two or more points in common with* X.

The subvariety $\mathrm{Sec}\,(X, i) = \mathrm{pr}_1\,(\pi(\lambda^{-1}(\mathbf{Sb}(X))))$ is referred to as *the secant variety of* X *in* $\mathbf{P}_k^N$. Also as in [10] we let

$$S(X, i) = \pi(\lambda^{-1}(\mathbf{Sb}(X))). \tag{4.3}$$

The arguments used to prove Proposition 8.11 and its corollaries in [10] give the following proposition and corollaries.

PROPOSITION 4.4. *If the variety* X *has a non-singular point* x *such that*

$$\mathbf{T}(X)_x = \mathbf{Sb}(X)_x,$$

then X *is a linear subspace of* $\mathbf{P}_k^N$.

COROLLARY 4.4.1. $\dim\,(\mathbf{Sb}(X)) \leqslant 2 \dim\,(X)$, *with strict inequality if and only if* X *is a linear subspace of* $\mathbf{P}_k^N$.

COROLLARY 4.4.2. $\dim\,(\mathrm{Sec}\,(X)) \leqslant 2 \dim\,(X) + 1$.

COROLLARY 4.4.3. *If* X *is not a linear subspace of* $\mathbf{P}_k^N$, *then*

$$\dim\,(S(X, i)) = 2 \dim\,(X) + 1.$$

5. The Zariski secant scheme

For all k-points $x \in X$, the *tangent cone* $\bar{C}_{X,x}$ of X at x is contained in $\mathrm{Sec}\,(X)$. In fact, this follows by Proposition 8.5 in [10]. But if x is a singular point, then $\mathrm{Sec}\,(X)$ need not contain the *Zariski tangent space* $Z_{X,x}$ of X at x.

Recall that if $I(X) = (F_1, \ldots, F_m)k[X_0, \ldots, X_N]$, where $F_1, \ldots, F_m$ are homogeneous polynomials, and if $x = (a_0 : \ldots : a_N)$, then $Z_{X,x}$ is the linear subspace of $\mathbf{P}_k^N$ defined by

$$\sum_{i=0}^{N} X_i \frac{\partial F_j}{\partial X_i}(a_0, \ldots, a_N) = 0, \quad j = 1, \ldots, m. \tag{5.1}$$

If x is a regular point, $\bar{C}_{X,x} = Z_{X,x}$.

(¹) For related notions, see [19].

With notation as in (1.1) and in Proposition 4.1, we now put

$$Z(X, i) = \pi(\lambda^{-1}(\mathbf{P}(\Omega^1_{X/k}))), \tag{5.2}$$

which is a closed subset of $\mathbf{P}^N_k \times_k \mathbf{P}^N_k$. Finally the subscheme of $\mathbf{P}^N_k$

$$\operatorname{Zarsec}(X, i) = \mathrm{pr}_1 (Z(X, i) \cup S(X, i)),$$

where the union and the image are scheme-theoretic, is referred to as the *Zariski-secant scheme* of the embedded variety X.

We need the following

PROPOSITION 5.3. *$\mathrm{pr}_1 (Z(X, i))$ is the union of all Zariski tangent spaces $Z_{X,x}$ as x runs through all k-points of X.*

Proof. Let y be a k-point of $\mathrm{pr}_1 (Z(X, i))$. Then $y = \mathrm{pr}_1 (z)$ where z is a k-point of $Z(X, i)$. Now $x = \mathrm{pr}_2 (z) \in X$, so

$$y \in \pi_x(\lambda_x^{-1}(\mathbf{P}(\Omega^1_{X/k})_x)). \tag{5.3.1}$$

Thus to complete the proof, it suffices to show the following lemma:

LEMMA 5.3.2.

$$Z_{X,x} = \pi_x(\lambda_x^{-1}(\mathbf{P}(\Omega^1_{X/k})_x)).$$

Proof. We may assume that $x = (1 : 0 : \ldots : 0)$. Identifying $D_+(X_0)$ with $\mathbf{A}^N_k$, we put $U = X \cap \mathbf{A}^N_k$. Let

$$Z = Z_{X,x} \cap \mathbf{A}^N_k$$

be the Zariski tangent space of U at $x = (0, \ldots, 0) \in \mathbf{A}^N_k$. With notation as in (1.1) we now have

$$g_X^{-1}(U) = \mathbf{P}(\Omega^1_{U/k}),$$

so in particular

$$\mathbf{P}(\Omega^1_{X/k})_x = \mathbf{P}(\Omega^1_{U/k})_x \subset T_x = \mathbf{P}^{N-1}_k$$

is the projectivization of Z. Hence the claim follows from (4.1.2).

As in [10], let $\sigma = \xi + t \in A(T)$ (cf. section 1). Then $A(T)$ is free over $\mathbf{Z}[t] = A(\mathbf{P}^N_k)$ on the base

$$1, \sigma, \ldots, \sigma^{N-1}.$$

Moreover, we have the

LEMMA 5.4. *$\pi_*(\lambda^*(\sigma^\alpha t^\beta)) = s^\alpha t^\beta$ for all $0 \leqslant \alpha < N$ and for all β.*

This lemma is proved in [10], as Proposition 9.4.

Finally we note that

PROPOSITION 5.5. (i) $Z(X,i)=\bigcup_{\alpha=1}^{r} Z_\alpha(X,i)$ *where* $r=r(X,i)$ *and*

$$Z_\alpha(X,i)=\pi(\lambda^{-1}(P_\alpha)),$$

(ii) *Moreover,*

$$\mathrm{cl}_{\mathbf{P}_k^N\times_k\mathbf{P}_k^N}(Z_\alpha(X,i))=\sum_{j=M_1}^{M_2}\tilde{\beta}_{\alpha,j}t^j s^{2N-(\varrho_\alpha+1)-j}, \tag{5.5.1}$$

where $\tilde{\beta}_{\alpha,j}\in\mathbf{Z}$ *and* $M_1=\mathrm{Max}\,\{0, N-\varrho_\alpha\}$, $M_2=\mathrm{Min}\,\{N, 2N-\varrho_\alpha-1\}$.

Proof. (i) is immediate by Proposition 5.3, since

$$\mathbf{P}(\Omega^1_{X/k})=P_1\cup\ldots\cup P_r,$$

(ii): We delete the subscript α, and have

$$\mathrm{cl}_T(\mathbf{P}(\Omega^1_{X/k}))=\sum A_{uv}t^u\sigma^v,$$

where $0\leqslant v\leqslant N-1$, $0\leqslant u\leqslant N$, $u+v=2N-1-\varrho$. This sum with s instead of σ is just that of the right side of (5.5.1), which now follows by Lemma 5.4.

(5.5.1) implies that

$$\dim\,(Z(X,i))=\varrho+1,$$

since not all $\tilde{\beta}$'s are zero.

Chapter III. Embedding obstruction for singular projective varieties

6. Definition of the obstruction. Main results

Let $i: X\hookrightarrow\mathbf{P}_k^N$ be an embedded projective variety with singularities. Let π denote the blowing-up of the diagonal in $\mathbf{P}_k^N\times_k\mathbf{P}_k^N$, and let $\widetilde{X\times_k X}$ be the strict transform of $X\times_k X$:

$$\begin{array}{ccc}\widetilde{X\times_k X} & \hookrightarrow & \mathbf{Bl}\\ \downarrow & & \downarrow{\scriptstyle\pi}\\ X\times_k X & \hookrightarrow & \mathbf{P}_k^N\times_k\mathbf{P}_k^N\end{array}$$

With notations as in section 1, we have the (See Lemma 1.3.)

LEMMA 6.1. $\mathrm{cl}_{\mathbf{Bl}}\,(\widetilde{X\times_k X})\cdot\mathrm{cl}_{\mathbf{Bl}}\,(E)=\mathrm{cl}_{\mathbf{Bl}}\,(\mathbf{T}(X))$.

Proof. First, we get that

$$\mathrm{cl}_{\mathbf{Bl}}\,(\widetilde{X\times_k X})\cdot\mathrm{cl}_{\mathbf{Bl}}\,(E)=\nu\,\mathrm{cl}_{\mathbf{Bl}}\,(\mathbf{T}(X)), \tag{6.1.1}$$

where $\nu = e_m(O)$ is the multiplicity of the local ring O of $\widetilde{X \times_k X}$ at the generic point of $\mathbf{T}(X)$. In fact, this is shown in the same way as Lemma 11.1.1 in [10].

To prove that $\nu = 1$, let U be the open subset of X consisting of all regular points. We then obtain the following diagram

$$\begin{tikzcd} \widetilde{U \times_k U} \arrow[r, hook, "\beta"] \arrow[d, "\pi_U"] & \widetilde{X \times_k X} \arrow[d, "\pi_X"] \\ U \times_k U \arrow[r, hook, "\alpha"] & X \times_k X \end{tikzcd}$$

where π_U, π_X are blowing-up with center at the diagonals, and α, β are the canonical open embeddings. Now the generic point of $\mathbf{T}(X)$ is in $\widetilde{U \times_k U}$, which is non-singular. Thus $\nu = 1$.

Now let $n = \dim(X)$ and $d = \deg(X)$ be the degree of X in $\mathbf{P}_k^N$ by the embedding i. For $n \leqslant l \leqslant 2n$ we define

$$\gamma_l = d^2 - \sum_{j=0}^{l-n} \binom{l+1}{l-n-j} p_j,$$

and for $l > 2n$ we let $\gamma_l = 0$.

Moreover, for $\varrho_\alpha - n \leqslant l \leqslant \varrho_\alpha$ define

$$\bar{\gamma}_{\alpha,} = \sum \binom{l+1}{l+n-\varrho_\alpha - j} q_{\alpha,j},$$

where the sum is taken over all j from 0 to $l+n-\varrho_\alpha$. For $l > \varrho_\alpha$ put $\bar{\gamma}_{\alpha,l} = 0$, while for $n \leqslant l < \varrho_\alpha - n$ we let $\bar{\gamma}_{\alpha,l} = 1$. Put

$$\bar{\gamma}_l = (\bar{\gamma}_{1,l}, \ldots, \bar{\gamma}_{r,l}). \tag{6.2}$$

Whenever $r(X, i) = 1$, we write $\bar{\gamma}_l = \bar{\gamma}_{1,l}$.

Finally we make the following definition:

Definition 6.3. $\boldsymbol{\gamma}_l = (\gamma_l, \bar{\gamma}_l)$ is referred to as the l'th part of the embedding-obstruction of the embedded variety X. $\Gamma_l = (\gamma_l, \gamma_{l+1}, \ldots)$ is referred to as the total l'th. embedding obstruction. $\boldsymbol{\gamma}_l$ or Γ_l are said to vanish if all their entries are zero. We then write $\boldsymbol{\gamma}_l = 0$ or $\Gamma_l = 0$.

The main aim of this section is to prove the following

THEOREM 6.4. *Let $m \geqslant n = \dim(X)$. Then X may be embedded into $\mathbf{P}_k^m$ via a projection from $\mathbf{P}_k^N$ if and only if $\Gamma_m = 0$.*

The proof rests on the following theorem, which is due to E. Lluis (see [13]):

Theorem 6.5. *X may be embedded into* $\mathbf{P}_k^m$ *via a projection from* $\mathbf{P}_k^N$ *if and only if*

$$\dim(\operatorname{Zarsec}(X, i)) \leq m. \tag{6.5.1}$$

Proof. (See also the proof of Proposition 12.1 in [10]). Let $r = N - m - 1$ and let P^r be a linear subspace of $\mathbf{P}_k^N$ of dimension r.

To prove the theorem, it suffices to show that the projection with center P^r induces an embedding of X into $\mathbf{P}_k^m$ if and only if

$$\operatorname{Zarsec}(X, i) \cap P^r = \emptyset. \tag{6.5.2}$$

More precisely, projecting with P^r as center one obtains a morphism

$$p: \mathbf{P}_k^N - P^r \to \mathbf{P}_k^m.$$

Let $Y = p(X)$ be the (closed) scheme-theoretic image of $X - P^r$, cf. EGA I 9.5. We then have a morphism

$$\varphi: X - P^r \to Y,$$

and the claim is that φ is an isomorphism of X with Y if and only if (6.5.2) holds.

First, assume (6.5.2). Then in particular

$$X \cap P^r = \emptyset,$$

so

$$\varphi: X \to Y.$$

Since moreover $P^r \cap \operatorname{Sec}(X, i) = \emptyset$, it follows that f is bijective on closed points, and hence bijective on the underlying topological spaces of X and Y: In fact, suppose that two closed points of X, say x_1 and x_2, are mapped to the same point $y \in Y$. Then $\overline{p^{-1}(y)} = P^{r+1}$ is a linear subspace of $\mathbf{P}_k^N$, of dimension $r+1$, which contains P^r as well as x_1 and x_2. Now the line joining x_1 and x_2 is a secant of X which lies in P^{r+1} and hence meets P^r, a contradiction.

Thus as f is proper, it is a homeomorphism of underlying topological spaces.

Hence it remains to show that the morphism of sheaves

$$\theta: O_Y \to f_*(O_X),$$

which corresponds to f is an isomorphism.

By EGA III (4.4.2) f is a finite morphism. Let $U = \operatorname{Spec}(A)$ be an open affine subset of Y. Then $f^{-1}(U) = \operatorname{Spec}(B)$, where

$$\theta_U: A \to B$$

makes B to a finite A-module. To show is that θ_U is bijective. For this, it suffices to show that for all primes p in A, the homomorphism

$$(\theta_U)_p\colon A_p \to B_p = B \otimes_A A_p$$

is bijective. Since f is bijective as map of topological spaces, so is the induced

$$\operatorname{Spec}(B) \to \operatorname{Spec}(A).$$

From this it is easily seen that if q is a prime in B and $p = \theta_U^{-1}(q)$, then

$$B_p = B_q.$$

Hence, in order to show that f is an isomorphism, it suffices to prove the following: Let $x \in X$, $y = f(x)$. Then the induced local homomorphism

$$\theta_x^{\#}\colon O_{Y,y} \to O_{X,x}$$

is bijective.

From the above it follows that $\theta_x^{\#}$ is injective and makes $O_{X,x}$ to a finite $O_{Y,y}$-module. By Nakayama's Lemma it therefore suffices to show that

$$m_{Y,y} O_{X,x} = m_{X,x},$$

i.e. that f is unramified at x.

This follows since the center of projection P^r does not meet Zarsec (X, i) and hence does not meet any Zariski tangent space of X in $\mathbf{P}_k^N$, see Lemma 5.3.2. Hence the projection p induces a closed embedding of the Zariski tangent space of X at x into that of Y at y. Thus f is unramified at x.

For the converse, assume that

$$\operatorname{Zarsec}(X, i) \cap P^r \neq \emptyset,$$

but that f is an isomorphism of X with Y. We show that this leads to a contradiction.

Let $y \in Y$ be a k-point. Then

$$f^{-1}(y) = \{x\},$$

since f is an isomorphism. Hence no secant of X meets P^r, so

$$\operatorname{Sec}(X) \cap P^r = \emptyset.$$

Moreover, f induces an isomorphism of the Zariski tangent space of X at x, $T_{X,x}$ onto that of Y at y, $T_{Y,y}$. Thus

$$T_{X,x} \cap P^r = \emptyset,$$

for all $x \in X$. Hence

$$\text{Zarsec}\,(X) \cap P^r = \varnothing,$$

which is a contradiction. This completes the proof of Theorem 6.5.

By Theorem 6.5, the proof of Theorem 6.4 now amounts to computing the dimension of Zarsec (X, i). For this we need the following lemma (see Lemma 7.1 in [10]).

Let $S \subset \mathbf{P}^N \times_k \mathbf{P}^N = \mathbf{P}$ be a subscheme of pure codimension d. Let h be the class of a hyperplane in $\mathbf{P}_k^N$. Then

$$A(\mathbf{P}) = \mathbf{Z}[s, t],$$

where

$$s = \mathrm{pr}_1^*\,(h),\ t = \mathrm{pr}_2^*\,(h).$$

Now put

$$N_1 = \text{Min}\,\{N, d\},\ N_0 = \text{Max}\,\{0, d-N\},$$

Then

$$\mathrm{cl}_{\mathbf{P}}\,(S) = \sum_{l=N_0}^{N_1} \alpha_l t^l s^{d-l}.$$

LEMMA 6.6. *If* Min $\{\dim\,(S), N\} - \dim\,(\mathrm{pr}_1\,(S)) = r$, *then*

$$\alpha_{N_1} = \ldots = \alpha_{N_1-r+1} = 0, \quad \alpha_{N_1-r} \neq 0. \tag{6.6.1}$$

Proof. If P^r denotes a generic linear r-dimensional subspace of $\mathbf{P}_k^N$, one gets

$$\mathrm{pr}_1\,(S) \cap P^{N-m-1} = \varnothing \Leftrightarrow \dim\,(\mathrm{pr}_1\,(S)) \leqslant m.$$

Hence since (see the proof of Lemma 7.1 in [10])

$$\mathrm{cl}_{\mathbf{P}}\,(S \cap \mathrm{pr}_1^{-1}\,(P^{N-m-1})) = \mathrm{cl}_{\mathbf{P}}\,(S)\, s^{m+1} = \sum \alpha_l t^l s^{d+m+1-l},$$

where the sum is taken over all l from Max $\{N_0, d-N+m+1\}$ = Max $\{0, d-N+m+1\}$ to N_1, we obtain

$$\dim\,(\mathrm{pr}_1\,(S)) \leqslant m \Leftrightarrow \alpha_l = 0 \quad \text{for} \quad l = N_1, \ldots, \text{Max}\,\{0, d-N+m+1\}. \tag{6.6.2}$$

Taking $m = \text{Min}\,\{\dim\,(S), N\} - r$ it thus suffices to show that

$$\text{Max}\,\{0, d-N+\text{Min}\,\{\dim\,(S), N\} - r + 1\} = N_1 - r + 1. \tag{6.6.3}$$

This is seen as follows: Max $\{0, d-N+\text{Min}\,\{\dim\,(S), N\}-r+1\}$ = Max $\{0, d-N+N-r+1+\text{Min}\,\{N-d, 0\}\}$ = Max $\{0, N_1-r+1\}$. Thus it remains to show that $N_1 - r + 1 \geqslant 0$, i.e. that

$$N_1 \geqslant \operatorname{Min}\{2N-d, N\} - \dim(\mathrm{pr}_1(S)) - 1. \tag{6.6.4}$$

For $d \geqslant N$, (6.6.4) is trivial. So assume $d < N$. Then $N_1 = d$ and $2N-d > N$, hence the claim amounts to

$$d \geqslant 2N - d - \dim(\mathrm{pr}_1(S)) - 1. \tag{6.6.5}$$

But this is clear: In fact,

$$S \subseteq \mathrm{pr}_1(S) \times_k \mathbf{P}^N_k,$$

so $d = \operatorname{codim}(S) \geqslant \operatorname{codim}(\mathrm{pr}_1(S) \times_k \mathbf{P}^N_k) = N - \dim(\mathrm{pr}_1(S))$.

This completes the proof of (6.6.4), and hence of Lemma 6.6.

Proof of Theorem 6.4: We first treat the trivial case when X is a linear subspace of $\mathbf{P}^N_k$. As

$$c_i(\mathbf{P}^n_k) = c_i((\Omega^1_{\mathbf{P}^n_k/k})^*) = \binom{n+1}{i}\tau^i,$$

where $\tau \in A(\mathbf{P}^n_k)$ is the class of a hyperplane, it follows in this case that

$$s(X) = \frac{1}{c(X)} = \frac{1}{(1+\tau)^{n+1}} = \sum_{i=0}^{n} \binom{-(n+1)}{i}\tau^i,$$

and hence

$$d_i = \binom{-(n+1)}{i}.$$

Thus if X a linear subspace of $\mathbf{P}^N_k$, then

$$\gamma_l = 1 - \sum_{j=0}^{l-n} \binom{l+1}{l-n-j}\binom{-(n+1)}{j}, \quad n \leqslant l \leqslant 2n,$$

$$\bar{\gamma}_l = \sum_{j=0}^{l-n+1} \binom{l+1}{l-n+1-j}\binom{-(n+1)}{j}, \quad n \leqslant l \leqslant 2n.$$

Now recall the *Vandermonde convolution formula*

$$\sum_{i=0}^{m} \binom{r-p}{m-i}\binom{p}{i} = \binom{r}{m}, \tag{6.7}$$

which immediately implies that Γ_m vanishes for $m \geqslant n$.

Hence we may assume that X is not a linear subspace of $\mathbf{P}^N_k$. By Corollary 4.4.3 we thus have

$$\dim(S(X, i)) = 2n+1.$$

Moreover, by Corollary 3.2 we may assume that $N > \operatorname{Max}\{\varrho, 2n+1\}$. Thus one obtains

$$\mathrm{cl}_{\mathbf{P}}(S(X,i)) = s^{N-(2n+1)} \sum_{j=N-(2n+1)}^{N} \beta_j t^j s^{N-j},$$

and (cf. (5.5.1))

$$\mathrm{cl}_{\mathbf{P}}(Z_\alpha(X,i)) = s^{N-(\varrho+1)} \sum_{j=N-\varrho}^{N} \bar{\beta}_{\alpha,j} t^j s^{N-j}.$$

By Lemma 6.6 and the assumption that $N > \operatorname{Max}\{\varrho, 2n+1\}$ it now suffices to prove the following

PROPOSITION 6.8. *We have*

$$\beta_j = \mu\gamma_{j-(N-2n)} \quad \textit{for} \quad N-n \leqslant j \leqslant N, \tag{6.8.1}$$

and

$$\bar{\beta}_{\alpha,j} = \bar{\gamma}_{j-(N-\varrho\alpha)} \quad \textit{for} \quad N-n \leqslant j \leqslant N, \tag{6.8.2}$$

where μ is a non-zero rational number.

Proof. With the same notation as in section 4, we write as in [10].

$$\mathrm{cl}_{\mathbf{Bl}}(\widetilde{X \times_k X}) = \mathbf{a}_0 + \boldsymbol{\eta}\mathbf{a}_1,$$

where $\boldsymbol{\eta} = \mathrm{cl}_{\mathbf{Bl}}(T)$ and $\mathbf{a}_i = \lambda^*(a_i) \in A(T)$. In fact, recall from section 1 that

$$A(T) = \mathbf{Z}[t, \xi],$$

where $t = f^*(h)$, h being the class of a hyperplane (usually we put $h = t$, when no confusion is possible), $\xi = p(O(1))$ satisfies

$$\xi^N + \ldots + \binom{N+1}{i} t^i \xi^{N-i} + \ldots + (N+1) t^N = 0,$$

and moreover

$$A(\mathbf{Bl}) = \mathbf{Z}[\mathbf{t}, \boldsymbol{\xi}, \boldsymbol{\eta}],$$

where $\mathbf{t} = \lambda^*(t)$, $\boldsymbol{\xi} = \lambda^*(\xi)$. Finally, $\boldsymbol{\eta}$ satisfies

$$\boldsymbol{\eta}^2 + \boldsymbol{\xi}\boldsymbol{\eta} = 0, \tag{6.8.3}$$

see Lemma 9.3 in [10]. Since X is not a linear subspace of $\mathbf{P}_k^N$, Corollary 4.4.1 gives that

$$\delta(X,i) = [k(\widetilde{X \times_k X}) : k(\mathbf{Sb}(X,i))] \neq 0.$$

Now $\lambda_*(\boldsymbol{\eta}) = 1$, hence

$$a_1 = \lambda_*(\mathrm{cl}_{\mathbf{Bl}}(\widetilde{X \times_k X})) = \delta\, \mathrm{cl}_T(\mathbf{Sb}(X, i)).$$

Moreover

$$a_1 = \sum_{l=N-2n}^{N} b_l \sigma^{2N-(2n+1)-l} t^l \tag{6.8.4}$$

since $\mathbf{Sb}(X, i)$ is a subvariety of T of codimension $2N-(2n+1)$, and since $A(T)$ is a free $\mathbf{Z}[t]$-module on $1, \sigma, \dots, \sigma^{N-1}$ (cf. section 1). Now

$$\mathbf{a}_1 = \delta\, \mathrm{cl}_{\mathbf{Bl}}(\lambda^{-1}(\mathbf{Sb}(X, i)) = \delta \sum_{l=N-2n}^{N} b_l \boldsymbol{\sigma}^{2N-(2n+1)-l} \mathbf{t}^l$$

We now claim the following:

$$\mathrm{cl}_{\mathbf{P}}(S(X, i)) = \mu \sum_{l=N-2n}^{N} b_l s^{2N-(2n+1)-l} t^l \tag{6.8.5}$$

where

$$\mu = \frac{1}{\delta\varepsilon},\ \varepsilon = [k(\lambda^{-1}(\mathbf{Sb}(X, i))) : k(S(X, i))].$$

In other words

$$\beta_{N-(2n+1)} = 0,\ \beta_j = \mu b_j \quad \text{for} \quad j \geqslant N-2n. \tag{6.8.6}$$

Proof of (6.8.5): By Lemma 5.4,

$$\pi_*(\mathbf{a}_1) = \sum_{l=N-2n}^{N} b_l s^{2N-(2n+1)-l} t^l.$$

Moreover, not all b_l are zero, since this would imply $\lambda_*(\mathrm{cl}_{\mathbf{Bl}}(\widetilde{X \times_k X})) = 0$, hence $\dim(\mathbf{Sb}(X)) < 2n$. Thus $\pi_*(\mathbf{a}_1) \neq 0$, and thus $\varepsilon \neq 0$.

Now

$$\mathbf{a}_1 = \lambda^*(\lambda_*(\mathrm{cl}_{\mathbf{Bl}}(\widetilde{X \times_k X}))) = \delta\, \mathrm{cl}_{\mathbf{Bl}}(\lambda^{-1}(\mathbf{Sb}(X, i))).$$

Similarly

$$\pi_*(\mathrm{cl}_{\mathbf{Bl}}(\lambda^{-1}(\mathbf{Sb}(X, i)))) = \varepsilon\, \mathrm{cl}_{\mathbf{P}}(S(X, i)).$$

Putting these identities together we obtain

$$\pi_*(\mathbf{a}_1) = \delta\varepsilon\, \mathrm{cl}_{\mathbf{P}}(S(X, i)),$$

and (6.8.5) follows.

Thus (6.8.1) follows from the

LEMMA 6.8.7. *For $N-n \leqslant j \leqslant N$ we have*

$$b_j = \gamma_{j-(N-2n)},$$

For other values of j, $b_j = 0$.

Proof. To show is that

$$b_{l+N-2n}=\begin{cases} d^2-\sum_{i=0}^{l-n}\binom{l+1}{l-n+i}p_i & \text{for} \quad n\leqslant l\leqslant 2n,\\ 0 \quad \text{otherwise.}\end{cases} \tag{6.8.8}$$

By Lemma 6.1 and (6.8.3) we obtain

$$\mathrm{cl}_{\mathbf{Bl}}(\mathbf{T}(X))=\boldsymbol{\eta}(\mathbf{a}_0-\boldsymbol{\xi}\mathbf{a}_1).$$

Here the notations are as in the proof of Proposition 6.8. Moreover

$$a_0=d^2(\sigma t)^{N-n} \quad \text{and} \quad b_{N-2n}=0. \tag{6.8.9}$$

In fact, recall the diagram

$$\begin{array}{ccccc} T & \xhookrightarrow{j} & \mathbf{Bl} & \xrightarrow{\lambda} & T\\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle f}\\ \mathbf{P}_k^N & \xhookrightarrow{\delta} & \mathbf{P} & \xrightarrow{pr_2} & \mathbf{P}_k^N \end{array}$$

where the two horizontal compositions are the identities.

Now

$$j_*(j^*(\mathbf{a}_1))=\boldsymbol{\eta}\mathbf{a}_1$$

by the projection formula, so

$$\pi_*(\boldsymbol{\eta}\mathbf{a}_1)=\pi_*(j_*(j^*(\mathbf{a}_1)))=\pi_*j_*(a_1),$$

since λj is the identity. Hence

$$\pi_*(\boldsymbol{\eta}\mathbf{a}_1)=\delta_*f_*(a_1).$$

Now as

$$f_*(\sigma^i)=\begin{cases}0 & \text{for} \quad i=0,\ldots,N-2\\ 1 & \text{for} \quad i=N-1,\end{cases}$$

we get by (6.8.4)

$$f_*(\mathbf{a}_1)=\sum_{l=N-2n}^{N} b_l f_*(\sigma^{2N-(2n+1)-l})t^l=b_{N-2n}t^{N-2n}$$

and thus

$$\pi_*(\boldsymbol{\eta}\,\mathbf{a}_1)=b_{N-2n}\,\delta_*(t^{N-2n}).$$

Since $A(\mathbf{P}_k^N)=\mathbf{Z}[t]$ is identified with the canonical subring of $A(\mathbf{P})=\mathbf{Z}[s,t]$ via pr_2^*, and $\mathrm{pr}_2\,\delta$ is the identity, we get

$$\delta_*(t^{N-2n})=\delta_*((\mathrm{pr}_2\,\delta)^*(t^{N-2n})=\delta_*\delta^*(t^{N-2n})=t^{N-2n}\,\mathrm{cl}_{\mathbf{P}}(\Delta)=t^{N-2n}(s^N+s^{N-1}t+\ldots+t^N)$$

(cf. [10], Lemma 9.4.3). Thus

$$\pi_*(\boldsymbol{\eta}\mathbf{a}_1) = b_{N-2n}(s^N t^{N-2n} + \ldots + s^{N-2n}t^N)$$

On the other hand, as $\pi_X\colon \widetilde{X \times_k X} \to X \times_k X$ is birational, we get

$$\pi_*(\mathbf{a}_0 + \boldsymbol{\eta}\mathbf{a}_1) = \mathrm{cl}\,(X \times_k X) = d^2(st)^{N-n}.$$

Now write

$$\mathbf{a}_0 = \sum_{i=1}^{2n} \alpha_i \mathbf{t}^{N-2n+i} \boldsymbol{\sigma}^{N-i}$$

which is possible since $1, \sigma, \ldots, \sigma^{N-1}$ forms a base for $A(T)$ over $\mathbf{Z}[t]$. Then

$$\pi_*(\mathbf{a}_0) = \sum_{i=1}^{2n} \alpha_i t^{N-2n+i} s^{N-i}.$$

This gives

$$\sum_{i=1}^{2n} \alpha_i t^{N-2n+i} s^{N-i} + b_{N-2n} \sum_{i=0}^{2n} t^{N-2n+i} s^{N-i} = d^2(st)^{N-n},$$

which finally implies that

$$\begin{aligned} &b_{N-2n} = 0, \\ &a_1 + b_{N-2n} = 0, \\ &\cdots\cdots\cdots\cdots \\ &\alpha_N + b_{N-2n} = d^2, \\ &\cdots\cdots\cdots\cdots \\ &\alpha_{2n} = b_{N-2n} = 0, \end{aligned}$$

and (6.8.9) follows at once.

If now $\beta \leqslant N-1$, then

$$f_*(\sigma^\alpha \xi^\beta) = \begin{cases} 0 \quad \text{for} \quad \alpha + \beta < N-1, \\ \dbinom{\alpha - (N+1)}{\alpha + \beta - (N-1)} t^{\alpha+\beta-(N-1)} \quad \text{for} \quad \alpha + \beta \geqslant N-1, \end{cases} \tag{6.8.10}$$

In fact, we have

$$\sigma^\alpha \xi^\beta = \sum_{j=0}^{\alpha} \binom{\alpha}{j} t^{\alpha-j} \xi^{j+\beta},$$

so

$$f_*(\sigma^\alpha \xi^\beta) = \sum_{i=0}^{\alpha} \binom{\alpha}{j} t^{\alpha-j} f_*(\xi^{j+\beta}).$$

Now

$$f^*(\xi^{N-1+i}) = \binom{-(N+1)}{i} t^i \quad \text{for} \quad i \geqslant 0. \tag{6.8.11}$$

In fact, the relation

$$\xi^N + (N+1)t\xi^{N-1} + \ldots + \binom{N+1}{i} t^i \xi^{N-i} + \ldots + (N+1)t^N = 0$$

gives

$$\xi^{N-1+i} + (N+1)t\xi^{N-2+i} + \ldots + \binom{N+1}{i} t^i \xi^{N-1} + \ldots = 0.$$

Thus if we put

$$f^*(\xi^{N-1+i}) = \alpha_i t^i,$$

we get

$$\alpha_i + (N+1)\alpha_{i-1} + \ldots + \binom{N+1}{i}\alpha_0 = 0.$$

Hence

$$\left(\sum_{i=0}^{\infty} \alpha_i X^i\right)(1+X)^{N+1} = 1,$$

and (6.8.11) is immediate.

By means of (6.8.11) we obtain the following, where $m = \alpha + \beta - (N-1)$:

$$f_*(\sigma^\alpha \xi^\beta) = \left(\sum_{i=0}^{m} \binom{\alpha}{m-i} \binom{-(N+1)}{i}\right) t^m.$$

Thus the Vandermonde convolution formula (6.7) with $p = -(N+1)$, $r = \alpha + p = \alpha - (N+1)$ yields

$$f_*(\sigma^\alpha \xi^\beta) = \binom{\alpha - (N+1)}{m} = \binom{\alpha - (N+1)}{\alpha + \beta - (N-1)},$$

provided $m \geqslant 0$. If $m < 0$, i.e. if $\alpha + \beta < N-1$, then $f_*(\alpha^\alpha \xi^\beta) = 0$ since f_* is of degree $-(N-1)$. Thus (6.8.10) is proved.

Now by (6.8.9) we get

$$\mathrm{cl}_T(\mathbf{T}_X)\,\xi^{s+n-1} = d^2(\sigma t)^{N-n}\xi^{s+n-1} - \sum_{l=N-2n+1}^{N} b_l \xi^{s+n} \sigma^{2N-2n-1-l} t^l.$$

On the other hand,

$$f_*\,(\mathrm{cl}_T(\mathbf{T}_X)\,\xi^{s+n-1}) = \begin{cases} 0 & \text{for} \quad -n+1 \leqslant s < 0 \\ p_s t^{N-n+s} & \text{for} \quad 0 \leqslant s \leqslant n, \end{cases}$$

and hence

$$p_s t^{N-n+s} = d^2 f_*(\sigma^{N-n}\xi^{s+n-1})\, t^{N-n} - \sum_{l=N-2n+1}^{N} b_l f_*(\sigma^{2N-2n-1-l}\xi^{s+n})\, t^l,$$

where we let $p_s = 0$ for $s < 0$. Now by (6.8.10)

$$f_*(\sigma^{N-n}\xi^{s+n-1})\,t^{N-n}=\begin{cases}\displaystyle\binom{-(n+1)}{s}t^{N+s-n} & \text{for} \quad n\geqslant s\geqslant 0\\ 0 \quad \text{for} \quad -n+1\leqslant s<0,\end{cases}$$

and

$$f_*(\sigma^{2N-2n-1-}\ \xi^{s+n})\,t^{l}=\begin{cases}\displaystyle\binom{N-2n-2-l}{N+s-n-l}t^{N+s-n} & \text{for} \quad l\leqslant N+s-n\\ 0 \quad \text{for} \quad l>N+s-n,\end{cases}$$

since by assumption $N>2n+1$. This gives

$$p_s=d^2\binom{-(n+1)}{s}-\sum_{l=N-2n+1}^{N+s-n}b_l\binom{N-2n-2-l}{N+s-n-l}, \tag{6.8.12}$$

for all $0\leqslant s\leqslant n$, and

$$0=-\sum_{=N-2n+1}^{N+s-n}b_l\binom{N-2n-2-l}{N+s-n-l}, \tag{6.8.13}$$

for all $-n+1\leqslant s<0$.

(6.8.13) immediately implies that

$$b_{N-2n+1}=\ldots=b_{N-1-n}=0.$$

Hence (6.8.12) takes the form

$$p_s=d^2\binom{-(n+1)}{s}-\sum_{l=N-n}^{N-n+s}b_l\binom{N-2n-2-l}{N+s-n-l}.$$

Now put $i=l-(N-2n)$, $\tilde\beta_i=b_l=b_{i+N-2n}$. Then we get

$$p_s=d^2\binom{-(n+1)}{s}-\sum_{i=n}^{n+s}\tilde\beta_i\binom{-(i+2)}{s+n-i}.$$

Since

$$\binom{-m}{n}=(-1)^n\binom{m+n-1}{n}=(-1)^n\binom{m+n-1}{m-1},$$

this can be written as

$$p_s=(-1)^s d^2\binom{n+s}{n}+(-1)^{n+s+1}\sum_{i=n}^{n+s}(-1)^i\tilde\beta_i\binom{s+n+1}{i+1},$$

i.e., letting $j=i-n$:

$$\sum_{j=0}^{s}(-\tilde\beta_{j+n})(-1)^{s+j}\binom{s+n+1}{j+n+1}=p_s-(-1)^s\binom{n+s}{s}d^2=y_s, \tag{6.8.14}$$

for all $s=0,\ldots,n$.

Define

$$n_{rs}=\begin{cases}\binom{n+r}{n+s}(-1)^{r+s} & \text{for} \quad 1\leqslant s\leqslant r\leqslant n+1\\ 0 \quad \text{for} \quad 1\leqslant r<s\leqslant n+1.\end{cases}$$

Then

$$\{n_{rs}\}\cdot\{|n_{rs}|\}=\mathbf{E},$$

where E is the unit matrix.

Indeed, it is clear that this product can be expressed as

$$A=\{\alpha_{rs}\},$$

where for $r<s$

$$\alpha_{rs}=0,$$

and

$$\alpha_{rr}=1,$$

while finally for $r>s$ $(p=r-s)$:

$$\alpha_{rs}=\sum_{i=s}^{s+p}(-1)^{r+i}\binom{n+r}{n+i}\binom{n+i}{n+s}=\sum_{j=0}^{p}(-1)^{r+s+j}\binom{n+r}{n+s+j}\binom{n+s+j}{n+s}.$$

With $n+s=m$ one thus obtains

$$\alpha_{rs}=(-1)^{r+s}\sum_{j=0}^{p}(-1)^{j}\binom{m+p}{m+j}\binom{m+j}{m}=(-1)^{r+s}\sum_{j=0}^{p}(-1)^{j}\binom{m+p}{p}\binom{p}{j}=0.$$

Thus

$$-\begin{Bmatrix}\tilde{\beta}_n\\ \vdots\\ \tilde{\beta}_{2n}\end{Bmatrix}=\{|n_{rs}|\}\begin{Bmatrix}y_0\\ \vdots\\ y_n\end{Bmatrix}$$

i.e. for $n\leqslant j\leqslant 2n$:

$$-\tilde{\beta}_j=\sum_{l=0}^{j-n}\binom{j+1}{j-n-l}y_l,$$

i.e.

$$\tilde{\beta}_j=-\sum_{l=0}^{j-n}\binom{j+1}{j-n+l}p_l+\left\{\sum_{l=0}^{j-n}(-1)^l\binom{j+1}{j-n-l}\binom{n+l}{l}\right\}d^2.$$

Now

$$\sum_{l=0}^{j-n}(-1)^l\binom{j+1}{j-n-l}\binom{n+l}{l}=\sum_{l=0}^{j-n}\binom{j+1}{j-n-l}\binom{-(n+1)}{l}.$$

Using the Vandermonde formula (6.7) with $m=r=j-n$, $p=-(n+1)$, we get

$$\sum_{l=0}^{j-n}\binom{j+1}{j-n-l}\binom{-(n+1)}{l}=\binom{j-n}{j-n}=1.$$

Thus

$$\tilde{\beta}_j = b_{j+N-2n} = d^2 - \sum_{l=0}^{j-n} \binom{j+1}{j-n-l} p_l,$$

and the proof of Lemma 6.8.7 is complete.

To show (6.8.2), and thus complete the proof of Proposition 6.8, it now suffices to prove the following

LEMMA 6.8.15. *For all $0 \leqslant l \leqslant n$ and all $1 \leqslant \alpha \leqslant r(X, i)$ we have*

$$\bar{\beta}_{\alpha,\, l+N-n} = \sum_{s=0}^{l} \binom{l+\varrho-n+1}{s+\varrho-n} q_{\alpha,s},$$

and for other values of l, $\bar{\beta}_{\alpha,\, l+N-n} = 0$.

Proof. Note first that since $\sigma = \xi + t$, the relation

$$\sum_{i=0}^{N} \binom{N+1}{i} t^i \xi^{N-i} = 0,$$

gives that

$$\sigma^{N+1} = 0.$$

Together with

$$\sigma^N = \xi^N + N\xi^{N-1}t + \dots$$

this gives

$$f_*(t^\gamma \sigma^\beta) = \begin{cases} 0 & \text{for} \quad \beta < N-1 \quad \text{and} \quad \beta > N \\ t^\gamma & \text{for} \quad \beta = N-1 \\ -t^{\gamma+1} & \text{for} \quad \beta = N. \end{cases} \tag{6.8.16}$$

In fact, we have

$$f_*(\xi^\beta) = \begin{cases} 0 & \text{for} \quad \beta < N-1 \\ 1 & \text{for} \quad \beta = N-1 \\ -(N+1)t & \text{for} \quad \beta = N. \end{cases}$$

For simplicity we delete the subscript α. By Proposition 1.6 and Definition 1.7 we get

$$f_*\,(\mathrm{cl}_T\,(P)\,\xi^{s+\varrho-n}) = \begin{cases} q_s{}^{N+s-n} & \text{for} \quad s = 0, \dots, n \\ 0 & \text{otherwise.} \end{cases} \tag{6.8.17}$$

Moreover since $N > \varrho$, we have

$$\begin{aligned}\mathrm{cl}_T(P)\,\xi^{s+\varrho-n} &= \left(\sum_{i=N-\varrho}^{N} \bar{\beta}_i t^i \sigma^{2N-(\varrho+1)-i}\right)(\sigma - t)^{s+\varrho-n} \\ &= \left(\sum_{i=N-\varrho}^{N} \bar{\beta}_i t^i \sigma^{2N-(\varrho+1)-i}\right)\left(\sum_{j=0}^{s+\varrho-n}(-1)^j \binom{s+\varrho-n}{j} t^j \sigma^{s+\varrho-n-j}\right) \\ &= \sum_{i=N-\varrho}^{N}\ \sum_{j=0}^{s+\varrho-n}(-1)^j\binom{s+\varrho-n}{j}\bar{\beta}_i t^{i+j}\sigma^{2N+s-n-1-(i+j)}.\end{aligned}$$

By (6.8.16) we now have

$$f_*(t^{i+j}\sigma^{2N+s-n-1-(i+j)}) = \begin{cases} t^{N+s-n} & \text{for } \ i+j = N+s-n \\ -t^{N+s-n} & \text{for } \ i+j = N+s-n-1 \\ 0 & \text{otherwise.}\end{cases}$$

Thus

$$\begin{aligned}&f_*\,(\mathrm{cl}_T(P)\,\xi^{s+\varrho-n}) \\ &\quad = t^{N+s-n}\left\{-\sum_{i+j=N+s-n-1}\bar{\beta}_i\binom{s+\varrho-n}{j}(-1)^j + \sum_{i+j=N+s-n}\bar{\beta}_i\binom{s+\varrho-n}{j}(-1)^j\right\},\end{aligned}$$

where $N \geqslant i \geqslant N-\varrho$ and $s+\varrho-n \geqslant j \geqslant 0$. The coefficient is equal to

$$\begin{aligned}&\sum_{i=N-\varrho}^{N+s-n-1}\bar{\beta}_i\binom{s+\varrho-n}{N+s-n-1-i}(-1)^{N+s-n-i} + \sum_{i=N-\varrho}^{N+s-n}\bar{\beta}_i\binom{s+\varrho-n}{N+s-n-i}(-1)^{N+s-n-i} \\ &\qquad = \sum_{i=N-\varrho}^{N+s-n}(-1)^{N+s-n-i}\binom{s+\varrho-n+1}{N+s-n-i}\bar{\beta}_i.\end{aligned}$$

Hence we obtain the following system of equations

$$q_s = \begin{cases}\displaystyle\sum_{i=N-\varrho}^{N+s-n}(-1)^{N+s-n-i}\binom{s+\varrho-n+1}{N+s-n-i}\bar{\beta}_i & \text{for } \ s=0,\dots,n \\ 0 \quad \text{otherwise.}\end{cases}$$

For simplicity we write

$$\alpha_l = \bar{\beta}_{l+N-n}.$$

Then the above system takes the form

$$q_s = \begin{cases}\displaystyle\sum_{l=n-\varrho}^{s}(-1)^{s-l}\binom{s+\varrho-n+1}{s-l}\alpha_l & \text{for } \ s=0,\dots,n \\ 0 \quad \text{otherwise,}\end{cases}$$

i.e.

$$\begin{Bmatrix} 0 \\ \vdots \\ 0 \\ q_0 \\ \vdots \\ q_n \end{Bmatrix} = \{\nu_{rs}\} \begin{Bmatrix} \alpha_{n-\varrho} \\ \vdots \\ \alpha_{-1} \\ \alpha_0 \\ \vdots \\ \alpha_n \end{Bmatrix},$$

where

$$\nu_{rs} = \begin{cases} (-1)^{r-s}\binom{r}{s} & \text{for} \quad 1 \leqslant s \leqslant r \leqslant \varrho+1 \\ 0 \quad \text{for} \quad 1 \leqslant r < s \leqslant \varrho+1. \end{cases}$$

For $r > s$ one obtains

$$\sum_{i=s}^{r} (-1)^{i-s}\binom{i}{s}\binom{r}{i} = \sum_{j=0}^{r-s} (-1)^j \binom{s+j}{s}\binom{r}{s+j} = \sum_{j=0}^{r-s}(-1)^j\binom{r}{s}\binom{r-s}{j} = 0.$$

This yields

$$\{\nu_{rs}\} \cdot \{|\nu_{rs}|\} = \mathbf{E},$$

and we find

$$\begin{Bmatrix} \alpha_{n-\varrho} \\ \vdots \\ \alpha_n \end{Bmatrix} = \{|\nu_{rs}|\} \begin{Bmatrix} 0 \\ \vdots \\ 0 \\ q_0 \\ \vdots \\ q_n \end{Bmatrix}.$$

Thus $\alpha_{n-\varrho} = \dots = \alpha_{-1} = 0$, and for $l \geqslant 0$

$$\alpha_l = \sum_{i=0}^{l} \binom{l+\varrho-n+1}{\varrho-n+i} q_i.$$

This completes the proof of Lemma 6.8.15, hence of Proposition 6.8.

Thus the proof of Theorem 6.4 is completed.

7. Affine embedding theorems

In this section we give an analogous result to Theorem 6.4 in the affine case. Throughout this section, $\mathbf{A}_k^N$ is identified with the open subscheme $D_+(X_0)$ of $\mathbf{P}_k^N = \operatorname{Proj}(k[X_0, \dots, X_N])$ in the canonical way. The hyperplane at infinity $V_+(X_0)$ will be denoted by H_0.

Let Y be an affine variety over k, and $j\colon Y \hookrightarrow \mathbf{A}_k^N$ a closed embedding. Let X be the projective closure of Y, and $i\colon X \hookrightarrow \mathbf{P}_k^N$ the corresponding projective embedding. This notation will be kept troughout this section.

An embedding of Y into $\mathbf{A}_k^m$ is *induced by projection* if there is a morphism $p\colon X \to X' \subset \mathbf{P}_k^m = \operatorname{Proj}(k[Y_0, \ldots, Y_m])$ induces by a projection from $\mathbf{P}_k^N$ with center contained in H_0, and such that p induces an isomorphism $p'\colon Y \xrightarrow{\approx} X' \cap D_+(Y_0)$. It should be kept in mind that we thus exclude projections from a point at infinity of Y. In the above situation we also say that Y may be embedded into $\mathbf{A}_k^m$ via a projection from $\mathbf{A}_k^N$.

With the notations of (4.1.1) we now put

$$\operatorname{Sec}(Y, i) = \operatorname{pr}_1(\pi(\lambda^{-1}(\lambda(\widetilde{Y \times_k Y})))),$$

where as before the image of a subscheme by a morphism is the (closed) scheme-theoretic image.

Moreover, we have a canonical closed embedding

$$\mathbf{P}(\Omega^1_{Y/k}) \hookrightarrow \mathbf{P}(\Omega^1_{\mathbf{A}_k^N/k}),$$

and hence an embedding

$$\mathbf{P}(\Omega^1_{Y/k}) \hookrightarrow \mathbf{P}(\Omega^1_{\mathbf{P}_k^N/k}).$$

Let $\bar{\mathbf{P}}(\Omega^1_{Y/k})$ denote the closure of $\mathbf{P}(\Omega^1_{Y/k})$ in $\mathbf{P}(\Omega^1_{\mathbf{P}_k^N/k})$. Finally put

$$\operatorname{Zar}(Y, i) = \operatorname{pr}_1(\pi(\lambda^{-1}(\mathbf{P}(\Omega^1_{Y/k})))),$$

$$\operatorname{Zarsec}(Y, i) = \operatorname{Zar}(Y, i) \cup \operatorname{Sec}(Y, i),$$

and

$$S(Y, i) = \pi(\lambda^{-1}(\lambda(\widetilde{Y \times_k Y}))),$$

$$Z(Y, i) = \pi(\lambda^{-1}(\mathbf{P}(\Omega^1_{Y/k}))).$$

Before we continue, note the following

PROPOSITION 7.1. *$S(Y, i) = S(X, j)$ and $Z(Y, i) = \pi(\lambda^{-1}(\bar{\mathbf{P}}(\Omega^1_{Y/k})))$.*

Proof. The first part follows since $\widetilde{Y \times_k Y}$ is an open dense subscheme of $\widetilde{X \times_k X}$, the second since $\mathbf{P}(\Omega^1_{Y/k})$ is an open dense subscheme of $\bar{\mathbf{P}}(\Omega^1_{Y/k})$.

Now put $p_s(Y, i) = p_s(X, j)$ for all $s = 0, \ldots, n = \dim(Y)$, and define $q_{\alpha,s}(Y, i)$ by means of $\bar{\mathbf{P}}(\Omega^1_{Y/k})$ in the same way as $q_{\alpha,s}(X, j)$ is defined by $\mathbf{P}(\Omega^1_{X/k})$: Let

$$\bar{\mathbf{P}}(\Omega^1_{Y/k}) = \bar{P}_1 \cup \ldots \cup \bar{P}_r, \quad r = r(X, i),$$

and let $\bar{\varrho}_\alpha = \dim(\bar{P}_\alpha)$. Then $\bar{q}_{\alpha,s}$ is defined by

$$f_*(\operatorname{cl}_T(\bar{P}_\alpha)\,\xi^{\bar{\varrho}_\alpha - n + s}) = q_{\alpha,s}(Y, i)\,t^{N-n+s},$$

for $s = 0, \ldots, n$, and all $1 \leqslant \alpha \leqslant r(X, i)$.

By means of the invariants p_s and $q_{\alpha,s}$ for the affine, embedded variety $i\colon Y \hookrightarrow \mathbf{A}_k^N$, we now proceed to define $\gamma_l(Y, i)$, $\bar{\gamma}_l(Y, i)$, $\boldsymbol{\gamma}_l(Y, i)$ and $\Gamma_l(Y, i)$ as in Definition 6.3. One then obtains the following result:

THEOREM 7.2. *Let $m \geqslant n = \dim(Y)$. Then Y may be embedded into $\mathbf{A}_k^m$ via a projection from $\mathbf{A}_k^N$ if and only if $\Gamma_m(Y, i) = 0$.*

Proof. As with Theorem 6.4, the proof rests on the following

THEOREM 7.3. *Y may be embedded into $\mathbf{A}_k^m$ via a projection from $\mathbf{A}_k^N$ if and only if* $\dim(\mathrm{Zarsec}\,(Y, i)) \leqslant m$.

First note that Theorem 7.3 implies Theorem 7.2. In fact, this follows in exactly the same way as we show that Theorem 6.5 implies Theorem 6.4.

In order to prove Theorem 7.3, recall first that to give an embedding of Y into $\mathbf{A}_k^m$ induced by a projection amounts to giving a morphism

$$p\colon X \to X' \subset \mathbf{P}_k^m = \mathrm{Proj}\,(k[Y_0, \ldots, Y_m]),$$

induced by a projection from $\mathbf{P}_k^N$ with center contained in H_0, and such that p induces an isomorphism

$$p'\colon Y \approx X' \cap D_+(Y_0).$$

If $r = N - m - 1$, then the projection with center $P^r \subset H_0$ has the property above if and only if P^r does not meet any secant line of Y and does not meet any Zariski tangent space of Y in $\mathbf{P}_k^N$. Indeed, the proof of this is very similar to that of Theorem 6.5 and will therefore not be repeated here.

Thus P^r has the required property if and only if it does not meet the closure of the union of all the secant lines of Y and the closure of the union of all Zariski tangent spaces of Y in $\mathbf{P}_k^N$. Now note the

LEMMA 7.3.1. $\mathrm{Sec}\,(Y, i)$ *is the closure of the union of all secant lines of Y in $\mathbf{P}_k^N$, and* $\mathrm{Zar}\,(Y, i)$ *is the closure of the union of all Zariski tangent spaces of Y in $\mathbf{P}_k^N$.*

Proof. The same proof as that of Proposition 4.2 and Proposition 5.3.

To complete the proof of the theorem, we observe that

$$\dim \mathrm{Zarsec}\,(X) \cap H_0 = \dim \mathrm{Zarsec}\,(X) - 1. \tag{7.3.2}$$

Indeed, this follows since no irreducible component of $\mathrm{Zarsec}\,(X)$ is contained in H_0: This is clear for $\mathrm{Sec}\,(X)$. Let Z be an irreducible component of $\mathrm{Zar}\,(X)$, we want to show

$Z \nsubseteq H_0$. Assume the converse, and let Λ be an irreducible component of $\lambda^{-1}(\bar{\mathbf{P}}(\Omega^1_{Y/k}))$ which is mapped onto Z. Put $P = \lambda(\Lambda)$. Then P is irreducible, hence so is $\lambda^{-1}(P)$, and we conclude that $\Lambda = \lambda^{-1}(P)$. Further, let $X' = f(P)$. Then X' is an irreducible subset of X and $X' \cap Y \neq \emptyset$, since otherwise one would have

$$P \subseteq \bar{\mathbf{P}}(\Omega^1_{Y/k}) - \mathbf{P}(\Omega^1_{Y/k}),$$

which is impossible since P is an irreducible component of $\bar{\mathbf{P}}(\Omega^1_{Y/k})$. Now let U be a non-empty open subset of X' which does not contain the images of the generic points of $\bar{\mathbf{P}}(\Omega^1_{Y/k})$, except of course for the image of the generic point of P. We may assume that $U \subseteq Y$. Now if $\bar{f}\colon \bar{\mathbf{P}}(\Omega^1_{Y/k}) \to X$ is the morphism induced by f, then

$$\mathrm{pr}_1\,(\pi(\lambda^{-1}(\bar{f}^{-1}(U))))$$

is the union of all Zariski tangent spaces of X at points from U. On the other hand,

$$\mathrm{pr}_1\,(\pi(\lambda^{-1}(\bar{f}^{-1}(U)))) \subseteq \mathrm{pr}_1\,(\pi(\lambda^{-1}(P))) = \mathrm{pr}_1\,(\pi(\Lambda)) = Z.$$

In particular this gives $U \subseteq H_0$, a contradiction.

This completes the proof of (7.3.2), and hence of Theorem 7.3.

Theorem 7.2 immediately implies the following affine analogue of Lluis' embedding theorem (see R. G. Swan [21], Theorem 2.1 as well as the remark on page 31):

THEOREM 7.4. *With notation as before, Y can be embedded into $\mathbf{A}^m_k$, where $\bar{\varrho} =$ $\dim(\mathbf{P}(\Omega^1_{Y/k}))$ and*

$$m \geqslant \max\{2n+1, \bar{\varrho}\}.$$

Remark 7.4.1. Since Y is an affine variety,

$$\bar{\varrho} \leqslant z + n - 1,$$

where z is the maximum of the dimensions of the Zariski tangent spaces of Y.

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