AN EXISTENCE THEOREM FOR HARMONIC MAPPINGS OF RIEMANNIAN MANIFOLDS

BY

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1. Introduction

Let X be a compact, connected n-dimensional Riemannian manifold of class C^3, with interior Ω and nonvoid boundary Σ. Moreover, let $\mathcal{M}$ be a complete Riemannian manifold without boundary of dimension $N \geqslant 2$, and of class C^3. To every mapping U: $X \to \mathcal{M}$ of class C^1 one can associate an *energy* $E(U)$ defined by

$$E(U) = \int_X e(U)\, dR^n. \tag{1.1}$$

Here R^n stands for the n-dimensional Lebesgue measure on X induced by its metric while the *energy density*

$$e(U) = \tfrac{1}{2} \operatorname{tr}_X \langle U_*, U_* \rangle_m$$

is the trace of the pull-back of the metric tensor of $\mathcal{M}$ under the mapping U taken with respect to the metric tensor of X. A mapping $U: \Omega \to \mathcal{M}$ is said to be *harmonic* if it is of class C^2 and satisfies the Euler–Lagrange equations of the energy functional. In local coordinates, these can be written in the form

$$\Delta_X u^l + \Gamma^l_{ik}(u)\gamma^{\alpha\beta} D_\alpha u^i D_\beta u^k = 0, \quad 1 \leqslant l \leqslant N, \tag{1.2}$$

where

$$\Delta_X = \gamma^{-1/2} D_\alpha (\gamma^{1/2} \gamma^{\alpha\beta} D_\beta)$$

is the Laplace–Beltrami operator on X. Here and in the sequel we use the following notations: $\gamma_{\alpha\beta}$ are the coefficients of the metric of X, with respect to some local coordinate system, $(\gamma^{\alpha\beta})$ is the inverse of $(\gamma_{\alpha\beta})$, and $\gamma = \det(\gamma_{\alpha\beta})$. The coefficients of the metric of $\mathcal{M}$

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are denoted by g_{ik} with (g^{ik}) being the inverse of (g_{ik}), while Γ^l_{ik} are the Christoffel symbols. Greek indices $\alpha, \beta, \ldots$ are to be summed from 1 to n, Latin indices $i, j, \ldots$ from 1 to N. The Einstein summation convention is used.

Our object is to solve the Dirichlet problem, i.e. to find a harmonic mapping U whose restriction to Σ coincides with a given function $\Phi: \Sigma \to \mathcal{M}$.

In order to give a precise statement of our results we introduce the notion of *normal range* of a point $P \in \mathcal{M}$ as the complement of the cut locus of P in $\mathcal{M}$, i.e. the maximal domain of any normal coordinate system with center P. Our assumption on the Dirichlet data Φ will be that $\Phi(\Sigma)$ is contained in a geodesic ball

$$\mathcal{K}_M(P) = \{Q \in \mathcal{M};\ \mathrm{dist}\,(Q, P) \leqslant M\}$$

which lies within normal range of *all* of its points. For a discussion of this condition, which is implied by a strong convexity condition, we refer to section 2.

We can now formulate the following theorem which is a consequence of Theorems 2, 3, and 4 below.

THEOREM 1. *Assume that the image of $\Phi \in C^1(\Sigma, \mathcal{M})$ is contained in a ball $\mathcal{K}_M(Q)$ which lies within normal range of all of its points, and for which*

$$M < \pi/(2\sqrt{\varkappa}) \tag{1.3}$$

where $\varkappa \geqslant 0$ is an upper bound for the sectional curvature of $\mathcal{M}$. *Then these is a harmonic mapping U of class $C^2(\Omega, \mathcal{M}) \cap C^0(\mathrm{X}, \mathcal{M})$ such that $U|_\Sigma = \Phi$ and $U(\mathrm{X}) \subset \mathcal{K}_M(Q)$.*

In the case $n = 1$ this is a well-known result about geodesics. For $n = 2$ the theorem can be derived from the work of Morrey [16], taking our estimates (2.2) and (2.3) into account. However, since Morrey's main tool, a regularity theorem for minima of certain variational problems, is true only for $n = 2$ this approach cannot immediately be carried over to higher dimensions. The variational method due to Eliasson [4] and K. Uhlenbeck [19] is not applicable to the Dirichlet problem for harmonic maps, since it rests in an essential way on the assumption that the boundary of X is void.

In arbitrary dimensions, the first boundary value problem for harmonic maps was recently solved by R. S. Hamilton [6] provided that the sectional curvature of $\mathcal{M}$ is non-positive. The approach of Hamilton, as well as that of the preceding work by Eells and Sampson [3], and Hartman [7], is to deal with the parabolic system connected with (1.2).

The present authors [9] recently reproved parts of Hamilton's results using degree theory and a priori estimates, much in the spirit of [2]. In [10] they extended their method to include manifolds also of possibly positive curvature, assuming that $M < \pi/(4\sqrt{\varkappa})$ instead of (1.3).

In the present paper we shall use the direct method of the calculus of variations to construct a "weak" solution U of the problem

$$E(U) \to \min$$

with the side conditions $U|_\Sigma = \Phi$ and $U(\mathrm{X}) \subset \mathcal{K}_{M'}(Q)$, where $M < M' < \pi/(2\sqrt{\varkappa})$. Then an appropriate maximum principle implies that U is in fact a weak solution of (1.2), and satisfies $U(\mathrm{X}) \subset \mathcal{K}_M(Q)$. Next we derive a regularity theorem which shows that this weak solution is actually a classical solution. The proof of this result is based on the methods of [11], but an important idea is borrowed from Wiegner [22]. Finally we complete the proof of Theorem 1 by showing that the regularity of U holds also onto the boundary. In fact, we shall give more precise results on the boundary behavior than that stated in Theorem 1.

At the end of the paper we exhibit an example of a weakly harmonic mapping $U: \mathrm{X} \to \mathcal{M} = S^N$ which is discontinuous and satisfies $U(\mathrm{X}) \subset \mathcal{K}_M(Q)$ with $M = \pi/(2\sqrt{\varkappa})$.

Hence our regularity result is optimal in this respect. It is tempting to conjecture that the Dirichlet problem for harmonic mappings cannot be solved in general if the condition (1.3) is violated.

2. Auxiliary differential geometric estimates

The following functions will enter in our estimates:

$$a_\nu(t) = \begin{cases} t\sqrt{\nu}\,\mathrm{ctg}\,(t\sqrt{\nu}) & \text{if } \nu > 0, \quad 0 \leqslant t < \pi/\sqrt{\nu} \\ t\sqrt{-\nu}\,\mathrm{ctgh}\,(t\sqrt{-\nu}) & \phantom{\text{if }} \nu \leqslant 0, \quad 0 \leqslant t < \infty \end{cases}$$

$$b_\nu(t) = \begin{cases} \dfrac{\sin (t\sqrt{\nu})}{t\sqrt{\nu}} & \text{if } \nu > 0, \quad 0 \leqslant t < \pi/\sqrt{\nu} \\ \dfrac{\sinh (t\sqrt{-\nu})}{t\sqrt{-\nu}} & \phantom{\text{if }} \nu \leqslant 0, \quad 0 \leqslant t < \infty \end{cases}$$

LEMMA 1. *Let $u = (u^1, u^2, \ldots, u^N)$ be normal coordinates on $\mathcal{M}$ associated with an arbitrarily chosen normal chart $\mathcal{N}(Q)$ around Q such that Q has coordinates $(0, 0, \ldots, 0)$. Denote by $g_{ik}(u)$, $\Gamma^l_{ik}(u)$, and $\Gamma_{ikl}(u)$ the coordinates of the first fundamental form and the Christoffel symbols, respectively, in this coordinate system. Assume that the sectional curvature $\mathbf{K}$ of $\mathcal{M}$ satisfies*

$$\omega \leqslant \mathbf{K} \leqslant \varkappa \quad \text{with } -\infty \leqslant \omega \leqslant 0 \leqslant \varkappa < \infty.$$

Then for all u satisfying $|u| = (u^i u^i)^{1/2} < \pi/\sqrt{\varkappa}$ and for all $\xi \in \mathbf{R}^N$ we have the following estimates

$$\{a_\varkappa(|u|) - 1\} g_{ik}(u)\xi^i\xi^k \leqslant \Gamma_{ikl}(u) u^l \xi^i \xi^k \leqslant \{a_\omega(|u|) - 1\} g_{ik}(u)\xi^i\xi^k \tag{2.1}$$

$$\{\delta_{ik}-a_\omega(|u|)g_{ik}(u)\}\xi^i\xi^k \leqslant \Gamma^l_{ik}(u)u^l\xi^i\xi^k \leqslant \{\delta_{ik}-a_\varkappa(|u|)g_{ik}(u)\}\xi^i\xi^k \tag{2.2}$$

$$b^2_\varkappa(|u|)\,\xi^i\xi^i \leqslant g_{ik}(u)\,\xi^i\xi^k \leqslant b^2_\omega(|u|)\,\xi^i\xi^k. \tag{2.3}$$

Proof. The estimates (2.1) and (2.3) follow from Rauch's comparison theorem (for a proof, cf. [8], Lemma 6). To verify (2.2) we use Gauss' lemma (cf. [5], p. 136). For normal coordinates we obtain

$$g_{ik}(u)\,u^k = u^i$$

whence by differentiation

$$\Gamma^l_{ik}(u)\,u^l = \delta_{ik} - g_{ik}(u) - \Gamma_{ikl}(u)\,u^l. \tag{2.4}$$

Now (2.2) follows immediately from (2.1) and (2.4).

In the regularity proof below we shall need a bound for the second member of (1.2). For our purposes it is sufficient to note that for reasons of compactness, there exists, to every $Q \in \mathfrak{M}$ and $M \in \mathbf{R}^+$ such that $M < i(Q)$, a constant $c = c(Q, M)$ with the property that

$$\{\sum_l |\Gamma^l_{ik}(u)\,\xi^i\xi^k|^2\}^{1/2} \leqslant c|\xi|^2 \quad \text{for } \xi \in \mathbf{R}^N,\ |u| \leqslant M, \tag{2.5}$$

if the Γ^l_{ik} are calculated with respect to normal coordinates around Q, and where $\mathrm{i}(Q)$ denotes the cut locus distance of Q.

It is possible to estimate the constant c in terms of the tensor $R^{(1)}$, introduced by Kern,

$$R^{(1)}(X, Y, Z, W) = (D_X R)(Y, Z, W) - (D_Z R)(W, X, Y)$$

where $D_X R$ and $D_Z R$ denote covariant derivatives of the Riemann curvature tensor R of $\mathfrak{M}$, and X, Y, Z, and W are vector fields on $\mathfrak{M}$. These estimates are, however, rather complicated to state and to prove wherefore we refer to [14] for statements and proofs.

As was mentioned in the introduction, the condition that the ball $\mathcal{K}_M(P)$ be within normal range of each of its points can be thought of as a convexity condition. In fact, $\mathcal{K}_M(P)$ is convex provided that it is within normal range of its points and satisfies $M < \pi/(2\sqrt{x})$ [13, 4]. Also, if $\mathcal{K}$ is a compact set in $\mathfrak{M}$ which does not meet its cut locus, then there exists a neighborhood $\mathcal{U}$ of $\mathcal{K}$ with the same property [12, 3.5]. Hence, if $\mathcal{K}_M(\mathrm{P})$ is within normal range of its points then the same holds for $\mathcal{K}_{M+\varepsilon}(P)$ with $\varepsilon > 0$ sufficiently small.

It is also known that $\mathcal{K}_M(P)$ lies within normal range of all of its points if one of the following four conditions is satisfied. Here $\mathbf{K}$ denotes the sectional curvature.

(i) $\mathfrak{M}$ is simply connected, and $\mathbf{K} \leqslant 0$ ([5], p. 201).

(ii) $\mathfrak{M}$ is connected and orientable, N is even, $0 < \mathbf{K} \leqslant \varkappa$, and $M < \pi/(2\sqrt{\varkappa})$ ([5], pp. 227–228, and [13], pp. 3–4).

(iii) $\mathcal{M}$ is compact, connected, and non-orientable, N is even, $0<\mathbf{K}\leqslant\varkappa$, and $M<\pi/(4\sqrt{\varkappa})$ ([5], pp. 229–230).

(iv) $\mathcal{M}$ is simply connected, $0<\varkappa/4<\mathbf{K}\leqslant\varkappa$, and $M<\pi/(2\sqrt{\varkappa})$ ([5], p. 254).

Finally we note that in local coordinates the energy density of a mapping $U\in C^1$ is given by

$$e(U)=\tfrac{1}{2}g_{ik}(u)\gamma^{\alpha\beta}D_\alpha u^i D_\beta u^k. \tag{2.6}$$

3. Existence of weakly harmonic mappings

The Sobolev space $H_2^1(\Omega, \mathbf{R}^N)$ is constituted by measurable mappings u: $\mathrm{X}\to\mathbf{R}^N$ with the property that $u\circ\chi^{-1}\in H^1_{2,\,\mathrm{loc}}(W, \mathbf{R}^N)$ for every coordinate map χ of X with range $W\subset \mathbf{R}^n_+=\{x\in\mathbf{R}^n;\ x_n\geqslant 0\}$. The Hilbert space structure on $H_2^1(\Omega, \mathbf{R}^N)$ is defined by a scalar product with associated norm

$$\|u\|_1^2=\int_\Omega |u|^2 dR^n+\int_\Omega e_N(u)\,dR^n$$

where the invariant e_N is defined in local coordinates by

$$e_N(u)=\tfrac{1}{2}\gamma^{\alpha\beta}D_\alpha(u^i\circ\chi^{-1})\,D_\beta(u^i\circ\chi^{-1}).$$

The subspace $\mathring{H}_2^1(\Omega, \mathbf{R}^N)$ is the closure of $C_c^\infty(\Omega, \mathbf{R}^N)$. It is easy to check that all properties of Sobolev spaces of a local nature are retained. For example, every element of $H_2^1(\Omega, \mathbf{R}^N)$ has a trace on Σ in $L^2(\Sigma, \mathbf{R}^N)$ such that if two elements have the same trace then their difference belongs to $\mathring{H}_2^1(\Omega, \mathbf{R}^N)$.

Since the composition of an H_2^1 function with a C^1 mapping is another H_2^1 function, we can define $H_2^1(\Omega, \mathcal{M})$ unambiguously if we require that an element $U\in H_2^1(\Omega, \mathcal{M})$ have its image within normal range of some point $Q\in\mathcal{M}$, and that its representation in normal coordinates around Q belong to $H_2^1(\Omega, \mathbf{R}^N)$. The same applies to $\mathring{H}_2^1(\Omega, \mathcal{M})$ and $H_2^1\cap L^\infty(\Omega, \mathcal{M})$.

To simplify the notational apparatus we introduce the following

Conventions. An equality sign between two H_2^1 mappings means that equality might hold only R^n almost everywhere, while the abbreviation sup $|u|$ should be interpreted as ess sup $|u|$ with respect to R^n, etc. Moreover, if the choice of local coordinate system in X is immaterial for the purpose at hand, we shall not distinguish between a point x and its coordinate representation $(x_1, \ldots, x_n)$, between a subset $S\subset \mathrm{X}$ and $\chi(S)\subset\mathbf{R}^n_+$, or between $u|_S$ and $u\circ\chi^{-1}$, if $u\in H_2^1$ and $S\subset\chi^{-1}(W)$. This abuse of notation will not cause any confusion.

In order to circumvent the lack of (global) linear structure in $H_2^1(\Omega, \mathcal{M})$ we introduce, for any $Q\in\mathcal{M}$ and any $M'<i(Q)$, the set

$$\mathcal{B}_{M'} = \mathcal{B}_{M'}(Q) = \{U \in H_2^1(\Omega, \mathcal{M});\ \sup_\Omega \operatorname{dist}(U(x), Q) \leqslant M'\}.$$

Via a normal chart with center Q we can *identify* $\mathcal{B}_{M'}(Q)$ with the convex, weakly sequentially closed subset of $H_2^1(\Omega, \mathbf{R}^N)$ defined by $\{u \in H_2^1 \cap L^\infty(\Omega, \mathbf{R}^N), \sup_\Omega |u| \leqslant M'\}$. The energy functional E can then be extended to $\mathcal{B}_{M'}$, using the formulas (1.1) and (2.6).

If Ω is isomorphically imbeddable in $\mathbf{R}^n$ it is well known that E is lower semicontinuous on $\mathcal{B}_{M'}$, i.e. if $\{u_k\}_1^\infty \in \mathcal{B}_{M'}$ converges weaky to u in $H_2^1(\Omega, \mathbf{R}^N)$ then $\liminf_{k\to\infty} E(u_k) \geqslant E(u)$. It is easy to see, however, that the proof of this fact, see e.g. Theorem 1.8.2 in [17], can be carried over to the general case.

Lemma 2. *Assume that the ball $\mathcal{K}_{M'}(Q)$ lies within normal range of its center, and that*

$$M' < \pi/(2\sqrt{\varkappa})$$

where $\varkappa \geqslant 0$ is an upper bound for the sectional curvature of $\mathcal{M}$. Then to every $\varphi \in \mathcal{B}_{M'}(Q)$ there is a solution of the variational problem

$$E(u) \to \min,\ u \in \mathcal{B}_{M'} \cap \{u - \varphi \in \mathring{H}_2^1(\Omega, \mathbf{R}^N)\}.$$

Proof. The element φ itself being admissible, the clearly non-negative infimum $\mathcal{E}$ of the variational problem is finite. Moreover, from (2.3) and from the fact that the metric tensor $(\gamma^{\alpha\beta})$ is positive definite we deduce that for some constant $c_0 > 0$

$$E(u) \geqslant c_0 \int_\Omega e_N(u)\, dR^n, \quad u \in \mathcal{B}_{M'}.$$

This implies that

$$\|u\|_1^2 \leqslant \text{const}[(M')^2 + E(u)]$$

whence we see that a minimizing sequence is bounded in $H_2^1(\Omega, \mathbf{R}^N)$, and we may assume that it is weakly convergent to some $u \in \mathcal{B}_{M'}$. By the lower semicontinuity of E we must have $E(u) = \mathcal{E}$, and hence u is the desired minimizing element.

A straight-forward computation shows that the first variation of the functional E, at $u \in \mathcal{B}_{M'}(Q)$ in the direction of ψ, defined by

$$\delta E(u, \psi) = \lim_{\varepsilon \searrow 0} \varepsilon^{-1}\{E(u + \varepsilon\psi) - E(u)\},$$

exists for all $\psi \in \mathring{H}_2^1 \cap L^\infty(\Omega, \mathbf{R}^N)$ such that $u + \varepsilon\chi \in \mathcal{B}_{M'}(Q)$ for all small enough nonnegative ε, and is given by

$$\delta E(u, \psi) = \int_\Omega \delta e(u, \psi)\, dR^n.$$

Here the invariant variational derivative $\delta e(u, \psi)$ is given, in local coordinates, by

$$\delta e(u, \psi) = g_{ik}(u)\gamma^{\alpha\beta}D_\alpha u^i D_\beta \psi^k + \tfrac{1}{2}D_l\, g_{ik}(u)\gamma^{\alpha\beta}D_\alpha u^i D_\beta u^k \psi^l.$$

Taking the identities

$$D_l g_{ik} = \Gamma_{lik} + \Gamma_{lki}, \qquad \Gamma^j_{ik} = g^{jm}\Gamma_{imk} \tag{3.1}$$

into account, we can formulate this as

LEMMA 3. *The minimizing function u of Lemma 2 satisfies*

$$\delta E(u, \psi) = \int_\Omega \delta e(u, \psi)\, dR^n \geqslant 0$$

for all $\psi \in \mathring{H}^1_2 \cap L^\infty(\Omega, \mathbf{R}^N)$ such that $u + \varepsilon\psi \in \mathcal{B}_{M'}(Q)$ for some $\varepsilon > 0$, where

$$\begin{aligned}\delta e(u, \psi) &= \{g_{ik}(u)\gamma^{\alpha\beta}D_\alpha u^i D_\beta \psi^k + \Gamma_{lik}(u)\gamma^{\alpha\beta}D_\alpha u^i D_\beta u^k \psi^l\} \\ &= g_{ik}(u)\gamma^{\alpha\beta}D_\alpha u^i [D_\beta \psi^k + \Gamma^k_{jl} D_\beta u^j \psi^l].\end{aligned} \tag{3.2}$$

LEMMA 4. *Suppose that the function φ of Lemma 2 satisfies $\varphi \in \mathcal{B}_M$ for some $M < M'$. Then the solution u of the variational problem is also contained in $\mathcal{B}_M$, and satisfies*

$$\delta E(u, \psi) = 0 \quad \text{for all } \psi \in \mathring{H}^1_2 \cap L^\infty(\Omega, \mathbf{R}^N).$$

Proof. If $\eta \geqslant 0$ belongs to $C^\infty_c(\Omega, \mathbf{R})$ we see that $|u - \varepsilon u\eta| = |1 - \varepsilon\eta||u| \leqslant M'$ for $\varepsilon^{-1} > \sup_\Omega \eta$, and since $u - \varepsilon u\eta - \varphi \in \mathring{H}^1_2$ we may use $\psi = -u\eta$ as a test function in Lemma 3, and thus

$$\delta E(u, u\eta) \leqslant 0. \tag{3.3}$$

Now using the inequality (2.1) and the Gauss lemma, $g_{ik}(u)u^k = u^i$, we see that from (3.2) follows

$$\begin{aligned}\delta e(u, u\eta) &= g_{ik}(u)u^k\gamma^{\alpha\beta}D_\alpha u^i D_\beta \eta + \eta\gamma^{\alpha\beta}[g_{ik}(u)\, D_\alpha u^i D_\beta u^k + u^l\Gamma_{lik}(u)\, D_\alpha u^i D_\beta u^k] \\ &\geqslant \tfrac{1}{2}\gamma^{\alpha\beta}D_\alpha |u|^2 D_\beta \eta + a_\varkappa(M')\eta g_{ik}(u)\gamma^{\alpha\beta}D_\alpha u^i D_\beta u^k \geqslant \tfrac{1}{2}\gamma^{\alpha\beta}D_\alpha |u|^2 D_\beta \eta.\end{aligned}$$

In view of (3.3) the function $|u|^2 \in H^1_2 \cap L^\infty(\Omega, \mathbf{R})$ is a weak subsolution of the Laplace–Beltrami operator on Ω, and it follows from an obvious extension of Stampacchia's maximum principle ([11], Lemma 2.1) that

$$\sup_\Omega |u|^2 \leqslant \sup_{\partial\Omega} |u|^2 = \sup_{\partial\Omega} |\varphi|^2 \leqslant M^2.$$

The first part of the lemma thus being proved, the second part follows from Lemma 3 upon realizing that if $\sup_\Omega |u| \leqslant M$ then $u \pm \varepsilon\psi \in \mathcal{B}_{M'}$ for every $\psi \in H_2^1 \cap L^\infty(\Omega, \mathbf{R}^N)$ and for all small enough ε.

If U is a harmonic mapping with range in a ball $\mathcal{K}_M(Q)$ with $M < i(Q)$ then we can use the equations (1.2) and Stokes' formula to see that the representation u of U satisfies

$$\int_\Omega \delta\tilde{e}(u, \varphi)\, dR^n = 0 \quad \text{for all } \varphi \in \mathring{H}_2^1 \cap L^\infty(\Omega, \mathbf{R}^N) \tag{3.4}$$

where

$$\delta\tilde{e}(u, \varphi) = \gamma^{\alpha\beta} D_\alpha u^i D_\beta \varphi^i - \Gamma^l_{ik}(u)\gamma^{\alpha\beta} D_\alpha u^i D_\beta u^k \varphi^l.$$

It is therefore natural to make the following.

Definition. A mapping $U \in H_2^1(\Omega, \mathcal{M})$ is *weakly harmonic* if, for some point $Q \in \mathcal{M}$, $U(\Omega)$ lies within normal range of Q, and if its representation u in normal coordinates around Q satisfies (3.4).

A simple computation, involving the formulas (3.1), shows that $\delta\tilde{e}(u, \varphi) = \delta e(u, \psi)$, if $\varphi^j = g_{jl}\psi^l$. Thus a mapping is weakly harmonic if and only if it is a critical point of the energy functional E. From this fact it is also obvious that the definition of weakly harmonic mapping is coordinate invariant, i.e. if $U(\Omega)$ lies within normal range of some other point $Q' \in \mathcal{M}$ then its representation u' in normal coordinates around Q' satisfies, mutatis mutandis, the relation (3.4).

We can formulate the results of this section in

THEOREM 2. *Assume that the image of $\Phi \in H_2^1(\Omega, \mathcal{M})$ is contained in a ball $\mathcal{K}_M(Q)$ for which*

$$M < \min\{\pi/(2\sqrt{\varkappa}),\ i(Q)\},$$

$\varkappa \geqslant 0$ being an upper bound for the sectional curvature in $\mathcal{M}$. Then there is a weakly harmonic mapping $U \in H_2^1(\Omega, \mathcal{M})$ with $U(\Omega) \subset \mathcal{K}_M(Q)$ such that the traces of U and Φ on Σ coincide.

Remark. It is easily seen that it is sufficient to assume $\varkappa$ to be a bound for the sectional curvature in $\mathcal{K}_M(Q)$. The same remark applies to Theorems 1, 3, and 4.

Proof. We need only apply Lemmata 2–4 with an M' satisfying $M < M' < \min\{\pi/(2\sqrt{\varkappa}), i(Q)\}$ and take into account the remark above about extremals of E and weakly harmonic mappings.

4. Regularity of weakly harmonic mappings

THEOREM 3. *Let $\mathcal{K}_M(Q)$ be a ball of $\mathcal{M}$ which is within normal range of all of its points, and for which*

$$M < \pi/(2\sqrt{\varkappa})$$

where $\varkappa \geqslant 0$ *is an upper bound for the sectional curvature of* $\mathcal{M}$. *Then a weakly harmonic mapping* U *with* $U(\Omega) \subset \mathcal{H}_M(Q)$ *is harmonic.*

Remark. By well-known results from the theory of linear elliptic partial differential equations one immediately concludes that if X and $\mathcal{M}$ are of class C^μ with $\mu > 2$ then a harmonic mapping U belongs to $C^\mu(\Omega, \mathcal{M})$, if μ is not an integer, and else to $C^{\mu-\varepsilon}(\Omega, \mathcal{M})$ for all $\varepsilon > 0$.

Proof. We start by observing that it is sufficient to prove that U is continuous in Ω. In fact, it then follows from (2.5) and the results of [11], [15], or [18], that U is Hölder continuous in Ω, and by methods which by now have become standard (see e.g. [15] and [17]) one raises the regularity of U as far as the coefficients of the system will allow.

To prove continuity at an arbitrary point x_0 of Ω we for once introduce normal coordinates in a suitably small neighborhood Ω^*, with smooth boundary, such that x_0 has coordinates $(0, 0, \ldots, 0)$. In Ω^* we consider the Green function G of the operator $L = -D_\alpha(\sqrt{\gamma}\gamma^{\alpha\beta} D_\beta)$. It is well known, cf. e.g. [20] and [21], that, for $n \geqslant 3$, G satisfies

$$0 \leqslant G(x, y) \leqslant K_1 |x-y|^{2-n}, \quad x, y \in \Omega^*, \tag{4.1}$$

$$G(x, y) \geqslant K_2 |x-y|^{2-n}, \quad \text{if } |x|, |y| < \varrho \quad \text{and} \quad B_{2\varrho}(0) \subset \Omega^*, \tag{4.2}$$

$$|\nabla_x G(x, y)| \leqslant K_3 |x-y|^{1-n}, \quad x, y \in \Omega^*. \tag{4.3}$$

Here K_1 and $K_2 > 0$ depend on n and on the ellipticity constants of L. K_3 will of course depend on Ω^* as well as on the coefficients $\sqrt{\gamma}\gamma^{\alpha\beta}$.

We shall also have occasion to use a slightly mollified Green function $G^\sigma(x, y)$ defined by

$$G^\sigma(x, y) = \fint_{B_\sigma(y)} G(x, z)\, dz, \quad B_\sigma(y) = \{|z-y| < \sigma\} \subset \Omega^*,$$

where we have used the notation $\fint_S v\, dx = [\text{mes } S]^{-1} \int_S v\, dx$. G^σ obviously belongs to $\mathring{H}_2^1 \cap L^\infty(\Omega^*, \mathbf{R})$ and satisfies

$$G^\sigma(x, y) \leqslant 2^{n-2} K_1 |x-y|^{2-n}, \quad \text{if } \sigma < \tfrac{1}{2}|x-y|, \tag{4.4}$$

and

$$\int \sqrt{\gamma}\, \gamma^{\alpha\beta} D_\alpha \zeta D_\beta G^\sigma(., y)\, dx = \fint_{B_\sigma(y)} \zeta\, dx \quad \text{for } \zeta \in \mathring{H}_2^1(\Omega^*, \mathbf{R}). \tag{4.5}$$

For the sake of brevity we use the notation

$$q(v) = \gamma^{\alpha\beta}(D_\alpha v^i D_\beta v^i - v^l \Gamma^l_{ik}(v)\, D_\alpha v^i D_\beta v^k) \tag{4.6}$$

for any representation v of the given mapping U with respect to normal coordinates with center in $\mathcal{K}_M(Q)$. We have $|v| \leqslant 2M < \pi/\sqrt{\varkappa}$, whence by (2.2)

$$q(v) \geqslant 2a_\varkappa(|v|)e(u). \tag{4.7}$$

We fix a normal coordinate system around the center Q of the ball $\mathcal{K}_M(Q)$ and reserve the letter u for the representation of U with respect to these coordinates. By (2.1) we get

$$q(u) \geqslant 2a_\varkappa(M)e(u) > 0. \tag{4.8}$$

Now in the defining relation (3.4) we use as a test vector $\varphi = uG^\sigma(.,y)$ and obtain

$$\int_{\Omega^*} q(u)\,G^\sigma(.,y)\sqrt{\gamma}\,dx = -\tfrac{1}{2}\int_{\Omega^*} \sqrt{\gamma}\,\gamma^{\alpha\beta}D_\alpha|u|^2 D_\beta G^\sigma(.,y)\,dx. \tag{4.9}$$

Let $w \in H_2^1(\Omega^*, \mathbf{R})$ be the solution of the Dirichlet problem

$$\int_{\Omega^*} \sqrt{\gamma}\,\gamma^{\alpha\beta}D_\alpha w D_\beta \zeta\,dx = 0 \quad \text{for all } \zeta \in C_c^\infty(\Omega^*, \mathbf{R}) \quad w - |u|^2 \in \mathring{H}_2^1(\Omega^*, \mathbf{R}). \tag{4.10}$$

In view of (4.5) we get, after putting $\zeta = G^\sigma(.,y)$ in (4.10) and subtracting the result from (4.9),

$$2\int_{\Omega^*} q(u)\,G^\sigma(.,y)\sqrt{\gamma}\,dx = \fint_{B_\varrho(y)} \{w - |u|^2\}\,dx. \tag{4.11}$$

Invoking Fatou's lemma, (4.6), (4.8), and a theorem of Lebesgue we find

$$2\int_{\Omega^*} q(u)\,G(.,y)\sqrt{\gamma}\,dx \leqslant w(y) - |u(y)|^2 \quad \text{a.e. } y \in \Omega^*. \tag{4.12}$$ [1]

Also, on account of the maximum principle, $|w| \leqslant M^2$, and we get from (4.8) and (4.11)

$$\int_{\Omega^*} e(u)\,G(.,y)\sqrt{\gamma}\,dx \leqslant \frac{M^2}{2a_\varkappa(M)} \quad \text{for all } y \in \Omega^*. \tag{4.13}$$

In particular,

$$\lim_{R\to 0}\int_{B_R(0)} e(u)G(.,0)\sqrt{\gamma}\,dx = 0. \tag{4.14}$$

Now introduce new normal coordinates around $Q_t = Q_{t,R} \in \mathcal{K}_M(Q)$, $0 \leqslant t \leqslant 1$, $0 < R < R_0$. These points are defined by their having standard coordinates $t\bar{u}_R = t\fint_{T_{2R}} u\,dx$ where $T_{2R} = B_{2R}(0) \setminus B_R(0)$.

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[1] Having proved this inequality one can use (4.2) to see that this implies that we actually have equality here. This is nothing but the Riesz representation for subsolutions.

R_0 is chosen so that $B_{4R_0}(0) \subset \Omega^*$. The representation of $U|_{\Omega^*}$ in these coordinates is denoted by $v = v_{t,R}$.

If, in the relation (3.4) proclaiming that v is weakly harmonic, we use as test vector $\varphi = vG^\sigma(.,\ y)\eta$, where $|y| < R$, $0 \leqslant \sigma < 1 - R$, $\eta = \eta_R \in C_c^\infty(B_{2R}(0), \mathbf{R})$, $\eta \equiv 1$ in $B_R(0)$, and $|\nabla\eta| \leqslant KR^{-1}$ with K independent of R, we get

$$\int \gamma^{\alpha\beta} D_\alpha[|v|^2\eta]\, D_\beta\, G^\sigma(.,y)\, \sqrt{\gamma}\, dx - \int \gamma^{\alpha\beta} D_\alpha\, \eta D_\beta\, G^\sigma(.,y)\, |v|^2 \sqrt{\gamma}\, dx$$
$$+ 2\int q(v)\, G^\sigma(.,y)\, \eta\, \sqrt{\gamma}\, dx + 2\int \gamma^{\alpha\beta} G^\sigma(.,y)\, D_\alpha v^i D_\beta \eta\, v^i \sqrt{\gamma}\, dx = 0. \tag{4.15}$$

The first integral is equal to $⨍_{B_\sigma(y)} |v|^2 dx$, if $B_\sigma(y) \subset B_R(0)$. In the second one we write

$$|v(x)|^2 = \operatorname{dist}^2 (Q_1, Q_t) + [\operatorname{dist}^2 (U(x), Q_t) - \operatorname{dist}^2 (Q_t, Q_1)]$$

where by the triangle inequality

$$|\operatorname{dist}^2 (U(x), Q_t) - \operatorname{dist}^2 (Q_t, Q_1)| \leqslant 4M\, |\operatorname{dist} (U(x), Q_t) - \operatorname{dist} (Q_t, Q_1)|$$
$$\leqslant 4M \operatorname{dist} (U(x), Q_1) \leqslant K\, |u(x) - \bar{u}|.\text{[1]}$$

Hence by (4.5) we find that the second integral differs from $\operatorname{dist}^2 (Q_1, Q_t)$ by a term which in view of the Poincare' inequality, (4.2), and (4.3) can be estimated by

$$KR^{-n} \int_{T_{2R}} |u(x) - \bar{u}|\, dx \leqslant K\left\{R^{-n} \int_{T_{2R}} |u(x) - \bar{u}|^2 dx\right\}^{1/2}$$
$$\leqslant K\left\{R^{2-n} \int_{T_{2R}} |\nabla u(x)|^2 dx\right\}^{1/2} \leqslant K\left\{\int_{B_{2R}(0)} e(u)\, G(.,0)\, \sqrt{\gamma}\, dx\right\}^{1/2}$$

where K is independent of R for small enough R. The Cauchy-Schwarz inequality, (4.2) and (4.4) yield that the fourth integral in (4.15), finally, can be majorized by

$$K\left\{\int_{T_{2R}} \gamma^{\alpha\beta} G^\sigma(.,y)\, D_\alpha v^i D_\beta v^i \sqrt{\gamma}\, dx\right\}^{1/2} \leqslant K\left\{\int_{B_{2R}(0)} e(u)\, G(.,0)\, \sqrt{\gamma}\, dx\right\}^{1/2}.$$

Now we can let $\sigma \to 0$ in (4.15), taking into account the estimates above, (4.4), (4.13), and a theorem of Lebesgue to arrive at

$$|v(y)|^2 \leqslant \operatorname{dist}^2(Q_1, Q_t) - 2\int q(v)\, G(.,y)\eta\, \sqrt{\gamma}\, dx + K\left\{\int_{B_{2R}} e(u) G(.,0)\, \sqrt{\gamma}\, dx\right\}^{1/2} \tag{4.16}$$

for almost every $y \in B_\varrho(0)$, $\varrho < R/2$.

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[1] In fact, by (2.3) we have $\operatorname{dist} (U(x), Q_1) \leqslant b_\omega(M)\, |u(x) - \bar{u}|$ if ω is a non-positive lower bound for the sectional curvature of $\mathcal{M}$ in $\mathcal{K}_M(Q)$.

The first integral on the right hand side is divided into two parts in both of which we use (4.7):

$$\int_{B_{2\varrho}(0)} qG(.,y)\,\eta\sqrt{\gamma}\,dx+\int_{B_{2R}\setminus B_{2\varrho}(0)} qG(.,y)\,\eta\sqrt{\gamma}\,dx$$
$$\geqslant 2a_\varkappa(\sup_{B_{2\varrho}}|v|)\int_{B_{2\varrho}} e(u)G(.,y)\,\eta\sqrt{\gamma}\,dx+2a_\varkappa(2M)\int_{B_{2R}\setminus B_{2\varrho}} e(u)G(.,y)\,\eta\sqrt{\gamma}\,dx$$

For all $x\in B_{2R}\setminus B_{2\varrho}$ and $y\in B_\varrho(0)$, $\varrho<R/2$, we have $|x|\leqslant|x-y|+|y|\leqslant|x-y|+\varrho\leqslant 2|x-y|$, and by (4.1) and (4.2) it follows that

$$\int_{B_{2R}\setminus B_{2\varrho}} e(u)G(.,y)\,\eta\sqrt{\gamma}\,dx\leqslant K\int_{B_{2R}\setminus B_{2\varrho}} e(u)G(.,0)\,\eta\sqrt{\gamma}\,dx\leqslant K\int_{B_{2R}} e(u)G(.,0)\sqrt{\gamma}\,dx.$$

By (4.14) this integral tends to zero as $R\to 0$, and it can be absorbed into the last term of (4.16) whence we get

$$|v(y)|^2\leqslant \operatorname{dist}^2(Q_1,Q_t)-2a_\varkappa(\sup_{B_{2\varrho}}|v|)\int_{B_{2\varrho}} e(u)G(.,y)\,\eta\sqrt{\gamma}\,dx$$
$$+K\left\{\int_{B_{2R}} e(u)G(.,0)\sqrt{\gamma}\,dx\right\}^{1/2},\quad y\in B_\varrho(0) \tag{4.17}$$

where K is independent of R (at least for small R), t, y, and ϱ.

Our first aim now is to prove that for all $t\leqslant 1$ and all small enough R, we have $|v(y)|^2=|v_{t,R}(y)|^2\leqslant M_1^2$, for an arbitrary $M_1>M$ such that $a_\varkappa(M_1)>0$. To this end introduce $h:[0,1]\to\mathbf{R}^+$ by

$$h(t)=\limsup_{R\to 0}\,\limsup_{\varrho\to 0}\,\sup_{B_\varrho}|v_{t,R}(y)|,$$

Since $t\mapsto|v_{t,R}(y)|=\operatorname{dist}(U(y),Q_{t,R})$ for almost every y and every R is a Lipschitz function with constant dist $(Q_1,Q)\leqslant M$ it is immediate that h is Lipschitz continuous. Obviously $h(0)\leqslant M<M_1$. Hence, if $h(t)$ were greater than M_1 for some $t\leqslant 1$ there would be a t_0 such that $h(t_0)=M_1$. Choosing $\varepsilon>0$ so that $a_\varkappa(M_1+\varepsilon)>0$ we find that $\sup_{B_{2\varrho}}|v_{t_0,R}|<M_1+\varepsilon$ for all small enough R and all $\varrho<\varrho_0$, for some $\varrho_0=\varrho_0(R)>0$. Then we get from (4.17) for such R

$$\limsup_{\varrho\to 0}\,\sup_{B_\varrho}|v|^2\leqslant\operatorname{dist}^2(Q_{1,R},Q_{t_0,R})+K\left\{\int_{B_{2R}} e(u)G(.,0)\sqrt{\gamma}\,dx\right\}^{1/2}. \tag{4.18}$$

Noting that $Q_{t_0,R}$ lies on the geodesic ray from Q to $Q_{1,R}$ we see that $\operatorname{dist}^2(Q_{1,R},Q_{t_0,R})\leqslant M^2$ and thus, in view of (4.14), we reach the desired contradiction:

$$h^2(t_0)=\limsup_{R\to 0}\,\limsup_{\varrho\to 0}\,\sup_{B_\varrho}|v|^2\leqslant M^2<M_1^2.$$

Having thus proved that $h(1) \leqslant M_1$ we may return to (4.17) and there take $t=1$. This leads to (4.18) with $t=1$, which implies that U is continuous at x_0 since for all small enough R it follows that

$$\begin{aligned}
&\lim_{\varrho\to 0} \sup_{y,z\in B_\varrho} \operatorname{dist}(U(y), U(z)) \\
&\quad \leqslant \lim_{\varrho\to 0} \{\sup_{y\in B_\varrho} \operatorname{dist}(U(y), Q_{1,R}) + \sup_{z\in B_\varrho} \operatorname{dist}(U(z), Q_{1,R})\} \\
&\quad \leqslant 2 \lim_{\varrho\to 0} \sup_{B_\varrho} |v_{1,R}(x)| \leqslant K\left\{\int_{B_{2R}} e(u)G(.,0)\sqrt{\gamma}\,dx\right\}^{1/4},
\end{aligned}$$

where the last term can be made arbitrarily small.

The theorem is proved.

5. Boundary regularity

We shall now state and prove a result on the boundary behavior of harmonic maps.

It is obvious that Theorem 1 is a corollary of Theorems 2, 3, and 4. One also sees that, in fact, the regularity assumption on the boundary value mapping Φ in Theorem 1 can be weakened to $\Phi \in H_2^1(\Omega, \mathcal{M}) \cap C^0(\mathrm{X}, \mathcal{M})$.

THEOREM 4. *Let $U \in C^2(\Omega, \mathcal{M}) \cap H_2^1$ be a harmonic mapping with range in a ball $\mathcal{K}_M(Q)$ which is within normal range of its points, and for which*

$$M < \pi/(2\sqrt{\varkappa})$$

where $\varkappa \geqslant 0$ is an upper bound for the sectional curvature of $\mathcal{M}$. If the trace of U on Σ is continuous, Hölder continuous, or has Hölder continuous first derivatives, then the same holds for U in X.

Remark. If X and $\mathcal{M}$ are of class C^μ, with $\mu > 2$, and if the trace of U on Σ is of class $C^{\nu,\alpha}$, $0 < \alpha < 1$, with $\nu + \alpha \leqslant \mu$, then U is of class $C^{\nu,\alpha}$ on X. This follows from Theorem 4 and the linear theory.

Proof. The statement about Hölder continuity follows immediately from the results of [11], once it is known that U is continuous on X. The Hölder continuity being proved the rest of the theorem can be derived from the results and methods of [15].

Hence to finish the proof we need only show that U is continuous at every point of Σ. To that end, let $x_0 \in \Sigma$ and let $Q_t \in \mathcal{M}$, $0 \leqslant t \leqslant 1$, be the points on the geodesic line from $Q = Q_0$ to $Q_1 = U(x_0)$ such that dist $(Q, Q_t) = t$ dist (Q, Q_1). For each t and R, R small enough, we consider the solution $w = w_{t,R}$ of the boundary value problem $Lw = 0$ in $B_R(x_0) =$

$\{x \in X;\ \text{dist}\,(x, x_0) < R\}$, $w - |v|^2 \in \mathring{H}_2(B_R(x_0), \mathbf{R})$ where $v = v_t$ is the representation of U in the chosen normal coordinate system around Q_t.

Well-known results from the theory of linear elliptic partial differential equations tell us that w is continuous on $\Sigma \cap B_R(x_0)$; in particular, $\lim_{y \to x_0} w(y) = w(x_0) = \text{dist}^2\,(U(x_0), Q_t)$.

Repeating the argument that led to (4.11), and using (4.7), we see that

$$\fint_{B_\varrho(y)} \{|v(x)|^2 - w(x)\}\, dx \leqslant -2a_\varkappa(\sup_{B_R(x_0)} |v|) \int_{B_R(x_0)} e(u) G^\sigma(.,y) \sqrt{\gamma}\, dx \tag{5.1}$$

if G^σ is the mollified Green function of $B_R(x_0)$.

We claim that $h(t) = \lim\sup_{x \to x_0} |v_t(x)| \leqslant M_1$ for all $t \in [0, 1]$ if $M_1 > M$ is chosen so that $a_\varkappa(M_1) > 0$. In fact, if this were not so there would be a $t_0 > 0$ such that $h(t_0) = M_1$, since $h(t)$ clearly is Lipschitz continuous and $h(0) < M$. Hence for R small enough we have $a_\varkappa(\sup_{B_R(x_0)} |v_{t_0}|) > 0$ and after discarding the right hand side and letting $\sigma \to 0$ in (5.1) we find that

$$|v_{t_0}(y)|^2 \leqslant w_{t_0}(y), \quad y \in B_R(x_0),$$

which implies that $h^2(t_0) \leqslant \lim\sup_{y \to x_0} w(y) = \text{dist}^2(U(x_0), Q_t) \leqslant M^2$, a clear contradiction.

Thus, knowing that $h(1) \leqslant M_1$, we may return to (5.1) and note that for R small enough $a_\varkappa(\sup_{B_R} |v_1|) > 0$, and for such R we find that

$$|v_1(y)|^2 \leqslant w_{1,R}(y).$$

But $w_{1,R}(y) \to \text{dist}(U(x_0), Q_1) = 0$ as $y \to x_0$, and this proves our assertion, since $|v_1(y)| = \text{dist}(U(y), U(x_0))$.

6. An example

In this section, we shall show that the example given in [11, p. 67] can be construed to furnish an example of a discontinuous weakly harmonic mapping $U\colon X \to \mathcal{M}$ with $U(X) \subset \mathcal{K}_M(Q)$ and $M = \pi/(2\sqrt{\varkappa})$.

For this purpose let $n = N \geqslant 3$, and choose $\mathcal{M} = S_R^N$, the N-dimensional sphere of radius R imbedded in the natural way in $\mathbf{R}^{N+1}$. Clearly, $\mathcal{M}$ has constant sectional curvature $\varkappa = R^{-2}$. Choose as one chart on $\mathcal{M}$ the stereographic projection σ of $\mathcal{M}$ from the north pole $P = (0, 0, \ldots, R)$ onto the equator plane $\{u \in \mathbf{R}^{N+1}, u^{N+1} = 0\}$ which in a natural way can be identified with $\mathbf{R}^N$. Set $u = \sigma(u)$, $u = \sigma^{-1}(u) = \tau(u)$. Then

$$\tau^i(u) = \frac{2u^i}{1 + \varkappa |u|^2} \quad \text{for } 1 \leqslant i \leqslant N$$

$$\tau^{N+1}(u) = \left(1 - \frac{2}{1 + \varkappa |u|^2}\right) R$$

whence we get that

$$g_{ik}(u) = 2a(|u|)\delta_{ik}$$

where $a(t) = 2(1+\varkappa t^2)^{-2}$. Note that $a'(R) = -2a(R)R^{-1}$.

Let Ω be any bounded domain in $\mathbf{R}^n$ containing the origin and having smooth boundary Σ; we set $X = \Omega \cup \Sigma$ and provide X with the standard Euclidean metric.

Then the energy integral of any mapping $U: \Omega \to \mathcal{M}\backslash\{P\}$ is given by

$$E(U) = \int_\Omega a(|u|)\,|\nabla u|^2 dx$$

if u is the representation of U in the coordinates given above.

Consider the "equator mapping" U_e: $X \to \mathcal{M} \subset \mathbf{R}^{N+1}$ defined by

$$U_e(x) = \left(\frac{R}{|x|}x_1, \frac{R}{|x|}x_2, \ldots, \frac{R}{|x|}x_n, 0\right) \in \mathbf{R}^{N+1}.$$

Since the equator is mapped into itself by σ, we find that the representation $u_e = \sigma \circ U_e$ is given by $u_e(x) = xR|x|^{-1}$, which shows that $U_e \in H_2^1 \cap L^\infty(\Omega, \mathcal{M})$. Obviously, $U_e(X) \subset \mathcal{K}_M(P)$ with $M = \pi/2\sqrt{\varkappa}$. It is straight-forward to check that U_e is a critical point of E, since $a'(R) = -2a(R)R^{-1}$, cf. [11, p. 67]. Thus U_e is a weakly harmonic mapping having the stated properties.

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