SUBALGEBRAS OF L^∞ CONTAINING H^∞

BY

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1. Introduction

Let H^∞ be the algebra of bounded analytic functions on $D=\{z: |z|<1\}$ and let L^∞ be the Banach algebra of bounded measurable functions on $T=\{z: |z|=1\}$ with the uniform norm. Then H^∞ can be regarded as a uniformly closed subalgebra of L^∞ by identifying each $f\in H^\infty$ with its boundary function.

If A is a closed subalgebra of L^∞, let $\mathfrak{M}[A]$ denote its maximal-ideal space. K. Hoffman [13] has shown that each $\varphi\in\mathfrak{M}[H^\infty]$ has a unique norm-preserving extension to a bounded linear functional on L^∞. For example, if $z\in D$ then evaluation at z is an element of $\mathfrak{M}[H^\infty]$ and its extension is given simply by the Poisson kernel. Now if A is a closed subalgebra of L^∞ containing H^∞, then the usual Gelfand topology on $\mathfrak{M}[A]$ agrees with the weak-* topology that $\mathfrak{M}[A]$ inherits as a compact subset of the dual space of L^∞. Consequently, each $f\in L^\infty$ is continuous on $\mathfrak{M}[H^\infty]$ and harmonic on D. Moreover, if A and B are closed algebras such that $H^\infty\subset A\subset B\subset L^\infty$, then $\mathfrak{M}[H^\infty]\supset\mathfrak{M}[A]\supset\mathfrak{M}[B]\supset\mathfrak{M}[L^\infty]$. Our main result is the following theorem:

THEOREM 1. *Let A be a closed subalgebra of L^∞ containing H^∞. Let A_I be the closed subalgebra of A generated by H^∞ and $\{f^{-1}\in A: f\in H^\infty\}$. Then $\mathfrak{M}[A_I]=\mathfrak{M}[A]$.*

When combined with a recent result of S. Y. Chang [7], Theorem 1 proves a conjecture of R. Douglas [9]. To state Douglas' conjecture, we let Q be a subset of L^∞ and write $[H^\infty, Q]$ for the uniformly closed subalgebra of L^∞ generated by H^∞ and Q. An algebra of the form $[H^\infty, Q]$, where $Q\subset\{u: |u|=1 \text{ a.e. on } T \text{ and } \bar{u}\in H^\infty\}$ is called a *Douglas algebra.* Since each positive function in $(L^\infty)^{-1}$ is the modulus of a function in $(H^\infty)^{-1}$, we see that A_I is a Douglas algebra whenever $H^\infty\subset A\subset L^\infty$ and we see that if A is a Douglas algebra, then $A=A_I$. Douglas' conjecture was that every uniformly closed subalgebra A of L^∞ containing H^∞ is a Douglas algebra, or, equivalently, that every such algebra A satisfies $A=A_I$. Now S. Y. Chang has proved that if A is a closed algebra lying between H^∞ and

L^∞, and if B is a Douglas algebra with $\mathfrak{M}[B]=\mathfrak{M}[A]$, then $A=B$. In light of this result, Theorem 1 has the following consequence.

THEOREM 2. *Every closed algebra A, such that $H^\infty \subset A \subset L^\infty$, is generated by H^∞ and $\{\bar{u}\in A$: u is an inner function in $H^\infty\}$.*

Douglas' conjecture arose from the study of operator algebras generated by Toeplitz operators. It has been discussed by several authors: S. Axler [1], [2], S. Y. Chang [6], A. M. Davie, T. W. Gamelin, and J. Garnett [8], R. Douglas [9], R. Douglas and W. Rudin [10], D. Sarason [15], [16], [17], T. Weight [18]. I would like to thank J. Garnett for invaluable discussions.

2. Interpolating Blaschke products

We call a sequence $\{z_n\}_{n=1}^\infty$ in D an *interpolating sequence* if for every bounded sequence $\{w_n\}_{n=1}^\infty$, there is an f in H^∞ such that $f(z_n)=w_n$ for all n. Every interpolating sequence must satisfy $\Sigma(1-|z_n|)<\infty$, and so is the zero sequence of a Blaschke product. We call a Blaschke product whose zeros form an interpolating sequence, an *interpolating Blaschke product*.

A finite measure μ on the upper half-plane, H^+, is called a *Carleson measure* if there is a constant C for which $\mu(S)\leqslant C\delta$, whenever S is a square of the form $S=\{x+iy: x_0\leqslant x\leqslant x_0+\delta, 0<y\leqslant\delta\}$. The analogous definition is made for D, where squares are replaced by sectors of the form $S=\{re^{i\theta}: 1-\delta\leqslant r<1 \text{ and } \theta_0\leqslant\theta\leqslant\theta_0+\delta\}$. Any rectifiable curve Γ in H^+ or D induces a measure on H^+ or D, respectively, by defining the measure of a Borel set S to be the length of $\Gamma\cap S$. The *hyperbolic distance* between two points is defined by

$$\varrho(z,w)=\begin{cases}\left|\dfrac{z-w}{z-\bar{w}}\right| & \text{on } H^+\\[2ex] \left|\dfrac{z-w}{1-\bar{w}z}\right| & \text{on } D.\end{cases}$$

Interpolating sequences can be characterized in the following way [12]. On H^+, a sequence $\{z_n\}_{n=1}^\infty$ is an interpolating sequence if and only if there exists an $\varepsilon>0$ such that $\varrho(z_n,z_m)\geqslant\varepsilon$ for $n\neq m$ and the measure $\Sigma(\operatorname{Im} z_n)\delta_{z_n}$ is a Carleson measure. Here δ_{z_n} denotes the point mass at z_n. On D, we replace the measure with $\Sigma(1-|z_n|)\delta_{z_n}$.

3. Proof of Theorem 1

We claim that it suffices to prove the theorem for algebras of the form $A_u=[H^\infty, u, \bar{u}]$ where u is a unimodular function in L^∞. To see this, note that any algebra A containing

H^∞ is generated by its invertible elements. Now, if $f \in A^{-1}$, there exists $g \in (H^\infty)^{-1}$ such that $|f| = |g|$ a.e. Then $u = fg^{-1}$ and $\bar{u} = gf^{-1}$ are unimodular, and we see that A is generated by H^∞ and $G = \{u \in A: u$ is unimodular and $\bar{u} \in A\}$. It is easy to see that $\mathcal{M}[A_u] = \{\varphi \in \mathcal{M}[H^\infty]: |\varphi(u)| = 1\}$ and $\mathcal{M}[A] = \{\varphi \in \mathcal{M}[H^\infty]: |\varphi(u)| = 1$ for all u in $G\}$. Now $(A_u)_I \subset A_I \subset A$ for all u in G, so that $\mathcal{M}[A] \subset \mathcal{M}[A_I] \subset \bigcap_{u \in G} \mathcal{M}[(A_u)_I]$. If $\mathcal{M}[(A_u)_I] = \mathcal{M}[A_u]$ for all u in G, we have $\mathcal{M}[A] \subset \mathcal{M}[A_I] \subset \bigcap_{u \in G} \mathcal{M}[A_u] = \mathcal{M}[A]$. Thus $\mathcal{M}[A] = \mathcal{M}[A_I]$. This proves the claim.

For the remainder of this discussion, let u be a fixed, nonconstant, unimodular function in L^∞ and let $A_u = [H^\infty, u, \bar{u}]$. For each α, $0 < \alpha < 1$, we wish to find an interpolating Blaschke product B_α, such that

$$\text{There exists } \beta < 1 \text{ such that if } B_\alpha(z) = 0, \text{ then } |u(z)| \leqslant \beta. \tag{3.1}$$

$$\text{If } |u(z)| < \alpha, \quad \text{then } |B_\alpha(z)| \leqslant \frac{1}{10}. \tag{3.2}$$

Assuming we can do this for the moment, we prove our theorem as follows. Let $B = [H^\infty, \{\bar{B}_\alpha\}\ 0 < \alpha < 1\}$. Suppose $\varphi \in \mathcal{M}[A_u]$, and suppose $\varphi(B_\alpha) = 0$ for some α. K. Hoffman [13, p. 206] has shown that since B_α is interpolating, φ is in the closure of the zeros $\{z_n\}$ of B_α. By (3.1) above, $|u(z_n)| \leqslant \beta < 1$, so that $|\varphi(u)| \leqslant \beta < 1$, contradicting the assumption that $\varphi \in \mathcal{M}[A_u]$. So $\varphi(B_\alpha) \neq 0$ for each $\varphi \in \mathcal{M}[A_u]$ and each B_α. Thus each B_α is invertible in A_u. We see now, that $B \subset (A_u)_I$, so that $\mathcal{M}[B] \supset \mathcal{M}[(A_u)_I] \supset \mathcal{M}[A_u]$. For the opposite inclusions, note that $\mathcal{M}[B] = \{\varphi \in \mathcal{M}[H^\infty]: |\varphi(B_\alpha)| = 1$ for each $\alpha\}$. Suppose $\varphi \in \mathcal{M}[B]$ and $|\varphi(u)| < \alpha < 1$. By the corona theorem [4], there exists a net $\{z_\gamma\}$ in D, such that z_γ converges to φ. There exists a γ_0, such that if $\gamma > \gamma_0$, then $|u(z_\gamma)| < \alpha$. By (3.2) above, $|B_\alpha(z_\gamma)| \leqslant 1/10$ for $\gamma > \gamma_0$, and we see that $|\varphi(B_\alpha)| \leqslant 1/10$. This contradiction implies that $|\varphi(u)| = 1$, so that $\varphi \in \mathcal{M}[A_u]$. We have now shown that $\mathcal{M}[B] = \mathcal{M}[(A_u)_I] = \mathcal{M}[A_u]$.

It remains to find B_α, given u and α. We will surround the places where $|u| < \alpha$ by contours Γ which induce a Carleson measure. This construction comes from the proof of the corona theorem [4], by L. Carleson. We will then uniformly distribute, in the ϱ-metric, a sequence $\{z_n\}$ on the contours. Our interpolating Blaschke product will have $\{z_n\}$ as its zeros. Our method is very similar to S. Ziskind's [19], except that we work with a bounded harmonic function with unimodular boundary values and we give several technical simplifications.

4. Preliminaries to the construction

The construction is best explained in the upper half-plane H^+. So suppose u is a bounded harmonic function on H^+ with unimodular boundary values, and fix α, $0 < \alpha < 1$.

Lemma 1. *There exists an* $\alpha' < 1$, *such that if* $\inf_R |u(z)| < \alpha$ *for some rectangle of the form* $R = \{x+iy: x_0 \leq x \leq x_0 + \delta, \delta/2 \leq y \leq \delta\}$, *then* $\sup_R |u(z)| < \alpha'$.

Proof. By a translation and a dilation, we can assume $x_0 = 0$ and $\delta = 1$. The result now follows by a normal families argument.

Let S be a square of the form $\{x+iy: x_0 \leq x \leq x_0 + \delta, 0 < y \leq \delta\}$. Find $\beta < 1$ such that $(1-\beta)/(1-\alpha') < 10^{-4}$. For a set $U \subset \mathbf{R}$, let $|U|$ denote its measure.

Lemma 2. *Suppose* $x_0 \leq \operatorname{Re} a \leq x_0 + \delta$ *and* $\delta/2 \leq \operatorname{Im} a \leq \delta$ *and* $|u(a)| > \beta$. *Let* $E = \{x+iy \in S: |u(x+iy)| < \alpha'\}$. *Let* E^* *be the vertical projection of* E *on* $\{x+iy: y=0\}$. *Then* $|E^*| \leq \delta/2$.

This lemma is essentially proved in [12] and in [4].

Proof. By a translation and a dilation, again, we can assume $x_0 = 0$ and $\delta = \pi/2$. Let $\mathcal{S}$ be the strip $\{x+iy: 0 < y < \pi\}$ and let $\varphi(z) = e^z$. Now φ maps $\mathcal{S}$ one-to-one and conformally onto the upper half-plane H^+. First we assume E is bounded by a finite number of Jordan curves. Let ω and ω_1 be the harmonic measures for E relative to $\mathcal{S} \backslash E$ and $\varphi(E)$ relative to $H^+ \backslash \varphi(E)$, respectively. Let ω^* and ω_1^* be the harmonic measures for E^* relative to $\mathcal{S}$ and $\varphi(E)^*$ relative to H^+, respectively. Here $\varphi(E)^*$ is the *circular* (clockwise) projection of $\varphi(E)$ on $\{\operatorname{Im} z = 0\}$. Hall's lemma [11, p. 208] and an elementary estimate show that

$$\omega(a) = \omega_1(\varphi(a)) \geq 2/3\omega_1^*(\varphi(\bar{a} + \pi i)) \geq 2 \times 10^{-4} |E^*|.$$

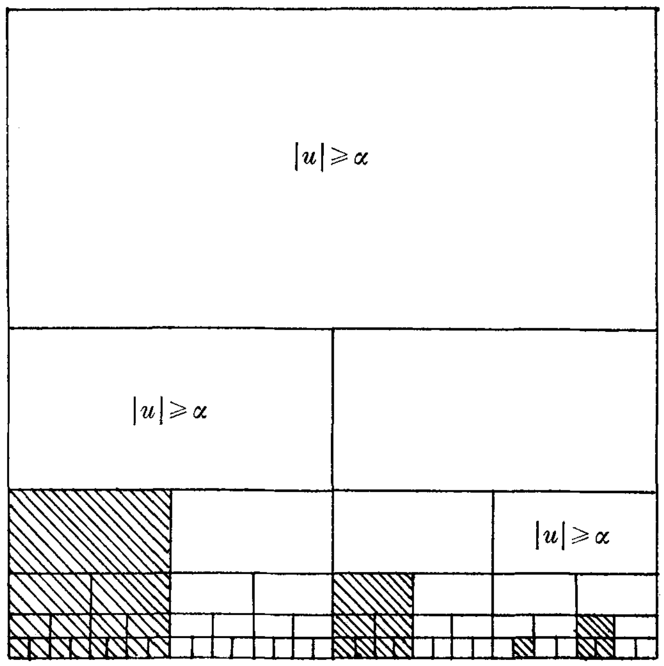
Notice that on the boundary of $\mathcal{S} \backslash E$, $|u(z)| \leq 1 - \omega(z) + \alpha' \omega(z)$. By the maximum principle, the inequality persists on $\mathcal{S} \backslash E$. So $\beta < 1 - \omega(a) + \alpha' \omega(a)$ and we conclude that $|E^*| < \pi/4$.

Now for an arbitrary E, we can cover the compact set $\{z \in E: |u(z)| \leq \alpha' - 1/n, \operatorname{Im} z \geq 1/n\}$ by a finite number of balls contained in E. Let E_n be the complement of the unbounded component of the union of these balls. Apply the above reasoning to each E_n and E_n^*.

5. The construction of Γ

Let $S^{(0)} = \{x+iy: 0 \leq x \leq 1, 0 < y \leq 1\}$. Partition the bottom half of $S^{(0)}$ into two squares with sides of length 1/2. Partition the bottom half of each of these squares into two more squares with sides of length 1/4. Continue the process indefinitely. We wish to describe two procedures which we will apply to a subcollection of the squares in $S^{(0)}$. For S a square in $S^{(0)}$, let T_S be the top half of S.

Case 1. $\sup_{T_S} |u| > \beta$.



Case 1.

If $\tilde{S}$ is a square contained in S and $|u(z)|<\alpha$ for some z in $T_{\tilde{S}}$, shade $\tilde{S}$ unless it is already shaded. Note that by Lemma 1, $\sup_{T_{\tilde{S}}}|u|<\alpha'<\beta$ and by Lemma 2,

$$\sum_{\tilde{S}\text{ shaded}} |\tilde{S}^*|\leqslant \tfrac{1}{2}|S^*|. \tag{5.1}$$

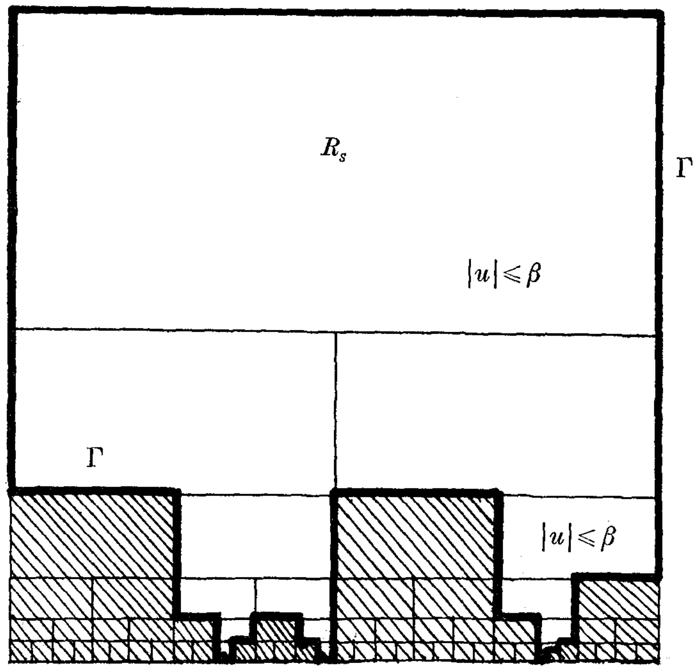
Case 2. $\sup_{T_S}|u|\leqslant\beta$.

If $\tilde{S}$ is a square contained in S and $\sup_{T_{\tilde{S}}}|u|>\beta$, shade $\tilde{S}$ unless it is already shaded. Let $R_S=S\setminus\bigcup_{\tilde{S}\text{ shaded}}\tilde{S}$ and note that

$$|\partial R_S|\leqslant 6|S^*|. \tag{5.2}$$

Proceed as follows. Apply the appropriate case to $S^{(0)}$, obtaining shaded squares $S_1^{(1)}, S_2^{(1)}, S_3^{(1)}, \ldots$. On each $S_j^{(1)}$, apply the appropriate case, obtaining doubly shaded squares $S_1^{(2)}, S_2^{(2)}, S_3^{(2)}, \ldots$. Repeat this process indefinitely. Observe that we alternate cases in passing from one shaded square to a shaded descendant. Define Γ as the union of the boundaries of the R_S obtained from applications of Case 2. To see that Γ induces a Carleson measure, it suffices to check $|\Gamma\cap S|\leqslant C|S^*|$ where S is a square in the grid on $S^{(0)}$. By (5.1) and (5.2), we see that

$$|\Gamma\cap S|\leqslant\sum_{n=0}^{\infty}6\times 2^{-n}|S^*|=12|S^*|.$$



Case 2.

Note that any point in $S^{(0)}$ for which $|u(z)| < \alpha$ will be in some R_S. Also, $|\partial R_S \cap \{\operatorname{Im} z = 0\}| = 0$. This follows since u has unimodular vertical limits a.e. and any point in $\partial R_S \cap \{\operatorname{Im} z = 0\}$ is a point where $\lim_{y \to 0} \sup |u(x+iy)| \leqslant \beta < 1$.

6. The construction of $\mathbf{B}_\alpha$

In this section, we will first consider the behavior of a Blaschke product whose zeros are located on $\Gamma \subset S^{(0)}$. Choose $\varepsilon < 1/10$ and place points a_n on Γ so that each z in Γ satisfies $\varrho(z, a_n) < 2\varepsilon$ for some n and so that $\varrho(a_n, a_m) \geqslant \varepsilon$ if $n \neq m$. It is shown in [19] that $\{a_n\}$ is an interpolating sequence. Let B be the Blaschke product whose zero sequence is $\{a_n\}$. We wish to verify (3.1) and (3.2) on $S^{(0)}$. By construction (3.1) holds. If $z \in S^{(0)}$ and $|u(z)| < \alpha$, then z is in some R_S. But $|B| < \varepsilon$ on $\partial R_S \backslash \{\operatorname{Im} z = 0\}$ and $\partial R_S \cap \{\operatorname{Im} z = 0\}$ has harmonic measure zero as a subset of ∂R_S, since it has length zero. We conclude that $|B| \leqslant \varepsilon < 1/10$ on R_S by Theorem 1.63 of [14], and (3.2) holds.

We now wish to construct B_α. Let u be a unimodular function in $L^\infty(T)$. For $k = 0, 1, ..., 7$, let

$$\psi_k(z) = \frac{e^{i\pi/4}+1}{e^{i\pi/4}-1}\left(\frac{z-e^{ik\pi/4}}{z+e^{ik\pi/4}}\right)$$

and let $\{a_{n,k}\}$ be the zeros obtained by applying the procedure described above to the function $u \circ \psi_k^{-1}$. We can find a subset $\{z_m\}$ of $\bigcup_{k,n} \psi_k^{-1}(a_{n,k})$ such that $\varrho(z_m, z_l) \geqslant \varepsilon$, for $m \neq l$, and such that for each m, there is an l for which $\varrho(z_m, z_l) < 3\varepsilon$. Let B_α be the Blaschke product whose zero sequence is $\{z_m\}$. Then B_α is an interpolating Blaschke product, and we have that (3.1) and (3.2) hold in $|z| \geqslant 1/2$. If $|u(z_0)| < \alpha$ for some z_0 with $|z_0| < 1/2$, increase the zero sequence of B_α with a finite number of distinct zeros in a neighborhood of z_0, so that $|B_\alpha(z)| \leqslant 1/10$ for all $|z| < 1/2$. This proves the theorem.

7. Further results

In view of S. Y. Chang's result, we have shown that every subalgebra of L^∞ containing H^∞ is generated by H^∞ and $\{\bar{B} \in A$: B is an interpolating Blaschke product$\}$. Which algebras are of the form $[H^\infty, \bar{B}]$, where B is an interpolating Blaschke product? If f is a simple function, it is possible to see that $[H^\infty, f] = [H^\infty, \bar{B}]$ for some interpolating Blaschke product B. If A is generated by H^∞ and a countable collection of L^∞ functions, then $A = [H^\infty, U\bar{V}]$, where U and V are inner functions and $\bar{U}V \in A$. Such an algebra is contained in some algebra of the form $[H^\infty, \bar{B}]$, but it is not clear whether $A = [H^\infty, \bar{B}]$ or not.

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