MOSAICS, PRINCIPAL FUNCTIONS, AND MEAN MOTION IN VON NEUMANN ALGEBRAS

BY

RICHARD W. CAREY and JOEL D. PINCUS

University of Kentucky, Lexington Ky., USA

State University of New York, Stony Brook, N.Y., USA

§ 1. Introduction

If U and V are bounded self-adjoint operators on Hilbert space with $VU - UV = (1/i)C$ in trace class, then the so called principal function $g(y, x)$ associated with the algebra generated by U and V was defined by the relation

$$\det[(V-l)(U-z)(V-l)^{-1}(U-z)^{-1}] = \exp \frac{1}{2\pi i}\iint g(y,x)\frac{dy}{y-l}\frac{dx}{x-z} \tag{1.1}$$

and studied in a series of papers.

There are two main points which underlie our interest in the principal function.

(A) $g(y, x)$ is explicitly computable from the symbols $S_{\pm}(U; V)$ of V with respect to U. This is explained fully in [15] and is described in example 9.1 below. Furthermore the unitary invariants of these symbols and $g(y, x)$ remain unaltered by trace class perturbations of U and V.

(B) $g(y, x) \in L^1(R^2, dy\,dx)$

Because of relations like $\det[e^W e^Q e^{-W} e^{-Q}] = \exp \operatorname{Tr}[W, Q]$, which is valid for smooth operator valued functions W and Q of U and V, there is a natural bilinear form associated with the study of the principal function. This useful fact was discovered by one of the authors [32].

Indeed it has been known from the beginning of this theory that relations like

$$\operatorname{Tr}[U^j V^k, V] = \frac{i}{2\pi}\iint \frac{\partial}{\partial x} x^j y^k g(y,x)\,dx\,dy$$

follow at once from (1.1) by taking residues at ∞ in l and z.

This research was partially supported by grants from the National Science Foundation. The second author was a visiting professor at the Rockefeller University when this work was completed, and wishes to express his thanks for the hospitality extended to him at that institution.

A symmetrized version of this last equation is easily obtained by means of an algebraic lemma due to N. Wallach; namely, $\operatorname{Tr}[W, Q] = (i/2\pi) \iint \{W, Q\} g(y, x)\, dx\, dy$ as we will show in section 5. In this equation $\{W, Q\}$ is the Poisson bracket of the complex valued functions $W(x, y)$, $Q(x, y)$.

The results (A) and (B) above were obtained by introducing a certain operator valued function of two complex variables, the so called determining function $E(l, z)$ of $T = U + iV$. This is a complete unitary invariant for the non normal part of T which can sometimes be computed.

This function was discovered originally as the solution of a certain Riemann–Hilbert problem associated with T, and the connection of the principal function with symbols and the Lebesgue summability of the principal function expressed in (A) and (B) follow from an analysis of the boundary values of $E(l, z)$ as l and z approach the spectrum of V and U.

Recently, other authors [24] have taken up this theory and focused attention purely on the bilinear form $\operatorname{Tr}[W, Q]$ without using $E(l, z)$.

If E_x and F_y denote the spectral resolutions of U and V, then

$$\frac{i}{2\pi} \int_\Delta \int_\Omega g(y, x)\, dx\, dy = \operatorname{Tr}[E(\Delta)\, F(\Omega)\, E(\Delta)\, C].$$

The right hand side thus leads to a measure Σ defined on the plane, and—since the above relations were known—it was natural to show that $\operatorname{Tr}[W, Q] = (i/2\pi) \iint \{W, Q\}\, d\Sigma$ directly, without using the $E(l, z)$ function.

However this rewriting of the theory does not seem to give the absolute continuity of $d\Sigma$ and it gives up the computability of $g(y, x)$ in terms of the symbols or other local data.

In the case where more than two operators are involved the situation is different. In the multivariable case the focus on the trace form and the associated topological development is quite interesting because there has so far been no counterpart to $E(l, z)$ introduced in this situation.

Here an extension of the connection (made by one of us) of the principal function $g(y, x)$ with the index of $T - (x + iy)1$ is established in terms of trace forms.

We wish to explain a little more fully why it is nice to have a function $g(y, x)$ rather than just a measure $d\Sigma$.

Suppose T is a completely hyponormal operator and $T^*T - TT^*$ has one dimensional range. One result due to the authors [11] (whose proof depends on Theorem 7.10 of the present paper) is the following

THEOREM. *z is an eigenvalue of T^* if and only if there exists a ball B_z about z such that*

$$\iint_{B_z} \frac{1-g(y,x)}{|x+iy-z|^2}\,dx\,dy < \infty .$$

The proof of this result relies on the fact that the trace class perturbation problem $T_z^* T_z \to T_z T_z^*$ ($T_z = T - z1$) has an associated spectral displacement function which is related to the principal function $g(y, x)$ in a simple way. This connection between principal functions and scattering theory is one of the main points of the present study.

There are also quite a few other results relating the spectral multiplicity of U and of V by means of the principal function [31] as well as invariant subspace results [11] and some more subtle relationships which will be reported elsewhere.

An analogy may perhaps be in order. M. G. Krein [29], extending and sharpening results of the physicist I. M. Lifshitz, established the existence of the so-called spectral displacement function $\delta(\cdot)$, by considering $\mathrm{Tr}\,[(A-l)^{-1}-(A+D-l)^{-1}]$ where A is self-adjoint and D is self adjoint and trace class.

He showed that $\delta(\cdot)$ is Lebesgue summable and that $\mathrm{Tr}\,[f(A+D)-f(A)] = \int f'(\sigma)\delta(\sigma)\,d\sigma$ for a certain class of functions $f(\cdot)$. Furthermore, $\det S(\sigma) = \exp(-2\pi i\delta(\sigma))$ where $S(\sigma)$ is the scattering operator associated with the perturbation problem $A \to A+D$.

In scattering problems the existence of a summable phase shift $2\pi\delta(\sigma)$ has considerable importance. Naturally, in potential scattering problems the study of the phase shift depends upon a study of the boundary values of Greens functions.

We have said all this to partly justify the technical complexities of the present paper. We have as our main goal here the extension of our results to the case of von Neumann algebras, and although it is indeed the case that a simple transliteration of the results for $\mathrm{Tr}\,[W, Q]$ to $\tilde{\tau}[W, Q]$ where $\tilde{\tau}$ is a relative trace gives results like $\tilde{\tau}[W, Q] = (i/2\pi)\iint \{W, Q\}\,d\Sigma$ for some measure Σ; the knowledge that $d\Sigma = g(y, x)\,dx\,dy$ for a summable function would be lost without the more extensive development which we give here.

Two additional remarks are in order before we turn to an outline of the paper.

The theory of mosacis and principal functions extends naturally to the treatment of pairs of operators $\{W, V\}$ and $\{W, U\}$ where W, U are unitary and V is self adjoint. In these situations one encounters some new structure if the spectrum of W or U is the whole unit circle. For instance in the type I case there are simple examples which show that the index class of the corresponding C^*-algebras is not determined by the symbols alone but requires knowledge also of a certain spectral displacement function.

In addition there are results about compressions of unitary operators and symmetric operators. These results will be presented elsewhere.

We remark also that replacing τ by a center valued trace leads in a natural way to a theory of center valued principal functions.

In this paper we show how the determining function theory of operators with trace class self-commutator $T^*T - TT^*$ extends to the context of von Neumann algebras equipped with a normal trace (see [31], [34], [10], [12]).

We construct the so called *mosaics* of T, $B(\nu, \mu)$ in this new context and show how they serve as *local data* from which the principal function can be computed. The mosaics themselves are operator valued functions which occur in the study of the boundary behaviour of the determining function either on the spectrum of Re (T) or Im (T). The principal function is the relative trace of the mosaic. We also show how the index and transformation properties of the principal function familiar from the previous investigations of the type I case remain valid in the type II case.

The principal function $g(\nu, \mu)$ coincides with the Breuer ([6], [7]) index of $T-(\mu+i\nu)$ when $T-(\mu+i\nu)$ is (relatively) Fredholm, i.e. when $\mu+i\nu$ is not in the (relative) essential spectrum of T. But $g(\nu, \mu)$ is everywhere defined, and is not (like the index of $T-(\mu+i\nu)$) defined only on the components of the complement of the essential spectrum of T; furthermore, we will see that $g(\nu, \mu)$ is invariant under unitary transformations of T belonging to the algebra, and under perturbations of T by elements of the trace ideal of the algebra.

There is a basic functional calculus associated with the operator T, which we will now describe, and some nice formulae for the trace of certain commutators which explicitly involve the principal function.

Let us think of T in the form $T=U+iV$, where U and V are self-adjoint elements of a von Neumann algebra $\mathcal{M}$ equipped with a normal trace, τ.

Let $M(R^2)$ be the space of complex measures ω on R^2 such that $\|\omega\| = \iint (1+|t|)(1+|s|)d|\omega(t, s)| < \infty$. Under convolution $M(R^2)$ is a commutative semi-simple B^* algebra. Let $F(x, y) = \iint e^{itx+isy}d\omega(s, t)$, the characteristic function of ω. Let $\hat{M}(R^2)$ be the set of all characteristic functions of measures in $M(R^2)$.

We can associate with $F \in \hat{M}(R^2)$ and the pair $\{U, V\}$ an element in $\mathcal{M}$ defined by the integral (iterated *or* multiple)

$$F(U, V) = \iint F(x, y)\, dE_x dF_y,$$

where E_x and F_y are respectively the spectral resolutions of U and of V.

Note here that any function $F'(x, y)$ which coincides with $F(x, y)$ on $\sigma(U) \times \sigma(V)$ will define the same operator $F(U, V)$ since then $F(U, V) = \iint F'(x, y)dE_x dF_y$. Thus, the restriction that $F(x, y)$ be in $\hat{M}(R^2)$ only has force on sp $(U) \times$ sp (V).

The association $F(x, y) \mapsto F(U, V)$ defines a functional calculus of the Mikhlin–Weyl type; $[F_1(U, V), F_2(U, V)]$, $F_1(U, V)F_2(U, V)-(F_1F_2)(U, V)$, and $F(U, V)^* - \bar{F}(U, V)$ are all in the trace ideal.

We will show that, as has previously been known when $\mathfrak{M}$ is of type I [11], that

$$\tilde{\tau}[F_1(U, V), F_2(U, V)] = \frac{i}{2\pi}\iint\left(\frac{\partial F_1}{\partial\mu}\frac{\partial F_2}{\partial\nu} - \frac{\partial F_1}{\partial\nu}\frac{\partial F_2}{\partial\mu}\right) g(\nu, \mu)\, d\nu\, d\mu. \tag{1.2}$$

whenever F_1 and F_2 are in $\hat{M}(R^2)$. (For a weaker result when $\mathfrak{M}$ is of type I which does not identify the right-hand side of (1.2) specifically see also [24].)

As a relatively simple, but important, consequence of (1.2) it becomes possible to find the principal function associated with a new pair of self-adjoint operators A and B obtained from U and V by a change of variables. That is, suppose $A = \iint \alpha(x, y)\, dE_x dF_y = \alpha(U, V)$ and $B = \iint \beta(x, y)\, dE_x dF_y = \beta(U, V)$ where α, β are real valued functions in $\hat{M}(R^2)$ on $\mathrm{sp}\,(U) \times \mathrm{sp}\,(V)$. Let $\Omega = \alpha + i\beta$.

The principal function $\hat{g}(\alpha, \beta)$ associated with the pair $\{A, B\}$ is easily seen to be given by

$$\hat{g}(\alpha, \beta) = \sum\, [\mathrm{sgn}\ (\mathrm{Jacobian}\ \Omega)](\nu_i, \mu_i) g(\nu_i, \mu_i) \tag{1.3}$$

the summation being extended over those points (ν_i, μ_i) for which $\Omega(\nu_i, \mu_i) = (\alpha, \beta)$.

To see this, note that if we use the functional calculus based on A and B, then

$$\tilde{\tau}[F_1(A, B), F_2(A, B)] = \frac{i}{2\pi}\iint \frac{\partial(F_1, F_2)}{\partial(\alpha, \beta)} \hat{g}(\alpha, \beta)\, d\alpha\, d\beta.$$

But

$$\begin{aligned}\tilde{\tau}[F_1(A, B), F_2(A, B)] &= \tilde{\tau}[F_1(\alpha(U, V), \beta(U, V), F_2(\alpha(U, V), \beta(U, V)]\\ &= \frac{i}{2\pi}\iint \frac{\partial(F_1, F_2)}{\partial(\alpha, \beta)}(\alpha(\mu, \nu), \beta(\mu, \nu))\frac{\partial(\alpha, \beta)}{\partial(\mu, \nu)} g(\nu, \mu)\, d\nu\, d\mu.\end{aligned}$$

Hence,

$$\iint \frac{\partial(F_1, F_2)}{\partial(\alpha, \beta)}[\sum[\mathrm{Sgn}\ (\mathrm{Jacobian}\ \Omega)(\mu_i, \nu_i)] g(\nu_i, \mu_i) - \hat{g}(\beta, \alpha)]\, d\alpha\, d\beta = 0.$$

for all $F_1, F_2 \in \hat{M}(R^2)$. The equality (1.3) follows since F_1 and F_2 are arbitrary.

One of the most interesting changes of variables is the change from cartesian to polar coordinates.

Suppose that $x = r\cos\theta$, $y = r\sin\theta$ and that $T = WQ$ is the polar decomposition of T.

Thus $Q=(T^*T)^{1/2}$ and W is a partial isometry with initial space equal to the range of T^* and final space equal to the range of T.

If $0 \notin \mathrm{sp}(U)\times \mathrm{sp}(V)$, then we can see readily that $\int\int (x+iy)(x^2+y^2)^{-1/2} dE_x dF_y$ will differ from W by a trace class operator. For, let $k(x, y)=(x^2+y^2)^{1/2}$, $l(x, y)=(x^2+y^2)^{-1/2}$, and $m(x, y)=x+iy$ on some neighborhood of $\mathrm{sp}(U)\times\mathrm{sp}(V)$. Call the trace ideal I_τ.

Then $l\circ k(U, V)-l(U, V)k(U, V)\in I_\tau$, $l(U, V)k(U, V)-1\in I_\tau$, and $m\circ l(U, V)=m(U, V)l(U, V)$ (mod I_τ). Since $m(U, V)=T$ (mod I_τ), we have $m\circ l(U, V)=WQl(U, V)$ (mod I_τ) or $m\circ l(U, V)=W$ (mod I_τ)—since $Ql(U, V)=1$ (mod I_τ).

This simple reasoning fails if $O\in\mathrm{sp}(U)\times\mathrm{sp}(V)$. However, $\hat{g}(r^2, \theta)=g(r\sin\theta, r\cos\theta)$ has an interesting meaning in terms of a functional calculus built on the minimal unitary dilation space for W. In particular, we prove as one of our main results Theorem 7.1 which gives an analogue of (1.2) in terms of dilated operators.

We have already given one illustration of the utility of this result. The eigenvalue criterion for T presented earlier is a consequence of this relation.

Another observation shows how the principal function is a kind of two dimensional spectral displacement function, and gives for example a trace expression for the Laplace transform of the principal function that is suggestive and sometimes quite useful. This result is particularly interesting when the essential spectrum of T is thick, and the principal function is not the index of $T-(x+iy)$.

Let $H(r)\in C_0^1(R^1)$ and let $H_1(r)=\partial H/\partial r$. Then by the functional calculus for self-adjoint operators both $H(T^*T)$ and $H(TT^*)$ are well defined. It is also easy to see that $H(TT^*)-H(T^*T)\in I_\tau$. Theorem

$$\tilde{\tau}\{H(TT^*)-H(T^*T)\}=\frac{1}{\pi}\int\int H_1(x^2+y^2)\,g(y, x)\,dx\,dy. \tag{1.3}$$

Section 2 introduces the determining function and its basic properties.

In section 3 we establish an exponential representation for Aronszjan–Weinstein matrices associated with a self-adjoint perturbation problem. We then apply this result to obtain a weak * measurable family of operators $B(\nu, \mu)\in\mathcal{M}$, $0\leqslant B(\nu, \mu)\leqslant 1$ where μ and ν are real. The principal function is defined in terms of the *mosaic* $B(\nu, \mu)$.

These operators arise from a study of certain boundary values associated with the determining function, and have been studied before in the type I situation in relation to certain symbols associated with $U+iV$ [13], [15]. In particular it is known in the type I case how to construct an operator with a given mosaic [13].

In section 4 we establish our functional calculus modulo the trace ideal.

In section 5 we prove (1.2), and establish a continuity property of the principal function which enables us to evaluate $g(\nu, \mu)$ for some examples with *thick* essential spectrum, i.e. positive two-dimensional Lebesgue measure. We also establish a *cut-down* property of the principal function, and we show that a known result in the type I case [34] extends to the present situation, that is we show that if $T=U+iV$ is completely semi-normal and if $[U, V]\in I_\tau$, then sp (T) is the essential closure of the support of the principal function.

In section 6 we consider a pair of operators $\{W, P\}$ where W is a partial isometry, P is self-adjoint, and $WP-PW\in I_\tau$. By means of a natural unitary dilation theory associated with W, a new functional calculus is noted, and a version of (1.2) is then established with a summable function $\tilde{g}(\lambda, \tau)$ defined on the cylinder $-\infty<\lambda<\infty$, $|\tau|=1$.

In section 7 we specialize these considerations by taking W and $P^{1/2}$ to be the partial isometry and modulus entering into the polar factorization of T. We then establish a fundamental relation between $\tilde{g}(\lambda,\tau)$ and $g(\nu,\mu)$; namely, $\tilde{g}(\lambda^2,\tau)=g(\nu,\mu)$ if $\mu+i\nu=\lambda\tau$.

In section 8 we prove that $g(\nu,\mu)=\text{Index}\,(U+iV-(\mu+i\nu))$ in the sense of Breuer [6] when $\mu+i\nu$ belongs to the complement of the essential spectrum.

In section 9 we give some examples and show how, when T is taken to be a superposition of translation operators, $g(\nu,\mu)$ is connected with the mean motion of a certain associated exponential polynomial. We will state a conjecture about this connection.

If $\mathcal{A}$ is a C^*-algebra, we will sometimes use the notation $\mathcal{A}^s$ to denote the self-adjoint elements of $\mathcal{A}$. Moreover, if $\mathcal{A}$ is a subalgebra of $\mathcal{L}(\mathcal{H})$, the algebra of bounded linear operators on a Hilbert space $\mathcal{H}$, and $M\in\mathcal{A}^s$, we let M_{ac} (resp. M_s) denote the absolutely (resp. singular) continuous part of M. The absolutely continuous space relative to M shall be written as $\mathcal{H}_{ac}(M)$; the singular subspace which is the orthogonal complement of $\mathcal{H}_{ac}(M)$ shall be denoted $\mathcal{H}_s(M)$.

Recall that a von Neumann algebra $\mathcal{M}$ is a C^* algebra which is also the dual space of a Banach space $\mathcal{M}_*$ [36]. A factor is a von Neumann algebra with trival center. The set of projections, $\mathcal{P}(\mathcal{M})$, the hermitian idempotents of $\mathcal{M}$ becomes a complete lattice under the order relation $\leqslant$, defined by $E\leqslant F\Leftrightarrow EF=E$. The lattice $\mathcal{P}(\mathcal{M})$ has an equivalence relation $\sim$ and an order relation $<$ defined in the following way:

$E\sim F\Leftrightarrow$ there is a $U\in\mathcal{M}$ such that $E=U^*U$ and $F=UU^*$.

$E<F\Leftrightarrow$ there is an $E'\in\mathcal{P}(\mathcal{M})$ such that $E\sim E'\leqslant F$.

It is easy to see that $E<F$ and $F<E$ imply $E\sim F$.

An element $E\in\mathcal{P}(\mathcal{M})$ is called finite relative to $\mathcal{M}$ if for every $F\in\mathcal{P}(\mathcal{M})$ the relations $E\sim F$ and $E\geqslant F$ imply $E=F$.

A functional τ on $\mathcal{M}^+$, the positive portion of $\mathcal{M}$, with values $\geqslant 0$, finite or infinite, is called a trace on $\mathcal{M}$ when

(i) if $A, B \in \mathfrak{M}^+$, $\tau(A+B)=\tau(A)+\tau(B)$

(ii) if $A \in \mathfrak{M}^+$ and λ is a non-negative real number, $\tau(\lambda A)=\lambda\tau(A)$ where $0\cdot(+\infty)=0$.

(iii) if $A \in M^+$ and U is unitary in $\mathfrak{M}$, $\tau(U^*AU)=\tau(A)$.

A trace is faithful if $\tau(A)=0$ implies $A=0$; semi-finite if for every non-zero $A \in \mathfrak{M}^+$, there exists a non-zero element B in $\mathcal{A}^+$ with $\tau(B)<+\infty$ and $B \leqslant A$. A trace is normal if for every uniformly bounded increasing directed set $\{A_\alpha\} \subset \mathfrak{M}^+$, $\tau(\text{l.u.b.}_\alpha A_\alpha)=\text{l.u.b.}_\alpha \tau(A_\alpha)$.

The basic result about traces is the following well known fact: see, for example [36].

The set of $A \in \mathfrak{M}^+$ with $\tau(A)<+\infty$ is the positive portion of a two sided ideal $\mathfrak{J}_\tau$ of $\mathfrak{M}$. There exists a unique linear functional $\tilde{\tau}$ and $\mathfrak{J}_\tau$ which coincides with τ in $\mathfrak{J}_\tau \cap \mathfrak{M}^+$, and one has $\tilde{\tau}(AX)=\tilde{\tau}(XA)\,(A \in \mathfrak{J}_\tau, X \in \mathfrak{M}^+)$.

If E is a Banach space, with dual space E^*, the weak *—topology of E^* is denoted by $\sigma(E^*, E)$. In particular, the weak *—topology of a von Neumann-algebra $\mathfrak{M}$ is denoted by $\sigma(\mathfrak{M}, \mathfrak{M}_*)$; the strong topology is denoted by $s(\mathfrak{M}, \mathfrak{M}_*)$ while the strong*-topology is written $s^*(\mathfrak{M}, \mathfrak{M}_*)$. The commutative C^*-algebra of complex valued continuous functions on $\mathbf{R}^n$, $(n=1, 2)$ vanishing at ∞, is denoted $C_0(\mathbf{R}^n)$ while the subalgebra of continuously differentiable functions is written $C_0^1(\mathbf{R}^n)$.

When (Ω, μ) is a σ-finite measure space and E is a Banach space, $L^1(\Omega, \mu; E)$ is the Banach space of E-valued μ-Bochner integrable functions f with the norm $\|f\| = \int |f(t)\, d\mu(t)$. For a von Neumann algebra with separable predual $\mathfrak{M}_*$ the notation $L^\infty(\Omega, \mu; \mathfrak{M})$ will be used to denote the Banach space of $\mathfrak{M}$-valued essentially bounded weak* μ-measurable functions. The importance of using weak* measurable as opposed to Bochner measurable is that $L^\infty(\Omega, \mu; \mathfrak{M})$ is naturally isomorphic to the von Neumann algebra tensor product $L^\infty(\Omega, \mu; \mathbf{C}) \,\overline{\otimes}\, \mathfrak{M}$ having $L^1(\Omega, \mu; \mathfrak{M}_*)=L^1(\Omega, \mu; \mathbf{C}) \otimes_\gamma \mathfrak{M}_*$ as predual. (γ is the greatest cross norm.) See [36].

Finally we note that if $h \in L^\infty(\Omega, \mu; \mathfrak{M})$ and $\|h(\cdot)\| \in L^1(\Omega, \mu; \mathbf{C})$, then $\int h(t)\, d\mu(t)$ is well defined as a weak* integral and represents an element in $\mathfrak{M}$.

§ 2. Determining functions

In this chapter we review the basic definitions and constructions of the determining function theory, adapted slightly to our present purpose.

The material after Propositon 2.1 can be omitted on a first reading.

Let $\mathcal{A}$ be a C^*-algebra with identity 1. Let $T \in \mathcal{A}$. Set $U=\frac{1}{2}(T+T^*)$, $V=-\frac{1}{2}i(T-T^*)$. With $D=[T, T^*]=TT^*-T^*T$ note that $[V, U]=VU-UV=-iC$ where $C=\frac{1}{2}D$. Let $\mathcal{A}^+$ be the set of all positive elements in $\mathcal{A}$; $\mathcal{A}^+$ is a convex cone in $\mathcal{A}$. If $M \in \mathcal{A}^s$, then $M = M_+ - M_-$ where $M_\pm \in \mathcal{A}^+$. Consider the C^*-subalgebra of $\mathcal{A}$ generated by C and the identity. Call this subalgebra $\mathcal{C}$. By the Gelfand–Naimark theorem, $\mathcal{C}$ is isometrically isomorphic with $C[\mathrm{sp}\,(C)]$, the continuous complex valued functions on the compact set sp (C). Form the map

$$\hat{C}(\lambda)=\begin{cases} -i\sqrt{-\lambda} & \lambda<0 \\ 0 & \lambda=0 \\ \sqrt{\lambda} & \lambda>0 \end{cases}$$

and note that it defines an element in $C[\mathrm{sp}\,(C)]$. Thus, there exists a unique element $\hat{C}$ of $\mathcal{C}$ whose Gelfand transform is $\hat{C}(\lambda)$. The relations $\hat{C}^2=C$, $\hat{C}\hat{C}^*=\hat{C}^*\hat{C}=C_++C_-=|C|$ are clear.

The determining function of the pair $\{V, U\}$ is defined to be

$$E(l,z)=I+\frac{1}{i}\hat{C}(V-l)^{-1}(U-z)^{-1}\hat{C} \tag{2.1}$$

for $z\notin \mathrm{sp}\,(U)$, $l\notin \mathrm{sp}\,(V)$.

We note that $E(l, z)$, for each fixed l and z, is in the C^*-subalgebra of $\mathcal{A}$ generated by T and I, the identity.

Lemma 2.1. *$E(l, z)$ is an invertible element of $\mathcal{A}$.*

Proof. We will show that $E^{-1}(l, z)=I+i\hat{C}(U-z)^{-1}(V-l)^{-1}\hat{C}$. We can write

$$\begin{aligned} &E(l,z)\cdot\{I+i\hat{C}(U-z)^{-1}(V-l)^{-1}\hat{C}\}\\ &\quad=\left[I+\frac{1}{i}\hat{C}(V-l)^{-1}(U-z)^{-1}\hat{C}\right][I+i\hat{C}(U-z)^{-1}(V-l)^{-1}\hat{C}]\\ &\quad=I+i\hat{C}(U-z)^{-1}(V-l)^{-1}\hat{C}+\frac{1}{i}\hat{C}(V-l)^{-1}(U-z)^{-1}\hat{C}\\ &\qquad+\hat{C}(V-l)^{-1}(U-z)^{-1}\hat{C}^2(U-z)^{-1}(V-l)^{-1}\hat{C}=I, \end{aligned}$$

since $\hat{C}^2=C$, and $(V-l)^{-1}(U-z)^{-1}C(U-z)^{-1}(V-l)^{-1}=i(V-l)^{-1}(U-z)^{-1}-i(U-z)^{-1}(V-l)^{-1}$. Similarly, $\{I+i\hat{C}(U-z)^{-1}(V-l)^{-1}\hat{C}\}\,E(l, z)=I$.

Lemma 2.2.

$$E^{-1}(l,z)\,E(l,\omega)=I-i(\omega-z)\hat{C}(U-z)^{-1}(V-l)^{-1}(U-\omega)^{-1}\hat{C}. \tag{2.2}$$

Proof. By Lemma 2.1, we have

$$\begin{aligned} E^{-1}(l,z)\,E(l,\omega)=I&+i\hat{C}(U-z)^{-1}(V-l)^{-1}\hat{C}-i\hat{C}(V-l)^{-1}(U-\omega)^{-1}\hat{C}\\ &-\hat{C}(U-z)^{-1}(V-l)^{-1}C(V-l)^{-1}(U-\omega)^{-1}\hat{C}. \end{aligned} \tag{2.3}$$

But,

$$\begin{aligned}(\omega-z)(U-z)^{-1}(V-l)^{-1}(U-\omega)^{-1} &= (U-\omega)^{-1}(V-l)^{-1}-(U-z)^{-1}(V-l)^{-1}-i(U-\omega)^{-1}(V-l)^{-1}C(V-l)^{-1}(U-\omega)^{-1}\\ &\quad +i(U-z)^{-1}(V-l)^{-1}C(V-l)^{-1}(U-\omega)^{-1}.\end{aligned}$$

Accordingly

$$\begin{aligned}&-(U-z)^{-1}(V-l)^{-1}+(V-l)^{-1}(U-\omega)^{-1}+i(U-z)^{-1}(V-l)^{-1}C(V-l)^{-1}(U-\omega)^{-1}\\ &\quad =(\omega-z)(U-z)^{-1}(V-l)^{-1}(U-\omega)^{-1}.\end{aligned}$$

Insertion of this relation into the right-hand side of (2.3) completes the proof.

Let $\mathcal{A}^{**}$ be the second dual of the C^* algebra $\mathcal{A}$. It is known that $\mathcal{A}^{**}$ is a von Neumann algebra [36], and $\mathcal{A}$ is a C^* subalgebra of $\mathcal{A}^{**}$ under a canonical embedding. We assume throughout this paper that these identifications have been made.

Let the polar decomposition of C relative to $\mathcal{A}^{**}$ be given by $C=(\operatorname{sgn} C)|C|$, where $\operatorname{sgn} C$ is the associated partial isometry. Note that $[\operatorname{sgn} C, \hat{C}]=0$, and $\hat{C} \operatorname{sgn} C=\hat{C}^*$.

PROPOSITION 2.1.

$$\begin{aligned}&i(E(l_1, z_2) \operatorname{sgn} CE^*(l_2, \bar{z}_2)-E(l_1, \bar{z}_1) \operatorname{sgn} CE^*(l_2, z_1))\\ &\quad =(l_1-\bar{l}_2)(z_2-\bar{z}_1)\hat{C}(V-l_1)^{-1}(U-z_2)^{-1}(U-\bar{z}_1)^{-1}(V-\bar{l}_2)^{-1}\hat{C}^*.\end{aligned}$$

The necessary details of the proof is a repetition of the preceeding algebra, and will be omitted.

This proposition enables us to derive a certain positivity result. Let

$$K(l_2, z_1; l_1, z_2)=i\,\frac{E(l_1, z_2) \operatorname{sgn} CE^*(l_2, \bar{z}_2)-E(l_1, \bar{z}_1) \operatorname{sgn} CE^*(l_2, z_1)}{(l_1-\bar{l}_2)(z_2-\bar{z}_1)}.$$

Note that $K(l_2, z_1; l_1, z_2) \in \mathcal{A}$ and is analytic in l_1 and z_2; conjugate analytic in l_2 and z_1.

By the Gelfand–Naimark theorem $\mathcal{A}$ has a faithful representation π as a C^* subalgebra of $\mathcal{L}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.

Let $\{z_\alpha, l_\alpha\}_{\alpha=1}^n$ be a finite set of non-real pairs and let $\{f_\alpha\}_{\alpha=1}^n$ be a set of vectors in $\mathcal{H}$. Then, by Proposition 2.1

$$\sum_{\alpha,\beta=1}^{n}(\pi K(l_\beta, z_\alpha; l_\alpha, z_\beta)f_\beta, f_\alpha)_{\mathcal{H}}=\Big(\sum_\beta \pi(U-z_\beta)^{-1}(V-\bar{l}_\beta)^{-1}\hat{C}^* f_\beta, \sum_\alpha \pi(U-z_\alpha)^{-1}(V-\bar{l}_\alpha)^{-1}\hat{C}^* f_\alpha\Big)_{\mathcal{H}} \geqslant 0. \tag{2.4}$$

Thus $\pi(K)$ defines a positive definite matrix function.

This fact enables us to construct a reproducing kernel Hilbert space which provides one means of constructing operators from determining functions.

The following proposition is very easy to prove.

PROPOSITON 2.2.

$$E^*(\bar{l}, \bar{z}) \operatorname{sgn} C E(l, z) = \operatorname{sgn} C \tag{2.5}$$

$$\frac{i}{(l-\bar{l})(z-\bar{z})}\{E^*(\bar{l}, z) \operatorname{sgn} C E(l, z) - E^*(l, z) \operatorname{sgn} C E(\bar{l}, z)\} \in \mathcal{A}^+ \tag{2.6}$$

$$\|I - E(l, z)\| \leqslant \Gamma |\operatorname{Im} l|^{-1} |\operatorname{Im} z|^{-1} \tag{2.7}$$

for some positive constant Γ.

Although it is not crucial for the purposes of the present paper, it will give the reader a more complete picture if we note that the determining functions can be characterized purely as functions of two complex variables by means of certain symmetry and positivity properties.

After determining functions were introduced [31], the possibility of finding such a characterization arose in discussions between L. de Branges and the second author, and the resulting characterization was subsequently found, in the case of positive self-commutator, by both de Branges and the second author working separately. We give a C^* version of these old results.

Suppose that a function $\tilde{E}(l, z)$ defined as analytic for $\operatorname{Im} l \neq 0$, $\operatorname{Im} z \neq 0$ and taking values in a C^* algebra $\mathcal{A}$ is given such that

$$\|1 - \tilde{E}(l, z)\| = O(|\operatorname{Im} l|^{-1} |\operatorname{Im} z|^{-1}) \tag{α_1}$$

and $C \equiv i \cdot \lim_{l, z \to \infty} |z||l|(1 - \tilde{E}(l, z))$ exists and is self-adjoint in $\mathcal{A}$. In $\mathcal{A}^{**}$ let the polar decomposition of C be given by $C = (\operatorname{sgn} C)|C|$, where $\operatorname{sgn} C$ is the usual partial isometry and $[\operatorname{sgn} C, |C|] = 0$.

Suppose, in addition, that π is a faithful * representation of E, and

$$\tilde{E}^*(\bar{l}, \bar{z}) \operatorname{sgn} C \tilde{E}(l, z) = \operatorname{sgn} C. \tag{α_2}$$

Now with $\tilde{E}_\pi = \pi(E)$ let

$$\pi K(l_2, z_1; l_1, z_2) \equiv i \frac{\tilde{E}_\pi(l_1, z_2) \operatorname{sgn} C \tilde{E}_\pi^*(l_2, \bar{z}_2) - \tilde{E}_\pi(l_1, \bar{z}_1) \operatorname{sgn} C \tilde{E}_\pi^*(l_2, z_1)}{(l_1 - \bar{l}_2)(z_2 - \bar{z}_1)}$$

and assume that

$$\sum_{\alpha, \beta = 1}^{n} \pi K(l_\beta, z_\alpha; l_\alpha, z_\beta) f_\beta, f_\alpha) \geqslant 0 \tag{β}$$

for finite sets $\{l_\alpha\}_{\alpha=1}^n$, $\{z_\beta\}_{\beta=1}^n$, $\{f_\alpha\}_{\alpha=1}^n$.

For convenience we will also restrict ourselves to bounded operators, and assume that

(γ) $\tilde{E}_\pi(l, z)$ is continuous on the real l axis outside of a compact interval and that $\tilde{E}_\pi(l, z)$ is continuous on the real z axis outside of a compact interval.

THEOREM 2.1. *Let $\mathcal{E}$ be the C^* algebra generated by $\{\tilde{E}(l, z), \operatorname{sgn} C\}$ satisfying* (α_1), (α_2), (β), *and* (γ). *Let π be a faithful $_*$ representation of $\mathcal{E}$. There exists a Hilbert space $\hat{\mathcal{H}}_\pi$ and a completely non-commuting pair of bounded self-adjoint operators U and V in $\mathcal{L}(\hat{\mathcal{H}}_\pi)$ whose determining function coincides with $\pi(\tilde{E}(l, z))$ on $\hat{\mathcal{H}}_\pi$.*

In this statement the assertion that U and V are completely non-commuting means that there is no non-trivial invariant subspace for both U and V on which they commute.

Two other facts should be noted.

COROLLARY 2.1. *There exists an embedding π of $\mathcal{E}$ into a C^* algebra on a Hilbert space $\mathcal{B}$, and a completely non-normal operator $T \in \mathcal{B}$ such that $E_T(l, z) = \pi[\tilde{E}(l, z)]$.*

THEOREM 2.2. *Suppose $\{\pi_1, \mathcal{H}_1\}$ and $\{\pi_2, \mathcal{H}_2\}$ are unitarily equivalent representations of $\{E(l, z), \operatorname{sgn} C\}$. Let T_1 and T_2 be operators which implement $\pi_i[E(l, z)]$, $i = 1, 2$ as in Corollary* 2.1; *then T_1 and T_2 are unitarily equivalent.*

The fact that the determining function of a completely non-normal operator T is a complete unitary invariant for T was noted in [34].

For the sake of completeness, we will give a schematic description of the proof of Theorem 2.1 (which can be omitted on first reading).

Denote the closure of the range of πC by h. Using (β) and the theory of reproducing kernel Hilbert spaces, it can be shown that there exists a unique Hilbert space $\tilde{\mathcal{H}}_\pi$, whose elements are h-valued analytic functions $F(z_1, z_2)$ defined for non-real z_1 and z_2 such that for every vector $b \in h$ and non-real numbers w_1 and w_2, $\pi K(w_1, w_2; z_1, z_2) b$ is in $\tilde{\mathcal{H}}_\pi$ as a function of z_1 and z_2, and

$$(F(w_1, w_2), b)_h = \langle F(t_1, t_2), \pi K(w_1, w_2; t_1, t_2) b \rangle_{\tilde{\mathcal{H}}_\pi}.$$

We identify h with the subspace of constant functions. It is known (see the appendix in [33]) that the maps

$$F(z_1, z_2) \mapsto \frac{F(z_1, z_2) - F(z_1, \omega_2)}{z_2 - \omega_2},$$

$$F(z_1, z_2) \mapsto \frac{F(z_1, z_2) - \tilde{E}_\pi(z_1, z_2) \tilde{E}_\pi(\omega_1, z_2)^{-1} F(\omega_1, z_2)}{z_1 - \omega_1}$$

for $\operatorname{Im}\omega_2 \neq 0$, $\operatorname{Im}\omega_1 \neq 0$, are resolvents of bounded self-adjoint transformations in $\tilde{\mathcal{H}}_\pi$ which we will call H_2 and H_1 respectively. If $G(z_1, z_2) = [F(z_1, z_2) - F(z_1, \omega_2)]/(z_2 - \omega_2)$, then $(H_2 - \omega_2) G(z_1, z_2) = F(z_1, z_2)$. Since

$$\lim_{y_2\to\infty} iy_2 G(z_1, iy_2) = \lim_{y_2\to\infty} iy_2 \frac{F(z_1, iy_2) - F(z_1, \omega_2)}{iy_2 - \omega_2} = -F(z_1, \omega_2),$$

we see that

$$F(z_1, z_2) = (z_2 - \omega_2) G(z_1, z_2) - \lim_{y_2\to\infty} iy_2 G(z_1, iy_2),$$

so that

$$H_2 G(z_1, z_2) = z_2 G(z_1, z_2) - \lim_{y_2\to\infty} iy_2 G(z_1, iy_2).$$

Similarly

$$H_1 G(z_1, z_2) = z_1 G(z_1, z_2) - \tilde{E}_\pi(z_1, z_2) \lim_{y_1\to\infty} iy_1 G(iy_1, z_2),$$

and a calculation shows that

$$\begin{aligned}
(H_1 H_2 - H_2 H_1) G(z_1, z_2) &= H_1\{z_2 G(z_1, z_2) - \lim_{y_2\to\infty} iy_2 G(z_1, iy_2)\} \\
&\quad - H_2\{z_1 G(z_1, z_2) - \tilde{E}_\pi(z_1, z_2) \cdot \lim_{y_1\to\infty} iy_1 G(iy_1, z_2)\} \\
&= z_1 z_2 G(z_1, z_2) - z_1 \lim_{y_2\to\infty} iy_2 G(z_1, iy_2) \\
&\quad - \tilde{E}_\pi(z_1, z_2) \lim_{y_1\to\infty} iy_1\{z_2 G(iy_1, z_2) - \lim_{y_2\to\infty} iy_2 G(iy_1, iy_2)\} \\
&\quad - z_1 z_2 G(z_1, z_2) + z_2 \tilde{E}_\pi(z_1, z_2) \lim_{y_1\to\infty} iy_1 G(iy_1, z_2) \\
&\quad + \lim_{y_2\to\infty} iy_2\{z_1 G(z_1, iy_2) - \tilde{E}_\pi(z_1, iy_2) \lim_{y_1\to\infty} iy_1 G(iy_1, iy_2)\} \\
&= -i[1 - \tilde{E}_\pi(z_1, z_2)]\, \Omega(G)
\end{aligned}$$

where

$$\Omega(G) = \lim_{\substack{y_1\to\infty \\ y_2\to\infty}} iy_1 iy_2 G(iy_1, iy_2).$$

Let $G(z_1, z_2) = \pi K(\omega_1, \omega_2; z_1, z_2) b$, a generic element of $\mathcal{H}_\pi$; then

$$\begin{aligned}
\Omega(G) &= \lim_{\substack{y_1\to\infty \\ y_2\to\infty}} iy_1 iy_2 \frac{\{\tilde{E}_\pi(iy_1, iy_2) \operatorname{sgn} C \tilde{E}_\pi^*(\omega_2, \overline{iy_2}) - \tilde{E}_\pi(iy_1, \bar{\omega}_2) \operatorname{sgn} C \tilde{E}_\pi^*(\omega_1, \omega_2)\} b}{(iy_1 - \bar{\omega}_1)(iy_2 - \bar{\omega}_2)} \\
&= \operatorname{sgn} C[1 - \tilde{E}_\pi^*(\omega_1, \omega_2)]\, b.
\end{aligned}$$

Let $k(G) \equiv [1-\tilde{E}_\pi^*(\omega_1, \omega_2)]b$. So k is a linear map from $\mathcal{H}_\pi$ to h. Then for $d \in h$, we have

$$\langle k^*d, G(z_1, z_2)\rangle_{\mathcal{H}_\pi} = \langle d, k(K(\omega_1, \omega_2; z_1, z_2)\,b)\rangle_h = \langle d, (1-\tilde{E}_\pi^*(\omega_1, \omega_2)\,b)\rangle_h$$
$$= \langle (1-\tilde{E}_\pi(\omega_1, \omega_2))\,d, b\rangle_h.$$

Thus the adjoint k^*: $d \to (1-\tilde{E}_\pi(\omega_1, \omega_2))d$ is a continuous linear map from h into $\tilde{\mathcal{H}}_\pi$, and therefore we can write

$$(H_1H_2 - H_2H_1)\,G(z_1, z_2) = -\,ik^*\,[\operatorname{sgn} CkG]\,(z_1, z_2).$$

In what follows we make repeated use of the relation in (2.5).

Suppose now that we chose an operator J so that $[J, \operatorname{sgn} C] = 0$ and $J^2 = \operatorname{sgn} C$, $J^*J = I$. Then with $q \in h$ and $k^*Jq = [1-\tilde{E}_\pi(z_1, z_2)]Jq$,

$$\begin{aligned}
&(H_1-\omega_1)^{-1}(H_2-\omega_2)^{-1}k^*Jq \\
&= -(H_1-\omega_1)^{-1}\frac{\tilde{E}_\pi(z_1, z_2)-\tilde{E}_\pi(z_1, \omega_2)}{z_2-\omega_2}\,Jq \\
&= -\left\{\frac{\tilde{E}_\pi(z_1, z_2)-\tilde{E}_\pi(z_1, \omega_2)}{(z_2-\omega_2)(z_1-\omega_1)} - \tilde{E}_\pi(z_1, z_2)\,\tilde{E}_\pi(\omega_1, z_2)^{-1}\frac{(\tilde{E}_\pi(\omega_1, z_2)-\tilde{E}_\pi(\omega_1, \omega_2))}{(z_2-\omega_2)(z_1-\omega_1)}\right\}Jq \\
&= -\frac{\tilde{E}_\pi(z_1, z_2)\,\tilde{E}_\pi(\omega_1, z_2)^{-1}\tilde{E}_\pi(\omega_1, \omega_2)-\tilde{E}_\pi(z_1, \omega_2)}{(z_2-\omega_2)(z_1-\omega_1)}\,Jq \\
&= -\frac{\tilde{E}_\pi(z_1, z_2)\operatorname{sgn} C\tilde{E}_\pi^*(\bar{\omega}_1, \bar{z}_2)-\tilde{E}_\pi(z_1, \omega_2)\operatorname{sgn} C\tilde{E}_\pi^*(\bar{\omega}_1, \bar{\omega}_2)}{(z_2-\omega_2)(z_1-\omega_1)}\cdot\operatorname{sgn} C\tilde{E}_\pi(\omega_1, \omega_2)\,Jq \\
&= -\frac{1}{i}\,\pi K(\bar{\omega}_1, \bar{\omega}_2, z_1, z_2)\operatorname{sgn} C\tilde{E}_\pi(\omega_1, \omega_2)\,Jq.
\end{aligned}$$

Thus

$$\frac{1}{i}\,Jk(H_1-\omega_1)^{-1}(H_2-\omega_2)^{-1}k^*Jq = Jk\pi K(\bar{\omega}_1, \bar{\omega}_2, z_1, z_2)\operatorname{sgn} C\tilde{E}_\pi(\omega_2, \omega_2)\,Jq,$$

but $k\pi K(\bar{\omega}_1, \bar{\omega}_2; z_1, z_2)\,b = [1-\tilde{E}_\pi^*(\bar{\omega}_1, \bar{\omega}_2)]\,b$. Hence, we have

$$\begin{aligned}
\frac{1}{i}\,Jk(H_1-\omega_1)^{-1}(H_2-\omega_2)^{-1}k^*Jq &= J[1-\tilde{E}_\pi^*(\bar{\omega}_1, \bar{\omega}_2)]\operatorname{sgn} C\tilde{E}_\pi(\omega_1, \omega_2)\,Jq \\
&= J\operatorname{sgn} C\tilde{E}_\pi(\omega_1, \omega_2)\,Jq - J\operatorname{sgn} C\tilde{E}_\pi^{-1}(\omega_1, \omega_2)\,\tilde{E}_\pi(\omega_1, \omega_2)\,Jq \\
&= (J^*\tilde{E}_\pi(\omega_1, \omega_2)\,J-1)\,q = J^*(\tilde{E}_\pi(\omega_1, \omega_2)-1)\,Jq.
\end{aligned}$$

Thus we see that $\tilde{E}_\pi(\omega_1, \omega_2)$ is unitarily equivalent to $1-iJk(H_1-\omega_1)^{-1}(H_2-\omega_2)^{-1}k^*J$ on h. In particular $J^2C = kk^*$.

We have already noted that essentially these reproducing kernel constructions arose out of conversations between the second author and L. de Branges. But the definition of determining function is slightly changed from that in [31].

We turn now to the modifications which are necessary in order to prove Theorem 2.1.

Let the polar decomposition of k be given by $k=|k|W$ where $|k|=(kk^*)^{1/2}=|C|^{1/2}$ and W is the canonical partial isometry. Then

$$k^*=W^*|k|, \text{ and } k^* \operatorname{sgn} Ck=W^*|k|\operatorname{sgn} C|k|W=W^*CW.$$

Thus, $C=Wk^* \operatorname{sgn} CkW^*=|k|\operatorname{sgn} C|k|$, and

$$\hat{C}=|k|J=J|k|=JkW^*=Wk^*J.$$

Let us now note the following simple facts:

(a) closure (Range k) reduces $\tilde{E}_\pi$

(b) $\tilde{E}_\pi$ restricted to the orthogonal complement of (Range k) is the identity. To show this, note that since $\tilde{E}_\pi(l,z)=1-iJk(H_1-l)^{-1}(H_2-z)^{-1}k^*J$ and $[J,|k|]=0$, the range of k is invariant for $\tilde{E}$. But $\tilde{E}^*_\pi(\bar{l},\bar{z})=\operatorname{sgn} C\tilde{E}_\pi^{-1}(l,z)\operatorname{sgn} C$, and $\tilde{E}^{-1}(l,z)=1+iJk(H_2-z)^{-1}(H_1-l)^{-1}k^*J$. Thus we can conclude that the closure of the range of k is invariant also for $E^*_\pi(\bar{l},\bar{z})$. This proves (a). In order to prove (b), let $x\perp\overline{\mathcal{R}(k)}$. Now $\tilde{E}_\pi(l,z)x=x-y$ where $y\in\mathcal{R}(k)$. But $\tilde{E}_\pi(l,z)x\in\mathcal{R}(k)^\perp$, and hence $\tilde{E}_\pi(l,z)x=x$.

Let $R_i(W)$ denote the initial space of W. We define an isometric dilation space in terms of W by setting

$$\hat{\mathcal{H}}=\tilde{\mathcal{H}}_\pi\oplus R_i(W)^\perp\oplus R_i(W)^\perp\oplus\ldots$$

Define $\hat{W}$ on $\hat{\mathcal{H}}$ by: $x=\langle x_0,x_1,x_2\ldots\rangle$ $\hat{W}x=\langle Wx_0,P_ix_0,x_1,\ldots\rangle$ where P_i is the projection of $\tilde{\mathcal{H}}_\pi$ onto $R_i(W)^\perp$.

We can also extend k, H_1 and H_2 to $\hat{\mathcal{H}}$ by identifying them with the zero operator off of $\tilde{\mathcal{H}}_\pi\oplus 0\oplus 0\oplus\ldots$.

Now, since $\hat{W}$ is an isometric map

$$\begin{aligned} J^*\tilde{E}_\pi(l,z)J &= 1_{\tilde{\mathcal{H}}_\pi}-iJk\hat{W}^*\hat{W}(H_1-l)^{-1}\hat{W}^*\hat{W}(H_2-z)^{-1}\cdot\hat{W}^*\hat{W}k^*J\\ &= 1_{\tilde{\mathcal{H}}_\pi}-i\hat{C}(\hat{W}H_1\hat{W}^*-l)^{-1}(\hat{W}H_2\hat{W}^*-z)^{-1}\hat{C},\end{aligned}$$

If we extend J to $\hat{H}$ by setting $Jx=x$, (x in $R_i(W)^\perp$) and do the same for $\tilde{E}_\pi(l,z)$, we shall have

$$\tilde{E}_\pi(l,z)=1_{\hat{\mathcal{H}}}+\frac{1}{i}J\hat{C}J^*(JWH_1\hat{W}^*J^*-l)^{-1}\cdot(J\hat{W}H_2\hat{W}^*J^*-z)^{-1}J\hat{C}J^*$$

and

$$[J\hat{W}H_1\hat{W}^*J^*, J\hat{W}H_2\hat{W}^*J^*] = -iJCJ^*.$$

Thus $\hat{E}_\pi(l, z)$ is the determining function of the pair $U = J\hat{W}H_2\hat{W}^*J^*$ and $V = J\hat{W}H_1\hat{W}^*J^*$ on $\hat{\mathcal{H}}$.

§ 3. Boundary values and mosaics

We first discuss some notational conventions and recall some known facts.

Let $L^1(\Omega, \mu)$ be the Banach space of all complex valued μ-integrable functions, with a positive σ-finite measure μ, on a measure space (Ω, μ) and let E be a Banach space. $L^1(\Omega, \mu)\otimes_\gamma E$ denotes the completion of the algebraic tensor product of $L^1(\Omega, \mu)$ and E in the topology induced by the greatest cross norm γ defined by $\gamma(f) = \inf \sum_{i=1}^n \|x_i\| \, \|y_i\|$ where the inf is taken over all representations of f. Then $L^1(\Omega, \mu)\otimes_\gamma E = L^1(\Omega; \mu, E)$ by Grothendieck [23], where $L^1(\Omega, \mu, E)$ is the Banach space of all E-valued Bochner μ-integrable functions on the measure space (Ω, μ). If $\mathcal{M}$ is a von Neumann algebra with a separable predual $\mathcal{M}_*$, then $L^1(\Omega, \mu, \mathcal{M}_*)^* = L^\infty(\Omega, \mu, \mathcal{M})$ where $L^\infty(\Omega; \mu, \mathcal{M})$ is the Banach space of all $\mathcal{M}$-valued essentially bounded weakly* μ-measurable functions on Ω [36]. From here on we shall assume that $\mathcal{M}_*$ is a separable Banach space.

In the present paper we shall be concerned with the case when $\Omega = R^1$ or R^2 and μ is the Lebesgue measure, dt or dA.

Diagonalization of self-adjoint operators

Suppose $M \in \mathcal{L}(\mathcal{H})^s$. Let us consider the diagonalization of M_{ac}. In this respect the operator M_{ac} acting on $\mathcal{H}_{ac}(\mathcal{M})$ may be identified with the following system: $\mathcal{H}_{ac}(M)$ is the direct sum of function spaces $\mathcal{H}_\alpha = L^2(\Delta_\alpha; \mathcal{D}_\alpha)$, $\alpha = 1, 2, 3, \ldots$, and ∞, where the Δ_α are disjoint Borel subsets of the real axis and the $\mathcal{D}_\alpha$ are separable Hilbert spaces with dimension $\mathcal{D}_\alpha = \alpha$. The symbol $L^2(\Delta_\alpha; \mathcal{D}_\alpha)$ denotes the set of all $\mathcal{D}_\alpha$ valued strongly measurable functions $f(\lambda)$ on Δ_α satisfying $\int_{\Delta_\alpha} \|f(\lambda)\|^2 d\lambda < \infty$. The function

$$m(\lambda) = \begin{cases} \alpha & \lambda \in \Delta_\alpha \\ 0 & \lambda \notin U_\alpha \Delta_\alpha \end{cases}$$

is the spectral multiplicity for M_{ac}.

We shall assume that $\mathcal{D}_1 \subset \mathcal{D}_2 \subset \ldots \subset \mathcal{D}_\infty \equiv \mathcal{D}$ so that all $\mathcal{H}_\alpha$ (hence $\mathcal{H}$ itself) may be regarded as subspaces of $L^2(\Delta; \mathcal{D})$ where $\Delta = U_\alpha \Delta_\alpha$. The action of M_{ac} is that of multiplication by the coordinate in each of the $\mathcal{H}_\alpha$: $Mf(\lambda) = \lambda f(\lambda)$, $f \in \mathcal{H}_\alpha$.

M-smoothness

Let M be a self-adjoint operator on $\mathcal{H}$ and suppose $K \in \mathcal{L}(\mathcal{H}, \chi)$. Set $R_z = (M - z)^{-1}$ for $\operatorname{Im} z \neq 0$.

Definition 3.1. K is said to be M-smooth if

$$\underset{\operatorname{Im} z \neq 0, \|x\| \neq 0}{\text{supremum}} \left|(\{R_z - R_z^*\} K^* x \mid K^* x)\right| / 2\pi \|x\|^2 \equiv \|K\|_M < \infty. \tag{3.1}$$

The notion of M-smoothness originates with Kato in [26] and [28] where it was shown, among other things, that if K is M-smooth, then K necessarily annihilates every vector in $\mathcal{H}_s(M)$. Such smoothness is equivalent to the fact that K^*K has a representation as an integral operator with an essentially bounded kernel in a Hilbert space in which M is diagonal.

Observe that if ψ is an essentially bounded Lebesgue measurable function on R^1 with supremum $\|\psi\|_\infty$, then $\psi(M) = \int \psi(\lambda) dP_{ac} E_\lambda$ defines a bounded operator on $\mathcal{H}$ such that whenever $Q \in \mathcal{L}(\Omega, \chi)$ is $\mathcal{M}$-smooth, then so is $Q\psi(M)$ and $\|Q\psi(M)\|_M \leq \|Q\|_M \|\psi\|_\infty$.

THEOREM 3.1. *Let $\mathcal{A}$ be a von Neumann algebra and let A and D be elements in $\mathcal{A}^s$. Suppose D is factored in the form $D = K^* \bar{J} K$ where $\bar{J}$ is self-adjoint and unitary in $\mathcal{A}$ and $K \in \mathcal{A}$. Let $\mathcal{B}$ be the subalgebra generated by A, K and $\bar{J}$, then there exists a unique element B in $L^\infty(R^1, d\lambda, \mathcal{B})$ such that for almost all t, $0 \leq B(t) \leq 1$; $B - \frac{1}{2}(1 - \bar{J}) \in L^1(R^1; d\lambda/(1 + |\lambda|), \mathcal{B})$ and*

$$\Phi(l) \equiv \bar{J} + K(A - l)^{-1} K^* = \exp\left\{\frac{i\pi}{2}(1 - \bar{J}) + \int \frac{B(t) - \frac{1}{2}(1 - \bar{J})}{t - l} dt\right\} \tag{3.2}$$

for $\operatorname{Im} l > 0$. *The integral is taken in the weak* sense.*

Proof. The proof is based on two results of T. Kato which we now state:

Result one. Let f be holomorphic on a region P in the complex plane which is conformally equivalent to the closed unit disk. Let G be an element in $\mathcal{A}$ whose spectrum is contained in P. Then, if $W[G]$, the numerical range of G, is contained in P, it follows that $W[f(G)]$ is contained in the closed convex hull of $f(P)$ [Thm 7; 27].

Result two. Let $\mathcal{H} = L^2(R^1; \mathcal{D})$ and let M be the multiplication operator in $\mathcal{H}$. If $L \in \mathcal{L}(\mathcal{H}, \chi)$ is M-smooth, then there is an $\mathcal{L}(\chi, \mathcal{D})$-valued strongly measurable function $L(\cdot)$ defined on R^1 such that $\|L(t)\| \leq \|L\|_M$ and $L^* x = \{L(t)x\}$ for every $x \in \chi$ [26].

We have used the notation $L^* x = \{L(t)x\}$ to indicate that $L^* x \in \mathcal{H}$ is represented by the function $t \to \{L(t)x\}$.

We will prove the theorem by working in a faithful *-representation $\{\varrho, \mathcal{K}\}$ of $\mathcal{A}$.

Since $\mathcal{B}$ has a separable predual, $\mathcal{K}$ is a separable Hilbert space. Henceforth we will assume that A, K, and $\bar{J}$ are in $\mathcal{L}(\mathcal{K})$.

With these preliminaries stated, we now enter into our demonstration.

Observe that $\Phi(l)^{-1}$ exists and in fact $\Phi(l)^{-1} = \bar{J} - \bar{J}K\{A+D-l\}^{-1}K^*\bar{J}$. Thus $0 \notin \operatorname{sp}(\Phi(l))$ for $\operatorname{Im} l \neq 0$. Moreover the numerical range of $\Phi(l)$ is contained in the closed upper half plane when $\operatorname{Im} l > 0$.

Let $f(l) = \ln l$ where $\ln l$ denotes the branch of the logarithm determined by $-\pi/2 < \arg l < 3\pi/2]$. For fixed $\eta > 0$ consider the function $f_\eta(l) = f(l + i\eta)$. It is clear that $f_\eta(l)$ is holomorphic on $P = \{l: \operatorname{Im} l \geqslant 0\}$.

By result one we see that, for $\operatorname{Im} l > 0$, $W[f_\eta(\Phi(l)] \subset \{l: 0 \leqslant \operatorname{Im} l \leqslant \pi\}$. Therefore, $0 \leqslant \operatorname{Im} f_\eta(\Phi(l)) \leqslant \pi 1$. Consequently, since $\lim_{\eta \to 0^+} f_\eta(l) = f(l) = \ln l$, and the limit is uniform on a neighborhood of $\operatorname{sp}(\Phi(l))$, it follows that $f_\eta(\Phi(l))$ converges to $\ln \Phi(l)$ in the operator norm topology and

$$0 \leqslant \operatorname{Im} \ln \Phi(l) \leqslant \pi 1. \tag{3.3}$$

Therefore we can invoke an operator generalization of the Herglotz–Riesz representation theorem for analytic functions with imaginary part positive in the upper half plane [8] to obtain a bounded nondecreasing operator valued function $\Omega(\cdot)$ such that, for some bounded self-adjoint operator N (which is determined below)

$$\ln \Phi(l) = N + \int \left\{ \frac{1}{t-l} - \frac{t}{1+t^2} \right\} d\Omega(t) \tag{3.4}$$

for $\ln l > 0$.

The integral which occurs here is a Stieltjes operator integral and

$$\int \frac{d\Omega(t)}{1+t^2} < \infty .$$

Consider the map

$$t \to W(t) = \int_{-\infty}^{t} \frac{1}{1+s^2} d\Omega(s).$$

By the Naimark dilation theorem [8], there is a Hilbert space $\tilde{\mathcal{H}}$, an operator $K \in L(\tilde{\mathcal{H}}, \mathcal{K})$ and a resolution of unity $\{\tilde{E}_t\}$ in $\tilde{\mathcal{H}}$ such that

$$W(t) = K\tilde{E}_t K^* \tag{3.5}$$

Let $\tilde{M} = \int t d\tilde{E}_t$, where as usual the domain $\mathcal{D}(\tilde{M})$ is $\{x \in \tilde{\mathcal{H}}: \int t^2 d\|E_t x\|^2 < \infty$.

The subspace $\mathcal{H}$ generated by $\{(\tilde{M}-l)^{-1}K^*x: x \in \mathcal{K}, \operatorname{Im} l \neq 0\}$ is separable and invariant for the family $\{\tilde{E}_t\}$. Let M denote the restriction of $\tilde{M}$ to $\mathcal{H}$. (We note that we can choose

$\mathcal{H}$ to be the direct integral diagonalizing space of M so we identify $\mathcal{H}$ with $L^2(R^1, \mathcal{D})$.) It is not hard to see that $\ln \varphi(l) = N + K \int (1+lt)/(t-l)\, d\hat{E}_t K^* = N + K(1+lM)(M-l)^{-1}K^*$ from which, together with the inequality (3.3) we can see that K is M-smooth (see definition 3.1). Thus result two stated above allows us to conclude that the action of K^* is given by $K^*x = \{K(t)x\}(x \in \mathcal{K})$ where $t \to K(t)$ is a strongly measurable function with values in $L(\mathcal{K}, \mathcal{D})$, i.e. $K(t) \in L(\mathcal{K}, \mathcal{D})$ for almost all t and for each $x \in \mathcal{K}$, $t \to K(t)x$ is Lebesgue measurable.

Moreover, $\mathrm{ess}_t \sup \|(1+t^2)^{1/2}K(t)\| \leq 1$. With $h \in \mathcal{H}$ and $t \to \|h(t)\|^2$ summable, a small modification of Lemma 5.3 in [26] implies that $t \to K(t)^*h(t)$ is Bochner integrable and $Kh = \int K(t)^*h(t)\,dt$.

Accordingly, with $\operatorname{Im} l \neq 0$ and $h = \{(1+lt)(t-l)^{-1}K(t)x\} = (1+lM)(M-l)^{-1}Kx$, $x \in \mathcal{K}$ we have

$$K(1+lM)(M-l)^{-1}K^*x = \int \frac{1+lt}{t-l} K(t)^*K(t)\,x\,dt. \tag{3.6}$$

Thus, with $B(t) = (1+t^2)K(t)^*K(t)$ we have $0 \leq B(t) \leq 1$ for almost all t, (as an operator on $\mathcal{K}$), and

$$[\ln \varphi(l)]\,x = Nx + \int \left\{\frac{1}{t-l} - \frac{t}{1+t^2}\right\} B(t)\,x\,dt, \quad x \in \mathcal{K}. \tag{3.7}$$

We shall now estimate $\|B(t) - \frac{1}{2}(1-\bar{J})\|$ for large t. To do this, we first observe that since $B(t) = \pi^{-1} \operatorname{Im} \ln \Phi(t)$ for almost all t in a neighborhood of infinity, say Δ, we see that $B(t)$ coincides with the projection onto the negative spectrum of the self-adjoint operator $\bar{J} + K(A-t)^{-1}K^* \equiv \Phi(t)$. Since the spectrum of $\bar{J} \subset \{-1, 1\}$, we can choose Δ so that for all $t \in \Delta$, the spectrum of $\Phi(t)$ is a subset of the union of the intervals $(-3{,}2, -1/2)$ and $(1/2, 3/2)$; then for t in Δ

$$B(t) - \tfrac{1}{2}(I - \bar{J}) = \frac{1}{2\pi i} \int_\Gamma (\Phi(t) - l)^{-1} - (\bar{J} - l)^{-1}\,dl \tag{3.8}$$

where Γ is $\{l\colon |l+1| = \frac{1}{2}\}$.

But

$$(\Phi(t) - l)^{-1} - (\bar{J} - l)^{-1} = -(\bar{J} - l)^{-1}K(A-t)^{-1}K^*(\Phi(t) - l)^{-1}. \tag{3.9}$$

Hence

$$\|B(t) - \tfrac{1}{2}(I - \bar{J})\| = O\left(\frac{1}{|t|}\right) \quad \text{as} \quad |t| \to \infty. \tag{3.10}$$

Now

$$\int \left\{\frac{1}{t-l} - \frac{t}{1+t^2}\right\} dt = i\pi \quad \text{if} \quad \operatorname{Im} l > 0;$$

therefore it follows that

$$\ln \Phi(l) = N + i\pi/2(I-\bar{J}) - \int \{B(t) - \tfrac{1}{2}(I-\bar{J})\} \frac{t}{1+t^2}\,dt + \int \{B(t) - \tfrac{1}{2}(I-\bar{J})\} \frac{dt}{t-l}. \tag{3.11}$$

Since $\lim_{y\to\infty} \Phi(iy) = \bar{J}$, and $\ln \bar{J} = \frac{1}{2}\pi i(1-\bar{J})$ we conclude that

$$N = \int \{B(t) - \tfrac{1}{2}(I-\bar{J})\} \frac{t}{1+t^2}\,dt. \tag{3.12}$$

Hence

$$\ln \Phi(l) = i\pi/2(I-\bar{J}) + \int \{B(t) - \tfrac{1}{2}(I-\bar{J})\} \frac{dt}{t-l}. \tag{3.13}$$

Taking exponentials yields

$$\Phi(l) = \exp\left[i\pi/2(I-\bar{J}) + \int \{B(t) - \tfrac{1}{2}(I-\bar{J})\} \frac{dt}{t-l}\right]. \tag{3.14}$$

To complete the proof of the theorem, we must show that the operator $B(t)$ which we have constructed is in $\varrho(\mathcal{B})$ is unique and is weak* measurable.

The fact that $B(t)$ is in $\varrho(\mathcal{B})$ follows easily by observing that

$$\int \frac{\varepsilon B(\nu)}{(\nu-t)^2+\varepsilon^2}\,d\nu = \operatorname{Im} \ln [\bar{J} + K(A-(t+i\varepsilon))^{-1}K^*] \tag{3.15}$$

and that because $0 \leqslant B(t) \leqslant 1$ and $\mathcal{K}$ is separable

$$\frac{1}{\pi} \lim_{\varepsilon\downarrow 0} \int \frac{\varepsilon B(\nu)x}{(\nu-t)^2+\varepsilon^2}\,d\nu = B(t)x$$

on a set of full t-measure which can be chosen independently of $x \in \mathcal{K}$.

Finally the weak * measurability follows from the weak measurability of $B(t)$ and [36; Prop. 1.15.2] or [20; Chap. 1., sec. 4, Thm 1]. This finishes the proof.

By taking adjoints it follows immediately that the relation extends to $\operatorname{Im} l < 0$ if we change $\frac{1}{2}i\pi(1-\bar{J})$ to its negative.

Let $P_+ = \frac{1}{2}(1+\bar{J})$, $P_- = \frac{1}{2}(1-\bar{J})$. These are the projections onto the positive and negative eigenspaces respectively of $\bar{J}$.

We will now obtain an estimate for $P_+B(t)P_+$ and $P_-(1-B(t))P_-$ when t is large. In order to do this we note that, for large t,

$$B(t) = \frac{1}{2\pi i}\int_\Gamma (\Phi(t)-l)^{-1}\,dl \quad \text{where now } \Gamma = (l: |l+1| = \tfrac{1}{2}).$$

Thus

$$P_+B(t)P_+ = \frac{1}{2\pi i}\int_\Gamma P_+\{(\Phi(t)-l)^{-1} - (\bar{J}-l)^{-1}\}P_+\,dl,$$

since

$$\frac{1}{2\pi i}\int_\Gamma P_+(\bar{J}-l)^{-1}P_+\,dl=0.$$

By (3.9) we have

$$P_+B(t)P_+=-\frac{1}{2\pi i}\int_\Gamma P_+\{\bar{J}-l)^{-1}K(A-t)^{-1}K^*(\Phi(t)-l)^{-1}P_+\}\,dl.$$

Also,

$$\frac{1}{2\pi i}\int_\Gamma P_+(\bar{J}-l)^{-1}K(A-t)^{-1}K^*(\bar{J}-l)^{-1}P_+\,dl=\frac{1}{2\pi i}\int_\Gamma (P_+-l)^{-2}P_+K(A-t)^{-1}K^*P_+\,dl$$

since $[\bar{J},P_+]=0$ and $\bar{J}P_+=P_+$. But the last integral is the zero operator, by the residue theorem.

Consequently,

$$P_+B(t)P_+=\frac{1}{2\pi i}\int_\Gamma P_+\{(\bar{J}-l)^{-1}K(A-t)^{-1}K^*(\bar{J}-l)^{-1}K(A-t)^{-1}K^*(\Phi(t)-l)^{-1}\}P_+\,dt.$$

Accordingly $P_+B(t)P_+=O(1/|t|^2)$ as t goes to ∞. Therefore $P_+B(t)P_+$ is weak* integrable. That is, for each f in $\mathcal{B}_*$ the complex valued function $t\to\langle P_+B(t)P_+,f\rangle$ is integrable. Similarly, $P_-(1-B(t))P_-=O(1/|t|^2)$ as t approaches ∞, and so $P_-(1-B(t))P_-$ is also weak* integrable.

We will now establish the relations:

$$\int P_+B(t)P_+\,dt=P_+KK^*P_+ \tag{3.16}$$

$$\int P_-(1-B(t))P_-\,dt=P_-KK^*P_-. \tag{3.17}$$

We begin by noting that

$$\int\frac{\varepsilon B(\gamma)}{(\gamma-t)^2+\varepsilon^2}\,d\gamma=\operatorname{Im}\ln(\Phi(t+i\varepsilon))=\operatorname{Im}\frac{1}{2\pi i}\int_\Gamma \ln z(\Phi(t+i\varepsilon)-z)^{-1}dz$$

where Γ is now the contour pictured in fig. 1

We have

$$\operatorname{Im}\frac{1}{2\pi i}\int_\Gamma \ln z(J-z)^{-1}dz=\pi P_-$$

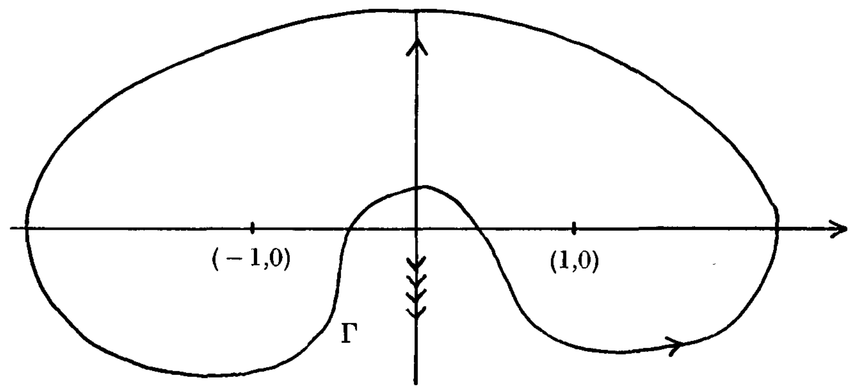

Fig. 1.

hence,

$$\operatorname{Im} \frac{1}{2\pi i} \int \ln z P_+ (J - z)^{-1} P_+ \, dz = 0.$$

Accordingly, using identity (3.9)

$$\int \frac{\varepsilon P_+ B(\gamma) P_+}{(\gamma - t)^2 + \varepsilon^2} \, d\gamma = - \operatorname{Im} \frac{1}{2\pi i} \int_\Gamma dz \ln z P_+ (J - z)^{-1} K (A - (t + i\varepsilon))^{-1} K^* (\Phi(t + i\varepsilon) - z)^{-1} P_+ .$$

Multiplying both sides by ε and letting $\varepsilon \uparrow \infty$, we obtain via the Lebesgue dominated convergence theorem applied to the right hand side

$$\begin{aligned} \lim_{\varepsilon \uparrow \infty} \int \frac{\varepsilon^2 P_+ B(\gamma) P_+}{(\gamma - t)^2 + \varepsilon^2} \, d\gamma &= \operatorname{Im} \frac{1}{2\pi i} \int_\Gamma i \ln z P_+ (J - z)^{-1} K K^* (J - z)^{-1} P_+ \, dz \\ &= \operatorname{Im} \frac{1}{2\pi i} \int_\Gamma i \ln z (1 - z)^{-2} dz P_+ K K^* P_+ = P_+ K K^* P_+ . \end{aligned}$$

Now, since $\|P_+ B(t) P_+\| = O(1/|t|^2)$ (as t approaches ∞) for each f in $\mathfrak{B}_*$ we have, by the Lebesgue dominated convergence theorem,

$$\lim_{\varepsilon \uparrow \infty} \int \frac{\varepsilon^2 \langle P_+ B(\gamma) P_+ , f \rangle \, d\gamma}{(\gamma - t)^2 + \varepsilon^2} = \int \langle P_+ B(\gamma) P_+ , f \rangle \, d\gamma .$$

Accordingly,

$$\int P_+ B(\gamma) P_+ \, d\gamma = P_+ K K^* P_+ .$$

Similarly,

$$\int P_- (1 - B(\gamma)) P_- \, d\gamma = P_- K K^* P_- .$$

We will now study the operators $B(t)$ when K is assumed to be in the Hilbert–Schmidt ideal.

Suppose $\mathcal{M}$ is a von Neumann algebra and τ is a normal trace on $\mathcal{M}^+$. Recall that the set of $T \in \mathcal{M}^+$ with $\tau(T) < \infty$ is the positive portion of a two-sided ideal $\mathfrak{J}_\tau$ of $\mathcal{M}$, and that there exists a unique linear functional $\tilde{\tau}$ on $\mathfrak{J}_\tau$ which coincides with τ on $\mathfrak{J}_\tau \cap \mathcal{M}^+$ such that $\tilde{\tau}(TA) = \tilde{\tau}(AT)$ $(T \in \mathfrak{J}, A \in \mathcal{M})$. Furthermore, the linear functional $A \mapsto \tilde{\tau}(TA)$ or $\mathcal{M}$ is $\sigma(\mathcal{M}, \mathcal{M}_*)$ continuous [36]. The ideal $\mathfrak{J}_\tau \mathfrak{J}_\tau = \{A \cdot B \colon A, B \in \mathfrak{J}_\tau\}$ is denoted $\mathbf{C}_2(\mathcal{M}, \tau)$.

In this context Theorem 3.1 becomes a bit more definitive.

PROPOSITION 3.1. *Suppose $\mathcal{M}$ is a von Nemann algebra equipped with a normal trace τ. Let A, J, K and $B(t)$ be as in Theorem* 3.1 *and suppose $K \in \mathbf{C}_2(\mathcal{M}, \tau)$. Then $B(t)$ satisfies the following properties:*

$$P_+ B(t) P_+ > P_-[1 - B(t)] P_- \in \mathfrak{J}_\tau \cap \mathcal{M}^+ \tag{3.18}$$

$$[1 - B(t)]^{1/2} P_- \in \mathbf{C}_2(\mathcal{M}, \tau) \tag{3.19}$$

$$P_+ B(t)^{1/2} \in \mathbf{C}_2(\mathcal{M}, \tau) \tag{3.20}$$

$$B(t)[1 - B(t)] \in \mathfrak{J}_\tau \cap \mathcal{M}^+ \tag{3.21}$$

$$B(t) - P_- \in \mathbf{C}_2(\mathcal{M}, \tau). \tag{3.22}$$

Proof. We begin by showing that the functions $t \to \tau[P_+ B(t) P_+]$ and $t \to \tau[P_-(1 - B(t)] P_-]$ are Lebesgue measurable.

By [20; Chap. I, Sec. 3.4, Cor. 3] there exists a central projection E in the weak closure of $\mathfrak{J}_\tau$ such that $ET = T$ for all $T \in \mathfrak{J}_\tau$. Moreover, by [20; Chap. I, Sec. 3.4, Cor. 5] there exists a directed set $\{E_\iota\}_{\iota \in I}$ of elements in $\mathfrak{J}_\tau \cap \mathcal{M}^+$ such that weak $\lim_{\iota \in I} E_\iota = E$. Since $\mathcal{M}_*$ is a separable space, the $\sigma(\mathcal{M}, \mathcal{M}_*)$ topology on norm bounded subsets of $\mathcal{M}$ is a metric topology [22; Vol. I, Chap. V, Sec. 51, Th. 1] and consequently is first countable. Thus we can select a sequence of operators $[E_n]_{n=1}^\infty \subset \{E_\iota\}_{\iota \in I}$ such that $\text{l.u.b.}_{n \geq 1} E_n = E$. For $n = 1, 2, \ldots$, the map $A \to \tau[E_n A E_n]$ is $\sigma(\mathcal{M}, \mathcal{M}_*)$-continuous since τ is normal [36, Thm 1.13.2]. Therefore $\tilde{\tau}[E_n P_+ B(t) P_+ E_n]$ is Lebesgue measurable since $t \to B(t)$ is weak * measurable. Again by the normality of τ it follows that $\lim_{n \to \infty} \tau[E_n P_+ B(t) P_+ E_n] = \tau[E P_+ B(t) P_+ E] = \tau[P_+ B(t) P_+]$ for almost all t. Consequently, $t \to \tau[P_+ B(t) P_+]$ is measurable. By a repetition of this reasoning it follows that $t \to \tau[P_-[1 - B(t)] P_-]$ is also measurable. Now the map $A \to \tau(E_n A E_n)$ defines an element f_n in $\mathcal{M}_*$ by [36, Thm. 1.13.2]. Thus by definition of the weak* integral, $\tau\{E_n \int P_+ B(t) P_+ \, dt E_n\} = \int \langle P_+ B(t) P_+, f_n \rangle dt = \int \tau(E_n P_+ B(t) P_+ E_n) dt$. By the normality of τ and relation (3.16) $\lim_{n \uparrow \infty} \int \tau(E_n P_+ B(t) P_+ E_n) dt = \tau(P_+ K K^* P_+)$. By the Fatou lemma we see that $t \to (P_+ B(t) P_+)$ is integrable. Therefore, applying the Lebesgue dominated convergence theorem we see that $\int \tau(P_+B(t)P_+)dt = \tau(P_+KK^*P_+)$. Similarly, we deduce that $\int \tau P_-(1-B(t))P_-dt = \tau(P_-KK^*P_-)$.

Assertion (3.18) follows at once whereas (3.19) and (3.20) are immediate consequences of (3.18). It remains to consider (3.21) and (3.22). Note that $P_+B(t)^{1/2}$ and $P_+B(t) \in \mathrm{C}_2(\mathcal{M}, \tau)$ as does $P_-[1-B(t)]^{1/2}$. Accordingly, $P_+B(t)P_-[1-B(t)]^{1/2} + P_+B(t)P_+[1-B(t)]^{1/2} \in \mathfrak{J}_\tau$. But then $P_+B(t)[1-B(t)]^{1/2} \in \mathfrak{J}_\tau$ since $P_+B(t)P_+ \in \mathfrak{J}_\tau$. Thus, $P_+B(t)[1-B(t)] \in \mathfrak{J}_\tau$. Using the same ideas it is easy to see that $P_-B(t)[1-B(t)]^{1/2} \in \mathfrak{J}_\tau$. Relation (3.21) follows at once by addition. It is plain that (3.22) follows from (3.19) and (3.20).

Remark 3.1. When $K \in \mathrm{C}_2(\mathcal{M}, \tau)$ and τ is faithful, a result of K. Asano [1] tells us that the principle value integral

$$P\int \frac{B(t)-P_-}{t-\nu}\,dt = \lim_{\varepsilon \downarrow \infty}\left[\int_{-\infty}^{\nu-\varepsilon} \frac{B(t)-P_-}{t-\nu}\,dt + \int_{\nu+\varepsilon}^{\infty} \frac{B(t)-P_-}{t-\nu}\,dt\right]$$

(the limit being taken with respect to the $\mathrm{C}_2(\mathcal{M}, \tau)$ norm) exists for almost all ν in the completion of $\mathrm{C}_2(\mathcal{M}, \tau)$. If $\mathcal{M}$ is of type I, then $\mathrm{C}_2(\mathcal{M}, \tau)$ is a complete Hilbert algebra contained in $\mathcal{M}$. Thus we can conclude, upon taking exponentials from (3.2) that

$$\lim_{\varepsilon \downarrow 0} K\{A-(\nu+i\varepsilon)\}^{-1}K^*$$

exists in $\mathrm{C}_2(\mathcal{M}, \tau)$, and a fortiori in $\mathcal{M}$. A proof of the familiar Kato–Rosenblum theorem can be given based on this observation. For example see Birman and Entina [2].

The phase shift

We now define a scalar valued function by setting

$$\delta(t) = \tilde{\tau}[P_+B(t)P_+ - P_-(1-B(t))P_-]. \tag{3.23}$$

It is plain that $-\tilde{\tau}(P_-) \leqslant \delta(t) \leqslant \tilde{\tau}(P_+)$. Also for nonreal l,

$$\begin{aligned}\tilde{\tau}\left(\int \frac{B(t)-P_-}{t-l}\,dt\right) &= \tilde{\tau}\left(P_+\int \frac{B(t)-P_-}{t-l}\,dtP_+\right) + \tilde{\tau}\left(\int \frac{P_-(B(t)-P_-)P_-}{t-l}\,dt\right)\\ &= \int \frac{\tilde{\tau}(P_+B(t)P_+)}{t-l}\,dt + \int \frac{\tilde{\tau}(P_-(B(t)-1)P_-)}{t-l}\,dt = \int \frac{\delta(t)}{t-l}\,dt. \end{aligned}\tag{3.24}$$

The residue theorem implies that the first integral is in $\mathfrak{J}_\tau$.

The following result identifies $\delta(t)$ with the so-called spectral displacement function corresponding to the perturbation problem $A \to A + K\bar{J}K^*$.

LEMMA 3.1. *For* $l \notin \operatorname{sp}(A) \cup \operatorname{sp}(A+K^*\bar{J}K)$,

$$\tilde{\tau}\{(A-l)^{-1}-(A+K\bar{J}K^*-l)^{-1}\}=\int\frac{\delta(t)}{(t-l)^2}\,dt.$$

Proof. With $\eta>0$ and $0<t<1+\eta$, form

$$F(t,l)=\exp(-i\pi tP_-)\exp(i\pi tP_-+T(l))$$

where $T(l)=\int(B(t)-P_-)/(t-l)\,dt$ when $\operatorname{Im} l>0$.

For $\varepsilon>0$, there exists $L>0$ such that $|l|>L$ implies

$$\|F(t,l)-1\|=\|\exp[i\pi tP_-+T(l)]-\exp i\pi tP_-\|<\varepsilon$$

whenever $0<t<1+\eta$.

Thus $\ln[F(t,l)]$ is defined for $|l|>L$, and is real analytic in t in trace norm.

Hence $\tilde{\tau}(\ln[F(t,l)])$ is analytic in t. By the Baker–Campbell–Hausdorff theorem we know that

$$\ln[F(t,l)]=T(l)+G(t,l)$$

for small enough t and $|l|>L$, where $G(t,l)$ is in $\mathfrak{J}_\tau$ and $\tilde{\tau}(G(t,l))=0$.

Thus, for such l and t, we have

$$\tilde{\tau}(\ln[F(t,l)])=\tilde{\tau}(T(l)).$$

By the analyticity in t, we can conclude that

$$\tilde{\tau}(\ln[F(1,l)])=\tilde{\tau}(T(l)),$$

and hence that

$$\tilde{\tau}(\ln[1+\bar{J}K(A-l)^{-1}K^*])=\tilde{\tau}(T(l)),\quad |l|>L.$$

But a straightforward calculation shows that

$$\frac{d}{dl}\tilde{\tau}(\ln[1+\bar{J}K(A-l)^{-1}K^*])=\tilde{\tau}[(A-l)^{-1}-(A+K^*\bar{J}K-l)^{-1}];\tag{3.26}$$

see [20; Chap. I, Sec. 6.11, Lemma 3].

Hence, we have shown that

$$\tilde{\tau}\{(A-l)^{-1}-(A+K^*JK-l)^{-1}\}=\int\frac{\delta(\nu)}{(\nu-l)^2}\,d\nu$$

for $|l| > L$, and by analytic continuation in l this will be true everywhere off sp $(A) \cup$ sp $(A + K\bar{J}K^*)$. This lemma can be extended to a class of functions considered by M. G. Krein, who obtained these results for type I [29].

COROLLARY 3.1. *Let $f(\lambda) = \text{const.} + \int (e^{it\lambda} - 1)/(it)\, d\omega(t)$ where $\omega(t)$ is a complex-valued function of bounded variation, and $\int d|\omega(t)| < \infty$. Then*

$$\tau[f(A + K^*\bar{J}K) - f(A)) = \int f'(\lambda)\,\delta(\lambda)\,d\lambda. \tag{3.27}$$

Since this extension of the lemma is obtained in a familar way, we omit the proof. We do note however that (3.27) suggests a sense in which "$\tilde{\tau}(E^1_\lambda - E^0_\lambda) = \delta(\lambda)$" where E^1_λ is the spectral resolution corresponding to $A + K^*\bar{J}K$ and E^0_λ is the spectral resolution corresponding to A. Indeed if $\mathcal{M}$ is of finite type this "result" is correct and constitutes a possible definition of $\delta(\lambda)$ in that case.

Relation (3.27) can also be interpreted in the following way:

Let $\langle \cdot \rangle$ be a linear form on polynomials which annihilates constants. It is plain that $\langle R \rangle = L(dR)$ where d is a derivation (the derivative) and L is a linear functional.

A necessary and sufficient condition that there exists a von Neumann algebra $\mathcal{M}$ equipped with a normal trace τ, together with a pair of * homomorphisms π_1 and π_2 from polynomials into $\mathcal{M}$ commuting modulo $\mathcal{J}_\tau$, for which

$$\langle R \rangle = \tilde{\tau}[\pi_1(R) - \pi_2(R)]$$

is that L be represented on R^1 by an absolutely continuous real signed measure having compact support. Relation (3.27) shows necessity. Sufficiency can be shown by choosing $\mathcal{M} = \mathcal{L}(\mathcal{H})$ and selecting a suitable perturbation problem [9].

Corollary 3.1 and integration by parts on R^1 for Lebesque–Stieltjes integrals imply that, with η_1 and η_2 small and positive, and

$$f_\varepsilon(\lambda, \nu) = \frac{\varepsilon}{\pi}\,\frac{1}{(\lambda - \nu)^2 + \varepsilon^2}$$

for all λ in a neighborhood Δ of the point ν in which the operator A is Fredholm (either in the classical sense or in the sense developed by M. Breuer [6], [7]) and $f_\varepsilon(\lambda, \nu)$ continued smoothly to zero outside this neighborhood, we have

$$\int f^1_\varepsilon(\lambda, \nu - \eta_2) - f^1_\varepsilon(\lambda, \nu - \eta_1)]\,\delta(\lambda)\,d\lambda = \int \tilde{\tau}[E^1_{\lambda+\eta_2} - E^1_{\lambda-\eta_1}]\,f_\varepsilon(\lambda,\nu)\,d\lambda + \int \tilde{\tau}[E^0_{\lambda+\eta_2} - E^0_{\lambda-\eta_1}]\,f_\varepsilon(\lambda,\nu)\,d\lambda, \tag{3.28}$$

the classical Fatou theorem gives us the following result:

THEOREM 3.2. *Let $\nu \in \Delta$ where $E^1(\Delta)$ and $E^0(\Delta)$ are in $\mathfrak{J}_\tau$. Then for almost all η_1 and η_2 in some positive neighborhood of zero*

$$\delta(\nu+\eta_2)-\delta(\nu-\eta_1)=\tilde{\tau}[E^1(\nu-\eta_1,\, \nu+\eta_2)-E^0(\nu-\eta_1,\, \nu+\eta_2)].$$

COROLLARY 3.2. *$\delta(t)$ is supported on the interval $(-(\|A\|+\|K\|^2),\, (\|A\|+\|K\|^2))$.*

Proof. If $|\nu|>\|A\|+\|K\|^2$, then $\nu \notin \operatorname{sp}(A)$. Suppose $\nu>0$. Then for any $\eta_2>0$ and sufficiently small $\eta_1>0$ we see that $\delta(\nu+\eta_2)-\delta(\nu-\eta_1)=0$. Thus $\delta(t)$ is constant for $t>\nu$. Since δ is integrable, this constant value must be zero. Similar reasoning shows $\delta(t)=0$ for $\nu<-(\|A\|+\|K\|^2)$.

Mosaics

In this section we develop the machinery necessary for the construction of the principal function of a pair of operators $\{U, V\}$ with trace-class commutator. In doing this we find it natural to follow these ideas into a broader context; however definitive information beyond the trace class situation is lacking.

Our object in the next few paragraphs will be to study the boundary values of the determining function on the spectrum of the operator U. (We could just as well consider the operator V.) By examining a certain combination of products of the determining function, we shall obtain for each $\eta>0$ an element of the unit ball of $L^\infty(R^2; dA, \mathcal{M})$, i.e. a dA-essentially bounded $\mathcal{M}$-valued weak* measurable function defined on R^2; since (norm) bounded subsets of $L^\infty(R^2; dA, \mathcal{M})$ are compact in the $\sigma(L^\infty(R^2; dA, \mathcal{M}), L^1(R^2, dA, \mathcal{M}_*))$ topology, $L^1(R^2; dA, \mathcal{M}_*)$ being the predual of $L^\infty(R^2; dA, \mathcal{M})$, for any sequence of η's converging to zero we obtain a weak subsequence of elements $\{B_{n_\varkappa}\}_{\varkappa=1}^\infty$ in $L^\infty(R^2; dA, \mathcal{M})$ with limit say B. Such a limit will be called a mosaic of the pair $\{U, V\}$. If $\mathcal{M}$ is a von Neumann algebra equipped with a normal trace τ and $[U, V] \in \mathfrak{J}_\tau$ the principal function $g(\nu, \lambda)$ is defined as

$$g(\nu, \lambda)=\tilde{\tau}[P_+B(\nu, \lambda)P_+]-\tilde{\tau}[P_-[1-B(\nu, \lambda)]P_-]$$

where $P_+(P_-)$ is the spectral projection of the operator $i[V, U] \in \mathcal{M}^s$ corresponding to the non-negative (respectively negative) real axis. Uniqueness of B (i.e., independence of the sequence $\{\eta_\varkappa\}_{\varkappa=1}^\infty$) is known only when $\mathcal{M}$ is type I!

Construction of the mosaic

Let $T=U+iV \in \mathcal{M}$ and let $E(l, z)$ be the determining function of T. We can use (2.2) to conclude that $E^*(l, z)\tilde{J}E(l, z)=\tilde{J}-i(z-\bar{z})\hat{C}^*\{U-\bar{z}\}^{-1}\{V-l\}^{-1}\{U-z\}^{-1}\hat{C}$ where $\tilde{J}\equiv \operatorname{sgn} C$.

In order to apply Theorem 3.1 to the present context it becomes convenient for us to introduce the notation $K_z = \sqrt{-i(z-\bar{z})}\,\hat{C}^*(U-\bar{z})^{-1}$, and to set $A = V$. (Here Im $z>0$.) Then

$$\theta(l, z) \equiv E^*(\bar{l}, z)\bar{J}E(l, z) = \bar{J} + K_z\{A-l\}^{-1}K_z^*. \tag{3.30}$$

By Theorem 3.1 there is a weak* measurable function $B(\cdot, z)$ taking values in the positive part of the unit sphere of $L^\infty(R^1; dt, \mathcal{M})$ such that $\|B(t, z) - P_-\| = O(1/|t|)$ as t approaches ∞ and

$$\theta(l, z) = \exp\left\{i\pi P_- + \int \frac{B(t, z) - P_-}{t-l}\, dt\right\} \tag{3.31}$$

as long as z is a fixed point in the upper half plane, and Im $l > 0$.

For each fixed $\eta > 0$ we form the map B: $R_1 \to L^\infty(R^1; dt; \mathcal{M})$ given by $\lambda \to B(\cdot, \lambda + i\eta)$, We construct a product measurable representation of this map, i.e. a weak*-product measurable function $B_\eta(\ ,\ )$ such that for almost all λ $B_\eta(\cdot, \lambda) = B(\cdot, \lambda + i\eta)$ as elements of $L^\infty(R^1, dt; \mathcal{M})$. To this end let $\{f_n\}$ be a countable dense subset of $\mathcal{M}_*$. Then for each choice of η the $L^\infty(R^1, dt; \mathbf{C})$—valued maps $\lambda \to \langle B(\cdot, \lambda + i\eta), f_m\rangle$ $m = 1, 2, 3, \dots$ have product measurable representations, say $T_m(\ ,\ )$ (see, for example, Lemma 16 and Theorem 17 pp. 196–200 of [22] volume one). Thus, for (t, λ) in a set of full product measure the sequence $\{T_m(t, \lambda)\}_{m=1}^\infty$ defines a $\mathbf{C}$-valued map $B_\eta(t, \lambda)$ on the dense set $\{f_m\}_{m=1}^\infty$ by setting $B_\eta(t, \lambda) f_m = T_m(t, \lambda)$, $m = 1, 2, 3, \dots$.

In order for $B_\eta(t, \lambda)$ to define an element of the dual space it suffices to verify uniform continuity on $\{f_m\}_{m=1}^\infty$ as well as linearity of the corresponding everywhere defined extension.

Continuity is established in the following way:

$$\begin{aligned} |B_\eta(t, \lambda) f_m - B_\eta(t, \lambda) f_n| &\leq \operatorname*{ess\,sup}_{(t, \lambda)} |B_\eta(t, \lambda) f_m - B_\eta(t, \lambda) f_n| \\ &\leq \operatorname*{ess\,sup}_{\lambda} [\operatorname*{ess\,sup}_{t} |T_m(t, \lambda) - T_n(t, \lambda)|] \\ &\leq \operatorname*{ess\,sup}_{\lambda} (\operatorname*{ess\,sup}_{t} \langle B(t, \lambda + i\eta), f_m - f_n\rangle \leq \|f_m - f_n\| \end{aligned}$$

since $0 \leq B(t, \lambda + i\eta) \leq 1$.

The linearity of the extended map can also be easily demonstrated. We will continue to use the notation $B_\eta(t, \lambda)$ for the extended map.

Thus, since $\mathcal{M}_*^* = \mathcal{M}$ by definition, we have a map $(t, \lambda) \to B_\eta(t, \lambda) \in \mathcal{M}$ and the weak* measurability of this map is clear. Furthermore, for almost all λ, $B_\eta(\cdot, \lambda) = B(\cdot, \lambda + i\eta)$ as elements in $L^\infty(R^1, dt; \mathcal{M})$. Thus $B_\eta(\cdot, \cdot)$ is a product measurable representation of $\lambda \to B(\cdot, \lambda + i\eta)$. It is clear that all such representations define the same element of $L^\infty(R^2, dA, \mathcal{M})$.

The family $\{B_\eta(\cdot,\cdot), \eta>0\}$ lies in the unit sphere of $L^\infty(R^2; dA, \mathcal{M})$ which is weak* compact by the Alaoglu theorem. Since $\mathcal{M}_*$ is separable, $L^1(R^2, dA: \mathcal{M}_*)^* = L^\infty(R^2, dA; \mathcal{M})$ (see [36]); therefore, by the Eberlein–Šmulian theorem we can select a positive sequence $\{\eta_j\}_{j=1}^\infty$ with limit zero such that $B_{\eta_j}(\cdot,\cdot)$ converges weak * in $L^\infty(R^2, dA: \mathcal{M})$ to a limit $B(\cdot,\cdot)$. Thus for some choice of $\eta_j>0$ $j=1,2,3,\dots$

$$\lim_{\eta_j \downarrow 0} \iint \langle B(t, \lambda+i\eta_j), f(t,\lambda)\rangle\, dt\, dt = \iint \langle B(t,\lambda), f(t,\lambda)\rangle\, dt\, d\lambda \tag{3.32}$$

when $f \in L^1(R^2, dA; \mathcal{M}_*)$.

We will also show now how $B(t,\lambda)$ is related to U_{ac}.

PROPOSITION 3.2.

$$\iint P_+ B(t,\lambda) P_+ \, d\lambda \leqslant 2\pi |C|^{1/2} P_+ P_{\mathrm{ac}}(U) P_+ |C|^{1/2} \tag{3.33}$$

$$\iint P_- [1 - B(t,\lambda)] P_- \, dt\, d\lambda \leqslant 2\pi |C|^{1/2} P_- P_{\mathrm{ac}}(U) P_- |C|^{1/2} \tag{3.34}$$

as positive operators. Moreover, $P_+ B(v,\lambda) P_+ = 0$ *and* $P_- B(v,\lambda) P_- = P_-$ *whenever* $\lambda \notin \mathrm{sp}\,(U_{\mathrm{ac}})$. *If* P_+ *or* $P_- = 0$, *then we have equality in* (3.33) *and* (3.34).

Proof. We shall establish (3.33). The proof of (3.34) is only a repetition of the same ideas.

We can use (3.16). In the present situation this gives us, for fixed $\eta>0$

$$\int P_+ B_\eta(t,\lambda) P_+ \, dt = P_+ K_z K_z^* P_+. \quad (z = \lambda + i\eta).$$

But

$$P_+ K_z K_z^* P_+ = 2\eta P_+ \hat{C}^* (U - \bar{z})^{-1} (U - z)^{-1} \hat{C} P_+ = 2\eta P_+ \hat{C}^* \int \frac{dE_t}{(t-\lambda)^2 + \eta^2} \hat{C} P_+,$$

where E_t is the spectral resolution of U.

Choose $s(\cdot)$ and $r(\cdot)$ in $L^1(R^1; dt, \mathbf{C})$ where $s(\cdot)$ is continuous and positive and $0 \leqslant r(t) \leqslant 1$. Also choose $m \in \mathcal{M}_*^+$. We evidently have

$$\int s(\lambda)\, d\lambda \left[\int r(t) P_+ B_\eta(t,\lambda) P_+ \, dt\right] \leqslant 2P_+ \hat{C}^* \left[\int s(\lambda) \left(\int \frac{\eta\, dE_t}{(t-\lambda)^2 + \eta^2}\right) d\lambda\right] \hat{C} P_+.$$

Now taking the weak* limit as $\eta_j \downarrow 0$ we have

$$\begin{aligned}\int s(\lambda)\, d\lambda \left[\int r(t) P_+ B(t,\lambda) P_+ \, dt\right] &\leqslant 2P_+ \hat{C}^* \left[\lim_{\eta\downarrow 0} \int \left(\int \frac{\eta s(\lambda)\, d\lambda}{(t-\lambda)^2 + \eta^2}\right) dE_t\right] \hat{C} P_+ \\ &= 2\pi P_+ \hat{C}^* s(U) \hat{C} P_+.\end{aligned}$$

Let us take $s(\lambda)=(\alpha/\pi)\,(1/((\lambda-\mu)^2+\alpha^2))$ with $\alpha>0$ and μ a fixed real number.

Applying the functional $m\in\mathfrak{M}_*^+$ to both sides of this last inequality, letting α tend to zero and noting that $r(t)$ is arbitrary

$$\int\langle P_+\,B(t,\mu)\,P_+,m\rangle\,dt\leqslant 2\pi\frac{d}{d\lambda}\langle P_+\,\hat{C}^*P_a(U)\,E_\lambda\,\hat{C}P_+,m\rangle\Big|_{\lambda=\mu}.$$

Since $P_+\hat{C}^*=\hat{C}^*P_+=|C|^{1/2}P_+$, relation (3.33) follows upon integrating this expression.

We also note that this last inequality tells us that $P_+B(\nu,\lambda)P_+=0$ whenever $\lambda\notin\operatorname{sp}(U_{ac})$. Similarly, it can be shown that $P_-[1-B(\nu,\lambda)]P_-=0$ for $\lambda\notin\operatorname{sp}(U_{ac})$.

Inequalities (3.33) and (3.34) can be strengthened to equalities when C is positive (or negative).

Suppose that $P_-=0$; then

$$E^{-1}(l,\lambda-i\eta)\,E(l,\lambda+i\eta)=1+2\eta C^{1/2}(U-(\lambda-i\eta))^{-1}(V-l)^{-1}(U-(\lambda+i\eta))^{-1}C^{1/2}$$
$$=\exp\left\{\int\frac{B_\eta(t,\lambda)}{t-t}\,dt\right\}.\tag{3.35}$$

Therefore, by taking residues at infinity in l, we get

$$\int B_\eta(t,\lambda)\,dt=2\eta C^{1/2}(U-(\lambda-i\eta))^{-1}(U-(\lambda+i\eta))^{-1}C^{1/2}.$$

Note that $B_\eta(t,\lambda)$ is compactly supported in t on a set that is independent of η and of λ. Indeed, $B_\eta(t,\lambda)$ is supported in an interval Δ which contains $\operatorname{sp}(V)\cup\operatorname{sp}(V+2\eta(U-(\lambda-i\eta))^{-1}C(U-(\lambda+i\eta))^{-1})$ and $2\eta\|(U-(\lambda-i\eta))^{-1}C(U-(\lambda+i\eta))^{-1}\|$ is uniformly bounded independent of λ and η. For, observe that, by inverting (3.35),

$$E^{-1}(l,\lambda+i\eta)\,E(l,\lambda-i\eta)$$
$$=1-2\eta C^{1/2}(U-(\lambda-i\eta))^{-1}(V+2\eta(U-(\lambda+i\eta))^{-1}C(U-(\lambda-i\eta))^{-1}-l)^{-1}(U-(\lambda+i\eta))^{-1}C^{1/2}.$$

Analyticity in l outside $\operatorname{sp}(V)$ then implies that $2\eta\|(U-(\lambda+i\eta))^{-1}C(U-(\lambda-i\eta))^{-1}\|$ is bounded uniformly in η and λ.

Now take $r(t)$ to be the characteristic function of the interval Δ so that with any $s\in L^1(R^1;dt,\mathbf{C})$ as above we get

$$\iint s(\lambda)\,B(t,\lambda)\,dt\,dy=\lim_{\eta\downarrow 0}\iint s(\lambda)\,B_\eta(t,\lambda)\,dt\,d\lambda$$
$$=\lim_{\eta\downarrow 0}2\eta\int s(\lambda)\,C^{1/2}(U-(\lambda-i\eta))^{-1}(U-(\lambda+i\lambda))^{-1}C^{1/2}d\lambda$$
$$=2\pi C^{1/2}s(U)\,C^{1/2}.$$

In particular, by taking $s(\lambda)=(1/\pi)(\eta/((\lambda-\mu)^2+\eta^2))$, we get for almost all μ

$$\int B(\nu,\mu)\,d\nu=\frac{d}{d\lambda}(C^{1/2}E_\lambda C^{1/2})\bigg|_{\lambda=\mu}. \tag{3.36}$$

Remark 3.2. Suppose $\mathcal{M}$ is type I and that C is trace class; the limits

$$\lim_{\eta\downarrow 0} E(l,\lambda\pm i\eta)=E(l,\lambda\pm i0)$$

exist in the Hilbert–Schmidt norm except possibly on a set of Lebesgue measure zero which is independent of l.

In this case it is known [15] that if $[U,A]$ is trace class, then $s-\lim_{t\to\pm\infty}e^{-itU}Ae^{itU}P_{\mathrm{ac}}(U)=S_\pm(U;A)$ exist, and

$$E^*(l,\lambda+i0)\bar{J}E(l,\lambda+i0)=\bar{J}+2\frac{d}{d\lambda}(\hat{C}^*E_\lambda S_+(U;V-l)^{-1}\hat{C}) \tag{3.37}$$

where the derivative is taken in the trace norm.

Applying Theorem 3.1 to (3.37) gives us

$$E^*(l,\lambda+i0)\bar{J}E(l,\lambda+i0)=\exp\left\{i\pi P_-+\int\frac{\tilde{B}(t,\lambda)-P_-}{t-l}\,dt\right\}$$

for $\operatorname{Im} l>0$, for some operator $\tilde{B}(t,\lambda)$.

It is not difficult to show that the operator $\tilde{B}(t,\lambda)$ which enters into this relation coincides with the weak limits $B(t,\lambda)$ introduced in the previous paragraphs.

In this case we note that the inequalities (3.33) and (3.34) also become equalities.

§ 4. Functional calculus modulo an ideal

In this section we develop a functional calculus corresponding to a pair of self-adjoint operators $\{U,V\}$ whose commutator $UV-VU$ belongs to a prescribed semi-normed ideal $\mathfrak{J}$ in $\mathcal{M}$. The prototype examples are those where $\mathcal{M}$ is equipped with a normal trace τ and $\mathfrak{J}$ is one of the associated $\mathbf{C}_p$ ideals, $p=1,2,\ldots$. Recall that $\mathbf{C}_p$ is the ideal generated by the positive elements P in $\mathcal{M}$ for which $\int|\lambda|^p d\tau(E_\lambda)<\infty$ if $P=\int\lambda dE_\lambda$. The class $\mathbf{C}_1$, the trace ideal, is of particular interest since it leads to the construction of the functional which is expressed in terms of the principal function of $\{U,V\}$. This in turn leads to the basic transformation law satisfied by the principal function.

Let $M(R^2)$ be the space of finite complex valued Borel measures ω on R^2 satisfying $\|\omega\|\equiv\int\int_{R^2}(1+|t|)(1+|s|)d|\omega(t,s)|<\infty$. The characteristic function of ω is the scalar function

$$F(x,y)=\iint_{R^2}e^{itx+isy}d\omega(t,s).$$

The set of all such F will be denoted here by $\hat{M}(R^2)$. The map $\omega \to F$ of $M(R^2)$ onto the function algebra $\hat{M}(R^2)$ is a *-isomorphism if we define the norm of F to be $\|F\| = \|\omega\|$. Note that if $F \in \hat{M}(R^2)$, then $F(x, y)$ has continuous partial derivatives which are bounded on R^2. Moreover, if F is a function on R^2 having compact support and partial derivatives satisfying a Hölder condition with exponent greater than $\frac{1}{2}$, then $F \in \hat{M}(R^2)$. In fact, for such an F, the corresponding measure ω is absolutely continuous.

In a similar way we define $\hat{M}(R^1)$ as the set of characteristic functions of measures ω on R^1 with $\|\omega\| = \int (1 + |t|) d|\omega(t)| < \infty$; if F is the characteristic function of ω we again define $\|F\| = \|\omega\|$.

Suppose now that $\mathfrak{J}$ is a semi-normed closed ideal in $\mathfrak{M}$ whose semi-norm $|||\ \ |||$ is related to the norm $\|\ \ \|$ in $\mathfrak{M}$ by

$$|||ABC||| \leqslant \|A\| \, \|C\| \, |||B|||$$
$$|||B^*||| = |||B|||$$

for A, C in $\mathfrak{M}$ and B in $\mathfrak{J}$. For an example, take $\mathfrak{J} = \mathfrak{J}_\tau$ where τ is any trace on $\mathfrak{M}$.

LEMMA 4.1. *Let A and B be elements in $\mathfrak{M}^s$ with $A - B$ in $\mathfrak{J}$. Suppose $F \in \hat{M}(R^1)$. Then $F(A) - F(B) \in \mathfrak{J}$ and $|||F(A) - F(B)||| \leqslant \|F\| \, |||A - B|||$.*

Proof. Of course $F(A)$ and $F(B)$ are defined by the functional calculus for self-adjoint operators.

We begin with the observation that for any $t \in R^1$, $|||e^{itA} - e^{itB}||| \leqslant |t| \, |||A - B|||$; this follows immediately from the relation

$$\frac{d}{dt} e^{itB} e^{-itA} = ie^{itB}(B - A) e^{-itA},$$

the derivative being taken in the uniform norm of $\mathfrak{M}$. With $F(x) = \int e^{itx} d\omega(t)$ we evidently have

$$F(A) - F(B) = \int \{e^{itA} - e^{itB}\} d\omega(t).$$

Hence,

$$|||F(A) - F(B)||| \leqslant \int |||e^{itA} - e^{itB}||| d|\omega(t)| \leqslant \int (1 + |t|) d|\omega(t)| \, |||A - B||| = \|F\| \, |||A - B|||.$$

Since $F(A)$ and $F(B) \in \mathfrak{M}$ the lemma is proved.

Next we establish a similar result involving commutators.

PROPOSITION 4.1. *If A and $B \in \mathfrak{M}^s$ and $[A, B] \in \mathfrak{J}$, then $[A, F(B)] \in \mathfrak{J}$ for any $F \in \hat{M}(R^1)$ Moreover, $|||[A, F(B)]||| \leqslant \|F\| \, |||[A, B]|||$.*

Proof. We will use Cayley transforms. Let α be a positive real number. With $iC = [A, B]$, $W_\alpha = 1 - 2\alpha i(A + \alpha i)^{-1}$, and $D_\alpha = (A + \alpha i)^{-1}C(A - \alpha i)^{-1}$, we have $D_\alpha W_\alpha = (2\alpha)^{-1}[B, W_\alpha]$. Thus, $(B - 2\alpha D_\alpha) W_\alpha = W_\alpha B$, and therefore $F(B - 2\alpha D_\alpha) W_\alpha = W_\alpha F(B)$. This last equality can be expressed in the form

$$\frac{1}{2\alpha i}(A + \alpha i)\{F(B - 2\alpha D_\alpha) - F(B)\}(A - \alpha i) = [A, F(B)].$$

Accordingly,

$$\begin{aligned} |||[A, F(B)]||| &\leqslant \|A + \alpha i\| \, \|A - \alpha i\| \, \|F\| \, |||D_\alpha||| \\ &\leqslant \|A + \alpha i\| \, \|(A + \alpha i)^{-1}\| \, \|A - \alpha i\| \, \|(A - \alpha i)^{-1}\| \, \|F\| \, |||C|||. \end{aligned}$$

But $\|(A \pm \alpha i)^{-1} = 1/\alpha$ and so

$$|||[A, F(B)]||| \leqslant \|A/\alpha + i\| \, \|A/\alpha - i\| \, \|F\| \, |||C|||.$$

Letting α tend to $+\infty$ we arrive at

$$|||[A, F(B)]||| \leqslant \|F\| \, |||C|||.$$

COROLLARY 4.1. *If A and $B \in \mathfrak{M}^s$ and $[A, B] \in \mathfrak{J}$, then $[F(A), G(B)] \in \mathfrak{J}$ for any F, G in $\hat{M}(R^1)$. Moreover, $|||[F(A), G(B)]||| \leqslant 4\|F\| \, \|G\| \, |||[A, B]|||$.*

Proof. This follows at once by decomposing F and G into real and imaginary parts and then applying Proposition 4.1.

Suppose $F \in \hat{M}(R^2)$ so that

$$F(x, y) = \int\int e^{itx + isy} d\omega(t, s)$$

for some $\omega \in M(R^2)$. $\mathfrak{M}$e associate with the pair $\{A, B\}(A, B \in \mathfrak{M}^s)$ an element $F(A, B) \in \mathfrak{M}$ defined by the iterated integral

$$F(A, B) = \int \left(\int F(x, y) \, dE_x \right) dF_y. \tag{4.1}$$

We shall see shortly that in fact $F(A, B) \in C^*$-algebra generated by A and B. It is clear that $F(A, B)$ depend only upon the values of $F(x, y)$ on $\mathrm{sp}\,(A) \times \mathrm{sp}\,(B)$.

The iterated integral in (4.1) may be regarded as an iterated or multiple integral [19].

Since A and $B \in \mathfrak{M}^s$ the families of orthogonal projections E_x and F_y are constant outside the compact sets sp (A) and sp (B) respectively. Also, $|F(x, y)-F(\lambda, \mu)| \leqslant L(|x-\lambda|^{\alpha}+|y-\mu|^{\alpha})$ for some constants $L>0$ and $\alpha>\frac{1}{2}$; a result of Birman and Solomyak [3] asserts that the uniform limits of Riemann–Stieltjes integral sums of the form

$$\sum_{p=1}^{n}\left(\int F(x, y_p)\, dE_x\right) dF(\delta_p), \quad (y_p \in \delta_p)$$

exist and equal $F(A, B)$, and further that $F(A, B)$ is the uniform limit of the Riemann–Stieltjes sums of the form

$$\sum_{p,q} F(x_p, y_p)\, E(\gamma_p)\, F(\delta_p), \quad (x_q \in \gamma_q;\, y_p \in \delta_p).$$

LEMMA 4.2.

$$F(A, B)=\iint e^{itA} e^{isB}\, d\omega(t, s).$$

Proof. Of course the above integral is to be regarded as a multiple Bochner-integral. By the Jordan decomposition of the real and imaginary parts of ω, it will suffice to prove the result when ω is a positive measure.

We will initially assume that ω is compactly supported.

For fixed n, let $\square_j^{(n)}$ be a sequence of disjoint rectangles with $\bigcup_j \square_j^{(n)} \supset$ support ω, and such that $\lim_{n\to\infty}$ (max diameter $\square_j^{(n)})=0$. Choose $(t_j^{(n)}, s_j^{(n)})$ to be the center of $\square_j^{(n)}$. Then

$$\iint e^{itA} e^{isB}\, d\omega = \lim_{n\to\infty} \sum_{j=1}^{n} \exp(it_j^{(n)}A) \exp(is_j^{(n)}B)\, \omega(\square_j^{(n)})$$

in the uniform operator topology.

Let $\omega_{j,n}$ be the measure which assigns the point mass $\omega(\square_j^{(n)})$ to the points $(t_j^{(n)}, s_j^{(n)})$ in the plane. Define $\omega_n=\sum_{j=1}^{n} \omega_{j,n}$. Then ω_n converges weakly to ω.

Let

$$F^n(x, y)=\sum_{j=1}^{n} \exp(it_j^{(n)}A) \exp(is_j^{(n)}B)\, \omega(\square_j^{(n)}).$$

Because ω has finite absolute first moments, it follows by [30; Thm. 6.8] that

$$\lim_{n\to\infty} F^n(x, y)=F(x, y), \tag{4.2}$$

and

$$\lim_{n\to\infty} \frac{\partial}{\partial x} F^n(x, y)=\frac{\partial}{\partial x} F(x, y), \tag{4.3}$$

with both limits being uniform on compact subsets of R^2.

Now

$$\lim_{n\to\infty}\sum_{j=1}^{n}\exp(it_j^{(n)}A)\exp(is_j^{(n)}B)\,\omega(\square_j^{(n)})=\lim_{n\to\infty}\iint F_n(x,y)\,dE_x\,dF_y.$$

The uniform convergence in (4.2) and (4.3) enables us now to use a result due to Ju. L. Daleckii and S. G. Krein [19] to conclude that

$$\lim_{n\to\infty}\int\left\{\int F_n(x,y)\,dE_x\,dF_y\right\}=\int\left\{\int F(x,y)\,dE_x\right\}dF_y.$$

This establishes the lemma for functions F whose associated measure has compact support. The case of arbitrary measures in $M(R^2)$ is easily reduced to this one since the compactly supported measures are dense in $M(R^2)$ and

$$\left\|\int\left[\int F(x,y)\,dE_x\right]dF_y\right\|\leqslant\Gamma\cdot\|F\|$$

where Γ is a constant which depends only on the size of $\mathrm{sp}\,(A)\times\mathrm{sp}\,(B)$ [19].

PROPOSITION 4.2. *Let F and $G\in\hat{M}(R^2)$ and $[A,B]\in\mathfrak{J}$. Then*

$$F(A,B)^*-\bar{F}(A,B)\in\mathfrak{J}\tag{4.4}$$

and

$$|||F(A,B)^*-\bar{F}(A,B)|||\leqslant\|F\|\,|||[A,B]|||$$

$$F(A,B)G(A,B)-(F\cdot G)(A,B)\in\mathfrak{J}\tag{4.5}$$

and

$$|||F(A,B)G(A,B)-(F\cdot G)(A,B)|||\leqslant\|F\|\,\|G\|\,|||[A,B]|||$$

$$[F(A,B),G(A,B)]\in\mathfrak{J}\tag{4.6}$$

and

$$|||[F(A,B),G(A,B)]|||\leqslant\|F\|\,\|G\|\,|||[A,B]|||.$$

In particular $F(A,B)$ has self-commutator in $\mathfrak{J}$.

Let C and D be elements of $\mathfrak{M}$ such that $C-F(A,B)\in\mathfrak{J}$ and $D-G(A,B)\in\mathfrak{J}$. Suppose P is a polynomial in two commuting variables; then with $Q(x,y)=P(F(x,y),G(x,y))$, $Q\in\hat{M}(R^2)$ and $Q(A,B)-P(C,D)\in\mathfrak{J}$, where $P(C,D)$ is any choice of ordering for the operator valued polynomial. (4.7)

Proof. Let F and G be the characteristic functions of the measures ω and μ respectively. Then $\bar{F}(x,y)=\iint e^{-ixt-iys}\,d\bar{\omega}$. Hence

$$F(A,B)^*-\bar{F}(A,B)=\iint e^{-isB}e^{-itA}-e^{-itA}e^{-isB}\,d\bar{\omega}.$$

By Corollary 4.1 we note that

$$|||e^{-isB}e^{-itA}-e^{-itA}e^{-isB}|||=|||[e^{-isB}, e^{-isA}]|||\leqslant |s|\,|t|\,|||[A, B]|||.$$

Accordingly,

$$|||F(A, B)^* - \bar{F}(A, B)||| \leqslant \|F\|\,|||[A, B]|||.$$

Next we consider the proof of (4.5). We note that $F\cdot G=\widehat{\omega * \mu}$. Therefore

$$F(A,B)\cdot G(A,B)-(F\cdot G)(A,B)=\iint\iint[e^{itA}e^{isB}e^{iaA}e^{ibB}-e^{i(t+a)A}e^{i(s+b)B}]\,d\omega(t,s)\,du(a,b). \tag{4.8}$$

But the operator integrand in (4.8) is equal to $e^{itA}[e^{isB}, e^{iaA}]e^{ibB}$. Hence

$$\begin{aligned}|||F(A,B)G(A,B)-(F\cdot G)(A,B)||| &\leqslant \iint\iint |||[e^{isB}, e^{iaA}]|||\,d|\omega(t,s)|\,d|\mu(a,b)|\\ &\leqslant \iint\iint |s|\,|a|\,d|\omega(t,s)|\,d|\mu(a,b)|\,|||[A,B]|||\\ &\leqslant \|F\|\,\|G\|\,|||[A,B]|||.\end{aligned}$$

The proof of (4.6) follows immediately from (4.5). It remains to consider (4.7). The fact that $Q\in\hat{M}(R^2)$ follows at once since $\hat{M}(R^2)$ is an algebra. We now consider $Q(A, B)-P(C, D)$. Since $Q(x, y)=\sum a_{m,n}F(x, y)^n G(x, y)^m$, if $P(x, y)=\sum a_{nm}x^n y^m$, it is clear that

$$Q(A,B)=\sum_{n,m} a_{nm}(F^n\cdot G^m)(A,B).$$

Therefore,

$$Q(A,B)-P(C,D)=\sum_{n,m} a_{nm}\{(F^nG^m)(A,B)-F^n(A,B)\,G^m(A,B)\}$$

modulo $\mathfrak{J}$. But by (4.5) we see that each term in this last sum is in $\mathfrak{J}$.

The reader may note that although the operator $F(A, B)$ depends solely on the values of the function F in a neighborhood of sp $(A)\times$ sp (B), the estimates in Proposition 4.2 involve the first moments, i.e., $\int|t|\,d\omega(t, s)$, $\int|s|\,d|\omega(t, s)|$; these estimates may be considered somewhat crude when compared with estimates which monitor the size of F and its derivatives locally on sp $(A)\times$ sp (B).

§ 5. The tracial bilinear form—cartesian

Let $\mathfrak{M}$ be a von Neumann algebra equipped with a normal trace τ. Let $T=U+iV$, where U and $V\in\mathfrak{M}^s$ and $[U, V]\in\mathfrak{J}_\tau$, the trace ideal.

For F and $G\in\hat{M}(R^2)$, the operators $F(U, V)$ and $G(U, V)$ are defined as in section 4

We will prove the following result:

THEOREM 5.1.

$$\tilde{\tau}[F(U, V), G(U, V)] = \frac{i}{2\pi}\iint \{F, G\}(\mu, \nu)\, g(\nu, \mu)\, d\nu\, d\mu \tag{5.1}$$

where

$$\{F, G\}(\mu, \nu) = \frac{\partial F}{\partial \mu}\frac{\partial G}{\partial \nu} - \frac{\partial F}{\partial \nu}\frac{\partial G}{\partial \mu}.$$

This result is the extension to the von Neumann algebra context of a result in [32].

It is clear from this theorem that $g(\nu, \mu)$ is invariant under inner automorphisms of $\mathcal{M}$ as well as under perturbations of U and V by elements in $\mathfrak{J}_\tau$.

We point out here that it is also true that if W is a partial isometry whose defect projections belong to $\mathfrak{J}_\tau$, then $\{WUW^*, WVW^*\}$ has the same principal function as $\{U, V\}$.

The proof of Theorem 5.1 depends upon a sequence of lemmas and propositions.

PROPOSITION 5.1. *Let $E(l, z)$ be the determining function of $\{U, V\}$. Then, for $|l|$ and $|z|$ sufficiently large*

$$\tilde{\tau}(\ln[E(l, z)]) = \frac{1}{2\pi i}\iint g(\nu, \mu)\frac{d\nu}{\nu - l}\frac{d\mu}{\mu - z}, \tag{5.2}$$

where $g(\nu, \mu)$ is a summable function with compact support.

Proof. Note that for z fixed and l chosen sufficiently large (depending on Im z), $\ln[E^{-1}(l, \bar{z})E(l, z)]$ is defined, since by (2.5) and (3.30) we have

$$E^{-1}(l, \bar{z})E(l, z) = I + \bar{J}K_z(V - l)^{-1}K_z^*.$$

From (3.25) and (3.26) we have

$$\tilde{\tau}(\ln[E^{-1}(l, \bar{z})E(, z)]) = \int g(\nu, z)\frac{d\nu}{\nu - l}$$

where $g(\nu, z) = \tilde{\tau}(P_+B(\nu, z)P_+) - \tilde{\tau}(P_-(1 - B(\nu, z))P_-)$. Now we note that if A and $B \in \mathfrak{J}_\tau$ and $|||A||| + |||B||| < \frac{1}{3}$, then

$$\tilde{\tau}(\ln[1 + A][1 + B]) = \tilde{\tau}(\ln[1 + A] + \ln[1 + B]).$$

Since $\ln[1 + A]^{-1} = -\ln[1 + A]$ we deduce that

$$\tilde{\tau}(\ln[E^{-1}(l, \bar{z})E(l, z)]) = \tilde{\tau}(\ln[E(l, z)]) - \tilde{\tau}(\ln[E(l, \bar{z})]) = \int g(\nu, z)\frac{d\nu}{\nu - l}$$

provided we choose l sufficiently large. Thus for $\eta > 0$, Im $z > 0$, and l sufficiently large, depending only on η

$$\tilde{\tau}(\ln[E(l, z + i\eta)] - \ln[E(l, \overline{z + i\eta}]) = \int g(\nu, z + i\eta)\frac{d\nu}{\nu - l}.$$

Let us define, for such fixed l

$$P_\eta(l,z)=\begin{cases}\tilde\tau(\ln[E(l,z+i\eta)]), & \operatorname{Im} z>0\\ \tilde\tau(\ln[E(l,z-i\eta)]), & \operatorname{Im} z<0.\end{cases}$$

Observe that the scalar function

$$\lambda\to\int g(\nu,\lambda+i\eta)\frac{d\nu}{\nu-l}$$

is differentiable in λ for $\eta>0$. Let

$$Q_\eta(l,z)=\left(\frac{1}{2\pi i}\right)\iint g(\nu,\mu+i\eta)\frac{d\nu}{\nu-l}\frac{d\mu}{\mu-z}$$

and consider the difference $P_\eta(l,z)-Q_\eta(l,z)$. Clearly, for each fixed large l

$$P_\eta(l,\lambda+io)-Q_\eta(l,\lambda+io)=P_\eta(l,\lambda-io)-Q_\eta(l,\lambda-io)$$

with both sides of this equation realizing the limiting values locally in $L^2(R^1;dt,\mathbf{C})$ with respect to λ.

Hence $P(l,z)-Q(l,z)$ is an entire function in z. But since both $P(l,z)$ and $Q(l,z)$ have limit zero when z becomes infinite, we can conclude that $P(l,z)=Q(l,z)$.

We therefore have for sufficiently large $|l|$

$$\tilde\tau(\ln[E(l,z+i\eta)])=\frac{1}{2\pi i}\iint g(\nu,\lambda+i\eta)\frac{d\nu}{\nu-l}\frac{d\lambda}{\lambda-z},\quad \operatorname{Im} z>0$$

and

$$\tilde\tau(\ln[E(l,z-i\eta)])=\frac{1}{2\pi i}\iint g(\nu,\lambda+i\eta)\frac{d\nu}{\nu-l}\frac{d\mu}{\lambda-z},\quad \operatorname{Im} z<0.$$

Thus we see that for each number $\eta>0$,

$$P_\eta^{\pm}(l,z)=\tilde\tau(\ln[E(l,z\pm i\eta)])\quad\begin{pmatrix}\operatorname{Im} z>0\\ \operatorname{Im} z<0\end{pmatrix}$$

have analytic continuations in the l-variable to all l with $\operatorname{Im} l\neq 0$. Furthermore these continuations are related in η; for with $\eta<\eta'$

$$P_{\eta'}^{\pm}(l,z)=P_\eta^{\pm}(l,z\pm i(\eta'-\eta))$$

since this equality remains valid for large enough l. Thus we can define τ $(\ln[E(l,z)])$ for all non-real l and z by setting

$$\tilde\tau(\ln[E(l,z)])=P^{\pm}_{|\operatorname{Im} z|}(l,z\mp i|\operatorname{Im} z|).$$

Therefore we shall have established the formula

$$\tilde\tau(\ln[E(l,z)])=\frac{1}{2\pi i}\iint g(\nu,\lambda)\frac{d\nu}{\nu-l}\frac{d\lambda}{\lambda-z}$$

where $g(\nu, \lambda)$ is a summable function, provided we can develop a sequence $\{g(\cdot, \cdot + i\eta_n)\}_{n=1}^{\infty}$ with $\eta_n \downarrow 0$ as $n \uparrow \infty$ such that

$$\lim_{n\uparrow\infty} \iint g(\nu, \lambda + i\eta_n) \frac{d\nu}{\nu - l} \frac{d\lambda}{\lambda - z} = \iint g(\nu, \lambda) \frac{d\nu}{\nu - l} \frac{d\lambda}{\lambda - z} \tag{5.3}$$

for each nonreal l and z.

We now enter into this final step.

Let $B_{\eta_n}(\nu, \lambda)$ converge in $\sigma(L^\infty(R^2; dA\,\mathfrak{M}), L^1(R^2; dA\,\mathfrak{M}_*))$ to $B(\nu, \lambda)$ for some sequence η_n of positive numbers converging to zero. We have seen in the discussion preceeding Proposition 3.2 that such a sequence exists.

Now by [20; Chap. I, Sec. 6.1, Prop. 2 and Cor.] there exists a family of pure states $\{m_i\}_{i\in J}$ in $\mathfrak{M}_*$ such that $\tau = \sum_{i\in I} m_i$ on $\mathfrak{M}^+$.

Since (3.16) and (3.17) tells us

$$\iint P_+ B_\eta(\nu, \lambda) P_+ \, d\nu d\lambda = P_+ C$$

and

$$\iint P_-(1 - B_\eta(\nu, \lambda)) P_- \, d\nu d\nu = -P_- C$$

we can conclude that for a denumerable subset of pure states $\{m_{i'}\}_{i\in J'}$ we have

$$\tau[P_+(B_\eta(\nu, \lambda) P_+] = \sum_{i\in J'} \langle P_+ B_\eta(\nu, \lambda) P_+, m_i\rangle$$

and

$$\tau[P_-(B_\eta(\nu, \lambda) - 1) P_-] = \sum_{i\in J'} \langle P_-(I - B_\eta(\nu, \lambda)) P_-, m_i\rangle.$$

It follows from (3.33) and (3.34), and the Beppo–Levi theorem that $P_+ B(\nu, \lambda) P_+$ and $P_-(1 - B(\nu, \lambda))P_-$ are in $L^1(R^2; dA, \mathfrak{J}_\tau)$. Define $g_\eta(\nu, \lambda) = \tilde{\tau}[P_+ B_\eta(\nu, \lambda) P_+] - \tilde{\tau}[P_-(1 - B_\eta(\nu, \lambda)) P_-]$ and $g(\nu, \lambda) = \tilde{\tau}[P_+ B(\nu, \lambda) P_+] - \tilde{\tau}[P_-(1 - B(\nu, \lambda)]$.

We shall verify (5.3) for the sequence $\{g_{\eta_n}(\nu, \lambda)\}_{n=1}^{\infty}$ and $g(\nu, \lambda)$.

For each $i \in J'$, we have by the convergence of $B_{\eta_n}(\nu, \lambda)$ in $\sigma(L^\infty(R^2; dA, \mathfrak{M}), L^1(R^2; dA, \mathfrak{M}_*))$ that

$$\lim_{n\to\infty} \iint \langle P_+ B_{\eta_n}(\nu, \lambda) P_+, m_i\rangle \frac{d\nu}{(\nu - l)^2} \frac{d\lambda}{(\lambda - z)^2} = \iint \langle P_+ B(\nu, \lambda) P_+, m_i\rangle \frac{d\nu}{(\nu - l)^2} \frac{d\lambda}{(\lambda - z)^2}.$$

And, hence by integration on l and z the analogous equality holds for the functions $(v - l)^{-1}(x - z)^{-1}$, since $\iint \langle P_+ B_{\eta_n}(\nu, \lambda) P_+, m_i\rangle d\nu d\lambda$ is uniformly bounded in n. Similarly

$$\lim_{n\to\infty} \iint \langle P_-[1 - B_{\eta_n}(\nu, \lambda)] P_-, m_i\rangle \frac{d\nu}{\nu - l} \frac{d\lambda}{\lambda - z} = \iint \langle P_-[-B(\nu, \lambda)] P_-, m_i\rangle \frac{d\nu}{\nu - l} \frac{d\lambda}{\lambda - z}$$

for nonreal l and z. Now, let δ be an arbitrary positive number. For each index M we can write

$$\left|\iint\{g_{\eta_n}(\nu,\lambda)-g(\nu,\lambda)\}\frac{d\nu}{\nu-l}\frac{d\lambda}{\lambda-z}\right|$$
$$\leqslant\sum_{i\leqslant M}\left|\iint\left[\langle P_+\{B_{\eta_n}(\nu,\lambda)-B(\nu,\lambda)\}P_+,m_i\rangle+\langle P_-\{B_{\eta_n}(\nu,\lambda)-B(\nu,\lambda)\}P_-,m_i\rangle\right]\frac{d\nu}{\nu-l}\frac{d\lambda}{\lambda-z}\right|$$
$$+\sum_{i>M}\left|\iint\left[\langle P_+\{B_{\eta_n}(\nu,\lambda)-B(\nu,\lambda)\}P_+,m_i\rangle+\langle P_-\{B_{\eta_n}(\nu,\lambda)-B(\nu,\lambda)\}P_-,m_i\rangle\right]\frac{d\nu}{\nu-l}\frac{d\lambda}{\lambda-z}\right|. \tag{5.4}$$

Suppose we choose M large enough so that with $\|\ \|_2$ denoting the norm in $L^2(R^2, dA; \mathbf{C})$

$$4\left\|\frac{1}{\nu-l}\frac{1}{\lambda-z}\right\|_2\cdot\sum_{i>M}\langle|C|,m_i\rangle<\delta/2;$$

we can then find an index N so that for all $n>N$ we have

$$\sum_{i\leqslant M}\left|\iint\langle P_+\{B_{\eta_n}(\nu,\lambda)-B(\nu,\lambda)\}P_+,m_i\rangle\frac{d\nu}{\nu-l}\frac{d\lambda}{\lambda-z}\right|$$
$$+\sum_{i>M}\left|\iint\langle P_-\{B_{\eta_n}(\nu,\lambda)-B(\nu,\lambda)\}P_-,m_i\rangle\frac{d}{\nu-l}\frac{d}{\lambda-z}\right|<\delta/2.$$

It is clear from the Schwartz inequality that the secnd term on the right-hand side of (5.4) is bounded by $\delta/2$. Thus for all $n\geqslant N$ we have that

$$\left|\iint\{g_{\eta_n}(\nu,\lambda)-g(\nu,\lambda)\}\frac{d\nu}{\nu-l}\frac{d\lambda}{\lambda-z}\right|<\delta.$$

Since δ was an arbitrary positive number, the result (5.3) is established.

The fact that $g(\nu,\lambda)$ has compact support follows from the analyticity of the left side of (5.2) for l and z large and outside the spectrum of V and U respectively.

Remark 5.1. We have obtained $B(\nu,\lambda)$ as a weak limit of a subsequence $B_{\eta_n}(\nu,\lambda)$. A priori another choice of convergent subsequence $B_{\eta'_n}(\nu,\lambda)$ could have a weak* limit $B'(\nu,\lambda)\neq B(\nu,\lambda)$. We have not proved equality. Nevertheless, $g'(\nu,\lambda)=g(\nu,\lambda)$ since it is clear both functions satisfy (5.2). Thus the principal function of the pair $\{V, U\}$ is uniquely defined.

PROPOSITION 5.2. *For nonreal l and z*

$$\frac{i}{2\pi}\iint g(\nu,\mu)\frac{d\nu}{\nu-l}\frac{d\mu}{(\mu-z)^2}=\tilde{\tau}[\{U-z\}^{-1}\{V-l\}^{-1}\{U-z\}^{-1}[V,U]].$$

Proof. Recall (2.2) which says

$$E^{-1}(l, z)\, E(l, \omega) = 1 - i(\omega - z)\, \hat{C}(U - z)^{-1}(V - l)^{-1}(U - \omega)^{-1}\hat{C} \tag{5.5}$$

for nonreal l, z and ω. With $|z - \omega|$ small and $\operatorname{Im} l$ large as needed (depending on $\operatorname{Im} z$ and $\operatorname{Im} \omega$) we can take logarithms and then traces of both sides of (5.5) to arrive at

$$\tilde{\tau}(\ln [E(l, \omega)]) - \tilde{\tau}(\ln [E(l, z)]) = \tilde{\tau}[\ln (1 - i(\omega - z)\, \hat{C}(U - z)^{-1}(V - l)^{-1}(U - \omega)^{-1}\hat{C}]).$$

This gives us at once the relation

$$\frac{1}{2\pi i} \iint g(\nu, \mu) \frac{d\nu}{\nu - l} \left\{ \frac{1}{\mu - \omega} - \frac{1}{\mu - z} \right\} d\mu = \tilde{\tau}(\ln [1 - i(\omega - z)\, \hat{C}(U - z)^{-1}(V - l)^{-1}(U - \omega)^{-1}\hat{C}]).$$

Use of the power series expansion of

$$\ln [1 - i(\omega - z)\, \hat{C}(U - z)^{-1}(V - l)^{-1}(U - \omega)^{-1}\hat{C}]$$

enables us to conclude that

$$\begin{aligned} &\frac{1}{2\pi i} \iint g(\nu, \mu) \frac{1}{\nu - l} \left\{ \frac{1}{\mu - z} - \frac{1}{\mu - \omega} \right\} d\mu \\ &\qquad = -i(\omega - z)\, \tilde{\tau}(\hat{C}(U - z)^{-1}(V - l^{-1}(U - \omega)^{-1}\hat{C}) + O(|\omega - z|^2), \end{aligned} \tag{5.6}$$

where $O(|\omega - z|^2)$ designates an operator whose norm approaches zero like $|\omega - z|^2$ as $|\omega - z| \downarrow 0$. If we divide both sides of (5.6) by $(\omega - z)$ and let ω approach z we shall have

$$-\frac{1}{2\pi} \iint g(\nu, \mu) \frac{1}{\nu - l} \frac{1}{(\mu - z)^2} d\nu\, d\mu = \tilde{\tau}(\hat{C}(U - z)^{-1}(V - l)^{-1}(U - z)^{-1}\hat{C}). \tag{5.7}$$

Our assertion now follows from the observation that

$$-i\tilde{\tau}(\hat{C}(U - z)^{-1}(V - l)^{-1}(U - z)^{-1}\hat{C}) = \tilde{\tau}((U - z)^{-1}(V - l)^{-1}(U - z)^{-1}[V, U]).$$

LEMMA 5.1. *For each pair of nonnegative integers n and m:*

$$\tilde{\tau}([U^n V^m, V]) = -\tilde{\tau}\Big(\sum_{0 \leqslant p \leqslant n-1} U^{n-p-1} V^m U^n [V, U]\Big).$$

Proof. This result is trivial. We simply note that $[U^n V^m, V] = [U^n, V]\, V^m$ and that

$$[U^n, V] = -\sum_{0 \leqslant p \leqslant n-1} U^p [V, U]\, U^{n-p-1}.$$

Thus

$$\tilde{\tau}([U^n V^m, V]) = -\tilde{\tau}\Big(\sum_{0 \leqslant p \leqslant n-1} U^{n-p-1} V^m U^p [V, U]\Big).$$

LEMMA 5.2. (N. Wallach). *Let $\langle \cdot, \cdot \rangle$ be an antisymmetric bilinear form in two commuting variables x and y such that $\langle P\circ R, Q\circ R\rangle = 0$ whenever P and Q are single variable polynomials and R is a polynomial in the two variables x and y. Further assume that $\langle P, y\rangle = 0$ for arbitrary plynomials P. Then $\langle \cdot, \cdot \rangle$ is the zero form.*

For the details of proof, the reader should consult [24].

Using this result we shall first demonstrate relation (5.1) when the restriction of the functions F and G to $\operatorname{sp}(U)\times\operatorname{sp}(V)$ are given as polynomials in x and y. To do this, it suffices to prove the following:

PROPOSITION 5.3. *Let R and S be polynomials in x and y. Then*

$$\tilde{\tau}((R(U, V), S(U, V)]) = \frac{i}{2\pi}\iint \{R, S\}(\mu, \nu)\, g(\nu, \mu)\, d\nu\, d\mu.$$

Proof. In view of Lemma 5.2, it is enough to prove that

$$\tilde{\tau}([U^{n_1}V^{m_1}\dots U^{n_p}V^{m_p}, V]) = \frac{i}{2\pi}\iint N\mu^{N-1}\nu^M g(\nu, \mu)\, d\nu\, d\mu, \tag{5.8}$$

where n_j, m_j are nonnegative integers and $N = \sum_{1\leqslant j\leqslant p} n_j$ and $M = \sum_{1\leqslant j\leqslant p} m_j$. (Note that if (5.8) were true the bilinear form $(R, S)\to\tilde{\tau}([R(U, V), S(U, V)]) - (i/2\pi)\int\{R, S\}(\mu, \nu)\times g(\nu, \mu)\, d\nu\, d\mu$ would satisfy the conditions of Lemma 5.2.) From $\tilde{\tau}([U^N V^M, V]) = \tilde{\tau}([U^{n_1}V^{m_1}\dots U^{n_p}V^{n_p}, V])$ and Lemma 5.1 it is plain that we will have established our result once we show

$$\begin{aligned}\tilde{\tau}[\sum_{0\leqslant p\leqslant N-1} U^{N-p-1}V^M U^p[V, U]] &= \frac{1}{2\pi i}\iint N\mu^{N-1}\nu^M g(\nu, \mu)\, d\nu\, d\mu \\ &= \frac{1}{2\pi i}\iint \{\mu^N\nu^M, \nu\}(\mu, \nu)\, g(\nu, \mu)\, d\nu\, d\mu.\end{aligned} \tag{5.9}$$

This step depends on Proposition 5.2:

By equating moments at infinity (with respect to l)

$$\tilde{\tau}(\{U - z\}^{-1}V^M\{U - z\}^{-1}[V, U]) = \frac{1}{2\pi i}\iint \nu^M \frac{1}{(\mu - z)^2} g(\nu, \mu)\, d\nu\, d\mu.$$

Relation (5.9) now follows directly from this last equation by extracting moments at infinity with respect to z.

We are now ready to prove Theorem 5.1 in its entirety.

Proof of Theorem 5.1. We begin with the observation that straightforward estimates when combined with Proposition 5.3 yield

$$\tilde{\tau}([e^{itU}e^{isV}, e^{iaU}e^{ibV}]) = \frac{i}{2\pi}\iint \{e^{i(t\mu+s\nu)}, e^{i(a\mu+b\nu)}\}\, g(\nu, \mu)\, d\nu\, d\mu$$

for all real values t, s, a and b. Thus, suppose

$$F(x, y) = \iint e^{itx+isy} d\omega$$

and

$$G(x, y) = \iint e^{itx+isy} d\sigma,$$

where ω and σ are measures in the class $M(R_2)$. By Lemma 4.2

$$F(U, V) = \iint e^{itU} e^{isV} d\omega$$

$$G(U, V) = \iint e^{itU} e^{isV} d\sigma.$$

Repetition of the Fubini theorem and our remarks above then yield

$$\begin{aligned}
\tilde{\tau}([F(U, V), G(U, V)]) &= \tilde{\tau}\left(\iiiint [e^{itU}e^{isV}, e^{iaU}e^{ibV}]\right) d\omega(t, s)\, d\sigma(a, b) \\
&= \iiiint \tilde{\tau}([e^{itU}e^{isV}, e^{iaU}e^{ibV}])\, d\omega(t, s)\, d\sigma(a, b) \\
&= \frac{i}{2\pi} \iiiint \left[\iint \{e^{i(t\mu+s\nu)}, e^{i(a\mu+b\nu)}\}(\mu, \nu)\, g(\nu, \mu)\, d\nu\, d\mu\right] d\omega(t, s)\, d\sigma(t, s) \\
&= \frac{i}{2\pi} \iint \left\{\iint e^{i(t\mu+s\nu)} d\omega(t, s), \iint e^{i(a\mu+b\nu)} d\sigma(a, b)\right\}(\mu, \nu)\, g(\nu, \mu)\, d\nu\, d\mu \\
&= \frac{i}{2\pi} \iint \{F(\mu, \nu), G(\mu, \nu)\}(\mu, \nu)\, d\nu\, d\mu.
\end{aligned}$$

A result in [13] implies that given any compactly supported function $g(\nu, \mu)$ in $L^1(R^2; dA, R^1)$ there is an Hilbert space $\mathcal{H}$ and operator $T \in \mathcal{L}(\mathcal{H})$ with $[T^*, T]$ trace class (relative to $\mathcal{L}(\mathcal{H})$) having $g(\nu, \mu)$ as its principal function. In view of this fact the content of Theorem 5.1 can be expressed in the following manner:

Let $\langle \cdot, \cdot \rangle$ be an antisymmetric bilinear form on polynomials in two commuting variables x and y such that $\langle P(R(x, y)), Q(R(x, y))\rangle = 0$ when P and Q are single variable polynomials and R is a polynomial in the two variables x and y; then $\langle R, S\rangle = L(\{R(x, y), S(x, y)\})$, where L is a linear functional.

A necessary and sufficient condition that there exists a von Neumann algebra equipped with a normal trace τ, and a $*$-homomorphism π from polynomials into the algebraic quotient ring $\mathcal{M}/\mathcal{J}_\tau$ such that for all polynomials R and S (with representatives $\tilde{\pi}(R)$ and $\tilde{\pi}(S)$ in $\mathcal{M}$)

$$\langle R, S\rangle = \tilde{\tau}([\tilde{\pi}(R), \tilde{\pi}(S)])$$

is that

$$L(R)=\frac{i}{2\pi}\iint R(\mu,\nu)\,d\omega(\nu,\mu),$$

where ω is an R^2 absolutely continuous real signed measure with compact support and finite total variation.

PROPOSITION 5.4. *Let $\mathfrak{M}$ be a von Neumann algebra with normal trace τ. Let $\{T_n\}$ be a sequence in $\mathfrak{M}$ such that $\lim_{n\to\infty} T_n = T$ in the strong * topology.*

Suppose further that $[T_n^*, T_n]\in \mathfrak{J}_\tau$ for each $n=1, 2, \ldots, [T^*, T]\in\mathfrak{J}_\tau$ and

$$\lim_{n\to\infty}[T_n^*, T_n]=[T_n^*, T]$$

in the sense of convergence in $\mathfrak{J}_\tau$. For each n, let g_n be the principal function of T_n and let g be the principal function of T. Then $g_n\to g$ in the $\sigma(C_0(R^2)^*, C_0(R^2))$ topology.

Proof. Since polynomials in x and y are dense in $C_0(R^2)$ it suffices by Theorem 5.1 to show for each pair of positive integers r and s that

$$\lim_{n\to\infty}\tilde{\tau}([(T_n^*)^r, T_n^s])=\tilde{\tau}([(T^*)^r, T^s]).$$

To do this, we note that for each $n=1, 2, \ldots$

$$[(T_n^*)^r, T_n^s]=\sum_{0\leqslant k\leqslant r-1}\ \sum_{0\leqslant l\leqslant s-1}(T_n^*)^{r-k-1}T_n^{s-l-1}[T_n^*, T_n]\,T_n^l(T_n^*)^k.$$

Therefore our proof will be complete once we know that: for $A_n, C_n\in\mathfrak{M}$

$$s-\lim_{n\to\infty}A_n=A,\quad s-\lim_{n\to\infty}C_n=C$$

and $B_n, B\in\mathfrak{J}_\tau$

$$\lim_{n\to\infty}|||B_n-B|||=0$$

implies

$$\lim_{n\to\infty}\tilde{\tau}(A_nB_nC_n)=\tilde{\tau}(ABC). \tag{5.10}$$

But this is easily seen by the $s(\mathfrak{M}, \mathfrak{M}_*)$ continuity of the mapping $(X, Y)\to X\cdot Y$ on norm bounded sets and noting

$$A_nB_nC_n-ABC=(A_n-A)BC+A_n(B_n-B)C_n-A_nB(C-C_n).$$

For then relation (5.10) follows at once from the $\sigma(\mathfrak{M}, \mathfrak{M}_*)$ continuity of the map $\mathfrak{M}\to\mathbf{C}$ given by $R\to\tilde{\tau}(RS)$ (S a fixed element in $\mathfrak{J}_\tau$) [36] and the estimate $|||RSX|||\leqslant \|R\|\,|||S|||\,\|X\|$ where $R, X\in\mathfrak{M}$, $S\in\mathfrak{J}_\tau$.

PROPOSITION 5.5. *Let U and V be elements in $\mathfrak{M}^s$ with $[V, U] \in \mathfrak{J}_\tau$ and principal function $g(\nu, \mu)$. Let F be a real function in $\hat{M}(R^1)$. Then the operator $U + iF(U)VF(U)$ has self commutator in $\mathfrak{J}_\tau$ and principal function*

$$g_F(\nu, \mu) = \begin{cases} g(\nu/F(\mu)^2, \mu) & F(\mu) \neq 0 \\ 0 & F(\mu) = 0. \end{cases}$$

Proof. Note that $[F(U)VF(U), U] = F(U)[V, U]F(U) \in \mathfrak{J}_\tau$.

For each pair of nonnegative integers n and m, we have by Theorem 5.1

$$\begin{aligned} \tilde{\tau}([\{F(U)VF(U)\}^n, U^m]) &= \tilde{\tau}([F(U)^{2n}V^n, U^m]) \\ &= \frac{i}{2\pi}\iint \{F(\mu)^{2n}\nu^n, \mu^m\}(\mu, \nu)\, g(\nu, \mu)\, d\nu\, d\mu \\ &= -\frac{i}{2\pi}\iint m\cdot n\nu^{n-1}\mu^{m-1}F(\mu)^{2n}g(\nu, \mu)\, d\nu\, d\mu \\ &= -\frac{i}{2\pi}\iint_{\{\mu:\, F(\mu)\neq 0\}} m\cdot n\nu^{n-1}\mu^{m-1}g(\nu/F(\mu)^2, \mu)\, d\nu\, d\mu. \end{aligned}$$

As an immediate corollary to Propositions 5.4 and 5.5 we note the following:

If Δ is a Borel set of Baire class 1, there is a bounded sequence $\{F_n\}$ of real functions in $\hat{M}(R')$, converging pointwise to the characteristic function χ_Δ, so that $F_n(U)$ converges strongly to $E(\Delta)$. The operators $T_n = U + iF_n(U)VF_n(U)$, $n = 1, 2, \ldots$ and $T = U + i\psi_\Delta(U)V\psi_\Delta(U)$ satisfy the hypothesis of Proposition 5.4; accordingly the principal function of the cut-down operator $E(\Delta)UE(\Delta) + iE(\Delta)VE(\Delta) = E(\Delta)(U + iV)E(\Delta)$ equals $g(\nu, \mu)\cdot\chi_{R^1\times\Delta}(\nu, \mu)$. (This result suggests a functional calculus based on L^∞ functions.)

From Theorem 5.1 it is clear that $g(\nu, \mu)$ is a unitary invariant of $T = U + iV$ as well as a trace class one. Taken together these invariance properties show that if W is a partially isometric operator in $\mathfrak{M}$ whose defect space projections belong to $\mathfrak{J}_\tau$, then T and WTW^* have the same principal function. The necessary details of proof will be left to the reader.

Suppose now that T is completely non-normal and semi-normal. Then it was shown in [34] that the essential closure of the support of the principal function is the spectrum of T in the type I case.

This theorem retains its validity in the present context. The proof, as before relies on a cut-down theorem of Putnam [35], and upon the cut-down property of the principal function which we have established.

THEOREM 5.2. *Suppose T is completely non-normal, hypo-normal, and $T^*T - TT^* \in \mathfrak{J}_\tau$. Then the essential closure of the support of the principal function of T^* is the spectrum of T^*.*

Proof. Let $[a, b] \times [c, d]$ be a square disjoint from the essential closure of the support of $g(\nu, \mu)$ and centered at a point ζ. Call the spectral projections associated to U and V over the intervals $[a, b]$ and $[c, d]$ respectively E and F, i.e. $U = \int \lambda dE_\lambda$, $V = \int \lambda dF_\lambda$ and $E = E_b - E_a$, $F = F_d - F_c$. Form the completely non-normal, hypo-normal operator $FETEF$.

We have seen that the principal function associated with $(FETEF$ is just $g(\nu, \mu)\chi[a, b](\mu)\chi[c, d](\nu)$ where $\chi(a, b)(\mu)$ and $\chi[c, d](\nu)$ are respectively the characteristic functions of $[a, b]$ and of $[c, d]$ evaluated at μ and at ν.

Thus, since $g(\nu, \mu) = 0$ on $[a, b] \times [c, d]$, $\tau[(FETEF)^*, FETEF] = 0$ by Theorem 5.1. But since the commutator is semi-definite, this means that it vanishes. Accordingly, $FETEF = 0$. But the Putnam "cut-down" result [35] tells us that sp $(FETEF) =$ sp $(T) \cap [a, b] \times [c, d]$; hence, $[a, b] \times [c, d] \cap$ sp $(T) = \phi$ and $\zeta \notin$ sp (T). The reverse inclusion is clear.

§ 6. The tracial bilinear form—another version

This section will be devoted first to the task of obtaining an analogue of Theorem 5.1 which corresponds to a change of variables from Cartesian to polar coordinates.

We will also obtain some further information about the functional calculus.

All of the considerations of the present section take place in a von Neumann algebra, $\mathcal{M}$, equipped with a normal trace τ, which we regard as a subalgebra of $\mathcal{L}(\mathcal{H})$ for some fixed Hilbert space $\mathcal{H}$.

We will again denote the trace ideal by $\mathcal{J}_\tau$.

Suppose W is a partial isometry for which

$$WV = (V + D)W$$

where V and D are bounded self-adjoint elements in $\mathcal{M} \subset \mathcal{L}(\mathcal{H})$. Note that the initial and final spaces $\mathcal{H}_i$ and $\mathcal{H}_f$ of W are invariant for V and $V + D$ respectively. Define the Hilbert space

$$\mathcal{K} = \dots \mathcal{H}_f^{\perp} \oplus \mathcal{H}_f^{\perp} \oplus \mathcal{H} \oplus \mathcal{H}_i^{\perp} \oplus \mathcal{H}_i^{\perp} \oplus \dots .$$

and let $\tilde{W}$ be defined on $\mathcal{K}$ as transforming

$$\tilde{x} = [\dots x_{-2}, x_{-1}, x_0, x_1, x_2, \dots]$$

into

$$\tilde{W}\tilde{x} = [\dots x_{-3}, x_{-2}, x_{-1} + Wx_0, P_R x_0, x_1, \dots]$$

where P_R is the projection $\mathcal{H} \to \mathcal{H}_i^{\perp}$. Further, let

$$\tilde{V}\tilde{x} = [\dots (V + D)x_{-2}, (V + D)x_{-1}, Vx_0, Vx_1, Vx_2, \dots]$$

and

$$\tilde{D}\tilde{x} = [\dots 0, 0, Dx_0, 0, 0, \dots].$$

It follows easily that $\tilde{W}\tilde{V} = (\tilde{V}+\tilde{D})\tilde{W}$, and that $\tilde{W}$ is the minimal unitary dilation of W (see [10] for the proof).

With $\hat{D}$ defined so that $\hat{D}^2 = \tilde{D}$, $\hat{D}\hat{D}^* = \hat{D}\hat{D} = |\tilde{D}|$ (recall the definition of $\hat{C}$) we define $\tilde{\phi}(l, \omega)\colon \mathcal{K} \to \mathcal{K}$ by setting

$$\tilde{\phi}(l, z) = 1_{|\mathcal{K}} + \hat{D}\tilde{W}(\tilde{W}-z)^{-1}\tilde{V}-l)^{-1}\hat{D}$$

for $(l, z) \notin \operatorname{sp}(\tilde{V}) \times \operatorname{sp}(\tilde{W})$. It is clear that $\mathcal{H} \subset \mathcal{K}$ is a reducing subspace for $\tilde{\phi}(l, \omega)$. This operator valued function is closely related to the determining function $E(l, z)$, [10], [12].

Let $\tilde{\mathcal{M}}$ denote the direct sum von Neumann algebra $\dots \oplus \circ \oplus \circ \oplus \mathcal{M} \oplus \circ \oplus \circ \oplus \dots$. Let $\mathcal{B}$ denote the W^*-algebra generated by $\tilde{W}$ and $\tilde{\mathcal{M}}$.

Note that $\tilde{\mathcal{M}}$ has a natural trace induced by the trace on $\mathcal{M}$, i.e., $\tilde{\tau}(A) = \tilde{\tau}(A_{|\mathcal{H}})$. Further, the compression of $\mathcal{B}$ to $\tilde{\mathcal{H}} = \oplus \circ \oplus \circ \oplus \mathcal{H} \oplus \circ \oplus \circ \oplus \dots$ is precisely $\tilde{\mathcal{M}}$. To see this, call $\hat{P}$ the projection from $\mathcal{K}$ into $\tilde{\mathcal{H}}$. It is plain that $\hat{P}\mathcal{B}\hat{P} = \{\hat{P}\mathcal{B}|\tilde{\mathcal{H}}\colon B \in \mathcal{B}\}$ contains $\tilde{\mathcal{M}}$. To show the reverse inclusion, we note that $\tilde{W}$ is a strong * dilation of W and $W \in \mathcal{M}$. It is then clear that $\tilde{\mathcal{M}} \supset \hat{P}\mathcal{B}\hat{P}$. We conclude that $\hat{P}\tilde{\phi}(l, z)|_{|\tilde{\mathcal{H}}} \in \tilde{\mathcal{M}}$.

PROPOSITION 6.1. *For* $\operatorname{Im} x \neq 0$, $|z| \neq 1$,

$$\tilde{\phi}(x, \omega)\, \tilde{\phi}(x, z)^{-1} = 1_{|\mathcal{K}} + (\omega - z)\, \hat{D}\tilde{W}(\tilde{W}-\omega)^{-1}(\tilde{V}-x)^{-1}(\tilde{W}-z)^{-1}\hat{D}. \tag{6.1}$$

Proof. Since, as can easily be seen by direct multiplication,

$$\tilde{\phi}(x, z)^{-1} = 1_{|\mathcal{K}} - \hat{D}\tilde{W}(\tilde{V}-x)^{-1}(\tilde{W}-z)^{-1}\hat{D},$$

we have

$$\begin{aligned}\tilde{\phi}(x, \omega)\, \tilde{\phi}(x, z)^{-1} = 1_{|\mathcal{K}} + \hat{D}\tilde{W}\{&(\tilde{W}-\omega)^{-1}(\tilde{V}-x)^{-1} - (\tilde{V}-x)^{-1}(\tilde{W}-z)^{-1} \\ &- (\tilde{W}-\omega)^{-1}(\tilde{V}-x)^{-1}\tilde{D}\tilde{W}(\tilde{V}-x)^{-1}(\tilde{W}-z)^{-1}\}\hat{D}.\end{aligned} \tag{6.2}$$

On the other hand

$$\begin{aligned}&(\omega-z)(\tilde{W}-\omega)^{-1}(\tilde{V}-x)^{-1}(\tilde{W}-z)^{-1} \\ &\quad = (\omega-z)(\tilde{W}-\omega)^{-1}[(\tilde{W}-z)^{-1}(\tilde{V}-x)^{-1} - [(\tilde{W}-z)^{-1}(\tilde{V}-x)^{-1}] \\ &\quad = (\omega-z)(\tilde{W}-\omega)^{-1}\{(\tilde{W}-z)^{-1}(\tilde{V}-x)^{-1} - (\tilde{W}-z)^{-1}(\tilde{V}-x)^{-1}\tilde{D}\tilde{W}(\tilde{V}-x)^{-1}(\tilde{W}-z)^{-1}\}.\end{aligned}$$

By the resolvent formula $(\omega-z)(\tilde{W}-\omega)^{-1}(\tilde{W}-z)^{-1} = (\tilde{W}-\omega)^{-1} - (\tilde{W}-z)^{-1}$ this last expression becomes

$$\begin{aligned}(\tilde{W}-\omega)^{-1}(\tilde{V}-x)^{-1} - (\tilde{W}-z)^{-1}(\tilde{V}-x)^{-1} - (\tilde{W}-\omega)^{-1}(\tilde{V}-x)^{-1}\tilde{D}\tilde{W}(\tilde{V}-x)^{-1}(\tilde{W}-z)^{-1} \\ + (\tilde{W}-z)^{-1}(\tilde{V}-x)^{-1}\tilde{D}\tilde{W}(\tilde{V}-x)^{-1}(\tilde{W}-z)^{-1}.\end{aligned}$$

Therefore,

$$\begin{aligned}(\omega-z)(\hat{W}-\omega)^{-1}(\hat{V}-x)^{-1}(\hat{W}-x)^{-1} &= (\hat{W}-\omega)^{-1}(\hat{V}-x)^{-1}-(\hat{W}-z)^{-1}(\hat{V}-x)^{-1}+[(\hat{W}-z)^{-1},(\hat{V}-x)^{-1}] \\ &\quad -(\hat{W}-\omega)^{-1}(\hat{V}-x)^{-1}\hat{D}\hat{W}(\hat{V}-x)^{-1}(\hat{W}-z)^{-1} \\ &= (\hat{W}-\omega)^{-1}(\hat{V}-x)^{-1}-(\hat{V}-x)^{-1}(\hat{W}-z)^{-1} \\ &\quad -(\hat{W}-\omega)^{-1}(\hat{V}-x)^{-1}\hat{D}\hat{W}(\hat{V}-x)^{-1}(\hat{W}-z)^{-1}. \end{aligned} \tag{6.3}$$

Insertion of (6.3) in (6.2) gives the result.

LEMMA 6.2. *For $n\geqslant 0$, $m\geqslant 0$, $\hat{P}[\hat{W}^n\hat{V}^m, V]_{|\tilde{\mathfrak{H}}}$ and $\hat{P}\hat{W}^n\hat{V}^m[\hat{W}, V]_{|\tilde{\mathfrak{H}}}$ belong to the trace ideal $\mathfrak{J}_\tau$ and*

$$\tilde{\tau}(\hat{P}[\hat{W}^n\hat{V}^m, \hat{V}]_{|\tilde{\mathfrak{H}}})=\tilde{\tau}(\hat{P}\sum_{0\leqslant p\leqslant n-1}\hat{W}^{n-p-1}\hat{V}^m\hat{W}^p(W,\hat{V}]_{|\tilde{\mathfrak{H}}}).$$

Proof. Since $\hat{P}$ commutes with $\hat{V}$, $\hat{P}\hat{W}^n{}_{|\tilde{\mathfrak{H}}}=W^n$ and $[\hat{W},\hat{V}]=\hat{D}\hat{W}$ we have $\hat{P}[\hat{W}^n\hat{V}^m,\hat{V}]_{|\tilde{\mathfrak{H}}}=[W^nV^m, V]$ and $\hat{P}\hat{W}^n\hat{V}^m[W, V]_{|\tilde{\mathfrak{H}}}=W^nV^mDW$. Accordingly, these operators are in the ideal $\mathfrak{J}_\tau$. The proof that

$$\tilde{\tau}(\hat{P}[\hat{W}^n\hat{V}^m, \hat{V}]_{|\tilde{\mathfrak{H}}})=\tilde{\tau}(\hat{P}\sum_{0\leqslant p\leqslant n-1}\hat{W}^{n-p-1}\hat{V}^m\hat{W}^p[\hat{W},\hat{V}])_{|\tilde{\mathfrak{H}}})$$

is essentially a repetition of the proof of Lemma 5.1 except that we multiply both sides by $\hat{P}$ before evaluating the trace.

LEMMA 6.3. *Let $N=(n_1, n_2, \ldots, n_p)$ and $M=(m_1, m_2, \ldots, m_p)$, $|N|=n_1+n_2+\ldots+n_p$ and $|M|=m_1+m_2+\ldots+m_p$, where the n_i's and m_i's are nonnegative integers. Then*

$$\hat{P}[\hat{W}^{n_1}\hat{V}^{m_1}\hat{W}^{n_2}\hat{V}^{m_2}\ldots\hat{W}^{n_p}\hat{V}^{m_p},\hat{V}]_{|\tilde{\mathfrak{H}}}$$

belongs to the trace ideal $\mathfrak{J}_\tau$ and

$$\tilde{\tau}(\hat{P}[\hat{W}^{n_1}\hat{V}^{m_1}\hat{W}^{n_2}\hat{V}^{m_2}\ldots W^{n_p}\hat{V}^{m_p},\hat{V}]_{|\tilde{\mathfrak{H}}})=\tilde{\tau}(\hat{P}[\hat{W}^N\hat{V}^M,\hat{V}]_{|\tilde{\mathfrak{H}}}). \tag{6.4}$$

Proof. The first part of the lemma follows as in Lemma 6.2. The equality (6.4) follows in a trivial fashion since $\hat{P}$ commutes with $\hat{V}$ and $\tilde{\tau}([T, V])=0$ whenever T is in the trace class.

In section 3 of the present paper we proved the existence of the principal function $g(\nu,\mu)$. The next result is analogous.

THEOREM 6.1. *There exists a unique real-valued summable function $g^P(\lambda, \tau)$ defined on the cylinder $(-\infty, \infty) \times \{\tau: |\tau| = 1\}$ such that*

$$\frac{1}{2\pi i}\iint g^P(\lambda, \tau)\frac{1}{\lambda - l}\frac{1}{(\tau - z)^2}\,d\lambda\,d\tau = \tilde{\tau}(\hat{D}\tilde{W}(\tilde{W} - z)^{-1}(\tilde{V} - l)^{-1}(\tilde{W} - z)^{-1}\hat{D}), \tag{6.5}$$

whenever $(l, z) \notin \operatorname{sp}(\tilde{V}) \times \operatorname{sp}(\tilde{W})$.

We do not wish to take the considerable space which would be necessary to give a full proof of this result starting from (6.1), since the proof is only a modification of the corresponding proof of Proposition 5.2 above.

By Lemma 3.1 for $l \notin \operatorname{sp}(\tilde{V}) \cup \operatorname{sp}(\tilde{V} + \tilde{D})$, there is a summable function $\delta(t)$ such that

$$\tilde{\tau}\{(\tilde{V} - l)^{-1} - (\tilde{V} + \tilde{D} - l)^{-1}\} = \int \frac{\delta(t)}{(t - l)^2}\,dt.$$

Here $\delta(t)$ is the spectral displacement function corresponding to the perturbation problem $\tilde{V} \to \tilde{V} + \tilde{D}$.

We will show now that $\delta(t)$ can be derived from the polar principal function $g^P(\lambda, \tau)$.

THEOREM 6.2.

$$\delta(\lambda) = \frac{1}{2\pi}\int_0^{2\pi} g^P(\lambda, e^{i\theta})\,d\theta.$$

Proof. We begin by observing that

$$\frac{1}{2\pi i}\iint g^P(\lambda, \zeta)\frac{1}{(\lambda - l)^2}\frac{1}{\zeta - z}\,d\zeta\,d\lambda = \tau\}\tilde{D}\tilde{W}(\tilde{V} - l)^{-1}(\tilde{W} - z)^{-1}(\tilde{V} - l)^{-1}\}.$$

This result is exactly analogous to Proposition 5.2 and is proved by the same reasoning. Now take $z = 0$. Then

$$\frac{1}{2\pi}\iint g^P(\lambda, e^{i\theta})\frac{1}{(\lambda - l)^2}\,d\lambda\,d\theta = \tilde{\tau}\{\tilde{D}\tilde{W}(\tilde{V} - l)^{-1}\tilde{W}^*(\tilde{V} - l)^{-1}\}.$$

But $\tilde{W}(\tilde{V} - l)^{-1}\tilde{W}^* = (\tilde{V} + \tilde{D} - l)^{-1}$. Thus

$$\frac{1}{2\pi}\iint g^P(\lambda, e^{i\theta})\frac{1}{(\lambda - l)^2}\,d\lambda\,d\theta = \tau\{\tilde{D}(\tilde{V} + D - l)^{-1}(\tilde{V} - l)^{-1}\} = \tau\{(\tilde{V} - l)^{-1}\tilde{D}(\tilde{V} + D - l)^{-1}\}.$$

But $(\tilde{V} + \tilde{D} - l)^{-1} - (\tilde{V} - l)^{-1} = -(\tilde{V} - l)^{-1}\tilde{D}(\tilde{V} + \tilde{D} - l)^{-1}$. Thus

$$-\int\left(\frac{1}{2\pi}\int_0^{2\pi} g^P(\lambda, e^{i\theta})\,d\theta\right)\frac{d\lambda}{(\lambda - l)^2} = \tau\{(\tilde{V} + \tilde{D} - l)^{-1} - (\tilde{V} - l)^{-1}\}.$$

But Lemma 3.1 asserts that

$$\dot{\tau}\{(\tilde{V}-l)^{-1}-(\tilde{V}+\tilde{D}-l)^{-1}\}=\int\frac{\delta(\lambda)}{(\lambda-l)^2}d\lambda.$$

Thus

$$\int\left(\frac{1}{2\pi}\int g^p(\lambda,e^{i\theta})\,d\theta\right)\frac{d\lambda}{(\lambda-l)^2}=\int\frac{\delta(\lambda)}{(\lambda-l)^2}d\lambda$$

for all non-real l, and we can conclude that $(1/2\pi)\int_0^{2\pi}g^p(\lambda,e^{i\theta})d\theta=\delta(\lambda)$ a.a. λ.

We wish now to derive a version of Corollary 5.1 for the pair $\{\tilde{W},\tilde{V}\}$. Before doing this, we develop an associated functional calculus.

Let $M(Z)$ be the collection of Borel measures μ on the integers such that $\int(1+|n|)\times d|\mu(n)|<\infty$. Let $M(Z\times R^1)$ be the collection of Borel measures μ on $Z\times R^1$ for which $\int\int_{Z\times R^1}(1+|n|)(1+|t|)d|\mu(n,t)|<\infty$. The norm of the characteristic functions of measures in $M(Z)$ or $M(Z\times R^1)$ is defined in the usual way.

PROPOSITION 6.2. *Let W be a partial isometry, with $WV-VW=DW$, and $V\in\mathfrak{M}^s$, $D\in\mathfrak{M}^s\cap\mathfrak{J}_\tau$. Let $H\in\hat{M}(R')$. Then $\hat{P}[\tilde{W},H(\tilde{V})]_{|\tilde{\mathfrak{H}}}\in\mathfrak{J}_\tau$, and $|||\hat{P}[\tilde{W},\tilde{H}(V)]_{|\tilde{\mathfrak{H}}}|||\leqslant\|H\|\,|||D|||$.*

Proof. We have $WV=(V+D)W$. Hence $WH(V)=H(V+D)W$. But, by Lemma 4.1 $H(V)-H(V+D)\in\mathfrak{J}_\tau$. Hence $[W,H(V)]\in\mathfrak{J}_\tau$. But $[W,H(V)]=\hat{P}[\tilde{W},H(\tilde{V})]_{|\tilde{\mathfrak{H}}}$. Moreover $|||H(V+D)-H(V)|||=\|H\|\,|||D|||$. Therefore $|||\hat{P}[\tilde{W},H(\tilde{V})]_{|\tilde{\mathfrak{H}}}|||\leqslant\|H\|\,|||D|||$.

PROPOSITION 6.3. *Let W be a partial isometry, let $V\in\mathfrak{M}^s$, $D\in\mathfrak{M}^s\cap\mathfrak{J}_\tau$ and assume that $WV-VW=DW$. Then for $F\in\hat{M}(Z)$, we have $\hat{P}[F(\tilde{W}),H(\tilde{V})]_{|\mathfrak{H}}\in\mathfrak{J}_\tau$, and $|||\hat{P}[F(\tilde{W}),H(V)]_{|\tilde{\mathfrak{H}}}|||\leqslant\|H\|\,\|F\|\,|||D|||$.*

Proof. For $n\geqslant 0$ we have

$$[\tilde{W}^n,H(\tilde{V})]=\sum_{0\leqslant j\leqslant n-1}\tilde{W}^j[\tilde{W},H(\tilde{V})]\,\tilde{W}^{n-j-1}.$$

Accordingly,

$$\hat{P}[\tilde{W}^n,H(\tilde{V})]_{|\tilde{\mathfrak{H}}}=\sum_{0\leqslant j\leqslant n-1}\hat{P}\tilde{W}^j\hat{P}\tilde{C}\hat{P}\tilde{W}^{n-j}{}_{|\tilde{\mathfrak{H}}}=\sum_{0\leqslant j\leqslant n-1}W^jCW^{n-j},$$

where $=\tilde{C}=H(\tilde{V}+\tilde{D})-H(\tilde{V}),=$and $\quad C=H(V+D)-H(V)$. Thus, $=|||\hat{P}[\tilde{W}^n,\ H(\tilde{V})]_{|\tilde{\mathfrak{H}}}|||\leqslant n|||C|||\leqslant n\|H\|\,|||D|||$.

Since $\tilde{W}^*=\tilde{W}^{-1}$, analogous considerations give us

$$|||\hat{P}[\tilde{W}^n,H(\tilde{V})]_{|\tilde{\mathfrak{H}}}|||\leqslant|n|\,\|H\|\,|||D|||,$$

for any integer n, positive or not.

Thus, if $F(e^{i\theta}) = \int_Z e^{in\theta} d\mu(n)$ and $F(\hat{W}) \equiv \int_Z \hat{W}^n d\mu(n)$, we will get

$$|||\hat{P}[F(\hat{W}), H(\hat{V})]_{|\tilde{\mathfrak{H}}}||| \leq \int |||\hat{P}(\hat{W})^n, H(\hat{V})]_{|\tilde{\mathfrak{H}}}||| d|\mu(n)|$$
$$\leq \|H\| \int |n| d|\mu(n)| \, |||D||| \leq \|H\| \, \|F\| \, |||D|||.$$

Now let $F \in \hat{M}(S' \times R^1)$, so that

$$F(e^{i\theta}, x) = \int_Z \int_{R^1} e^{in\theta} e^{itx} d\mu(n, t),$$

where $\int_Z \int_{R^1} (1 + |n|)(1 + |t|) d|\mu|(n, t) < \infty$. Define $F(\hat{W}, \hat{V}) = \int_Z \int_{R^1} \hat{W}^n e^{it\tilde{V}} d\mu(n, t)$. We will show that the map $F \to \hat{P}F(\hat{W}, \hat{V})_{|\tilde{\mathfrak{H}}}$ defines a complex *-homomorphism of $\hat{M}(S' \times R^1)$ into $\tilde{\mathfrak{M}}$ modulo the ideal $\mathfrak{J}_\tau$.

THEOREM 6.3. *Let F, $H \in \hat{M}(S' \times R^1)$, then*

$$(\hat{P}F(\hat{W}, \hat{V}))^*_{|\tilde{\mathfrak{H}}} - P\bar{F}(\hat{W}, \hat{V})_{|\tilde{\mathfrak{H}}} \in \mathfrak{J}_\tau, \tag{6.6}$$

$$\hat{P}F(\hat{W}, \hat{V})_{|\tilde{\mathfrak{H}}} \cdot \hat{P}H(\hat{W}, \hat{V})_{|\tilde{\mathfrak{H}}} - \hat{P}(FH)(\hat{W}, \hat{V})_{|\tilde{\mathfrak{H}}} \in \mathfrak{J}_\tau, \tag{6.7}$$

$$[\hat{P}F(\hat{W}, \hat{V})_{|\tilde{\mathfrak{H}}}, \hat{P}H(\hat{W}, \hat{V})_{|\tilde{\mathfrak{H}}}] \in \mathfrak{J}_\tau. \tag{6.8}$$

Thus, in particular, $\hat{P}F(\hat{W}, \hat{V})_{|\tilde{\mathfrak{H}}}$ has a trace class self-commutator.

Proof of (6.6).

$$\hat{P}F(\hat{W}, \hat{V})^*_{|\tilde{\mathfrak{H}}} - \hat{P}\bar{F}(\hat{W}, \hat{V})_{|\tilde{\mathfrak{H}}} = \iint_{Z \times R^1} \hat{P}e^{-it\tilde{V}} \hat{W}^{*n}{}_{|\tilde{\mathfrak{H}}} d\bar{\mu}(n, t) - \iint_{Z \times R^1} \hat{P}\hat{W}^{*n} e^{-it\tilde{V}}{}_{|\tilde{\mathfrak{H}}} d\bar{\mu}(n, t)$$
$$= \iint_{Z \times R^1} \hat{P}[e^{-it\tilde{V}}, \hat{W}^{*n}]_{|\tilde{\mathfrak{H}}} d\bar{\mu}(n, t).$$

But we know by Proposition 6.3, that

$$|||\hat{P}[\hat{W}^n, e^{it\tilde{V}}]_{|\tilde{\mathfrak{H}}}||| \leq (1 + |n|)(1 + |t|) |||D|||.$$

Hence

$$|||\hat{P}F(\hat{W}, \hat{V})^*_{|\tilde{\mathfrak{H}}} - \hat{P}\bar{F}(\hat{W}, \hat{V})_{|\tilde{\mathfrak{H}}}||| \leq \iint (1 + |n|)(1 + |t|) |||D||| d|\bar{\mu}|(n, t)| < \infty.$$

Proof of (6.7). Suppose that $F = \hat{\mu}$ and $H = \hat{\nu}$. Then $F \cdot H = \widehat{\mu * \nu}$, and

$$F \cdot H(\hat{W}, \hat{V}) = \iint_{Z \times R^1} \hat{W}^n e^{it\tilde{V}} d(\mu * \nu)(n, t) = \iint_{Z \times R^1} \left\{ \iint_{Z \times R^1} \hat{W}^{n+a} e^{i(t+b)\tilde{V}} d\mu(n, t) \right\} d\nu(a, b).$$

Therefore

$$F(\tilde{W}, \tilde{V})\tilde{H}(\tilde{W}, \tilde{V}) - (F \cdot H)(W, \tilde{V})$$
$$= \iint_{Z \times R^1} \iint_{Z \times R^1} \{\tilde{W}^n e^{it\tilde{V}} W^a e^{ib\tilde{V}} - \tilde{W}^{n+a} e^{i(t+b)\tilde{V}}\} \cdot d\mu(n, t)\, d\nu(a, b).$$

Note that the operator integrand in this expression is equal to $\tilde{W}^n[e^{it\tilde{V}}, \tilde{W}^a]e^{ib\tilde{V}}$.

Accordingly, since we have

$$|||\hat{P}W^n[e^{it\tilde{V}}, \tilde{W}^a]e^{ib\tilde{V}}{}_{|\tilde{u}}||| \leqslant (1 + |a|)(1 + |t|)|||D|||,$$

we get

$$|||\hat{P}F(\tilde{W}, \tilde{V})H(\tilde{W}, \tilde{V})_{|\tilde{u}} - \hat{P}(F \cdot H)(\tilde{W}, \tilde{V})_{|\tilde{u}}|||$$
$$\leqslant |||D||| \iint_{Z \times R^1} \iint_{Z \times R^1} (1 + |a|)(1 + |t|)\, d\mu|(n, t)\, d|\nu|(a, b) \leqslant |||D|||\, ||F \cdot H||.$$

It remains only to observe that (6.8) follows at once from (6.7).

THEOREM 6.4. *Let W be an intertwining partial isometry in $\mathfrak{M}$ for which $WV = (V + D)W$ where V and D are in $\mathfrak{M}^s$ and D is in the trace class. For F and H polynomials in two variables*

$$\tilde{\tau}(\hat{P}[F(\tilde{W}, \tilde{V}), H(\tilde{W}, \tilde{V})]_{|\tilde{u}}) = \frac{1}{2\pi i} \iint \{F, H\}(\zeta, \lambda)\, g^P(\lambda, \zeta)\, d\lambda\, d\zeta.$$

Proof. Using Lemma 5.2 it suffices (as in the proof of Theorem 5.3) to show that

$$\tilde{\tau}(P[\tilde{W}^{n_1}\tilde{V}^{m_1}W^{n_2}\tilde{V}^{m_2} \ldots \tilde{W}^{n_p}\tilde{V}^{m_p}, \tilde{V}]_{|\tilde{u}}) = \frac{1}{2\pi i} \iint \left(\frac{\partial}{\partial \tau} \zeta^{|N|}\right)(\lambda^{|M|})\, g^P(\lambda, \zeta)\, d\lambda\, d\zeta$$

where $|N| = n_1 + n_2 + \ldots + n_p$ and $|M| = m_1 + m_2 + \ldots + m_p$. By Lemma 6.3 the theorem will be proved once we show that $\tilde{\tau}(\hat{P}[\tilde{W}^{|N|}\tilde{V}^{|M|}, \tilde{V}]_{|\tilde{u}}) = (1/2\pi i) \iint |N|\zeta^{|N|-1}\lambda^{|M|}g^P(\lambda, \zeta)\, d\lambda\, d\zeta$, or equivalently

$$\tilde{\tau}\{\hat{P}(\sum_{0 \leqslant P \leqslant |N|-1} \tilde{W}^{|N|-P-1}\tilde{V}^{|M|}\tilde{W}^P[\tilde{W}, \tilde{V}]\}_{|\tilde{u}}) = \frac{1}{2\pi i} \iint |N|\zeta^{|N|-1}\lambda^{|M|}g^P(\lambda, \zeta)\, d\lambda\, d\zeta.$$

We shall make use of formula (6.5). By integrating around a large contour containing the spectrum of V in the interior, we get from (6.5)

$$\tilde{\tau}(\hat{P}\tilde{W}(\tilde{W} - z)^{-1}\tilde{V}^{|M|}(\tilde{W} - z)^{-1}[\tilde{W}, \tilde{V}]_{|\tilde{u}}) = \frac{1}{2\pi i} \iint \lambda^{|M|} \frac{1}{(\zeta - z)^2} g^P(\lambda, \zeta)\, d\lambda\, d\zeta.$$

We now similarly form

$$\frac{1}{2\pi i}\iint \lambda^{|M|} g^P(\lambda,\zeta)\,d\zeta\left(\oint \frac{z^{|N|}}{(\zeta-z)^2}\,dz\right)d\lambda = \tilde{\tau}\left(\oint z^{|N|}\tilde{P}(\tilde{W}-z)^{-1}\tilde{V}^{|M|}(W-z)^{-1}[W,\tilde{V}]_{|\tilde{\mathfrak{H}}}\,dz\right.$$

for a contour large enough to include the spectrum of W in its interior. Then, by taking residues, we get

$$\frac{1}{2\pi i}\iint |N|\,\zeta^{|N|-1}\lambda^{|M|}g^P(\lambda,\zeta)\,d\lambda\,d\zeta = \tilde{\tau}(\tilde{P}\{\sum_{0\leqslant P\leqslant |N|-1}\tilde{W}^{|N|-P-1}\tilde{V}^{|M|}\tilde{W}^P[\tilde{W},\tilde{V}]_{|\tilde{\mathfrak{H}}}).$$

Note that if $F=\hat{\omega}$, $H=\hat{\mu}$ $(\omega,\mu\in M(Z\times R^1))$ where ω and μ are supported on $Z^+\times R^1$ (Z^+ the nonnegative integers), then

$$\tilde{\tau}[F(W,V),G(W,V)] = \frac{1}{2\pi i}\iint \{F,H\}(\zeta,\lambda)\,g^P(\lambda,\zeta)\,d\lambda\,d\zeta.$$

§ 7. The equality of the polar and cartesian principal functions

Let $T\in\mathfrak{M}$ with $[T^*,T]\in\mathfrak{J}_\tau$ and define $T_z=T-z1$. Then T_z admits a representation is the form W_zQ_z where $(T_z^*T_z)^{1/2}=Q_z$ and W_z is the canonical partial isometry. Denote by $g_z^P(\lambda,\tau)$ the principal function associated with the pair $\{W_z,Q_z\}$ and let $g(\nu,\mu)$ be the principal function of $\{U,V\}$ where $U+iV$ is the cartesian form of T.

THEOREM 7.1. *Let $\delta+i\gamma=\sqrt{\lambda}e^{i\theta}$. Then for fixed $z=x+iy$, $g_z^P(\lambda,e^{i\theta})=g(\gamma+y,\delta+x)$ dA-almost everywhere.*

Proof. Since the assertion is for fixed $z=x+iy$, and since the cartesian principal function $g_z(\nu,\mu)$ corresponding to the pair $\{U-x1, V-y1\}$ is $g(\nu+y,\mu+x)$ where $g(\nu,\mu)$ is the principal function of $\{U,V\}$ we may assume, without loss of generality, that $z=0$.

Let $T=U+iV=WQ$ where $Q=|T|$ and let $P=T^*T=|T|^2$. Then if $TT^*-T^*T=D$, we have $WP-PW=DW\in\mathfrak{J}_\tau$. Furthermore,

$$P^{5/2}-(P+D)^{5/2}\in\mathfrak{J}_\tau. \tag{7.1}$$

To see this, note that $f(x)=x^{5/2}$ is continuously differentiable, and hence if we modify f outside sp $(P)\cup$ sp $(P+D)$ so that the modified function is differentiable and has compact support, then the modified function will be in $\hat{M}(R^1)$. The operators $f(P)$ and $f(P+D)$ of course only depend upon f on sp $(P)\cup$ sp $(P+D)$.

Let $A=WP^3$. Then

$$[A,A^*],\ [A,P],\ [A^*,P]\in\mathfrak{J}_\tau. \tag{7.2}$$

These relations follow at once. For, $[A, A^*] = WP^3P^3W^* - P^3W^*WP^3 = WP^6W^* - P^6$ which is in $\mathfrak{J}_\tau$ by an iteration of the relation $WP - PW = DW$. To prove $[A, P] \in \mathfrak{J}_\tau$, note that $[A, P] = WP^3P - PWP^3 = WP^4 - WP^4 + B$ where $B \in \mathfrak{J}_\tau$ (since $[W, P] \in \mathfrak{J}_\tau$).

Finally, $[A^*, P] = -[A, P]^*$, and therefore $[A^*, P] \in \mathfrak{J}_\tau$.

Now let Ω be a neighborhood of sp $(U) \times$ sp (V) which contains the support of $g(\nu, \mu)$.

Set

$$F(x, y) = \begin{cases} (x+iy)(x^2+y^2)^{5/2} & \text{on} \quad \Omega = \mathrm{sp}\,(U) \times \mathrm{sp}\,(V) \\ 0 & \qquad\quad \text{outside.} \end{cases}$$

Then $F(x, y) \in \hat{M}(R^2)$ and $\bar{F}(x, y) \in \hat{M}(R^2)$. Accordingly, we may consider the operator $F(U, V)$ defined by the functional calculus.

By the basic relations of that calculus, we can conclude that

$$\begin{cases} F(U, V) = A \text{ modulo } \mathfrak{J}_\tau \\ \bar{F}(U, V) = A^* \text{ modulo } \mathfrak{J}_\tau. \end{cases} \tag{7.3}$$

Now define

$$H(x, y) = \begin{cases} (x^2+y^2) & \text{on} \quad \mathrm{sp}\,(U) \times \mathrm{sp}\,(V) \\ 0 & \qquad\quad \text{outside a compact set} \end{cases}$$

so that $H \in M(R^2)$

The functional calculus enables us to conclude that

$$H(U, V) = P \text{ modulo } \mathfrak{J}_\tau. \tag{7.4}$$

Thus if p and q are polynomials in two variables, we must have by Theorem 5.1

$$\begin{aligned} \tilde{\tau}[p(A, P)\, q(A^*, P)] &= \tilde{\tau}[p \circ (F(U, V)), H(U, V))), q \circ (\bar{F}(U, V), H(U, V))] \\ &= \frac{i}{2\pi} \iint \{p \circ (F, H), q \circ (\bar{F}, H)\}\,(\mu, \nu)\, g(\nu, \mu)\, d\mu\, d\nu. \end{aligned} \tag{7.5}$$

However, since $W^*P^3 - P^3W^* \in \mathfrak{J}_\tau$,

$$\tilde{\tau}[p(A, P), q(A^*, P)] = \tilde{\tau}[p(WP^3, P), q(W^*P^3, P)].$$

In section 6 we defined the dilation space $\mathcal{K}$ and the projection $\hat{P}$ from $\mathcal{K}$ onto $\mathcal{H} = \ldots \oplus \circ \oplus \circ \oplus \mathcal{H} \oplus \circ \oplus \circ \oplus \ldots$. With the notation of that section we have

$$\tilde{\tau}[p(WP^3, P), q(W^*P^3, P)] = \tilde{\tau}(\hat{P}[p(\hat{W}\hat{P}^3, \hat{P}), q(\hat{W}^*P^3, \hat{P})]_{|\tilde{\mathcal{H}}}).$$

This equality follows from the fact that $\hat{W}$ is a strong * dilation of W and $\hat{P}$ kills any vector outside of $\mathcal{H}$.

Let now $J(\tau, r)$ be defined to be smooth enough and of compact support such that $J(\zeta, r) = \zeta r^3$, $|\zeta| = 1$ on a neighborhood of sp $(\hat{W}) \times$ sp $(\hat{P})$.

Then $J(\cdot,\cdot)\in\hat{M}(Z\times R^1)$, and we can use the functional calculus developed in Sec. 6 to conclude that $A-\hat{P}J(\hat{W},\hat{P})_{|\tilde{\mathfrak{H}}}\in\mathfrak{J}_\tau$ and $A^*-\hat{P}\bar{J}(\hat{W},\hat{P})_{|\tilde{\mathfrak{H}}}\in\mathfrak{J}_\tau$. Therefore, by Theorem 6.4

$$\begin{aligned}\tilde{\tau}[p(A,P),q(A^*,P)]&=\tilde{\tau}\hat{P}[p\circ(J(\hat{W},\hat{P}),\hat{P}),q\circ(\bar{J}(\hat{W},\hat{P}),\hat{P})]_{|\mathfrak{H}}]\\&=\frac{1}{2\pi i}\iint\{p\circ(J(\zeta,r),r),q\circ(\bar{J}(\zeta,r),r)\}(\zeta,r)\cdot g^P(r,\zeta)\,dr\,d\zeta.\end{aligned}\tag{7.6}$$

Now take $p(x,y)=x^ny^m$, $q(x,y)=s^sy^t$ where $n,m,s,t\in Z^+$, the nonnegative integers.

Then with $x+iy=r\zeta=r(\cos\theta+i\sin\theta)$ in a neighborhood of the support of $g(\nu,\mu)$, we have

$$p\circ(F,H)(\zeta,r)=[(r\zeta)r^5]^nr^{2m}$$
$$q\circ(F,H)(\zeta,r)=[(r\bar{\zeta})r^5]^sr^{2t}$$

or

$$p\circ(F,H)(\zeta,r)=r^{6n+2m}\zeta^n$$
$$q\circ(F,H)(\bar{\zeta},r)=r^{6s+2t}\bar{\zeta}^s.$$

Now if $\mu+i\nu=re^{i\theta}$, $\zeta=e^{i\theta}$, then

$$\frac{\partial(p\circ(F,H),q\circ(\bar{F},H))}{\partial(\mu,\nu)}=\frac{\partial(p\circ(F,H),q\circ(\bar{F},H))}{\partial(\theta,r)}\frac{\partial(\theta,r)}{\partial(\mu,\nu)}=\{p\circ(F,H),q\circ(\bar{F},H)\}(\zeta,r)\frac{1}{r}.$$

Hence by (7.5)

$$\tilde{\tau}[p(A,P),q(A^*,P)]=\frac{i}{2\pi}\iint\{p\circ(F,H),q\circ(\bar{F},H)\}(\zeta,r)\cdot g(r\zeta)\,dr\,d\zeta,$$

where we have written $g(r\zeta)=g(\nu,\mu)$. But, a calculation shows that $\{p\circ(F,H),q\circ(\bar{F},H)\}(\zeta,r)=i[(6n+2m)s+(6s+2t)n]r^{6n+2m+6s+2t-1}e^{i(n-s)\theta}$. Hence

$$\tilde{\tau}[p(A,P)\,q(A^*,P)]=\frac{-1}{\pi}[6ns+(ms+nt)]\iint r^{6n+2m+6s+2t-1}e^{i(n-s)\theta}g(re^{i\theta})\,dr\,d\theta.\tag{7.7}$$

We can, however, find a relation for $\tilde{\tau}[p(A,P),q(A^*,P)]$ in terms of the polar bracket formula and $g^P(r,e^{i\theta})$: $p\circ(J(\zeta,r),r=J(\zeta,r)^nr^m=\zeta^nr^{3n+m}$, and $q\circ(\bar{J}(\zeta,r),r)=\zeta^{-s}r^{3s+t}$, since $\bar{\zeta}=\zeta^{-1}$. ($|\zeta|=1$.)

Hence the Jacobian is

$$\begin{aligned}&\det\begin{pmatrix}(3n+m)r^{3n+m-1}\zeta^n & (3s+t)r^{3s+t-1}\zeta^{-s}\\ nr^{2n+m}\zeta^{n-1} & -sr^{3s+t}\zeta^{-s-1}\end{pmatrix}\\&=-[s(3n+m)+(3s+t)n]r^{3n+m+3s+t-1}\zeta^{n-s-1}.\end{aligned}$$

Thus, substituting $\zeta = e^{i\theta}$ in (7.6), we get

$$\tilde{\tau}[p(A, P), q(A^*, P) = \frac{-6ns + (ms + nt)]}{2\pi} \iint r^{3(n+s)+m+t-1} e^{i(n-s)\theta} g^P(r, e^{i\theta})\, dr\, d\theta. \tag{7.8}$$

Now let $\tilde{r} = r^2$ in (7.7), then

$$\tilde{\tau}[p(A, P), q(A^*, P)] = \frac{-[6ns + ms + nt]}{2\pi} \iint \tilde{r}^{3(n+s)+m+t-1} e^{i(n-s)\theta} g(\sqrt{\tilde{r}}\, e^{i\theta})\, d\tilde{r}\, d\theta. \tag{7.9}$$

Consequently, with $(n, m) \neq (0, 0)$ and $(s, t) \neq (0, 0)$

$$\iint r^{3(n+s)+m+t-1} e^{i(n-s)\theta} [g(\sqrt{r}\, e^{i\theta}) - g^P(r, e^{i\theta})]\, dr\, d\theta = 0. \tag{7.10}$$

With $t = 1$, n and $s \neq 0$ fixed, we will have for all $m = 0, 1, 2, \ldots$

$$\int r^m dr \int e^{i(n-s)\theta} \cdot r^{3(n+s)} [g(\sqrt{r}\, e^{i\theta}) - g^P(r, e^{i\theta})]\, d\theta = 0.$$

Since the polynomials are dense in the continuous functions over a compact set, we can conclude that indeed

$$r^{3(n+s)} \int e^{i(n-s)\theta} [g(\sqrt{r}\, e^{i\theta}) - g^P(r, e^{i\theta})]\, d\theta = 0$$

for almost all r. Hence,

$$\int e^{i(n-s)\theta} [g(\sqrt{r}\, e^{i\theta}) - g^P(r, e^{i\theta})]\, d\theta = 0$$

for almost all r. Since this holds for all non-zero n and s, we can in turn use this last result to conclude that $g(\sqrt{r} e^{i\theta}) = g^P(r, e^{i\theta})$ for almost all r and θ.

One elementary application has been stated in the introduction.

THEOREM 7.2. *Let $H(r) \in C_0^1(R^1)$ and let $H_1(r) = \partial H / \partial r$. Then $\tilde{\tau}\{H(TT^*) - H(T^*T)\} = 1/\pi \iint H_1(x^2 + y^2)\, g(y, x)\, dx\, dy$.*

Proof. Let $\delta(\lambda)$ be the spectral displacement function corresponding to the perturbation problem $TT^* \to T^*T$ introduced in (3.23).

According to (3.27)

$$\tilde{\tau}[H(TT^*) - H(T^*T)] = \int H_1(r)\, \delta(r)\, dr.$$

But $\delta(r) = 1/2\pi \int_0^{2\pi} g^P(r, e^{i\theta})\, d\theta$, by Theorem 6.2. Hence we have

$$\tilde{\tau}[H(TT^*) - H(T^*T)] = \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\infty} H_1(r)\, g^p(r, e^{i\theta})\, d\theta\, dr.$$

The theorem is proved if we now invoke Theorem 7.1 and make the change of variables $x=\sqrt{r}\cos\theta$, $y=\sqrt{r}\sin\theta$.

Remark 7.1. Let $T=WQ$ be the polar decomposition of an element in $\mathcal{M}$ with $[T^*, T]\in \mathfrak{J}_\tau$. Let $K\in\mathfrak{J}_\tau$ and consider the new polar form $T+K=\tilde{W}\tilde{Q}$. Then $Q^2-\tilde{Q}^2\in\mathfrak{J}_\tau$, but we cannot say the same for $W-\tilde{W}$. (In fact, let $T=0$ and take $K=L|K|$ where L is unitary in $\mathcal{M}$.) Nevertheless, Theorem 7.1 tells us that the polar principal functions for $T-z$ and $T+K-z$ are identical.

§ 8. Index results

In this section we shall assume that $\mathcal{M}$ is a von Neumann subalgebra of $\mathcal{L}(\mathcal{H})$.

Call $T\in\mathcal{M}$ finite if the projection onto the range of T, $\mathcal{R}(T)$, is finite and call T relatively compact if it is the limit in the norm of finite elements of $\mathcal{M}$.

M. Breuer [6], [7] has defined the notion of relatively Fredholm operator.

According to Breuer an operator T is relatively Fredholm if (1) the projection onto the null space of T, $\mathcal{N}(T)$, is finite and (2) there is a finite projection E of $\mathcal{M}$ such that the range of $1-E$ is contained in the range of T. The set of Fredholm elements of $\mathcal{M}$ will be denoted by $\mathfrak{F}(\mathcal{M})$.

This definition does not imply that the range of T is closed unless the finiteness of E relative to $\mathcal{M}$ implies the finiteness of E relative to the full algebra $\mathcal{L}(\mathcal{H})$.

We will assume throughout the following paragraphs of the present paper that $\mathcal{M}$ is equipped with a semifinite normal faithful trace τ. In addition we will assume that $E\in\mathcal{P}(\mathcal{M})$, E finite implies that $\tau(E)<+\infty$. If $E\in\mathcal{P}(\mathcal{M})$ is finite, we define $\dim(E)=\tau(E)$, and with Breuer make the definition:

$$\text{Index } T=\dim \mathcal{N}(T)-\dim \mathcal{N}(T^*).$$

Breuer proved the von Neumann algebra analogues of the familiar Atkinson results about Fredholm operators and index; namely, $\mathfrak{F}(\mathcal{M})$ is open in the norm topology, $T\in\mathfrak{F}(\mathcal{M})$ iff T is invertible modulo the two sided * ideal of compact elements of $\mathcal{M}$. Furthermore, if S, $T\in\mathfrak{F}(\mathcal{M})$, then Index $S^*=-$Index S, Index $ST=$Index $S+$Index T, and if Index $S=$Index T, then S, T lie in the same component of $\mathfrak{F}(\mathcal{M})$.

Assume now that T is a Fredholm operator in $\mathcal{M}$. Let E^0_λ be the spectral resolution corresponding to TT^*, and let E^1_λ be the spectral resolution corresponding to T^*T.

PROPOSITION 8.1. *$E^1_\lambda-E^0_\lambda$ is in trace class and $\tilde{\tau}(E^1_\lambda-E^0_\lambda)$ is constant in a positive neighborhood $\lambda=0$.*

Proof. Let $T = W(T^*T)^{1/2}$ be the polar decomposition of T. Then $WW^* = \mathcal{R}(T)$ and $W^*W = \mathcal{R}(T^*)$. Also $WT^*T = TT^*W$, and it follows that $WE^1_\lambda = E^0_\lambda W$.

But

$$E^1_\lambda - E^0_\lambda = E^1_{\{0\}} + E^1_\lambda - E^1_{\{0\}} - E^0_{\{0\}} + E^0_{\{0\}} - E^0_\lambda = E^1_{\{0\}} + E^1_\lambda - E^1_{\{0\}} - E^0_{\{0\}} - W(E^1_\lambda - E^1_{\{0\}})\,W^*.$$

If $\lambda > 0$ is sufficiently small, each of the projections which occurs in this expression is finite, and we can take the trace on both sides of this last equation to obtain

$$\tilde{\tau}(E^1_\lambda - E^0_\lambda) = \tilde{\tau}(E^1_{\{0\}}) + \tilde{\tau}(E^1_\lambda - E^1_{\{0\}}) - \tilde{\tau}(E^0_{\{0\}} - \tilde{\tau}(E^1_\lambda - E^1_{\{0\}}) = \tilde{\tau}(E^1_{\{0\}} - E^0_{\{0\}}).$$

The fact that $\tilde{\tau}(W(E^1_{\{0\}} - E^1_\lambda)\,W^*) = \tilde{\tau}(E^1_{\{0\}} - E^1_\lambda)$ follows from $\tilde{\tau}(W(E^1_{\{0\}} - E^1_\lambda)\,W^*)) = \tilde{\tau}(W^*W(E^1_{\{0\}} - E^1_\lambda))$ and $\overline{W^*W}\,|\,\mathcal{R}(T^*) = 1$.

Hence $\tilde{\tau}(E^1_\lambda - E^0_\lambda) = \tilde{\tau}(E^1_{\{0\}} - E^0_{\{0\}})$ for $\lambda > 0$ and sufficiently small.

Let us now consider the case where $T_z = T - zI$ is Fredholm. Let $E^1_\lambda(z)$ be the spectral resolution associated with $T^*_z T_z$ and let $E^0_\lambda(z)$ be the spectral resolution associated with $T_z T^*_z$. Then

$$[(T_z T^*_z - l)^{-1} - (T^*_z T_z - l)^{-1}] = \int \frac{d[E^0_\lambda(z) - E^1_\lambda(z)]}{\lambda - l}.$$

If we now denote by $\delta_z(\lambda)$ the phase shift associated as in section 3 with $V = T^*_z T_z \mapsto V + D = T_z T^*_z$, we will have by Lemma 3.1

$$\tilde{\tau}\{(T^*_z T_z - l)^{-1} - (T_z T^*_z - l)^{-1}\} = \int \frac{\delta_z(\lambda)}{(\lambda - l)^2}\,d\lambda.$$

Furthermore, since $0 \notin \mathrm{sp}_e(T_z T^*_z)$, Theorem 3.2 implies that $\delta_z(\lambda) = \tilde{\tau}(E^1_\lambda(z) - E^0_\lambda(z))$ for $\lambda \geqslant 0$ and sufficiently small; thus $\delta_z(\lambda)$ is constant near $\lambda = 0$ and equals the index of T_z.

On the other hand, the index of T_z is constant on the components of the complement of the essential spectrum of T. Select a component, say Ω; with $z \in \Omega$, there exists a number $b(z) > 0$ such that $\delta_z(\lambda) = \text{Index } (T - z)$ a.a. λ in the interval $[0, b(z)]$.

Take $\lambda \in [0, b(z)]$. Then $(1/2\pi)\int_0^{2\pi} g^P_z(\lambda, e^{i\theta})\,d\theta = \text{Index } T - z)$. Thus, if $0 < R < b(z)$.

$$\frac{1}{2\pi}\int_0^R \lambda\,d\lambda \int_0^{2\pi} g^P_z(\lambda, e^{i\theta})\,d\theta = \text{Index}\,(T - z)\,\frac{R^2}{2}.$$

Now we use Theorem 7.1. If $\delta + i\gamma = \sqrt{\lambda}\,e^{i\theta}$, $\lambda > 0$ for fixed $z = x + iy$, then $g^P_z(\lambda, e^{i\theta}) = g(\gamma + y, \delta + x)$.

Therefore, if $B_z(R)$ denotes the ball centered at z with radius R, we will have with $g(\zeta)=g(\nu, \mu)$ for $\zeta=\mu+i\nu$ and $\zeta=\lambda e^{i\theta}$

$$\iint_{B_z(R)} g(\zeta)\,dA = \iint_{B_0(R)} g(\zeta+z)\,dA = \int_0^R \lambda\,d\lambda \int_0^{2\pi} g_z^P(\lambda^2, e^{i\theta})\,d\theta = \text{Index}\,(T-z)\,\pi R^2.$$

Now divide both sides by πR^2 and go to the limit as R approaches zero.

The fundamental theorem of the calculus enables us to conclude that $g(z)=$ Index $(T-z)$.

We have in fact proved the following

THEOREM 8.1. *With z not in the essential spectrum of T,* Index $(T-z)=\delta_z(\lambda)=g(z-(\mu+i\nu))$ *when λ is positive and both λ and $\mu^2+\nu^2$ are sufficiently small.*

As a consequence of Theorem 8.1 we shall show that the bilinear form

$$(F, G)\to\tilde{\tau}([F(U, V), G(U, V)])$$

on $\hat{M}(R^2)$ is supported on $\text{sp}_e\,(U+iV)$. We first prove a lemma.

LEMMA 8.1. *Let E and A be elements of $\mathfrak{M}$ with E compact, self-adjoint and $[E, A]\in\mathfrak{J}_\tau$. Then $\tilde{\tau}([E, A])=0$.*

Proof. It is clear that we can assume that A is also self-adjoint and in that case it is enough to prove that the principal function of $E+iA$ is zero. But this follows easily from Theorem 8.1 since $\text{sp}_e\,(E+iA)$ has zero planar measure and for $z\notin\text{sp}_e\,(E+iT)$ we have Index $(E+iA-zl)=$ Index $(iA-zl)=0$ since E is compact and A is self-adjoint.

Now suppose F_1 and $F_2\in\hat{M}(R^2)$ and they define the same function on $\text{sp}_e\,(U+iV)$. Since U and V commute modulo the trace ideal $\mathfrak{J}_\tau$, a fortiori modulo the compact operators, by looking at the Gelfand transform we can see that $F_1(U, V)-F_2(U, V)$ is compact. Since $[\text{Re}\,\{F_1(U, V)-F_2(U, V)\}, G(U, V)]$ and $[\text{Im}\,\{F_1(U, V)-F_2(U, V)\}, G(U, V)]$ are trace class the previous lemma asserts that $\tilde{\tau}([F_1(U, V), G(U, V)])=\tilde{\tau}([F_2(U, V), G(U, V)])$. The same reasoning applies to considering the case of two functions G_1 and G_2 in $\hat{M}(R^2)$ which agree on $\text{sp}_e\,(U+iV)$.

§ 9. Examples

In this section we present two different kinds of examples. The first is for rings of type I, while the second is of type II. In the type I case, we outline a general procedure by means of which the principal function can be computed; this procedure is valid for all operators T in $\mathcal{M}$ with $[T^*, T]$ trace class and enables one to associate a principal function with certain pairs $\{U, V\}$ even when $[U, V]$ is not trace class.

The second example is more specific, yet quite interesting. For Toeplitz operators with almost periodic symbol f, the index (hence the principal function) at $z=\mu+i\nu$ is known to coincide with the mean motion of $f-z$ whenever $f-z$ is bounded below away from zero [17]. We conjecture that $g(\nu, \mu)$ coincides with the mean motion of $f-(\mu+i\nu)$ for almost all (ν, μ); using a result of Bohl [4] this is verified when f is a superposition of two exponentials.

Example 9.1. *Type* I_∞.

For details and proof of the following statements the reader should consult [15].

Suppose $\mathcal{M}$ is a factor of type I_∞, i.e. $\mathcal{M}=\mathcal{L}(\mathcal{H})$ where $\mathcal{H}$ is an infinite dimensional (separable) Hilbert space. Let $U\in\mathcal{M}^s$ and denote by $\mathcal{M}(U)$ the C^*-subalgebra of $\mathcal{M}$ generated by the collection of operators $T\in\mathcal{M}$ for which $TU-UT$ is trace class. Then, with $P_{\mathrm{ac}}(U)$ the projection of $\mathcal{H}$ onto $\mathcal{H}_{\mathrm{ac}}(U)$ (the absolutely continuous subspace of U), the limits

$$\lim_{t\to\pm\infty} e^{itU}Te^{-itU}P_\alpha(U)\equiv s_\pm(U; T)$$

exist in the strong operator topology for any T in $\mathcal{M}(U)$; moreover $s_\pm(U; T)\in\mathcal{M}(U)$. The mappings $T\to s_\pm(U; T)$ are conditional expectations which are also *-homomorphisms of $\mathcal{M}(U)$ onto the commutant of U. (Cf. [18, §2].)

Suppose $V\in\mathcal{M}(U)^s$ and assume that the spectral multiplicity function of U_{ac} is finite almost everywhere. In a direct integral space $\mathcal{H}$ in which U_{ac} is diagonal, (i.e., multiplication by the spectral variable) the operators $s_\pm(U; V)$ are decomposable. With

$$\mathcal{H}=\int\oplus\,\mathcal{H}_\lambda\,d\lambda$$

we have

$$s_\pm(U; V)=\int\oplus\, s_\pm(U; V)(\lambda)\,d\lambda. \tag{9.1}$$

Here we are identifying $s_\pm(U; V)$ with their restrictions to the subspace $\mathcal{H}_{\mathrm{ac}}(U)$. Of course $s_\pm(U; V)$ on $\mathcal{H}_{\mathrm{ac}}(U)^\perp$ are represented by the zero operator.

Our assumption on the multiplicity of U_{ac} means that $\dim \mathcal{H}_\lambda<\infty$ for almost all λ.

For any complex number l not in $\operatorname{sp}(V)$ we have

$$s_+(U; V-l)\,s_-(U; [V-l]^{-1})=\int\oplus\, s_+(U; V-l)(\lambda)\,s_-(U; [V-l]^{-1})(\lambda)\,d\lambda.$$

If we apply the usual determinant, det, we get for almost all λ,

$$\det\{s_+(U;\ V-l)(\lambda)s_-(U;[V-l]^{-1})(\lambda)\} = \exp\left\{\int g(\nu,\lambda)\frac{d\nu}{\nu-l}\right\},$$

where $g(\nu,\lambda)$ is a dA-integrable function having compact support. If the commutator $[U,\ V]$ is trace class, then $g(\nu,\lambda)$ is the principal function of $\{U,\ V\}$. By choosing a suitable branch of the argument, we have

$$g(\nu,\lambda) = \lim_{\varepsilon\downarrow 0} \arg\det\{s_+(U;\ V-(\nu+i\varepsilon))(\lambda)s_-(U;[V-(\nu+i\varepsilon)]^{-1})(\lambda)\}$$

for almost all ν, λ. If the roles of U and V are reversed we could obtain an expression for $g(\nu,\lambda)$ in terms of boundary values in sp (V_{ac}).

For singular integral operators V of the form

$$Vx(\lambda) = A(\lambda)x(\lambda) + \frac{1}{\pi i}P\int \frac{k^*(\lambda)k(\mu)x(\mu)}{\mu-\lambda}\,d\mu,$$

where the coefficients $A(\lambda)$ and $k(\lambda)$ are measurable essentially bounded (with respect to linear Lebesgue measure) operator valued functions of λ, $A(\lambda)$ being self-adjoint and $k(\lambda)$ Hilbert–Schmidt, the symbols of V with respect to the multiplication operator $Ux(\lambda) = \lambda x(\lambda)$ are equal to $s_\pm(U;\ V)(\lambda) = A(\lambda) \mp k^*(\lambda)k(\lambda)$. Thus, the principal function $g(\nu,\lambda)$ for the pair $\{U,\ V\}$ is given by

$$g(\nu,\lambda) = \lim_{\varepsilon\downarrow 0}\arg\det\{[A(\lambda)-k^*(\lambda)k(\lambda)-(\nu+i\varepsilon)](A(\lambda)+k^*(\lambda)k(\lambda)-(\nu+i\varepsilon)]^{-1}\}$$

for dA-almost all ν, λ.

Example 9.2. *Type* II_∞.

Let $\tilde{\mathcal{A}}$ denote the C^*-algebra on $L^2(R^1)$ generated by multiplications M_ϕ and translations T_λ where $M_\phi u(x) = \phi(x)u(x)$ and $T_\lambda u(x) = u(x-\lambda)$, for $u \in L^2(R^1)$. Let $\mathcal{A}$ be the C^*-algebra generated by translations on $L^2(R^+)$, with R^+ the nonnegative reals. Observe that after an inverse Fourier transform, a linear combination of translations $\sum_n a_n T_{\lambda_n}$ on $L^2(R^+)$ takes the form $PM_\phi = W_\phi$ where P is the projection onto the Hardy space $H^2(\mathbf{R}^1)$ and ϕ is the exponential polynomial $\sum_n a_n e^{i\lambda_n t}$. (The approximation theorem for $AP(R^1)$, the almost periodic functions, enables one to identify W_ϕ with a sum of translation operators.)

It is clear that $\mathcal{A}$ is isomorphic to the C^*-subalgebra of $\tilde{\mathcal{A}}$ generated by translations restricted to $\mathbf{R}^+$, $\{x_{\mathbf{R}^+}T_\lambda x_{\mathbf{R}^+}: \lambda \in \mathbf{R}^1\}$ where $x_{\mathbf{R}^+}$ is multiplication by the characteristic function of $\mathbf{R}^+ \subset R$. The closed two-sided ideal in $\mathcal{A}$ generated by commutators in $\mathcal{A}$ is the closed span of all finite sums $\sum_n M_{\phi_n} T_{a_n}$ where ϕ_n is a function of bounded variation with compact support in $\mathbf{R}^+$ and a_n is in $\mathbf{R}$ [16]. Moreover, the algebra $\mathcal{A}$ contains W_ϕ for each almost periodic function ϕ on $\mathbf{R}$ while every element in $\mathcal{A}$ can be written uniquely in the form $W_\phi + C$ where ϕ is in $AP(\mathbf{R}')$ and C is in the commutator ideal [16; Theorem 1].

Let $\tilde{\mathcal{A}}_0$ be the set of operators on $L^2(\mathbf{R}^1)$ which have the form

$$\sum_{1 \leqslant n \leqslant N} M_{\phi_n} T_{\lambda_n}, \lambda_j \in \mathbf{R}, \quad j = 1, 2, \ldots, N < \infty,$$

where $\phi_n \in L_c^\infty(\mathbf{R})$, the space of essentially bounded functions which have compact support. Consider the linear functional τ defined on $\tilde{\mathcal{A}}_0^+$ by

$$\tau[\sum M_{\phi_n} T_{\lambda_n}] = \int_{-\infty}^{\infty} \phi_0(t)\, dt$$

where $\phi_0(t)$ is the coefficient of the identity translation, i.e. $\lambda_0 = 0$. The weak closure $\mathcal{M}$ of $\tilde{\mathcal{A}}_0$ is a factor which contains $\tilde{\mathcal{A}}$ and the linear functional τ defined on $\tilde{\mathcal{A}}_0$ may be extended to a faithful, semifinite, normal trace on $\mathcal{M}$ [17], [38]. Hence $\mathcal{M}_+ = \{T \in \mathcal{M}: x_{\mathbf{R}^+} T x_{\mathbf{R}^+} = T\}$ is a type II_∞ factor and the restriction of τ to $\mathcal{M}_+$ is a faithful, normal, semifinite trace; (in fact, all such traces on $\mathcal{M}_+$ can differ at most by a constant positive multiple).

Let $\mathcal{K}_{m_+}$ denote the norm closure in $\mathcal{M}_+$ of the trace ideal.

Following Breuer [6], [7] Fred $(\mathcal{M}_+, \mathcal{K}_{m_+}) = \{T \in \mathcal{M}_+$ invertible mod $\mathcal{K}_{m_+}\}$. As was shown in [17; Theorem (2.2)] $W_\phi + C \in$ Fred $(\mathcal{M}_+, \mathcal{K}_{m_+})$ if and only if ϕ is invertible in $AP(\mathbf{R}^1)$.

If $\phi \in AP(R^1)$ put

$$\phi(t) = \pm |\phi(t)| \exp 2\pi i x(t),$$

where the sign of $\pm |\phi(t)|$ is to be chosen for every t in such a way that x becomes a continuous function of t. The function x is said to have (for $t \to +\infty$) a mean motion u if

$$x(t)/t \to u, \quad \text{i.e.,} \quad x(t) = ut + O(t); \quad t \to \infty.$$

The problem of the existence and determination of this constant u goes back to Lagrange's treatment of secular perturbations of the major planets. The existence of the mean motion has been established for exponential polynomials by Jessen and Tornehave [25] and R. Doss [21]. For exponential polynomials with three terms it was determined explicitly by Bohl [4]; this result is cited below. For general exponential polynomials

$$\phi(t) = \sum a_k \exp i(\lambda_k t + \beta_k)$$

where λ_k, η_k are real, $a_k > 0$ and the frequencies λ_k are linearly independent over the field of rational numbers, it was shown by H. Weyl [39] that the mean motion takes the form

$$\frac{1}{\pi}(W_1\lambda_1 + W_2\lambda_2 + \ldots + W_n\lambda_n)$$

where $\sum_k W_k = 1$ and W_j are independent of the initial phases η_k.

On the other hand if $\phi \in AP(\mathbf{R}^1)$ is bounded below away from zero, i.e. ϕ is invertible in $AP(\mathbf{R}^1)$, the mean motion exists and H. Bohr [5] has shown that

$$\phi(t) = \exp i(ut + \omega(t))$$

where $\omega \in AP(\mathbf{R}^1)$.

Now, if W_ϕ is a Fredholm operator so that ϕ is invertible, a simple homotopy argument shows that index $W_\phi = \mu$. Thus, if $\phi \in AP(R^1)$ and W_ϕ has a trace class self-commutator (for instance, take ϕ to be an exponential polynomial), then the principal function $g(\nu, \mu)$ corresponding to W_ϕ coincides with the mean motion of $(\phi(t) - z)$, $z = \mu + i\nu$ whenever $|\phi(t) - z| > a$, $a > 0$ for $\infty > t > -\infty$.

In particular the principal function for $e_\lambda = W_{\exp i\lambda t}$ equals λ times the characteristic function of the unit disc.

The determination of the principal function in situations where the range of the almost periodic function ϕ is space filling remains unsettled, although there is some evidence that it still agrees with the mean motion at points inside the essential spectrum. As an illustration of this contingency, consider in detail the case where ϕ is a superposition of two exponentials:

$$\phi(t) = a_1 \exp i(\lambda_1 t + \eta_1) + a_2 \exp i(\lambda_2 t + \beta_2).$$

For convenience we shall take $0 < \lambda_1 < \lambda_2$.

If the ratio of the frequencies $\varrho = \lambda_1/\lambda_2$ is rational, then ϕ is periodic, its range is closed and has zero area. Thus, the principal function being defined up to null sets, is determined as the mean motion of ϕ on the complement of the range.

The case of interest here then is when ϱ is irrational.

Let $z = |z| \exp i(\arg z)$. We consider $\phi(t) - z$. Following Bohl [4], the numbers $|z|$, a_1, a_2 determine a triangle with corresponding angles $\omega_1(|z|)$, $\omega_2(|z|)$, $\omega_3(|z|)$. Define

$$b(z) = \beta_2 - [\arg z + \pi] - (\beta_1 - [\arg z + \pi]/\varrho),$$

$$\zeta(z) = \frac{1}{\pi}(\omega_3(|z|) + \varrho\omega_2(|z|)),$$

$$\omega(z) = \frac{1}{2\pi}(\pi - \pi\zeta(z) + [b(\arg z) + \pi]\varrho).$$

It is plain that $0<\varrho<1$ and $0<\zeta(z)<1$. If ϱ is rational, let $\varrho=m/n$ where m and n are relatively prime positive integers. Let $H(z)$ denote the number of integers in the interval $(n\omega(z), n[\omega(z)+\zeta(z)])$. Finally, let $h(z)$ be the number of integers in the set $\{n\omega(z), n[\omega(z)+\zeta(z)]\}$.

Then, the mean motion of the almost periodic function $\phi(t)-z$ equals

$$\begin{cases}\dfrac{1}{\pi}(\lambda_1\omega_2(|z|)+\lambda_2\omega_3(|z|)) & \varrho \text{ irrational}\\[2mm] \dfrac{\lambda_2}{2n}(h(z)+2H(z)) & \varrho \text{ rational.}\end{cases}$$

When ϱ is irrational, we shall prove that the principal function $g(\nu, \mu)$ is almost everywhere equal to the mean motion. We do this by first observing how the mean motion in the irrational case, i.e. ϱ irrational, is a limiting situation of the rational one. For let $\{\lambda_{1j}\}$ and $\{\lambda_{2j}\}$ be sequences of rational numbers with the respective limits λ_1 and λ_2 such that $0<\varrho_j=\lambda_1/\lambda_{2j}=m_j/n_j<1$. It is clear that $n_j\to\infty$ as $j\to\infty$ since ϱ_j is rational and ϱ is not. The mean motion for $\phi_j(t)-z$ where

$$\phi_j(t)=a_1\exp i(\lambda_{1j}t+\beta_1)+a_2\exp i(\lambda_{2j}t+\beta_2)$$

agrees with

$$u_j(z)=(\lambda_{2j}/2n_j)(h_j(z)+2H_j(z)).$$

But, upon letting $j\to\infty$, we see that

$$\frac{H_j(z)}{n_j}\to\zeta(z).$$

Accordingly, $u_j(z)\to(1/\pi)(\lambda_1\omega_2(|z|)+\lambda_2\omega_3(|z|))$. The sequence of principal functions $\{g(\nu,\mu)_j\}$ $1\leqslant j<\infty$ corresponding to the operators $\{W_{\phi_j}\}$ $1\leqslant j<\infty$ likewise converge pointwise to the values $(1/\pi)(\lambda_1\omega_2(z)+\lambda_2\omega_3(z))$. Moreover, the fact that the sequence $\{g(\nu,\mu)_j\}$ $1\leqslant j<\infty$ is uniformly bounded in $L^\infty(R^2; dA, \mathbf{C})$ implies that the sequence $\{g(\nu,\mu)_j\}$ $1\leqslant j<\infty$ converges in the weak*-topology of $L^\infty(R^2; dA, \mathbf{C})$.

In view of the continuity property (Proposition 5.4) the identification of $g(\nu,\mu)$ with $(1/\pi)(\lambda_1\omega_2(|z|)+\lambda_2\omega_3(|z|))$ $(z=\mu+i\nu)$ will be settled once we establish the following lemma.

LEMMA 9.1. *The strong limit relationships*

$$s-\lim_{j\to\infty} W_{\phi_j}=W_{\phi_j},\quad s-\lim_{j\to\infty} W^*_{\phi_j}=W^*_{\phi_j}$$

hold. Moreover,

$$\lim_{j\to\infty}[W^*_{\phi_j}, W_{\phi_j}]=[W^*_\phi, W_\phi]\quad (\text{in } \mathfrak{J}_\tau) \tag{9.2}$$

Proof. The first part of the statement follows at once from the fact that $\lambda_j\to\lambda$ implies $s-\lim_{j\to\infty} e_{\lambda_j}=e_\lambda$ and $e^*_{\lambda_j}=e_{-\lambda_j}$, for any sequence of real numbers $\lambda_1, \lambda_2, \ldots$. It remains to consider the last part involving trace class convergence of the commutators. But in view of the considerations in the proof of Proposition 5.4, together with the preceding remarks, in order to establish (9.2) we need only observe that

$$[e_\lambda, e_\mu] = \begin{cases} 0 & \text{if } \lambda \text{ and } \mu \text{ have the same sign} \\ e_{\lambda+\mu}\chi_{(0,-\lambda)} & \text{if } \lambda < 0, \mu > 0 \\ -e_{\lambda+\mu}\chi_{(0,-\mu)} & \text{if } \lambda > 0, \mu < 0. \end{cases}$$

We have seen that the index group for the algebra $\mathcal{A}$ is the discrete reals. In this connection we observe that for subalgebras $\mathcal{A}' \subset \mathcal{A}$ generated by translations in a subgroup $\mathcal{G}$ of the reals, the corresponding index group will be $\mathcal{G}$. One way of seeing this is that the mean motion of an almost periodic function on R^1 belongs to the module generated by the corresponding frequencies. From this fact we should also expect that the values of the principal function also lie in $\mathcal{G}$. Thus, for instance if $\mathcal{G}$ is the group $\{m+n\theta: m\mu = 0, \pm 1, \pm 2, ...; \theta$ an irrational number$\}$ and $\mathcal{A}_\theta$ is the algebra generator by translations in $\mathcal{G}$, then the principal function map for the algebra $\mathcal{A}_\theta$ has its range in $\mathcal{G}$.

Other examples of type II_∞ factors are obtained by taking the tensor product of the algebra $\mathcal{M}$ with $\mathcal{L}(\mathcal{H})$ for dim $\mathcal{H} < \infty$. In this case the symbols are elements in $AP(R') \times \mathcal{L}(\mathcal{H})$ and the corresponding index is given as the mean motion of the determinants. In this context see [18] for a treatment of algebras generated by translations on R^n for $n \geqslant 1$.

References

[1]. Asano, K., Notes on Hilbert transforms of vector-valued functions in the complex plane and their boundary values. *Proc. Japan Acad.*, 43 (1967), 572–577.

[2]. Birman, M. Sh. & Entina, S. B., Stationary methods in abstract scattering theory. *Izv. Akad. Nauk. SSSR Ser. Math.*, 31 (1967), 401–430.

[3]. Birman, M. Sh. & Solomyak, M. Z., *Stieltjes double integral operators*. Topics in Mathematical Physics, Vol. I, Consultants Bureau, N.Y. (1967).

[4]. Bohl, P., Über ein in der Theorie der säkularen Störungen vorkommenden Problem. *J. Reine u. Angew. Math.*, 135 (1909), 189–283.

[5]. Bohr, H., Über fast periodische ebene Bewegungen. *Comment. Math. Helv.*, 4 (1934), 51–64.

[6]. Breuer, M., Fredholm theories in von Neumann algebras, I. *Math. Ann.*, 178 (1968), 243–254.

[7]. —— Fredholm theories in von Neumann algebras, II. *Math. Ann.*, 180 (1969), 313–325.

[8]. Brodskii, M. S., *Triangular and Jordan representations of linear operators*. American Math. Soc., Providence 1971.

[9]. Carey, R. W., A unitary invariant for pairs of self-adjoint operators. *J. Reine Angew. Math.*, 283 (1976), 294-312.

[10]. Carey, R. W. & Pincus, J. D., The structure of intertwining isometries, *Indiana Univ. Math. J.*, 22 (1973), 679–703.

[11]. —— An invariant for certain operator algebras. *Proc. Nat. Acad. Sci. USA*, 71 (1974) 1952–1956.

[12]. —— An exponential formula for determining functions. *Indiana Univ. Math. J.*, 23 (1974) 1031–1042.

[13]. —— Construction of seminormal operators with prescribed mosaic. *Indiana Univ. Math. J.*, 23 (1974) 1155–1165.

[14]. —— Eigenvalues of seminormal operators, examples. *Pacific J. Math.*, 51 (1974), 1–11.

[15]. —— Commutators, symbols and determining functions, *J. Functional Analysis*, 19 (1975), 50–80.

[16]. Coburn, L. A. & Douglas, R. G., Translation operators on the half-line, Proc. Nat. Acad. Sci. USA, 62 (1969), 1010–1013.

[17]. Coburn, L. A., Douglas, R. G., Schaeffer, D. G. & Singer, I. M., On C^*-algebras on a half-space, II, Index theory. *Publ. Math. IHES*, 40 (1971), 69–80.

[18]. Coburn, L. A., Moyer, R. D. & Singer, I. M., C^*-algebras of almost periodic pseudo-differential operators. *Acta Math.*, 130 (1973), 280–307.

[19]. Daleckii, Ju. L. & Krein, S. G., *Integration and differentiation of functions of Hermitian operators and applications to theory of perturbations.* AMS Translations, ser 2., volume 47.

[20]. Dixmier, J., *Les algebrès d'operateurs dans l'espace Hilbertien.* Gauthier–Villars, Paris, 1969.

[21]. Doss, R., On mean motion. *Amer. J. Math.*, 79 (1957) 389–396.

[22]. Dunford, N. & Schwartz, J., *Linear operators, I, II.* Interscience, New York, 1958.

[23]. Grothendieck, A., *Produits tensoriels topologiques et espaces nuclearies.* Mem Amer. Math. Soc. 16 (1955).

[24]. Helton, J. W. & Howe, R., Integral operators, commutator traces, index and homology. *Proceedings of a conference on operator theory*, Springer Verlag Lecture Notes, no. 345, 1973.

[25]. Jessen, B. & Tornehave, H., Mean motions and zeros of almost periodic functions. *Acta Math.*, 77 (1945) 137–279.

[26]. Kato, T., Smooth operators and commutators. *Studia Math.* (31) 1968, 535–546.

[27]. —— Some mapping theorems for the numerical range, *Proc. Japan Acad.*, 41 (1965), 652–655.

[28]. Kato, T., Wave operators and similarity for some non-self-adjoint operators. *Math. Ann.*, 162 (1966), 258–279.

[29]. Krein, M. G., Perturbation determinants and a formula for the traces of unitary and self adjoint operators. *Dokl. Akad. Nauk SSSR*, 144 (1962), 268–271.

[30]. Parthasarathy, K. R., *Probability measures on metric spaces*, Academic Press, New York, 1967.

[31]. Pincus, J. D., Commutators and systems of singular integral equations, I. *Acta Math.*, 121 (1968), 219–249.

[32]. —— On the trace of commutators in the algebra of operators generated by an operator with trace class self-commutator. Stony Brook preprint (1972).

[33]. —— Symmetric singular integral operators. *Indiana Univ. Math. J.*, 23 (1973), 537–556.

[34]. —— The spectrum of seminormal operators. *Proc. Nat. Sci. USA*, 68 (1971), 1684–1685.

[35]. Putnam, C., A similarity between hyponormal spectra and normal spectra. *Ill. J. Math.*, 16 (1972), 695–702.

[36]. Sakai, S., *C^*-algebras and W^*-algebras.* Springer Verlag, New York, 1971.

[37]. Schwartz, J., *W^*-algebras.* Gordon and Breach, New York, 1967.

[38]. Shaffer, D. G., An application of von Neumann algebras to finite difference equations. *Ann. of Math.*, 95 (1972), 116–129.

[39]. Weyl, H., Mean motion, I. *Amer. J. Math.*, 60 (1938), 889–896.

Received November 25, 1974