REGULARITY FOR A CLASS OF NON-LINEAR ELLIPTIC SYSTEMS

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In this article we demonstrate that the solutions of a certain class of non-linear elliptic systems are smooth in the interior of the domain. One example of this class of equations is the system

$$\text{div}\,(\varrho(|\nabla s|^2)\nabla s_k) = 0 \quad 1 \leqslant k \leqslant m, \tag{0.1}$$

where ϱ is a smooth positive function satisfying the ellipticity condition $\varrho(Q) + 2\varrho'(Q)q > 0$, ∇ denotes the gradient, and $|\nabla s|^2 = \sum_{k=1}^{m} |\nabla s_k|^2$. This type of system arises as the Euler–Lagrange equations for the stationary points of an energy integral which has an intrinsic definition on maps between two Riemannian manifolds; the equations are therefore of geometric interest. However, the method of proof also applies to the equations of non-linear Hodge theory, which have been studied by L. M. and R. B. Sibner [9]. These are systems of equations for a closed p-form ω, $d\omega = 0$ and

$$\delta(\varrho(|\omega|^2)\omega) = 0, \tag{0.2}$$

where ϱ must satisfy the same ellipticity condition given earlier. The proof is presented in a form which covers both cases.

We shall prove regularity in the interior for solutions of systems which do not depend explicitly on either the independent variable or the functions, but only on the derivatives of the functions. An extension to a more general class of systems of the same type with smooth dependence on dependent and independent variables will be important for integrals which arise in Riemannian geometry and probably can be carried out without any radically different techniques. Homogeneous Dirichlet and Neumann boundary value problems may be treated by reflection; however, the regularity up to the boundary for

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non-homogeneous boundary value problems cannot be proved using the techniques given in this paper.

The results of this paper are an extension of the Nash–De Giorgi-Moser results on the regularity of solutions of single non-linear equations to solutions of certain types of systems. We also allow a weakening of the ellipticity condition, so our results are new when applied to single equations. The method of proof is to exhibit an auxiliary function for the system which is subharmonic. Then estimates of Moser [7] for subsolutions and supersolutions are used to get a strong maximum principle. From here a perturbation theorem similar to theorems of Almgren [1] and Morrey [6] gives sufficient continuity for the linear theory of the regularity of solutions of elliptic systems to be applicable.

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Section 1: Statement of the theorem

We assume that an elliptic complex of a particularly simple kind has been given. Let V_i $(i=-1,0,1,2)$ be finite dimensional vector spaces and $A(i)$ a differential operator of first order in n independent variables with constant coefficients from functions with values in V_i to functions with values in V_{i+1}. Let $D_i=\partial/\partial x_i$. Then if u: $R^n\to V_i$, $A(i)u$: $R^n\to V_{i+1}$ is given by

$$A(i)\,u=\sum_{k=1}^{n} A_k(i)\,D_k\,u$$

where $A_k(i)\in L(V_i, V_{i+1})$. The symbol $\sigma(A(i))$, is a linear map from elements $\pi=(\pi_1,\pi_2,\ldots,\pi_n)$ of R^n into $L(V_i, V_{i+1})$ given by

$$\sigma(A(i),\pi)=\sum_{k=1}^{n}\pi_k\,A_k(i)$$

and the complex $\{A(i)\}$ is elliptic if the symbol sequence

$$V_{-1}\xrightarrow{\sigma(A(-1),\pi)}V_0\xrightarrow{\sigma(A(0),\pi)}V_1\xrightarrow{\sigma(A(1),\pi)}V_2$$

is exact for all $\pi\neq 0$.

The dual sequence consists of dual operators $A(i)^*$ from sections of V_{i+1}^* to V_i^*, $A(i)^*v=\sum_{k=1}^{n}A_k(i)^*D_kv$. We shall assume an inner product on the V_i has been given, so V_i^* may be identified with V_i. The dual complex is elliptic if the original complex is.

Let ω be a function on a domain $D\subset R^n$ with values in V_1. $Q=|\omega|^2$. Let ∇ and $\nabla^2=\Delta$

be the ordinary gradient and Laplace operators respectively. These may be considered as operators on V_i by coordinatewise action. In addition, ϱ will be a continuously differentiable non-negative real-valued function on the positive reals. We are interested in solutions of the equations

$$A(0)^*(\varrho(Q)\omega) = 0, \qquad A(1)\omega = 0. \tag{1.1}$$

In the case $\varrho = 1$, this is a linear system and local existence and regularity follow from theorems on elliptic complexes. From the linear theory, we know we may locally write $\omega = A(0)\varphi$ with $A(-1)^*\varphi = 0$, and the equations become a system for φ [3].

MAIN THEOREM. *Let $\{A(i)\}$ be an elliptic complex such that*

$$A(0)A(0)^* + A(1)^*A(1) = \Delta. \tag{1.2}$$

If in addition ϱ is continuous, differentiable for $Q > 0$, and satisfies the uniform ellipticity and growth conditions for some $K > 0$, $p \geqslant 0$, $\alpha > 0$ and $C \geqslant 0$:

$$K^{-1}(Q+C)^p \leqslant \varrho(Q) + 2Q\varrho'(Q) \leqslant K(Q+C)^p, \tag{1.3}$$

$$|\varrho'(Q_1)Q_1 - \varrho'(Q_2)Q_2| \leqslant K(Q_1+Q_2+C)^{p-\alpha}(Q_1-Q_2)^\alpha, \tag{1.4}$$

then any weak solution ω in a domain $D \subset R^n$ of the equations

$$A(0)^*(\varrho(Q)\omega) = 0; \quad A(1)\omega = 0$$

which lies in the space $L^{2p+2}(D)$ is Hölder continuous in the interior of D.

Given growth conditions (1.3) for ϱ, $\omega \in L^{2p+2}(D, V_1)$ is the natural space in which to obtain solutions of (1.1). The existence of solutions may often be obtained by a variational principle from the integral $\int_D G(Q)\,dx$, where $G' = \varrho$ and ω is subject to the constraint $A(1)\omega = 0$.

Condition (1.3) implies the following condition for a possibly larger constant K. This is a more useful form of the ellipticity condition.

$$\begin{aligned} &K(Q+C)^p \geqslant \varrho(Q) + 2\varrho'(Q)Q \geqslant K^{-1}(Q+C)^p \\ &K(Q+C)^p \geqslant \varrho(Q) \geqslant K^{-1}(Q+C)^p \\ &|Q\varrho'(Q)| \leqslant K(Q+C)^p \end{aligned} \tag{1.3$'$}$$

To apply the theorem to equation (0.2), we take $V_i = \Lambda_{p-1-i}R^n$, $A(i) = d$ (exterior differentiation) and $A(i)^* = d^* = \delta$. The application of the main theorem to (0.1) is only slightly more complicated. We let $\omega = \nabla s = (\nabla s_1, \nabla s_2, \ldots, \nabla s_n)$, which will have nm components. $V_{-1} = 0$, $V_0 = R^m$, $V_1 = R^m \times R^n$ and $V_2 = R^m \times (R^n \wedge R^n)$. Let $A(0) = \nabla = d$ (by which we mean m copies of d on functions) and $A(1) = d$ (also m copies on 1-forms). Then $A(0)^* = \delta$, which on 1-forms over R^n is simply div. Since $\omega = ds$, $A(1)\omega = d\omega = 0$ and equation (0.1) becomes $A(0)^*(\varrho(Q)\omega) = 0$. This gives the following two theorems.

THEOREM. *Let $s \in L_1^{2p+2}(D, R^m)$ be a weak solution in D of the equation*

$$\operatorname{div}(\varrho(|\nabla s|^2)\nabla s_k) = 0, \quad 1 \leqslant k \leqslant m.$$

If ϱ satisfies the ellipticity and growth conditions (1.3) *and* (1.4), *then s has Hölder continuous first derivatives in the interior of D.*

THEOREM. *Let ω be a p-form over a domain $D \subset R^n$ such that the coordinates of ω lie in $L^{2p+2}(D)$. If ω is a weak solution of the equations*

$$d\omega = 0; \ \delta(\varrho(|\omega|^2)\omega) = 0,$$

and if ϱ satisfies (1.3) *and* (1.4), *then ω is Hölder continuous in the interior of D.*

As a corollary, there is an important application to single equations which follows. This is a new result in the case $|df|$ is not bounded away from zero.

THEOREM. *Let $f \in L_1^{p+2}(D, R)$ be a weak solution to the equation $d^*|df|^p df = 0$, $0 < p < \infty$. Then f has Hölder continuous derivatives in the interior of D.*

We now show that if we assume hypotheses (1.2) and (1.3) and sufficient differentiability of ω, then some increasing function of Q is subharmonic. In fact, the following computation can be carried out if $(Q+C)^{p/2}\omega$ has square integrable weak derivatives. First we note that:

$$(\omega, \Delta(\varrho(Q)\omega)) = \sum_i (D_i(\omega, D_i(\varrho,(Q)\omega)) - (D_i\omega, D_i(\varrho(Q)\omega)))$$

$$= \sum_i D_i((\omega,\omega)\varrho'(Q)D_i(Q) + \varrho(Q)(\omega, D_i\omega)) \tag{1.5}$$

$$-\sum_i (\varrho(Q)(D_i\omega, D_i\omega) + \varrho'(Q)(D_i\omega,\omega)D_iQ). \tag{1.6}$$

Since $Q = (\omega, \omega)$, the first term (1.5) becomes

$$\sum_i D_i((\tfrac{1}{2}\varrho(Q) + Q\varrho'(Q))D_iQ) = \Delta H(Q),$$

where we define H by

$$H'(Q) = \tfrac{1}{2}\varrho(Q) + Q\varrho'(Q). \tag{1.7}$$

The term (1.6) can be estimated by (1.3)′

$$\sum_i \varrho(Q)(D_i\omega, D_i\omega) + 2\varrho'(Q)(D_i\omega, \omega)^2 \geqslant K^{-1}(Q+C)^p|\nabla\omega|^2,$$

where $|\nabla\omega|^2 = \sum_i (D_i\omega, D_i\omega)$. As a result

$$(\omega, \Delta(\varrho(Q)\omega)) \geqslant \Delta H(Q) - K^{-1}(Q+C)^p|\nabla\omega|^2.$$

From (1.2)

$$\Delta(\varrho(Q)\omega) = (A(0)A(0)^* + A(1)^*A(1))(\varrho(Q)\omega) = A(1)^*A(1)(\varrho(Q)\omega).$$

Since $A(1)\omega = 0$, $A(1)\varphi\omega = \sum_k A_k(1)\omega D_k\varphi = B_\omega\varphi$.

$$\begin{aligned} 0 &= (\omega, \Delta(\varrho(Q)\omega)) - (\omega, A(1)^*A(1)(\varrho(Q)\omega)) \\ &\geqslant \Delta H(Q) - B_w^* B_\omega(\varrho(Q)) - K^{-1}(Q+C)^p|\nabla\omega|^2 \\ &= L_\omega H(Q) - K^{-1}(Q+C)^p|\nabla\omega|^2, \end{aligned}$$

where we have defined the operator L_ω as

$$L_\omega = \sum_{k,j} D_k(a_{kj}D_j) = \Delta - B_\omega^*\left(\frac{\varrho'(Q)}{H'(Q)}A_\omega\right). \tag{1.8}$$

$$a_{kj} = \delta_{kj} - \frac{\varrho'(Q)}{H'(Q)}(A_k(1)\omega, A_j(1)\omega). \tag{1.9}$$

Since $\Delta = A(0)A(0)^* + A(1)^*A(1)$, for n-vectors $\pi = (\pi_1, \ldots, \pi_n)$

$$\begin{aligned} \pi^2(\omega,\omega) &= \sum_{j,k} (\omega(A_k(0)A_j(0)^* + A_k(1)^*A_j(1))\omega)\pi_k\pi_j \\ &\geqslant \sum_{j,k} (A_k(1)\omega, A_j(1)\omega)\pi_k\pi_j. \end{aligned}$$

This implies from (1.3)′ and (1.9) that if $\varrho'(Q)$ is negative

$$\sum_{j,k} \pi_k a_{kj}\pi_j \geqslant |\pi|^2$$

and

$$\geqslant |\pi|^2\left(1 - \frac{\varrho'(Q)}{H'(Q)}Q\right) \geqslant \pi^2/2K^2$$

in the case that $\varrho'(Q)$ is positive.

$\sum_{j,k}\pi_j a_{jk}\pi_k \leqslant (1+2K^2)|\pi|^2$ by the same reasoning. We have the following theorem as a result of this computation.

(1.10) THEOREM. *Let ω be a solution of equation* (1.1) *in D and assume* (1.2) *and* (1.3)′ *are valid. If $(Q+C)^{p/2}\omega$ has weak derivatives which are square integrable in D, then $H(Q)$ is subharmonic. There exists a symmetrix elliptic operator L_ω with bounded measureable coefficients given by*

$$L_\omega = \sum_{k,j} D_k(a_{kj} D_j),$$

where

$$\varphi^2(1+2K^2) \geqslant \sum_{k,j} a_{kj}\pi_k\pi_j \geqslant \pi^2/(2K^2),$$

such that for any smooth positive function ψ *with support on the interior of* D *we have*

$$\int_D \psi L_\omega H(Q)\,dx \geqslant K^{-1}\int_D \psi(Q+K)^p |\nabla\omega|^2 dx.$$

Section 2: Subsolutions of elliptic equations

In this section we shall state two results of Moser [7] on subsolutions and supersolutions of elliptic equations and give an application, Theorem 2.3, which shows that the strong maximum principle is true for subsolutions.

The estimates which we are interested in are for the uniformly elliptic operator L of second order in self-adjoint form

$$Lu = \sum_{j,k} D_j(a_{jk}(x)\,D_k u)$$

in a domain $D \subset R^n$. $a_{jk} = a_{kj}$ are bounded measureable functions and

$$\lambda|\pi|^2 \geqslant \sum_{k,j} a_{kj}\pi_k\pi_j \geqslant \lambda^{-1}|\pi|^2$$

for some constant λ. The constants in this section will all depend on λ. By a subsolution we mean a measureable function u on D with weak derivatives in L^2 such that for all smooth non-negative functions ψ with support in the interior of D

$$\int_D \psi Lu\,dx \geqslant 0.$$

We say u is a supersolution if $-u$ is a subsolution. In this section, $B(x, r)$ denotes the ball of radius r about the point x.

(2.1) THEOREM (Moser, [7]). *If* u *is a subsolution in* D, *then* u *is bounded in the interior of* D. *In particular, if* $B(x, 2r) \subset D$, *then for every* $\beta > 1$ *there exists a constant* $c(\beta)$ *such that*

$$\operatorname*{ess\,max}_{y\in B(x,r)} u(y) \leqslant c(\beta)\left(r^{-n}\int_{\substack{u(y)>0\\ y\in B(x,2r)}} u^\beta dx\right)^{1/\beta}.$$

(2.2) THEOREM. *If* $u>0$ *is a supersolution in* $B(x, 4r)$, *then for* $0<\beta<n/(n-2)$, *there exists a constant* $c'(\beta)$ *such that*

$$\left(r^{-n}\int_{B(x,3r)} u^\beta dx\right)^{1/\beta} \leqslant c'(\beta) \operatorname*{ess\,min}_{y\in B(x,r)} u(y).$$

We have restated these theorems of Moser to hold in a ball of arbitrary radius. We now give the analogy of (2.2) for subsolutions, which can be seen to be a strong maximum principle.

(2.3) THEOREM. *If $u \leqslant M$ is a subsolution in D, then u cannot take on its maximum at an interior point unless it is constant in D. In addition, if $B(x, 4r) \subset D$, then for $0 < \beta < n/(n-2)$ there exists a constant $c'(\beta)$ such that*

$$\left(r^{-n}\int_{B(x,3r)}(m-u)^{\beta}dx\right)^{1/\beta} \leqslant c'(\beta)(m - \operatorname*{ess\,max}_{y\in B(x,r)} u(y))$$

where $m = \operatorname*{ess\,max}_{y\in B(x,4r)} u(y)$.

Proof. We shall derive the inequality first. Since u is a subsolution, $m-u$ is a supersolution and positive in the ball of radius $4r$ about x. $\operatorname{ess\,min}_{y\in B(x,r)}(m-u)(y) = m - \operatorname{ess\,max}_{y\in B(x,r)} u(y)$ and the inequality follows from (2.2) applied to $m-u$. From the inequality, we see that if $M = \operatorname{ess\,max}_{y\in D} u(y)$ is taken on at an interior point x, and if r is the distance from this interior point x to the boundary of D, then $u = M$ almost everywhere in $B(x, r/4)$. It follows that $u = M$ almost everywhere in D.

We now show that a slightly weaker definition of a subsolution is possible. In fact, Lu can be defined as a distribution if u has weak derivatives which lie in any L^p space if $1 < p \leqslant 2$. In general it is not possible to work with these classes of subsolutions or supersolutions. However, if u is positive, $u = w^{k+1}$ for $0 < k \leqslant 1$, and w has weak derivatives in L^2, then u has weak derivatives in some L^p space, we can define Lu, and we obtain a regularity result for this class of subsolutions. The lemma we prove is similar to a step in the proof of (2.1). The a priori estimates are the same, but we need to check to be sure we can find test functions in the correct classes.

(2.4) LEMMA. *Let $u = w^{k+1}$, $0 < k < 1$, and assume that w has weak derivatives which lie in L^2 in $B(x, 2r)$. If for all smooth positive functions ψ with support in the interior of $B(x, 2r)$, u satisfies*

$$\int_{B(x,2r)} \psi L u\, dx \geqslant 0$$

then for $0 < \alpha < 1/(n-2)$, $w^{1+\alpha}$ has weak derivatives in L^2 in $B(x,r)$ and there exists a constant c_1 such that

$$\int_{B(x,r)} (D_i w^{1+\alpha})^2 dx \leqslant c_1\left(\int_{B(x,2r)} \sum_i (D_i w)^2 + r^{-2} w^2 dx\right)^{1+\alpha}.$$

Proof. We prove the lemma for the function $w(ry+x) = \tilde{w}(y)$. By this change of variables, we see that we may assume $r=1$ and $x=0$. By adding a small positive constant to u, we may assume that w is bounded away from 0. Also, from the Sobolev inequalities, $w \in L^{2(1+2\alpha)}(B(2))$, $D_i w^{k+1} = (k+1) w^k D_i w$ lies in some L^p space, and $D_j(a_{ij} D_i w^{k+1})$ is defined as a distribution as claimed. Since w is strictly positive,

$$- (k+1)^{-1} \int_{B(2)} \psi Lu\, dx = (k+1)^{-1} \int_{B(2)} \sum_{i,j} D_i \psi a_{ij} D_j w^{k+1} dx$$
$$= \int_{B(2)} \sum_{i,j} D_i \psi a_{ij} w^k D_j w\, dx$$
$$= \int_{B(2)} \sum_{i,j} (D_i(\psi w^k)\, a_{ij} D_j w - k \psi w^{1-k} D_i w a_{ij} D_j w)\, dx.$$

If ψ is a smooth positive function, then this integral is negative by assumption. From a closure argument, if ψw^k has weak derivatives in L^2 and $(\psi w^k) w^{-1}$ is bounded, then this integral is still negative. Let γ be a smooth positive function which is 1 on $B(1)$ and has support in $B(2)$. $F(w) = \min (w^{1+2\alpha}, Mw)$. Then the test function $\psi = \gamma w^{-k} F(w)$ is sufficiently differentiable for the inequality to hold. Substituting this into the above equation and differentiating, we find for $k \leqslant 1$ that

$$2\alpha \int_{w^{2\alpha} \leqslant M} \gamma w^{2\alpha} \sum_{i,j} D_i w a_{ij} D_j w\, dx \leqslant \int_{B(2)} F(w) \sum_{i,j} D_i \gamma a_{ij} D_j w\, dx$$
$$\leqslant \max \gamma \left(\int_{B(2)} F(w)^2 dx \right)^{1/2} \left(\int_{B(2)} (\sum_{i,j} D_i w a_{ij} D_j w)^2 dx \right)^{1/2}$$
$$\leqslant c_1' \left(\int_{B(2)} w^{2+4\alpha} dx \right)^{1/2} \left(\int_{B(2)} \sum_j (D_j w)^2 dx \right)^{1/2}$$

where $\int_{B(2)} w^{2+4\alpha} dx$ is bounded by the Sobolev theory and c_1' depends only on γ and the bound λ for the coefficients. Since this bound is independent of M, this gives the constant $c_1 = c_1' \lambda (1+\alpha)^{-2}$ from the computation

$$\int_{B(1)} \sum_i (D_i w^{1+\alpha})^2 dx \leqslant \lambda^{-1} (1+\alpha)^2 \int_{B(2)} w^{2\alpha} \sum_{i,j} D_i w a_{ij} D_j w\, dx.$$

(2.5) Lemma. *Theorem* (2.1) *holds if* $u = w^{k+1}$ *is subharmonic and* w *has weak derivatives which are square integrable in* D.

Proof. By successive applications of (2.4), we show that u has square integrable weak derivatives in the interior of D, and we may apply (2.1) in the interior of D.

Section 3: Weak differentiability and boundedness of ω

In this section we derive a preliminary regularity result which shows that ω is sufficiently differentiable for (1.10) to be valid and for (2.5) to apply to H. In this section and the following section, we shall get estimates in $B(x, r)$, the ball of radius r about a point x. By considering the expanded function $\tilde{\omega}(y) = (ry + x)$, which also solves the differential equation, we see that it will always be sufficient to prove these estimates for $x = 0$ and $r = 1$. The constants in this section depend on the constant K of (1.3)$'$. This result is similar to Morrey [5], Theorem 1.11.1, with a slight weakening of ellipticity conditions.

(3.1) LEMMA. *Let ω be a solution of (1.1) in $B(x, 2r)$, and assume that (1.2) and (1.3) are valid for this equation. If*

$$\int_{B(x,2r)} |\omega|^{2(p+1)} dx = \int_{B(x,2r)} Q^{p+1} dx < \infty,$$

then $(Q+C)^{p/2}\omega$ has weak derivatives which are square integrable in $B(x, r)$ and there exists a constant k_1 such that

$$\int_{B(x,r)} |\nabla((Q+C)^{p/2}\omega)|^2 dx \leqslant (p+1)^2 \int_{B(x,r)} (Q+C)^p |\nabla\omega|^2 dx$$

$$\leqslant k_1 r^{-2} \int_{B(x,2r)} Q^2 (Q+C)^p dx.$$

Proof. Assume $B(x, r) = B(1)$. We use a difference quotient method. From the theory of elliptic complexes, we may write $\omega = A(0)\varphi$ for $A(-1)^*\varphi = 0$, where φ has weak derivatives in $L^{2(p+1)}(B(3/2))$ with norm bounded by some constant times the norm of ω in $L^{2(p+1)}$. Let $\Delta_{h,i} u = (u(x + he_i) - u(x))h^{-1}$ be a difference quotient in the ith direction. Also recall that $A(0)$ has constant coefficients. Then if ψ is a smooth function with support in $B(3/2)$ which is 1 on $B(1)$

$$0 = -\int_{B(3/2)} \psi^2 (\Delta_{h,i}\varphi, \Delta_{h,i}(A(0)^*(\varrho(Q)\omega)))\, dx$$

$$= \int_{B(3/2)} (A(0)\psi^2 \Delta_{h,i}\varphi, \Delta_{h,i}(\varrho(Q)\omega))\, dx$$

$$\geqslant \int_{B(3/2)} \psi^2 (\Delta_{h,i}\omega, \Delta_{h,i}(\varrho(Q)\omega))\, dx - 2\max|\nabla\psi| \int_{B(3/2)} |\Delta_{h,i}\varphi|\,|\psi|\,|\Delta_{h,i}(\varrho(Q)\omega)|\, dx.$$

Let $\omega_\lambda = \omega + h\lambda\Delta_{h,i}\omega$ and $Q_\lambda = |\omega_\lambda|^2$. Then

$$(\Delta_{h,i}\omega, \Delta_{h,i}(\varrho(Q)\omega))$$

$$= \int_0^1 \varrho(Q_\lambda)|\Delta_{h,i}\omega|^2 + \varrho'(Q_\lambda)(\omega_\lambda, \Delta_{h,i}\omega)^2 d\lambda \geqslant K^{-1}\int_0^1 (Q_\lambda + C)^p d\lambda\, |\Delta_{h,i}\omega|^2.$$

Also

$$|\Delta_{h,i}(\varrho(Q)\,\omega)| = \left|\int_0^1 (\varrho(Q_\lambda)\,\Delta_{h,i}\,\omega + 2\varrho^2(Q_\lambda)(\omega_\lambda,\,\Delta_{h,i}\,\omega)\,\omega_\lambda)\,d\lambda\right| \leq 2K\int_0^1 (Q_\lambda + C)^p d\lambda\,|\Delta_{h,i}\,\omega|.$$

From these three inequalities we have

$$\int_{B(3/2)} \psi^2 \int_0^1 (Q_\lambda + C)^p d\lambda\,|\Delta_{h,i}\,\omega|^2 dx \leq 2K^2 \max|\nabla\psi| \int_{B(3/2)} \psi\,|\Delta_{h,i}\,\varphi| \int_0^1 (Q_\lambda + C)^p d\lambda\,|\Delta_{h,i}\,\omega|\,dx.$$

Using Hölder's inequality, we find

$$\int_{B(3/2)} \psi^2 \int_0^1 (Q_\lambda + C)^p d\lambda\,|\Delta_{h,i}\,\omega|^2 dx \leq 4K^4 \max|\nabla\psi|^2 \int_{B(3/2)} |\Delta_{h,i}\,\varphi|^2 \int_0^1 (Q_\lambda + C)^p d\lambda\,dx.$$

If we let λ go to zero and sum over the index i, we get

$$\int_{B(3/2)} \psi^2 (Q + C)^p |\nabla\omega|^2 dx \leq 4K^2 \max|\nabla\psi|^2 \int_{B(3/2)} Q^2 (Q + C)^p dx.$$

For the purposes of the following theorem, we assume that H has been chosen with $H(0) = C$, so in particular, H is nonnegative. Note that if $H(Q)$ is bounded, then Q is also bounded, from the definition of H (1.7).

(3.2) THEOREM. *Let ω be a solution of* (1.1) *in* $B(x, 4r)$ *and assume* (1.2) *and* (1.3) *are valid. If $\int_{B(x,4r)} Q^{p+1} dx < \infty$, then H has weak derivatives which lie in L^2 in $B(x, 2r)$, H is bounded in the interior of $B(x, 2r)$, and there exists k_2 with*

$$\max_{y \in B(x,r)} H(Q(y)) \leq k_2 r^{-n} \int_{B(x,4r)} (Q + C)^{p+1} dx.$$

Proof. Again we assume that $B(x, r) = B(1)$. By (3.1), $(Q + C)^{p/2}\omega$ has weak derivatives in $L^2(B(2))$. Checking the growth conditions on H and H', this implies that $H^{1/2}$ will have weak derivatives in $L^2(B(2))$. In addition, the hypotheses of (1.10) are satisfied, and H is a subsolution. By Lemma (2.5) we may apply (2.1) to get for $1 < \beta < n/(n-2)$

$$\max_{y \in B(1)} H(Q) \leq c(\beta)\left(\int_{B(2)} H(Q)^\beta dx\right)^{1/\beta},$$

which from the Sobolev inequalities is bounded by some constant times the term

$$\int_{B(2)} ((\nabla H(Q)^{1/2})^2 + H(Q))\,dx.$$

$$|\nabla H(Q)^{1/2}| = |H(Q)^{-1/2} H'(Q)(\omega,\,\nabla\omega)| \leq 2(p+1)^{1/2} K^{3/2} (Q + C)^{p/2} |\nabla\omega|,$$

where we have used (1.7) and (1.3)'. However, in (3.1) we have an estimate for the integral of this term squared over $B(2)$ in terms of $\int_{B(4)} (Q+C)^{p+1}dx$ as required.

As the last part of this section we derive a second estimate for $(Q+C)^{p/2}\nabla\omega$.

(3.3) LEMMA. *Let ω satisfy (1.1) in $B(x, 2r)$ and assume that (1.2) and (1.3) are valid. If $M = \max_{y\in B(x,2r)} Q(y)$, then there exists a constant k_3 such that*

$$\int_{B(x,r)} (Q+C)^p|\nabla\omega|^2 dx \leqslant k_3 r^{-2}\int_{B(x,2r)} H(M)-H(Q)\,dx.$$

Proof. Let $B(x, r) = B(1)$.

$$(3.4)\qquad H(M)-H(Q)=\int_0^1 H'(\lambda M+(1-\lambda)Q)\,d\lambda(M-Q)$$
$$\geqslant K^{-1}\int_0^1(\lambda M+(1-\lambda)Q+C)^p d\lambda(M-Q)\geqslant((p+1)(K))^{-1}(M+C)^p(M-Q).$$

Choose ψ to be a positive smooth function which is 1 on $B(1)$ and has support in $B(2)$. From (1.10) we have

$$\int_{B(2)}\psi^2(Q+C)^p|\nabla\omega|^2dx\leqslant K\int_{B(2)}\psi^2 L_\omega H dx$$
$$=K\int_{B(2)}\psi^2[\Delta(H-H_0)-2B^*_\omega(\sqrt{\varrho}B_\omega(\sqrt{\varrho}-\sqrt{\varrho_0}))]\,dx.$$
$$=K\int_{B(2)}[\Delta\psi^2(H-H_0)-2B^*_\omega(\sqrt{\varrho}B_\omega\psi^2)(\sqrt{\varrho}-\sqrt{\varrho_0})]\,dx,$$

where $H=H(Q)$, $\varrho=\varrho(Q)$, $H_0=H(M)$ and $\varrho_0=\varrho(M)$. We estimate

$$K\int_{B(2)}\Delta\psi^2(H-H_0)\,dx\leqslant K_1(\psi)\int_{B(2)}(H-H_0)\,dx,$$

where K_1 depends on K and ψ. To estimate the second part of the last integral, differentiate out

$$B^*_\omega(\sqrt{\varrho}B_\omega\psi^2)=\sum_i D_i((A_i(1)\,\omega, A_k(1)\,\omega)\sqrt{\varrho}D_k\psi^2)$$
$$\leqslant|\nabla^2(\psi^2)|Q\sqrt{\varrho}+2|\nabla\psi^2||\nabla\omega|\sqrt{Q}(\sqrt{\varrho}+\sqrt{\varrho}^{-1}Q\varrho')$$
$$\leqslant K_2(\psi)\,Q(Q+C)^{p/2}+K_3(\psi)\sqrt{Q}|\nabla\omega|(Q+C)^{p/2}\psi.$$

We use (1.3)' to get the last inequality. Also $\sqrt{\varrho}-\sqrt{\varrho_0}\leqslant\sqrt{K}(M+C)^{p/2}$, or use the mean value theorem to get

$$Q(\sqrt{\varrho}-\sqrt{\varrho_0})=Q\varrho^{-1/2}(\tilde{Q})'(\tilde{Q})(M-Q)\leqslant(M-Q)\sqrt{K^3}(M+C)^{p/2}.$$

Finally we can estimate

$$-\int_{B(2)} B^*_\omega(\sqrt{\varrho}B_\omega\psi^2)(\sqrt{\varrho}-\sqrt{\varrho_0})\,dx$$

$$\leqslant K_2(\psi)\sqrt{K^3}\int_{B(2)}(M+C)^p(M-Q)\,dx+K_3(\psi)K\int_{B(2)}|\nabla\omega|(Q+C)^{p/2}\sqrt{M-Q}\,dx.$$

Hölder's inequality on the last term will complete the estimate.

Section 4: A perturbation theorem

In this section we prove that if ω is sufficiently close to a constant ω_0, then ω will be Hölder continuous in a smaller domain. The proof is derived from similar proofs of theorems of Almgren [1] and Morrey [6]. The original estimate is improved as the domain is shrunk down to a point. There is an additional difficulty which occurs when $C=0$ in (1.3). In this case, the system fails to be elliptic at points where $Q=0$. The trick is to obtain sufficiently uniform estimates to cover this case. In general the proof would be very much simplified if we were to assume $C\neq 0$.

(4.1) THEOREM. *Let ω be a bounded solution of* (1.1) *in* $B(2)$ *and assume that* (1.2), (1.3) *and* (1.4) *are valid. Let* $|\omega|^2=Q\leqslant M$ *and assume that there exists a constant vector* ω_0 *such that for* $|\omega_0|^2=Q_0$, $Q_0+C\geqslant\eta(M+C)$. *Then there exist positive constants* ε, *and* k_4 *which depend on* K, p, α *and* $0<\eta<1$ *and not on* M *or* C *such that if*

$$\int_{B(2)}|\omega-\omega_0|^2dx\leqslant\varepsilon(M+C),$$

then ω is Hölder continuous in the ball of radius 1 *with*

(a) $\max\limits_{x\in B(1),\,y\in B(1)}|\omega(x)-\omega(y)|\leqslant k_4|x-y|^{1/2}(M+C)^{1/2}$

(b) $\max\limits_{x,y\in B(1)}|\omega(x)-\omega_0|\leqslant k_4(M+C)^{1/2}$

(c) $(Q+C)\geqslant\eta/2(M+C)$ for $x\in B(1)$.

The proof is via a series of lemmas. Throughout we shall assume that the hypotheses of Theorem (4.1) are satisfied and the constants K, C, p and α are the constants appearing in (1.3)′ and (1.4). The constants in this section will depend on these plus dimensionality constants such as norm estimates from the Sobolev inequalities and the volume of the unit ball in n space. Throughout M will denote the maximum of Q and we assume $Q_0=|\omega_0|^2\leqslant M$. Also we assume the constant α of (1.4) satisfies $\alpha\leqslant 2/(n-2)$. Condition (1.4) is used only to prove inequality (4.4).

The perturbation technique uses estimates on the linearized equations at a constant ω_0. We approximate the non-linear system $A(0)^*(\varrho(Q)\omega)=0$ by the linearization at ω_0, which is divided by the constant $\varrho(Q_0)$ to make the estimates uniform.

$$A(0)^{*\prime}(\tilde{\omega}) = A(0)^*(\tilde{\omega}+2\varrho'/\varrho(Q_0)(\tilde{\omega},\omega_0)\omega_0).$$

We call this linear operator with constant coefficients $A(0)^{*\prime}$ because it is just an adjoint taken with regard to the inner product $(\omega_1,\omega_2)'=(\omega_1,\omega_2)\leqslant 2\varrho'/\varrho(Q_0)(\omega_0,\omega_1)(\omega_0,\omega_2)$, which will always be positive definite for $Q_0 \neq 0$. We would like to find a solution to the system $A(0)^{*\prime}\tilde{\omega}=0$, $A(1)\tilde{\omega}=0$ with the tangential boundary values of $\omega-\omega_0$. However, it is an unnecessary complication to deal with boundary value problems for elliptic complexes and we go to the related system. Therefore we observe that we may write $\omega=A(0)\varphi$, $A(-1)^*\varphi=0$ and $\omega_0=A(0)\varphi_0$, $A(-1)^*\varphi_0=0$. Although we may not be able to do this globally, we certainly can find φ and φ_0 so that this is valid in the unit ball. Then φ is a solution of the elliptic system

$$A(0)^*(\varrho(Q)/\varrho(Q_0)A(0)\varphi+A(-1)A(-1)^*\varphi=0.$$

We look for solutions of the linearized system

$$A(0)^{*\prime}A(0)\tilde{\varphi}+A(-1)A(-1)^*\tilde{\varphi}=0$$

with the Dirichlet boundary values of $\varphi-\varphi_0$, let $\tilde{\omega}=A(0)\tilde{\varphi}$ and estimate $\tilde{\omega}-(\omega-\omega_0)$.

(4.2) LEMMA. *Let $\tilde{\omega}=A(0)\tilde{\varphi}$, where $\tilde{\varphi}$ is a solution of the linearized system*

$$(A(0)^{*\prime}A(0)+A(-1)A(-1)^*)\tilde{\varphi}=0$$

in $B(1)$, such that $\tilde{\varphi}$ has the Dirichlet boundary $\tilde{\omega}$ values of $\varphi\cdot\varphi_0$ on the boundary of $B(1)$. Then $\tilde{\omega}$ is smooth in the interior of $B(1)$ and there exists a constant c_1 such that

$$\left(\int_{B(1)}|\tilde{\omega}|^2dx\right)^{1/2},\quad \max_{x\in B(1/2)}|\tilde{\omega}|\quad\text{and}\quad \max_{x\in B(1/2)}|\nabla\tilde{\omega}|$$

are all bounded by

$$c_1\left(\int_{B(1)}|\omega-\omega_0|^2dx\right)^{1/2}.$$

Proof. Since the linearized system is elliptic with constant coefficients, the existence, uniqueness, and smoothness of $\tilde{\varphi}$ follow from the linear theory. From (1.3)$'$, the primed inner product satisfies $K^{-1}(v,v)\leqslant(v,v)'\leqslant 2K^2(v,v)$ and the ellipticity constants and bounds on the system for $\tilde{\omega}$ are uniform. So bounds on the derivatives of $\tilde{\varphi}$ and $\tilde{\omega}$ in the interior depend only the L_2(norm of $\tilde{\omega}$ in $B(1)$. Because $\tilde{\varphi}$ minimizes

$$\int_{B(1)} (A(0)\tilde{\varphi}, A(0)\tilde{\varphi})' + A(-1)^*\tilde{\omega}, A(-1)^*\tilde{\omega})\,dx$$

among all functions with the same boundary values, we have

$$\begin{aligned}\int_{B(1)} (\tilde{\omega}, \tilde{\omega})' dx &\leqslant \int_{B(1)} (A(0)\tilde{\varphi}, A(0)\tilde{\varphi})' + (A(-1)\tilde{\varphi}, A(-1)\tilde{\varphi})\,dx \\ &\leqslant \int_{B(1)} (A(0)(\varphi-\varphi_0), A(0)(\varphi-\varphi_0))' + |A(-1)(\varphi-\varphi_0)|^2 dx \\ &= \int_{B(1)} (\omega-\omega_0, \omega-\omega_0)' dx \leqslant 2K^2 \int_{B(1)} |\omega-\omega_0|^2 dx.\end{aligned}$$

Our next goal is to derive an estimate on the L^2 norm of $u=\tilde{\omega}-(\omega-\omega_0)\leqslant A(0)(\tilde{\varphi}-(\varphi-\varphi_0))$. Let $G(\omega, \omega_0)=\varrho(Q)/\varrho(Q_0)\omega-\omega-(\varrho'/\varrho)(Q_0)(\omega_0, \omega-\omega_0)\omega_0$. Note that $\varrho(Q_0)G(\omega, \omega_0)$ is simply $\varrho(Q)\omega$ minus the constant and linear terms in its Taylor series about ω_0.

(4.3) LEMMA. $$\int_{B(1)} u^2 dx \leqslant K^2 \int_{B(1)} |G(\omega, \omega_0)|^2 dx.$$

Proof. $A(0)^{*\prime}(\omega-\omega_0)=A(0)^*G(\omega, \omega_0)$ and $A(0)^{*\prime}u+A(-1)A(-1)^*\tilde{\varphi}=-A(0)^{*\prime}(\omega-\omega_0)+(A(0)^{*\prime}A(0)+A(-1)A(-1)^*)\tilde{\varphi}=A(0)^*G(\omega, \omega_0)$. Integrating this equation with the test function $\tilde{\varphi}-(\varphi-\varphi_0)$, which is zero on the boundary, we get

$$\int_{B(1)} (u, u)' + (A(-1)^*\tilde{\varphi}, A(-1)\tilde{\varphi})\,dx = \int_{B(1)} (u, G(\omega, \omega_0))\,dx.$$

Use the estimate $(u, u)' \geqslant K(u, u)$ and Hölder's inequality to get the result.

(4.4) LEMMA. *Let $u=\tilde{\omega}-(\omega-\omega_0)$ as defined above, then there exists a constant c_2 such that for $Q\leqslant M$, $Q_0\leqslant M$ in $B(2)$*

$$\int_{B(1)} |u|^2 dx \leqslant c_2 \frac{(M+C)^{2p}}{(Q_0+C)^{2p+\alpha}} \left(\int_{B(2)} |\omega-\omega_0|^2 dx\right)^{1+\alpha}.$$

Proof. From the previous lemma, it is sufficient to make estimates on $\int_{B(1)} |G(\omega,\omega_0)|^2 dx$. But the uniformity conditions (1.4) imply that $\varrho(Q)\omega$ has Hölder continuous derivatives with a Hölder continuous norm on the order of $(Q+C)^{p-\alpha/2}$, so we get from the definition of $\varrho(Q_0)G(\omega, \omega_0)$,

$$\varrho(Q_0)\,|G(\omega, \omega_0)| \leqslant 2^{p+2}K(Q_0+Q+C)^{p-\alpha/2}|\omega-\omega_0|^{1+\alpha},$$

or using (1.3) again

$$|G(\omega, \omega_0)| \leqslant 2^{p+2}K^2(Q_0+Q+C)^{p-\alpha/2}(Q_0+C)^p|\omega-\omega_0|^{1+\alpha}. \tag{4.5}$$

A straightforward pointwise inequality

$$(Q+Q_0+C)^{p/2}|\omega-\omega_0| \leqslant c(p)\,|(Q+C)^{p/2}\omega-(Q_0+C)^{k/2}\omega_0|$$

for $c(p)$ a combinatorial constant can be derived by approximating $(Q+C)^{p/2}\omega$ by its derivative at ω_0 with a remainder term. Let $F=(Q+C)^{p/2}\omega$, $F_0=F(Q_0)$. In final form

$$|G(\omega,\omega_0)| \leqslant c'(p)K^2(Q_0+Q+C)^q(Q_0+C)^p|F-F_0|^{1+\alpha}. \tag{4.6}$$

where $c'(p)=c(p)2^{2+p}$ and $q=(p-(p+1)\alpha)/2$. From the Sobolev imbedding theorems (taking $\alpha \pm n/(n-2)$), we have

$$\left(\int_{B(1)}|F-F_0|^{2(1+\alpha)}dx\right)^{1/1+\alpha} \leqslant c'(\alpha)\int_{B(1)}(|\nabla F|^2+|F-F_0|^2)\,dx \tag{4.7}$$

$$\leqslant c'(\alpha)(p+1)^2\int_{B(1)}(Q+C)^p|\nabla\omega|^2+(Q+C)^p|\omega-\omega_0|^2dx.$$

However, if we integrate the equation

$$\int_{B(2)}\psi^2(\omega-\omega_0, A(0)A(0)^*\varrho(Q)\,\omega)\,dx=0$$

by parts and use (1.1), (1.2) and (1.3)', assuming ψ has support in $B(2)$ we get an inequality similar to (3.1):

$$\int_{B(2)}\psi^2|\nabla\omega|^2(Q+C)^pdx \leqslant k_1(\psi)\int_{B(2)}(Q+C)^p|\omega-\omega_0|^2dx. \tag{4.8}$$

We replace Q and Q_0 by M where necessary and put (4.6), (4.7), (4.8) together to get the lemma.

(4.9) LEMMA. *Given a constant vector ω_0, $\omega_0^2=Q_0$, where $Q_0\leqslant M$, $Q\leqslant M$ and $(Q_0+C)=\eta(M+C)$ and $\int_{B(1)}|\omega-\omega_0|^2dx=\varepsilon(M+C)$. Then for $r\leqslant\frac{1}{4}$ there exists a constant vector ω, and a constant c_3 depending only on K, p, n such that*

(a) $Q_1=\omega_1^2\leqslant M$

(b) $|\omega_1-\omega_0|\leqslant c_3\left(\int_{B(1)}|\omega-\omega_0|^2dx\right)^{1/2}$

(c) $\int_{B(r)}|\omega-\omega_1|^2dx\leqslant c_3(r^{n+2}+\eta^{-2p}\varepsilon^{\alpha})\int_{B(2)}|\omega-\omega_0|^2dx.$

Proof. Notice that if (b) and (c) are true for ω_1 and $\omega_1^2>M$, they are also true for $\tilde\omega_1=\omega_1\sqrt{M}/|\omega_1|$, so we need only demonstrate (b) and (c). Let $\tilde\omega$ be the solution of the linearized equation (4.2) in $B(1/2)$. By expansion, we can assume that the balls in (4.2) have been replaced by balls of half the radii. Let

$$\omega_1 - \omega_0 = \left(\int_{B(1/2)} dx\right)^{-1} \int_{B(1/2)} \tilde{\omega}\, dx.$$

Then

$$|\omega_1 - \omega_0| \leqslant c_1 \left(\int_{B(1/2)} |\omega - \omega_0|^2 dx\right)^{1/2},$$

so (4.9) (b) is true with $c_3 \geqslant c_1$.

$$\begin{aligned}\int_{B(r)} |\omega - \omega_1|^2 dx &\leqslant 2\left(\int_{B(r)} |\tilde{\omega} - (\omega_1 - \omega_0)|^2 dx + \int_{B(1)} u^2 dx\right)\\ &\leqslant 2r^2 \left(\int_{B(r)} dx\right) \max_{x \in B(1/4)} |\nabla \tilde{\omega}|^2 + 2\int_{B(1/2)} u^2 dx\\ &\leqslant 2r^{n+2} \int_{B(1)} dx c_1^2 \int_{B(1/2)} |\omega - \omega_0|^2 dx + 2c_2 \frac{(M+C)^{2p}}{(Q_0 + C)^{2p+\alpha}} \left(\int_{B(1)} |\omega - \omega_0|^2 dx\right)^{1+\alpha},\end{aligned}$$

where (4.2) and (4.8) have been used. Choose

$$c_3 \geqslant \max\left(2\int_{B(1)} dx c_1^2, 2c_2, c_1\right).$$

By a successive approximation method we shall show that if $\varepsilon^\alpha \eta^{-2p}$ is small, this estimate will prove as we shrink, the size of the ball $B(r)$ down. We pick a fixed $r \leqslant 1/4$ which we shall determine later. Let $\omega^i = \omega(r^i x)$. Then ω^i solves (1.1) in $B(2)$, and we choose recursively the constant approximations. Let $\omega^0 = \omega$ and $\omega_0^0 = \omega_0$, and then $\omega_0^{i+1} = \omega_1^i$ from (4.9), or ω_0^{i+1} is the constant vector which satisfies by (4.9)

(4.10) (a) $|\omega_0^{i+1}|^2 \leqslant M$

(b) $|\omega_0^i - \omega_0^{i+1}| \leqslant c_3 \left(\int_{B(2)} |\omega^i - \omega_0^i|^2 dx\right)^{1/2}$

(c) $$\begin{aligned}\int_{B(1)} |\omega^{i+1} - \omega_0^{i+1}|^2 dx &= (r)^{-n} \int_{B(r)} |\omega^i - \omega_1^i|^2 dx\\ &\leqslant c_3 (r^2 + r^{-n} \varepsilon^\alpha \eta^{-2p}) \int_{B(1)} |\omega^i - \omega_0^i|^2 dx.\end{aligned}$$

(4.11) LEMMA. *Assume ω satisfies (1.1) in $B(1)$ and $Q \leqslant M$, $Q_0 \leqslant M$. There exist fixed constants $r < 1/4$ and c_3, such that if*

$$\int_{B(1)} |\omega - \omega_0|^2 dx \leqslant \varepsilon(M + C), \qquad (Q_0 + C) \geqslant \eta(M + C)$$

for some $8^2 c_3^2 \varepsilon \leqslant \eta$ and $\varepsilon^\alpha (\eta/2)^{-2p} < r^{n+2}$, then there exist constant vectors ω_0^i such that

(a) $|\omega_0^i|^2 + C \geqslant \eta/2(M+C)$

(b) $|\omega_0^{i-1} - \omega_0^i|^2 \leqslant c_3^2 r^{i-1}\left(\int_{B(2)} |\omega - \omega_0|^2\right)$

(c) $\int_{B(r^i)} |\omega - \omega_0^i|^2 dx \leqslant (r^i)^{n+1}\left(\int_{B(2)} |\omega - \omega_0|^2\right).$

Proof. We defined ω_0^i in the iteration (4.10) just described, choosing $rc_3 = 1/2$, so $c_3 r^{-n}\varepsilon^\alpha(\eta/2)^{-2p} < 1/2r$. Then if the hypothesis of the lemma are true for $i < j$, we can use (4.10) to get (4.11–b, c) for $i = j$. Then from b for $j \leqslant i$ we get $|\omega_0 - \omega_0^i| \leqslant \sum_j c_3\sqrt{\varepsilon}(\sqrt{r})^{j-1}\sqrt{M+C}$, and we choose the constants so we would still have $|\omega_0|^2 + C \geqslant |\omega_0|^2 + C - \eta/2(M+C) \geqslant \eta/2(M+C)$.

(4.12) COROLLARY. *Assume that ω satisfies* (1.1) *in* $B(2)$, $Q \leqslant M$, *and* $Q_0 + C \geqslant \eta(M+C) > 0$. *There exist fixed constants* $\varepsilon > 0$, $1/4 \geqslant r > 0$ *and* c_3 *which are independent of* M *and* C *such that if*

$$\int_{B(2)} |\omega - \omega_0|^2 dx < \varepsilon(M+C)$$

for a constant vector $|\omega_0|^2 = Q_0$, *then for* $x \in B(1)$ *there exist constant vectors* ω_x^i *with the following properties:*

(a) $\int_{B(x,r^i)} |\omega - \omega_x^i|^2 dx \leqslant r^{i(n+1)} \int_{B(2)} |\omega - \omega_0|^2 dx,$

(b) $|\omega_x^i - \omega_0| \leqslant 2c_3\left(\int_{B(2)} |\omega - \omega_0|^2 dx\right)^{1/2},$

(c) $|\omega_x^i - \omega_x^j| \leqslant 2c_3(\sqrt{r})^{\min(i,j)}\left(\int_{B(2)} |\omega - \omega_0|^2 dx\right)^{1/2},$

(d) $(|\omega_x^i|^2 + C) \geqslant \eta/2(M+C).$

Proof. By translation, we may assume $x = 0$, and since $B(x,1) \subset B(2)$, $\int_{B(x,1)} |\omega - \omega_0|^2 dx \leqslant \varepsilon(M+C)$. We apply the interaction process just described, letting $\omega_x^i = \omega_0^i$. We have just shown that for ε and r properly chosen

$$\int_{B(x,r^i)} |\omega - \omega_0^i|^2 dx = r^{in} \int_{B(1)} |\omega^i - \omega_0^i|^2 dx \leqslant r^{i(n+1)} \int_{B(1)} |\omega - \omega_0|^2 dx$$

$$|\omega_0^i - \omega_0^j| \leqslant \sum_{k=j}^{i-1} |\omega_0^{k+1} - \omega_0^k| \leqslant c_3 \sum_{k=j}^{i-1} \left(\int_{B(1)} |\omega^k - \omega_0^k|^2 dx\right)^{1/2}$$

$$\leqslant c_3\left(\sum_{k=j}^{i-1} (\sqrt{r})^k\right)\left(\int_{B(1)} |\omega - \omega_0|^2 dx\right)^{1/2}.$$

(4.12) (b–c) follow from the fact that $r \leqslant 1/4$, and (4.12-d) follows from (4.11-a).

We can now proceed to the proof of (4.1). By (4.12-c), ω_x^i is a Cauchy sequence for fixed $x \in B(1)$, and

$$\omega(x) = \lim_{i\to\infty} \omega_x^i = \lim_{i\to\infty} \left(\int_{B(x,r^i)} dy\right)^{-1} \int_{B(x,r^i)} \omega(y)\,dy$$

holds almost everywhere. In fact,

$$(4.13) \qquad |\omega(x) - \omega_x^i| = \lim_{j\to\infty} |\omega_x^j - \omega_x^i| \leqslant 2c_3(\sqrt{r})^i \left(\int_{B(2)} |\omega - \omega_0|^2 dx\right)^{1/2}$$

in particular, for $i=0$, this verifies (4.1-b) for $k_4 = 2c_3\sqrt{\varepsilon}$. Similarly for (1.4-c):

$$Q + C = |\omega(x)|^2 + C = \lim_{i\to\infty} |\omega_x^i|^2 + C \geqslant \eta/2(M+C).$$

For any points $x, y \in B(1)$ with $r^{i+1} \leqslant |x-y| = \varrho \leqslant r^i$:

$$(4.14) \qquad \begin{aligned} |\omega(x) - \omega(y)| &\leqslant |\omega_x^i - \omega_y^i| + |\omega(x) - \omega_x^i| + |\omega(y) - \omega_y^i| \\ &\leqslant |\omega_x^i - \omega_y^i| + 4c_3\sqrt{\varrho/r}\left(\int_{B(2)} |\omega - \omega_0|^2 dx\right)^{1/2} \end{aligned}$$

where we used (4.13) and the fact that $r^i \leqslant \varrho/r$. Let $w = (x+y)2^{-1}$ be the point half-way between x and y.

$$(4.15) \qquad \begin{aligned} |\omega_x^i - \omega_y^i|^2 &= \left(\int_{B(\varrho/2)} dz\right)^{-1} \int_{B(w,\varrho/2)} |\omega_x^i - \omega_y^i|^2 dz \\ &\leqslant 2\left(\int_{B(1)} dz\right)^{-1} (\varrho/2)^{-n} \left[\int_{B(w,\varrho/2)} |\omega_x^i - \omega(z)|^2 dz + \int_{B(w,\varrho/2)} |\omega_y^i - \omega(z)|^2 dz)\right] \\ &\leqslant c_4 \varrho^{-n} \left[\int_{B(x,r^i)} |\omega_x^i - \omega(z)|^2 dz + \int_{B(y,r^i)} |\omega_y^i - \omega(z)|^2 dz\right] \\ &\leqslant 2c_4 \varrho^{-n} (r^i)^{n+1} \int_{B(2)} |\omega - \omega_0|^2 dx. \end{aligned}$$

We chose $c_4 = 2^{n+1}(\int_{B(1)} dz)^{-1}$, and used the fact that $B(w, \varrho/2) \subset B(x, r^i) \cap B(y, r^i)$ with (4.12-c). From the choice of i, $r^{i(n+1)} \leqslant p^{n+1} r^{-(m+1)}$. Then (4.14) together with (4.15) gives (4.1-a) for the proper choice of k_4.

Section 5: Proof of the regularity theorem

The proof of the main theorem of this paper is based on the fact that, once given a ball of radius r on which $|\omega|^2 = Q$ is bounded, as we shrink the radius of the ball down, either the maximum decreases by a small factor, or ω is sufficiently close to a constant vector for the pertrubation theorem (4.1) to hold. The final result will be that ω is Hölder continuous. At points where $Q+C \neq 0$, one can see by the linear regularity theory that the Hölder exponent should be arbitrary, and our perturbation method gives directly a 1/2 estimate. However, at points $Q+C=0$, we can only show that the solution lies in some Hölder space (one might guess the exponent to be $(2p+1)^{-1}$) and the linear regularity theory cannot be used to carry the argument further, since the system fails to be elliptic at this point. Since we shall use the uniform estimates in Section 6, they are given in (5.4).

(5.1) PROPOSITION. *Let Q be bounded in $B(x, r)$, $M(\varrho)=\max_{y\in B(x,\rho)} Q(y)$. Then there exists a constant c_5 such that for $4\varrho \leq r$ either:*

(a) $M(\varrho)+C \leq (1-\lambda)(M(4\varrho)+C)$

or:

(b) *there exists a constant vector ω_0 such that*

$$\varrho^{-1}(Q_0+C)^p \int_{B(x,\varrho)} |\omega-\omega_0|^2 \leq \lambda c_5 (M(4\varrho)+C)^{p+1}$$

with:

(c) $Q_0+C \geq (1-c_5\sqrt{\lambda})^{2/(p+1)}(M(4\varrho)+C)$

Proof. By expansion and translation we may assume $x=0$ and $\varrho=1$. $M=M(4)$. If (5.1-a) is false, then $M-M(1) \leq \lambda(M+C)$. $\max_{x\in B(\rho)} H(Q)=H(M(\varrho))$ and from (1.7) and (1.3):

$$H(M)-H(M(1)) \leq \max_{M(1)\leq Q\leq M} H'(Q)(M-M(1)) \leq K(M+C)^p(M-M(1)) \leq K\lambda(M+C)^{p+1}.$$

First apply (3.4) and then the strong maximum principle (2.3) for H to get:

$$\int_{B(1)} (Q+C)^p |\nabla\omega|^2 dx \leq k_3 \int_{B(x,2)} H(M(2))-H(Q)\,dx \leq k_3 \int_{B(x,3)} H(M)-H(Q)\,dx \tag{5.2}$$
$$\leq k_3 c'(1)(H(M)-H(M(1))) \leq k_3 c'(1) K\lambda (M+C)^{p+1}.$$

As in (3.1), this also gives an estimate on $\int_{B(1)}|\nabla v|^2 dx$ if we set $v=(Q+C)^{p/2}\omega$. Let $v_0=\int_{B(1)} v(x)\,dx(\int_{B(1)} dx)^{-1}$ and choose $(|\omega_0|^2+C)^{p/2}\omega_0=v_0$. We have the pointwise inequality

$$(v-v_0, \omega-\omega_0)=((Q+C)^{p/2}\omega-(Q_0+C)^{p/2}\omega_0, \omega-\omega_0)$$
$$\geq \int_0^1 (|t\omega+(1-t)\omega_0|^2+C)^{p/2}|\omega-\omega_0|^2 dt \geq 2^{-p/2-1}(Q_0+C)^{p/2}|\omega-\omega_0|^2.$$

We then get

$$(Q_0+C)^p \int_{B(1)} |\omega-\omega_0|^2 dx \leq 2^{p+2} \int_{B(1)} |v-v_0|^2 dx \tag{5.3}$$
$$\leq c_5' \int_{B(1)} |\nabla v|^2 dx \leq \lambda c_5'(p+1)^2 k_5 c'(1) K(M+C)^{p+1}$$

from (5.2), which shows (5.1-b).

(5.1-c) will in general have content only when λ is very small. First we note that we can prove several more pointwise inequalities very directly:

$$(Q+C)^{(p+1)/2}-(Q_0+C)^{(p+1)/2}\leqslant |v-v_0|$$

$$(M+C)^{(p+1)/2}-(Q+C)^{(p+1)/2}\leqslant K(p+1)\,\frac{(H(M)-H(Q))}{(M+C)^{(p+1)/2}}.$$

Adding, integrating over $B(1)$, and finally applying (5.2) and (5.3) will give:

$$(M+C)^{(p+1)/2}-(Q_0+C)^{(p+1)/2}\left(\int_{B(1)} dx\right)\leqslant (\sqrt{\lambda}\,c_5''+\lambda c'(1)\,K)(M+C)^{(p+1)/2})$$

for proper choice of c_5'' from c_5'. If $\lambda\leqslant 1$, this gives (5.1-c).

The main regularity theorem now follows from this. If x lies in the interior of the domain, then by (3.2) we may assume that there exists a ball $B(x,r)$ in the interior of D on which $H(Q)$ and therefore Q are bounded, $Q\leqslant M$. The bound M will depend on the norm $\int_D(Q+C)^{p+1}dx$, the size of the ball $B(x,r)$ and the distance of the ball to the boundary of D. Assuming this bound, the following theorem completes the proof:

(5.4) THEOREM. *Let ω solve* (1.1) *in $B(x,r)$, assume* (1.2), (1.3) *and* (1.4) *are true and $Q\leqslant M$ in $B(x,r)$. Then there exists a smaller ball $B(x,r/2)$ and fixed constants γ and k depending only on K and p, such that*

$$|\omega(y)-\omega(z)|\leqslant k|y-z|^{\gamma}r^{-\gamma}(M+C)^{1/2}$$

for all $y,z\in B(x,r/2)$.

Proof. As usual, we may choose $x=0=y$, $r=2$, and by translation assume that Q is bounded on $B(1)$ by M. Using the constant c_5 of (5.1), we select λ so small that

$$(1-c_5\sqrt{\lambda})\geqslant 2^{-(p+1)/2}=\eta^{(p+1)/2}$$

and

$$\lambda c_5 2^{p+1}\leqslant\varepsilon$$

where ε is the constant of (4.1) with η chosen as $1/2$. (It looks like λ will be rather small!). For this choice of λ, either (5.1-a) applies, or the perturbation theorem (4.1) can be applied. Assume that (5.1-a) applies to ω on $B(4^{-i})$ for $i<j$, but that (5.1-b, c) hold on $B(4^{-j})$ unless $j=\infty$. The number γ is chosen to satisfy

(5.5) $$\sqrt{1-\lambda}=4^{-\gamma}$$

For $4^{-2-i}\leqslant|y|\leqslant 4^{-i}$ with $i<j$ we have

$$|\omega(0)-\omega(y)|\leqslant 2\sqrt{M}2^{-i}\leqslant 2(1-\lambda)^{i/2}(M+C)^{1/2}$$
$$=2(4^{-\gamma i})(M+C)^{1/2}\leqslant 2^{1+2\gamma}|y|^{\gamma}(M+C)^{1/2}.$$

For $|y|\ 4^{-2-j}$, the perturbation theorem yields, for the expanded function $\tilde{\omega}(z)=\omega(y)$ with $y=4^{-2-j}z$

$$|\tilde{\omega}(0)-\tilde{\omega}(z)|\leqslant k_4|z|^{1/2}(M(4^{-j})+C)^{1/2}$$
$$\leqslant k_4|z|^{1/2}2(1-\lambda)^{j/2}(M+C)^{1/2}$$

Applying this to ω, we get

$$|\omega(0)-\omega(y)|\leqslant k_4 4^{-\gamma j}y^{\gamma}|z|^{1/2-\gamma}2y^{\gamma}(M+C)^{1/2}$$
$$\leqslant k_4 2y^{\gamma}(M+C)^{1/2}.$$

Section 6: Bernstein's theorem

Growth conditions at infinity for a solution in all of R^n follow easily from the uniform estimates in (5.4). Here are making use of the fact that the constant k does not depend on the size of C, which was the difficult part of section 4. We let $M(r)=\max_{x\in B(r)}Q(r)$ as before. The number will be the same as in section 5, and the proof is a direct consequence of (5.4).

(6.1) THEOREM. *Let ω be a solution of* (1.1) *in R^n and assume* (1.2), (1.3) *and* (1.4) *are valid. Then there exists a constant $\gamma>0$ such that if*

$$\liminf_{r\to\infty} r^{-2\gamma}M(r)=0$$

then ω is a constant.

Proof. Choose γ as in (5.4). For $(x, y)\in B(r/2)$ we have

$$|\omega(x)-\omega(y)|\leqslant k|x-y|^{\gamma}r^{-\gamma}(M(r)+C)^{1/2}$$

Letting $r\to\infty$ we get

$$|\omega(x)-\omega(y)|\leqslant k|x-y|^{\gamma}\liminf_{r\to\infty} r^{-\gamma}(M(r)+C)^{1/2}.$$

If the limit is zero, $\omega(x)=\omega(y)$ for all x and y, and ω is constant.

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