RANDOM COVERINGS

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§ 1. Introduction

Let X be a given set and $\mathcal{C}$ a collection of subsets of X which are chosen according to some random procedure. For each positive integer m, let N_m equal the number of subsets necessary to cover $X m$ times. We study in this paper the distribution and expectation of the random variables N_m. We refer to this study as the random covering problem.

To make the problem precise, we define the random procedure for choosing subsets of $\mathcal{C}$. Let P be a given probability measure on the space $\mathcal{C}$. Let $\Omega = \mathcal{C} \times \ldots \times \mathcal{C} \times \ldots$ be endowed with the product measure $P \times \ldots \times P \times \ldots$. Ω is the sample space corresponding to the process of choosing independently subsets of $\mathcal{C}$ according to the probability law P. We assume that the N_m's are measurable functions on Ω. This assumption is readily verified in all ensuing examples.

The random covering problem has been studied in the following instances. If X consists of a finite number of points and $\mathcal{C}$ is the collection of singletons, then we have the classical occupancy problem (see [5, chapter 4] which discusses the case where the elements in $\mathcal{C}$ have equal probability). If X is the circle of unit circumference covered by arcs of length $\alpha (0 < \alpha < 1)$ thrown uniformly and independently on X, then the distribution of N_1 has been calculated by Stevens [10]. If X is the d-dimensional sphere ($d \geqslant 2$) covered by spherical caps of equal size, the centers of these caps being chosen uniformly and independently, then no exact formula for the distribution of N_1 is known (a notable exception occurs if the caps are hemispheres; see [7, 13]). The 2-dimensional case has an interesting application in virology and has been studied in [8] by certain approximation techniques and simulation methods.

Since exact formulas are difficult to obtain, it is natural to inquire whether one can describe the asymptotic behavior of the distribution and expectation of the N_m's as the size of the caps goes to 0. This is indeed the case, and we direct ourselves in this paper to this aspect of the random covering problem. We deal in fact with the following generalization. Let X be a C^4 connected compact d-dimensional Riemannian manifold, normalized so that its volume equals 1. (We say that a Riemannian manifold is of class C^k, $k \geq 1$, if it is a C^k manifold and the components g_{ij} of the metric tensor are of class C^{k-1} in any admissible coordinate system. The C^4 requirement is made to insure the validity of Theorems 2.3, 2.4 of section 2). For any two points p, $q \in X$, we define the Riemannian distance $\delta(p, q)$ to be the g.l.b. of the lengths of all piecewise C^1 curves joining p to q. It can be shown that $\delta(p, q)$ is a distance function on X rendering it into a metric space [12, p. 219].

For any $p \in X$ and $r > 0$, let $B(p, r) = \{q \mid \delta(p, q) < r\}$. $B(p, r)$ is called the open ball of radius r centered at p. Let $\mathfrak{C} = \mathfrak{C}_r$, $r > 0$, consist of all open balls of X of radius r. The balls of $\mathfrak{C}_r$ are in 1–1 correspondence with their centers, so that $\mathfrak{C}_r$ may be identified with X. The Riemannian volume is a probability measure on X, which we designate both by v and dp. The probability measure P assigned to $\mathfrak{C}_r$ is assumed to be the measure v on X via the above identification. Thus, the balls of $\mathfrak{C}_r$ are chosen uniformly and independently from X.

We relabel the random variables N_m as N_{rm}. Let $\alpha = (\pi^{d/2}/\Gamma((d/2)+1))\, r^d$; observe that α is the volume of the d-dimensional Euclidean ball of radius r [3, p. 125]. Define X_{rm} by:

$N_{rm} = (1/\alpha)\,(\log(1/\alpha) + (d+m-1)\log\log(1/\alpha) + (X_{rm}/\alpha))$. We shall prove

THEOREM 1.1. *For each $m > 0$, $\exists\ r_1 > 0$ and $C > 0$, r_1 and C depending only on m, such*

$$P(X_{rm} > x) \leq Ce^{-x/8}, \quad x \geq 0, \quad r \leq r_1, \tag{1.1}$$

$$P(X_{rm} < x) \leq Ce^{x}, \quad x \leq 0, \quad r \leq r_1. \tag{1.2}$$

THEOREM 1.2. *Let $E(N_{rm})$ be the expectation of N_{rm}. Then*

$$E(N_{rm}) = \frac{1}{\alpha}\left(\log\frac{1}{\alpha} + (d+m-1)\log\log\frac{1}{\alpha} + 0(1)\right) \quad \text{as} \quad r \to 0. \tag{1.3}$$

The point to Theorem 1.1 is that it provides estimates for the tails of the distribution of X_{rm} which are uniform in r. As shown in section 6, Theorem 1.2 is an immediate consequence of these uniform estimates. In view of Theorem 1.1, it seems natural to conjecture that there exists a limit law for X_{rm} as $r \to 0$. This is indeed the case if X is the circle, in which case the limit distribution can be identified [6] (see also [4] for a discrete analog of this result). In general, though, the existence and identification of the limit distribution remains an open problem and our methods do not seem powerful enough to settle it.

The proofs of inequalities (1.1), (1.2) are given respectively in sections 4 and 5. The method used for (1.1) is that of discretization. Namely, we replace X by a finite set S of points sufficiently dense in X and compare the probability of covering Xm times with that of covering Sm times. The discretization procedure does not seem strong enough to yield (1.2), and we use a different method based on moment estimates discussed in [9]. In section 2 we prove some geometrical results and in section 3 we obtain a probability estimate for the classical occupancy problem. These results are subsequently used to derive inequalities (1.1), (1.2). We employ the latter in section 6 to prove Theorem 1.2 and to generalize a result of Steutel [11] concerning the asymptotic behavior of $E(N_{rm})$, when X is the circle. We also obtain in section 6 results similar to Theorems 1.1, 1.2 for the family $\mathcal{C}_v$ consisting of all open balls of given volume v. It is shown that the results for $\mathcal{C}_v$ follow readily from those for $\mathcal{C}_r$.

§ 2. Geometrical prerequisites

We prove in this section various results concerning the distance function $\delta(p, q)$ and the volumes of the balls $B(p, r)$. These results will be used in section 4.5 to derive Theorem 1.1. The proofs of some of the theorems are lengthy, especially that of Theorem 2.4, and the reader is advised to take the theorems on faith upon a first reading. We assume throughout that X is connected.

THEOREM 2.1. *Let X be a C^1 d-dimensional compact Riemannian manifold. For each positive integer n, $\exists$ a set S_n of n points of X such that* $\sup_{p\in X} \delta(p, S_n) \leqslant C_0/n^{-1/d}$, *$\delta(p, S_n)$ denoting the distance between p and S_n and $C_0 > 0$ being a constant independent of n.*

Proof. We cover X by a finite number of coordinate patches $(\phi_1, C), \ldots, (\phi_s, C)$ where each ϕ_i is a homeomorphism from the d-dimensional cube C: $0 \leqslant x^1, \ldots, x^d \leqslant 1$ onto a closed subset $\phi_i(C)$ of X. Let $g_{i,jk}(x)dx^j dx^k$ be the line element for the coordinate patch (ϕ_i, C) (we are using the standard summation convention concerning upper and lower indices). For each i, $1 \leqslant i \leqslant s$, $\exists$ $M_i > 0$ such that

$$g_{i,jk}(x)\,\xi^j\xi^k \leqslant M_i \sum_{j=1}^{d} (\xi^j)^2, x \in C \quad \text{and} \quad \xi^1, \ldots, \xi^d \quad \text{arbitrary.} \tag{2.1}$$

For $x, y \in C$, let $p = \phi_i(x), q = \phi_i(y), \delta_e(x, y)$ = Euclidean distance between x and y, $\delta(p, q)$ = Riemannian distance between p and q. (2.1) implies readily that $\delta(p, q) \leqslant \sqrt{M_i}\,\delta_e(x, y)$.

Assume $n \geqslant s$ and let $m = [(n/s)^{1/d}]$, $[a]$ denoting the gretest integer $\leqslant a$. Let L be the set of m^d points $(i_1/m, \ldots, i_d/m)$, $0 \leqslant i_1, \ldots, i_d, \leqslant m-1$. Then $\delta_e(x, L) \leqslant \sqrt{d}/m$, $\delta_e(x, L)$ being the Euclidean distance between x and L. $\bigcup_{i=1}^{s} \phi_i(L)$ consists of k points, $k \leqslant m^d s$. Add to $\bigcup_{i=1}^{s} \phi_i(L)$ an arbitrary set of $(n-k)$ points and call the resulting set S_n. Any point $p \in X$ is contained in some $\phi_i(C)$. Since $(n/s)^{1/d} \leqslant 2m$ for $n \geqslant s$, we obtain $\sup_{p \in X} \delta(p, S_n) \leqslant c/n^{1/d}$, $n \geqslant s$, where $c = 2s^{1/d}\sqrt{d}\ \mathrm{Max}_{1 \leqslant i \leqslant s} \sqrt{M_i}$. For $1 \leqslant n < s$, choose S_n to be an arbitrary set of n points and set $c_n = n^{1/d} \sup_{p \in X} \delta(p, S_n)$. Theorem 2.1 then follows by letting $C_0 = \mathrm{Max}\ (c, c_1, \ldots, c_{s-1})$.

THEOREM 2.2. *Let X be a C^1 d-dimensional Riemannian manifold. For each positive integer n, $\exists$ a set S_n of n points of X such that*

$$\min_{\substack{p, q \in S \\ p \neq q}} \delta(p, q) \geqslant C_1/n^{1/d},$$

$C_1 > 0$ being a constant independent of n.

Proof. Let (ϕ, C) be a coordinate patch on X, C being the cube: $0 \leqslant x^1, \ldots, x^d \leqslant 1$ and let $g_{ij}dx^i dx^j$ be the corresponding line element. Choose $M > 0$ so that

$$g_{ij}(x)\,\xi^i\xi^j \geqslant M \sum_{i=1}^{d} (\xi^i)^2, x \in C \quad \text{and} \quad \xi^1, \ldots, \xi^d \quad \text{arbitrary.} \tag{2.2}$$

Let $m = [n^{1/d}] + 1$ and L the set of m^d points $(i_1/(m+1), \ldots, i_d/(m+1))$, $1 \leqslant i_1, \ldots, i_d \leqslant m$. Then (i) $\delta_e(x, y) \geqslant 1/(m+1)$ for $x, y \in L$ and $x \neq y$, (ii) $\delta_e(x, \partial C) = 1/(m+1)$ for $x \in L$ and ∂C the boundary of C. Let S_n consist of n distinct points of $\phi(L)$. Suppose $p = \phi(x), q = \phi(y) \in S_n$, $p \neq q$. Let Γ be any piecewise C^1 curve in X joining p to q and $l(\Gamma)$ its length. If $\Gamma \subset \phi(L)$, then (2.2) and (i) imply $l(\Gamma) \geqslant \sqrt{M}/(m+1)$. If $\Gamma \not\subset \phi(L)$, then Γ meets $\phi(\partial C)$ and (2.2), (ii) imply $l(\Gamma) \geqslant \sqrt{M}/(m+1)$. It follows that $\delta(p, q) \geqslant \sqrt{M}/(m+1) \geqslant \frac{1}{2}\sqrt{M}\, n^{-1/d}$ for $p, q \in S_n$, $p \neq q$, and Theorem 2.2 is proven with the choice $C_1 = \frac{1}{2}\sqrt{M}$.

In the sequel ω_d, $v(p, r)$ denote respectively $\pi^{d/2}/\Gamma(d/2+1)$, $v(B(p, r))$. The following lemma will be required in the proof of Theorem 2.3.

LEMMA 2.1. *Let $G(a)$ and $n \times n$ symmetric matrix valued function of class C^k, $k \geqslant 0$, on the m-dimensional open set A. Suppose that the associated quadratic form $\xi' G(a) \xi$ is positive definite in ξ for $a \in A$. Then $\exists T(a)$ of class C^k such that $T'(a)G(a)T(a) = E$, $a \in A$ (E denotes the identity matrix and T' the transpose of T).*

Proof. For $n = 1$, $G(a)$ is a positive number and we choose $T(a) = 1/\sqrt{G(a)}$. Suppose the lemma holds for n. We show that it holds for $n+1$. Let $G(a) = [g_{ij}(a)]$, $1 \leqslant i, j \leqslant n+1$.

Since $\xi' G(a)\xi$ is positive definite, $g_{11}(a)>0$ for $a\in A$. Let $\eta^1=\sqrt{g_{11}}\,(\xi^1+(g_{12}/g_{11})\xi^2+\dots+((g_{1,n+1})/g_{11})\xi^{n+1})$, $\eta^i=\xi^i$, $2\leqslant i\leqslant n+1$. Let $\eta=(\eta^2,\dots,\eta^{n+1})$. Then $\xi' G(a)\xi=\eta_1^2+\eta' H(a)\eta$ where $H(a)$ is a symmetric matrix of class C^k on A. Setting $\eta_1=0$, we find that $\eta' H(a)\eta$ is positive definite in η, $a\in A$. Hence $\exists$ $n\times n$ matrix $S(a)$ of class C^k such that $S'(a)H(a)S(a)=E$, $a\in A$. Let $\eta^1=\zeta^1$, $\eta=S(a)\zeta$, where $\zeta=(\zeta^2,\dots,\zeta^{n+1})$, and let $\tau=(\zeta^1,\dots,\zeta^{n+1})$. Then $\xi=T(a)\tau$, where $T(a)\in C^k$ on A. We have $\tau'[T'(a)G(a)T(a)]\tau=\xi' G(a)\xi=\tau' E\tau$, for $a\in A$ and all τ, so that $T'(a)G(a)T(a)=E$, $a\in A$.

THEOREM 2.3. *Let X be a C^4 d-dimensional compact Riemannian manifold. $\exists$ numbers ε, $R>0$ such that*

$$|v(p,r)-\omega_d r^d|\leqslant Cr^{d+1},\quad p\in X \quad\text{and}\quad r<R, \tag{2.3}$$

i.e. for small radius, the volume of a ball in X is almost that of a Euclidean ball of the same radius, (2.3) *giving a uniform bound for the difference.*

Remarks. (1) Theorem 2.3 is a special case of Theorem 2.4. The proof of the latter is however more complicated and introduces extraneous notions. We therefore first present a proof of Theorem 2.3. (2) Examination of the proof reveals that the term r^{d+1} may be replaced by r^{d+2}, if X is assumed to be of class C^5. The above estimate suffices however for our purposes.

Proof of Theorem 2.3. Let I, J be the respective cubes, $-1\leqslant x^1,\dots,x^d\leqslant 1$, $-2\leqslant x^1,\dots,x^d\leqslant 2$. We choose a finite number of coordinate patches $(\phi_1,J),\dots,(\phi_s\,J)$ such that $\phi_1(I),\dots,\phi_s(I)$ cover X. Let $g_{ij}(x)dx^i\,dx^j$ be the line element for the coordinate patch (ϕ_1,J). The g_{ij}'s are C^3 functions on J, and the Christoffel symbols

$$\Gamma^i_{jk}(x)=\tfrac{1}{2}\,g^{ir}\left(\frac{\partial g_{rj}}{\partial x^k}-\frac{\partial g_{jk}}{\partial x^r}+\frac{\partial g_{rk}}{\partial x^j}\right)$$

((g^{ij}) is the inverse matrix to (g_{ij})) are C^2 functions on J. Let $x(a,\xi,t)$ be the solution to the differential equations

$$\frac{d^2x^i}{dt^2}+\Gamma^i_{jk}(x)\frac{dx^j}{dt}\frac{dx^k}{dt}=0 \tag{2.4}$$

satisfying the initial conditions $x(0)=a$, $(dx/dt)(0)=\xi$; thus $x(a,\xi,t)$ is the geodesic emanating from a with initial velocity ξ. Let $|x|=\operatorname{Max}(|x^1|,\dots,|x^d|)$. It follows from standard theorems in differential equations ([2], chapter 1) that $\exists\varepsilon_1=0$ such that $x(a,\xi,t)\in J$ and is C^2 for $|a|<\frac{3}{2}$, $|\xi|<1$, $|t|=2\varepsilon_1$. $x(a,\xi/\varepsilon_1,\varepsilon_1 t)$ satisfies (2.4) with initial conditions $x(0)=a$, $dx/dt=\xi$, so that $x(a,\xi,t)\in J$ and is C^2 for $|a|<\frac{3}{2}$, $|\xi|<\varepsilon_1$, $|t|<2$.

Set $x(a, \xi) = x(a, \xi, 1)$ for $|a| < \frac{3}{2}$, $|\xi| < \varepsilon_1$. Then $x(a, 0) = a$ and $x(a, \xi) \in C^2$ for $|a| < \frac{3}{2}$, $|\xi| < \varepsilon_1$. Let e_j be the vector with components δ^i_j, $1 \leq i, j \leq d$. We have

$$\frac{\partial x}{\partial \xi^j}(a, 0) = \frac{dx}{dt}(a, e_j, t1)\big|_{t=0} = \frac{dx}{dt}(a, e_j t)\big|_{t=0} = e_j. \tag{2.5}$$

I.e. $\partial x^i/\partial \xi^j(a, 0) = \delta^i_j$, $1 \leq i, j \leq d$. The Jacobian $j(a, \xi) = \partial(x^1, ..., x^d)/\partial(\xi^1, ..., \xi^d) \in C^1$ and $j(a, 0) = 1$ for $|a| < \frac{3}{2}$, $|\xi| < \varepsilon_1$. Hence $\exists\, 0 < \varepsilon_2 < \varepsilon_1$ such that $j(a, \xi) > 0$ for $a \in I$, $|\xi| < \varepsilon_2$. Thus $x(a, \xi)$ is $1-1$ on $|\xi| < \varepsilon_2$ provided $a \in I$, and ξ serves as a local coordinate at $\phi_1(a)$.

The length of the geodesic $x = x(a, \xi, t)$, $0 \leq t \leq 1$, equals $\sqrt{g_{ij}(a)\xi^i\xi^j}$. For $a \in I$, $|\xi| < \varepsilon_2$, it is known that this length $= \delta(p, q)$ where $p = \phi_1(a)$, $g = \phi_1(x(a, \xi))$, [12, p. 310]. Let $R_1 = \mathrm{Min}_{a \in I, |\xi| \geq \varepsilon_2} \sqrt{g_{ij}(a)\xi^i\xi^j}$. If $a \in I$, $r < R_1$, then $g_{ij}(a)\xi^i\xi^j \leq r^2 \Rightarrow |\xi| < \varepsilon_2$. It follows that $g_{ij}(a)\xi^i\xi^j < r^2$ is the description of the ball $B(\phi_1(a), r)$ in the ξ coordinates. Let $G(x) = [g_{ij}(x)]$, $h(a, \xi) = \sqrt{\det G(x(a, \xi))}\, j(a, \xi)$. We have

$$v(\phi_1(a), r) = \int_{g_{ij}(a)\xi^i\xi^j \leq r^2} h(a, \xi)\, d\xi, \quad a \in I \quad \text{and} \quad r < R. \tag{2.6}$$

By Lemma 2.1 $\exists\, T(a)$ of class C^3 on J, such that $T'(a)G(a)T(a) = E$. Let $\xi = T(a)\eta$. (In section 5 η will be referred to as a local normal coordinate at $\phi_1(a)$). $|\det T(a)| = 1/\sqrt{\det G(a)} = 1/h(a, 0)$ and 2.6) becomes

$$v(\phi_1(a), r) = \int_{\|\eta\| \leq r} k(a, \eta)\, d\eta, \quad a \in I \quad \text{and} \quad r < R_1, \tag{2.7}$$

where $k(a, \eta) = h(a, T(a)\eta)/h(a, 0)$, $\|\eta\|^2 = \sum_{i=1}^d (\eta^i)^2$. $k(a, \eta) \in C^1$ for $|a| < \frac{3}{2}$, $\|\eta\| < R_1$, and $k(a, 0) = 1$. It follows that $\exists\, 0 < R_1' < R_1$, $C_1 > 0$ such that

$$|k(a, \eta) - 1| \leq C_1\|\eta\|, \quad a \in I \quad \text{and} \quad \|\eta\| < R_1'. \tag{2.8}$$

Let $C_1' = d/(d+1)\omega_d C_1$. (2.7), (2.8) yield

$$|v(\phi_1(a), r) - \omega_d r^d| = \left|\int_{\|\eta\| \leq r} (k(a, \eta) - 1)\, d\eta\right| \leq C_1 \int_{\|\eta\| \leq r} \|\eta\|\, d\eta = C_1' r^{d+1}, \tag{2.9}$$

$a \in I$ and $r < R_1'$.

Similarly, we prove (2.9) for ϕ_i, $1 \leq i \leq s$, replacing C_1', R_1' respectively by C_i', R_i'. Since $\phi_1(I), ..., \phi_s(I)$ cover X, we have proven (2.3) with the choice $C = \mathrm{Max}\,(C_1', ..., C_s')$, $R = \mathrm{Min}\,(R_1', ..., R_s')$.

We establish several lemmas which will be required in the proof of Theorem 2.4. In Lemma 2.2, v denotes Euclidean volume.

LEMMA 2.2. *Let A, B be two d-dimensional Euclidean balls of radius $r>0$. Assume that the distance ε between the centers satisfies $0\leqslant\varepsilon\leqslant 2r$. Then*

$$v(A-B)=2\omega_{d-1}\int_0^{\varepsilon/2}(r^2-x^2)^{(d-1)/2}\,dx, \tag{2.10}$$

$$\frac{\omega_{d-1}}{2^d}r^{d-1}\varepsilon\leqslant v(A-B)\leqslant\omega_{d-1}r^{d-1}\varepsilon. \tag{2.11}$$

Proof. We may assume that A, B are centered respectively at 0 and $p=(\varepsilon,0,\ldots,0)$. Let $S=A\cap B\cap\{x\,|\,x_1\geqslant\varepsilon/2\}=A\cap\{x\,|\,x_1\geqslant\varepsilon/2\}$. $A\cap B$ is symmetric with respect to the hyperplane $x_1=\varepsilon/2$, so that $v(A\cap B)=2v(S)$. Hence

$$v(A-B)=v(A)-v(A\cap B)=v(A)-2v(S)=v\{x\,|\,x\in A,\quad |x_1|\leqslant\varepsilon\} \tag{2.12}$$

Since $v\left\{x\,|\,x\in A,|x_1|\leqslant\dfrac{\varepsilon}{2}\right\}=2\omega_{d-1}\displaystyle\int_0^{\varepsilon/2}(r^2-x^2)^{(d-1)/2}\,dx,$ we have proven (2.10).

From (2.10), we get

$$v(A-B)\leqslant\omega_{d-1}r^{d-1}\varepsilon \tag{2.13}$$

which is the right inequality of (2.11). We now prove the left inequality of (2.11). For $0\leqslant 2x\leqslant\varepsilon\leqslant\sqrt{3}r$, $r^2-x^2\geqslant r^2/4$, so that (2.10) gives

$$v(A-B)\geqslant\frac{\omega_{d-1}}{2^{d-1}}r^{d-1}\varepsilon. \tag{2.14}$$

For $\sqrt{3}\,r\leqslant\varepsilon\leqslant 2r$,

$$v(A-B)\geqslant\frac{\omega_{d-1}}{2^{d-1}}r^{d-1}(\sqrt{3}r)\geqslant\frac{\omega_{d-1}}{2^d}r^{d-1}\varepsilon. \tag{2.15}$$

Thus in either case $v(A-B)\geqslant(\omega_{d-1}/2^d)r^{d-1}\varepsilon$.

The following lemma is a global version of the Implicit function theorem.

LEMMA 2.3. *Let A be a d-dimensional open set. Let a, η, $f(a,\eta)$ be d-dimensional vectors, $f(a,\eta)\in C^k(k\geqslant 1)$ for $a\in A$, $\|\eta\|<r$ $(r>0)$. Suppose that the Jacobian $|(\partial f^i/\partial\eta^j)(a,\eta)|\neq 0$, $a\in A$ and $\|\eta\|<r$. For any open set A_0 with compact closure contained in A, $\exists\ \delta(A_0)>0$ such that $x=f(a,\eta)$ has a unique solution $\eta=g(a,x)$ whenever $a\in A_0$ and $\|x-f(a,0)\|<\delta(A_0)$. Furthermore $g(a,x)\in C^k$ for $a\in A_0$, $\|x-f(a,0)\|<\delta(A_0)$.*

Proof. Let $p\in A$. Since $|(\partial f^i/\partial\eta^j)(a,\eta)|\neq 0$, $a\in A$ and $\|\eta\|<r$, we conclude from the Implicit function theorem that $\exists\delta_p>0$ such that $x=f(a,\eta)$ has a unique solution $\eta=$

$g(a, x)$, provided $\|a-p\|$, $\|x-f(p, 0)\|<\delta_p$, $g(a, x)$ being C^k for these values of a, x. Choose $0<\delta'_p<\delta_p$ so that $\|a-p\|<\delta'_p \Rightarrow \|f(a, 0)-f(p, 0)\|<\delta_p/2$. Then $\|a-p\|<\delta'_p$, $\|x-f(a, 0)\|<\delta_p/2 \Rightarrow \|a-p\|<\delta_p$, $\|x-f(p, 0)\|<\delta_p$. It follows that $x=f(a, \eta)$ has precisely one solution $\eta=g(a, x)$, provided $\|a-p\|<\delta'_p$, $\|x=f(a, 0)\|<\delta_p/2$, and that $g(a, x)\in C^k$ for these values of a, x.

By the Heine-Borel theorem, A_0 is covered by a finite number of balls $B(p_1, \delta'_{p_1})$, ..., $B(p_s, \delta'_{p_s})$. Let $\delta(A_0)=\text{Min}\,[\frac{1}{2}\delta p_1, \ldots, \frac{1}{2}\delta p_s]$. Then $\eta=g(a, x)$ is the unique solution $x=f(a, \eta)$, whenever $a\in A_0$, $\|x-f(a, 0)\|<\delta(A_0)$ and $g(a, x)\in C^k$ for $a\in A_0$, $\|x-f(a, 0)\|<\delta(A_0)$.

LEMMA 2.4. *Let x, $f(x)$ be d-dimensional vectors, $f(x)$ being C^1 and $|\partial f^i/\partial x^j(x)|\neq 0$ on $\|x\|\leqslant r$, $r>0$. Let $m=\text{Inf}_{\|x\|=r}\|f(x)-f(0)\|>0$. Then image of the ball $B(0, r)$, under the mapping $y=f(x)$, contains the ball $B(f(0), m)$.*

Proof. Assume, without loss of generality, that $f(0)=0$. Let S be the image of $B(0, r)$ under the mapping $y=f(x)$. We prove the following proposition $\mathcal{P}$. If $b\in S$ and $\|b\|<m$, then $B(b, (m-\|b\|)/2)\subset S$. Let $y\in B(b, (m-\|b\|)/2)$ and set $g(x)=\|f(x)-y\|^2$. $g(x)$ is C^1 for $\|x\|\leqslant r$. Now $b=f(c)$, where $\|c\|<$ r. Thus $g(c)=\|b-y\|^2<((m-\|b\|)/2)^2$. For $\|x\|=r$, $\|g(x)\|\geqslant(m-\|b\|-\|b-y\|)^2>((m-\|b\|)/2)^2$. We conclude that $g(x)$ attains its minimum on $\|x\|\leqslant r$ at some interior point p, $\|p\|<r$. Hence $(\partial g/\partial x^j)(p)=0$, $1\leqslant j\leqslant d$. We have

$$\frac{\partial g}{\partial x^j}(p)=2\sum_{i=1}^{d}\frac{\partial f^i}{\partial x^j}(p)\,(f^i(p)-y^i)=0. \tag{2.16}$$

Since $|\partial f^i/\partial x^j(p)|\neq 0$, we conclude from 2.16) that $y^i=f^i(p)$, $1\leqslant i\leqslant d$. I.e. $y=f(p)\in S$, thus proving $\mathcal{P}$.

Since $0\in S$, $\mathcal{P}$ asserts that $B(0, m/2)\subset S$. Suppose that $B_k=B(0, m(1-(2^{-k}))\subset S$ for some positive integer k. Let $y\in B_{k+1}$. Then $z=(2^{k+1}-2)\,y/(2^{k+1}-1)\in B_k$. $(m-\|z\|)/2>m2^{-(k+1)}$, $\|y-z\|=\|y\|/(2^{k+1}-1)<\text{m}2^{-(k+1)}$. Hence $\|y-z\|<(m-\|z\|)/2$, and we conclude from $\mathcal{P}$ that $y\in S$. Thus $B_{k+1}\subset S$, and by induction $B_k\subset S$, $1\leqslant k<\infty$. It follows that $B(0, m)=\bigcup_{k=1}^{\infty}B_k\subset S$.

THEOREM 2.4. *Let X be a C^4 d-dimensional compact manifold. Let $v(p, q, r)=v[B(q, r)-B(p, r)]$. Let $v_e(p, q, r)$ be the volume of the difference of two Euclidean balls of radius r, the Euclidean distance between their centers equalling $\delta(p, q)$. $\exists$ numbers C, $R>0$ such that*

$$|v(p, q, r)-v_e(p, q, r)|\leqslant Cr^d\delta(p, q),\ \delta(p, q)\leqslant 2r \quad \text{and} \quad r<R. \tag{2.17}$$

Remark. Dividing 2.16) by $v_e(p, q, r)$ and using the left inequality of 2.11), we obtain $\lim_{r\to 0}v(p, q, r)/v_e(p, q, r)=1$ uniformly in all pairs p, q satisfying $0<\delta(p, q)\leqslant 2r$.

Proof. We imitate the terminology and reasoning used to prove Theorem 2.3. Thus we choose coordinate patches $(\phi_1, J), ..., (\phi_s, J)$ such that $\phi_1(I), ..., \phi_s(I)$ cover X. For the coordinate patch (ϕ_1, J), $\exists\, \varepsilon > 0$ such that $x(a, \xi)$ is a C^2 function whose Jacobian $|\partial x^i/\partial \xi^j| > 0$ for $|a| < \frac{3}{2}$, $|\xi| < \varepsilon$. Then $\exists\, R_1 > 0$ such that $x(a, T(a)\eta)$ is C^2 for $|a| < \frac{3}{2}$, $\|\eta\| < R_1$. Rename $x(a, T(a)\eta)$ as $x(a, \eta)$ and let $j(a, \eta) = |\partial x^i/\partial \eta^j|$. Then $j(a, \eta) \in C^1$, $j(a, 0) = \det T(a)$, $j(a, \eta) \neq 0$ for $\|a\| < \frac{3}{2}$, $\|\eta\| < R_1$. Hence $x(a, \eta)$ is $1-1$ on $\|\eta\| < R_1$, provided $|a| < \frac{3}{2}$. Observe that for these values of a, η, $\delta[\phi_1(a), \phi_1(x(a, \eta))] = \|\eta\|$.

For $|a| < \frac{3}{2}$, $r > R_1$, $B(\phi_1(a), r)$ is described in the η coordinates by the Euclidean ball $B(r) = \{\eta \mid \|\eta\| < r\}$, and in the x coordinates by $x(a, B(r))$, $x(a, B(r))$ denoting for given α the image of $B(r)$ under the mapping $x = x(a, \eta)$. Since $x(a, 0) = a$, we may choose $0 < R_2 < R_1$ so that $\|x(a, \tau)\| < \frac{3}{2}$ for $|a| < \frac{5}{4}$, $\|\tau\| < R_2$. Then $y(a, \tau, \eta) = x(x(a, \tau), \eta)$ is C^2 for $|a| < \frac{5}{4}$, $\|\tau\|$, $\|\eta\| < R_2$. For $p = \phi_1(a)$, $q = \phi_1(x(a, \tau))$, we have

$$v(p, q, r) = \int_{y(a,\tau,B(r)) - x(a, B(r))} \sqrt{\det G(x)}\, dx, \quad |a| < \tfrac{5}{4}, \|\tau\|, r < R_2. \tag{2.18}$$

We express the above integral in the η coordinates. By Lemma 2.3, $\exists \delta > 0$ such that $x = x(a, \eta)$ has a unique solution $\eta = x^{-1}(a, x)$ satisfying $\|\eta\| < R_2$, provided $|a| < \frac{5}{4}$, $\|x - a\| < \delta$, $x^{-1}(a, x)$ being a C^2 function for these values of a, η. Since $y(a, 0, 0) = a$, we may choose $0 < R_3 < R_2$ so that $\|y(a, \tau, \eta) - a\| < \delta$ if $|a| < \frac{9}{8}$, $\|\tau\|$, $\|\eta\| < R_3$. Hence $z(a, \tau, \eta) = x^{-1}(a, y(a, \tau, \eta))$ is C^2 and $|z(a, \tau, \eta)| < R_2$, $|\partial z^i/\partial \eta^j(a, \tau, \eta)| \neq 0$ for $|a| < \frac{9}{8}$, $\|\tau\|$, $\|\eta\| < R_3$. Since $y(a, \eta, 0) = y(a, 0, \eta) = x(a, \eta)$ and $x^{-1}(a, x(a, \eta)) = \eta$ whenever $|a| < \frac{9}{8}$, $\|\tau\|$, $\|\eta\| < R_3$, we have $z(a, \eta, 0) = z(a, 0, \eta) = \eta$.

For $|a| < \frac{9}{8}$, $\|\tau\|$, $r < R_3$, we write

$$y(a, \tau, B(r)) - x(a, B(r)) = x(a, z(a, \tau, B(r))) - x(a, B(r)) = x(a, z(a, \tau, B(r)) - B(r)). \tag{2.19}$$

By the change of variables formula, (2.18) becomes

$$v(p, q, r) = \int_{z^*} k(a, \eta)\, d\eta \; |a| < \tfrac{9}{8}, \quad \|\tau\|, \quad r < R_3 \tag{2.20}$$

where $k(a, \eta) = \sqrt{\det G(x(a, \eta))} \cdot |j(a, \eta)|$ and where $z^* = z(a, \tau, B(r)) - B(r)$. Observe that $k(a, 0) = \sqrt{\det G(a)} \cdot |\det T(a) \cdot| = 1$.

Let $\mathfrak{z}(\tau, \eta) = \tau + \eta$ and $\mathfrak{z}^* = \mathfrak{z}(\tau, B(r)) - B(r)$. Then

$$v_e(p, q, r) = \int_{\mathfrak{z}^*} d\eta. \tag{2.21}$$

Let $l(a, \eta) = k(a, \eta) - 1$. $l(a, \eta)$ is C^1 for $|a| < \frac{3}{2}$, $\|\eta\| < R_1$. (2.20), (2.21) give

$$v(p, q, r) - v_e(p, q, r) = \left[\int_{z^*} k(a, \eta)\, d\eta - \int_{\mathcal{Z}^*} k(a, \eta)\, d\eta \right] + \int_{\mathcal{Z}^*} l(a, \eta)\, d\eta = I_1 + I_2, \quad (2.22)$$

where $p = \phi_1(a)$, $q = \phi_1(x(a, \tau))$ and $|a| < \frac{9}{8}$, $\|\tau\|$, $r < R_3/2$.

We estimate I_1, I_2. We first establish the bound (2.28) for the Euclidean volume of the symmetric difference of $z(a, \tau, B(r)) - B(r)$, $\mathcal{Z}(\tau, B(r)) - B(r)$.

Applying the mean value theorem to $z^i(a, \tau, \eta) - z^i(a, 0, \eta)$, we get

$$z^i(a, \tau, \eta) = \eta^i + \tau^i + \sum_{j=1}^{d} \left[\frac{\partial z^i}{\partial x^j}(a, \theta_i \tau, \eta) - \delta^i_j \right] \tau^j, \quad 1 \leqslant i \leqslant d, \quad (2.23)$$

where $0 \leqslant \theta_1, \ldots, \theta_d \leqslant 1$.

Now $z(a, \tau, 0) = \tau \Rightarrow \partial z^i / \partial \tau^j (a, \tau, 0) = \delta^i_j$, $1 \leqslant i, j \leqslant d$. Since $z(a, \tau, \eta)$ is a C^2 function, we conclude from (2.23) that $\exists 0 < R_4 < R_3/2$, $0 < C_1$ such that

$$\|\eta\|(1 - C_1\|\tau\|) \leqslant \|z(a, \tau, \eta) - \tau\| \leqslant \|\eta\|(1 + C_1\|\tau\|), \quad |a| \leqslant 1, \|\tau\|, \|\eta\| < R_4 \quad (2.24)$$

The right inequality of 2.24) implies

$$z(a, \tau, B(r)) \subset \mathcal{Z}(\tau, B(r')) \quad |a| \leqslant 1, \|\tau\|, r < R_4 \quad (2.25)$$

where $r' = (1 + C_1\|\tau\|)r$.

We conclude from the left inequality of (2.24) and Lemma 2.4 that

$$\mathcal{Z}(\tau, B(r'')) \subset z(a, \tau, B(r) \quad |a| \leqslant 1, \|\tau\|, r < R_4 \quad (2.26)$$

where $r'' = \text{Max}\,(0, [1 - C_1\|\tau\|]r)$.

Let $U \Delta V$ denote the symmetric difference of the sets U, V. (2.25), (2.26) give

$$z(a, \tau, B(r)) \Delta \mathcal{Z}(\tau, B(r)) \subset \mathcal{Z}(\tau, B(r')) - \mathcal{Z}(\tau, B(r'')). \quad (2.27)$$

The Euclidean volume of $\mathcal{Z}(\tau, B(r')) - \mathcal{Z}(\tau, B(r'')) = \omega_d((r')^d - (r'')^d) \leqslant C_2\|\tau\| r^d$ where $C_2 = 2d\omega_d C_1(1 + C_1 R_4)^{d-1}$. Hence

$$v_e[z(a, \tau, B(r)) \Delta \mathcal{Z}(\tau, B(r))] \leqslant C_2\|\tau\| r^d \quad |a| \leqslant 1, \|\tau\|, r < R_4 \quad (2.28)$$

where v_e denotes Euclidean volume.

With the aid of (2.28), we readily estimate I_1. Let U, V, W be d-dimensional measurable sets and $f(x)$ integrable on $U \cup V$. Since $U - W - [V - W] = [U - V] - W \subset U - V$, we obtain

$$\left| \int_{U-W} f(x)\, dx - \int_{V-W} f(x)\, dx \right| = \left| \int_{U-W-[V-W]} f(x)\, dx - \int_{V-W-[U-W]} f(x)\, dx \right| \leqslant \int_{U \Delta V} |f(x)|\, dx$$

(2.29)

$l(a, 0)=0$ and $l(a, \eta)$ is C^1 for $|a|<\frac{3}{2}$, $|\eta|<R_1$. Hence $\exists\ C_3<0$ such that

$$|l(a,\eta)| \leqslant C_3\|\eta\|\,|a| \leqslant 1, \quad \|\eta\|<R_2. \tag{2.30}$$

Since $\|z(a,\tau,\eta)\|$, $\|\mathfrak{Z}(\tau,\eta)\|<R_2$ for $|a|\leqslant 1$, $\|\tau\|$, $\|\eta\|<R_4$, we conclude from (2.28)–(2.30) that

$$|I_1| \leqslant C_2(C_3R_2+1)\|\tau\|r^d, \quad |a|\leqslant 1,\ \|\tau\|,\ r<R_4. \tag{2.31}$$

Lemma 2.2 and (2.30) imply

$$|I_2| \leqslant C_3\omega_{d-1}(\|\tau\|+r)\|\tau\|r^{d-1} \quad |a|\leqslant 1,\ \|\tau\|,\ r<R_4. \tag{2.32}$$

Let $C_4=C_2+C_2C_3R_2+3C_3\omega_{d-1}$. We conclude from (2.22), (2.31), (2.32) that

$$|v(p,q,r)-v_e(p,q,r)| \leqslant C_4\|\tau\|r^d = C_4r^d\delta(p,q) \quad |a|\leqslant 1,\ \|\tau\|\leqslant 2r<R_4, \tag{2.33}$$

where $p=\phi_1(a)$, $q=\phi_1(x(a,\tau))$.

Rename C_4, $R_4/2$ as C'_1, R'_1. (2.33) can also be established for each coordinate patch (ϕ_i, J), $1\leqslant i\leqslant s$, replacing M'_i, R'_i by M'_1, R'_1. Since $\phi_1(I), \dots, \phi_s(I)$ cover X, we have proven (2.17) with the choice $C=\mathrm{Max}\,(C'_1, \dots, C'_s)$, $R=\mathrm{Min}\,(R'_1, \dots, R'_s)$. ■

§ 3. A probability estimate for the classical occupancy problem

We derive in this section the following estimate, of some independent interest, which will be used in the proof of inequality (1.2).

THEOREM 3.1. *Let N balls be thrown independently at n urns labelled, $1, \dots, n$. Assume that for every throw the probability that the ball fall into the ith urn $=p_i$ where $p_1+\dots+p_n\leqslant 1$. The probability that all urns contain balls $\leqslant\prod_{i=1}^{n}(1-(1-p_i)^N)$.*

Proof. We say that i is hit if the ith urn contains a ball. Then

$$P(i+1,\dots,n \text{ are hit}) = \prod_{i=0}^{n} P(i+1 \text{ is hit}\,|\,1,\dots,i \text{ are hit}). \tag{3.1}$$

We prove

$$P(i+1 \text{ is hit}\,|\,1, \dots, i \text{ are hit}) \leqslant P(i+1 \text{ is hit}) = 1-(1-p_{i+1})^N, \quad 1\leqslant i\leqslant n-1 \tag{3.2}$$

(3.1) and (3.2) imply the theorem. We rewrite (3.2) in the equivalent form

$$P(i+1 \text{ is not hit}\,|\,1, \dots, i \text{ are hit}) \geqslant P(i+1 \text{ is not hit}), \quad 1\leqslant i\leqslant n-1 \tag{3.3}$$

For any two events A, B of positive probability, we have $P(A|B) \geqslant P(A) \Leftrightarrow P(AB) \geqslant P(A)P(B) \Leftrightarrow P(B|A) \geqslant P(B)$, so that (3.3) becomes

$$P(1, ..., i \text{ are hit} | i+1 \text{ is not hit}) \geqslant P(1, ..., i \text{ are hit}), \quad 1 \leqslant i \leqslant n-1 \tag{3.4}$$

Let $P(i, N; p_1, ..., p_n)$ = probability that urns $1, ..., i$, $1 \leqslant i \leqslant n$, are hit in N idependent throws, given that on any throw the jth urn is hit with probability p_j, $1 \leqslant j \leqslant n$. We may rewrite (3.4) as

$$P(i, N; q_i, ..., q_{n-1}) \geqslant P(i, N; p_1, ..., p_n), \quad 1 \leqslant i \leqslant n-1 \tag{3.5}$$

where $q_1, ..., q_{n-1}$ are the numbers $p_1/(1-p_{i+1}), ..., p_i/(i-p_{i+1}), p_{i+2}/(1-p_{i+1}), ..., p_n/(1-p_{i+1})$. (3.5) is proven as follows. Let π_j = probability that precisely j of the N thrown balls fall into the $(i+1)$th urn. Then

$$P(i, N; p_1, \ldots, p_n) = \sum_{j=0}^{N} \pi_j P(i, N-j; q_1, \ldots, q_{n-1}), \quad 1 \leqslant i \leqslant n-1. \tag{3.6}$$

Since $P(i, N; q_1, ..., q_n)$ clearly increases with N, we conclude from (3.6) that

$$P(i, N; p_1, \ldots, p_n) \leqslant \sum_{j=0}^{N} \pi_j P(i, N; q_1, \ldots, q_{n-1}) = P(i, N; q_1, \ldots, q_{n-1}), \quad 1 \leqslant i \leqslant n-1. \tag{3.7}$$

thus proving Theorem 3.1. ∎

§4. Proof of inequality (1.1)

Let $\omega_d r^d = 1/e$, $n(r) = [(1/\alpha)(\log(1/\alpha))^d]$, $0 < r \leqslant r_1$, where $\alpha = \omega_d r^d$. Since $\alpha \leqslant 1/e$, we have $(1/\alpha)(\log(1/\alpha))^d \geqslant e$ and $n(r) \geqslant 2$. Choose $S = S_n$ to be a set consisting of $n(r)$ points of X satisfying the requirements of Theorem 2.1, denote these points as $p_1, ..., p_n$. Let N'_{rm} = number of open balls of radius r which need to be thrown to cover Sm times. For each positive integer N, the event $[N'_{rm} > N]$ means that some point p_i has not been covered m times in the first N throws. Since $p_i \in B(q, r)$ iff $q \in B(p_i, r)$, we conclude that the probability that p_i be covered precisely j times in first N throws $= \binom{N}{j} \alpha_i^j (1-\alpha_i)^{N-j}$, where $\alpha_i = v(p_i, r)$. Hence the probability that p_i be covered $< m$ times in first N throws $= \sum_{j=0}^{m-1} \binom{N}{j} \alpha_i^j (1-\alpha_i)^{N-j}$, and

$$P(N'_{rm} > B) \leqslant \sum_{i=1}^{n} \sum_{j=0}^{m} \binom{N}{j} \alpha_i^j (1-\alpha_i)^{N-j} \leqslant \sum_{i=1}^{n} \sum_{j=0}^{m-1} \frac{(N\alpha_i)^j}{j!} e^{-\alpha_i N}. \tag{4.1}$$

According to Theorem 2.3, we may choose $0 < r_2 < r_1$, $C > 0$ so that

$$|\alpha_i - \alpha| \leqslant C_1 \alpha^{(d+1)/2}, \quad \alpha_i \leqslant 2\alpha \quad \text{for } 1 \leqslant i \leqslant n, \quad r \leqslant r_2. \tag{4.2}$$

It follows that if $N\alpha \geq \frac{1}{2}$, $r \leq r_2$, then

$$P(N'_{rm} > N) \leq 2^{m-1} en(N\alpha^{m-1} e^{-\alpha N} \exp(C_1(N\alpha)\alpha^{1/d}). \tag{4.3}$$

Let

$$N = \frac{1}{\alpha}\left(\log\frac{1}{\alpha} + (d+m-1)\log\log\frac{1}{\alpha} + x\right), \quad x \geq 0. \tag{4.4}$$

We have $P(N'_{rm} > N) = P(N'_{rm} > [N])$. For $\alpha \geq 1/e$, $x \geq 0$, $[N]\alpha \geq N\alpha - \alpha \geq e-1$. We may therefore insert $[N]$ into (4.3) and obtain

$$P(N'_{rm} > N) \leq 2^{m-1} e^2 n(N\alpha)^{m-1} e^{-\alpha N} \exp(C_1(N\alpha)\alpha^{1/d})$$

$$\leq 2^{m-1} e^{2-x} \exp(C_1(N\alpha)\alpha^{1/d}) \left(\frac{\log\frac{1}{\alpha} + (d+m-1)\log\log\frac{1}{\alpha} + x}{\log\frac{1}{\alpha}}\right)^{m-1} \tag{4.5}$$

for $x \geq 0$, $r \leq r_2$.

Choose $0 < r_3 < r_2$ so that

$$\exp(C_1(N\alpha)\alpha^{1/d}) \leq 2e^{x/2}, \quad \text{for} \quad r \leq r_3 \tag{4.5}$$

Then

$$P(N'_{rm} > N) \leq (2^m e^2) e^{-x/2} \left(\frac{\log\frac{1}{\alpha} + (d+m-1)\log\log\frac{1}{\alpha} + x}{\log\frac{1}{\alpha}}\right)^{m-1}$$

$$\leq (2^m e^2)(d+m+x)^{m-1} e^{-x/2} \leq C_2 e^{-x/4}, \quad x \geq 0, \quad r \leq r_3, \tag{4.7}$$

where $C_2 = 2^m e^2 \sup_{0 \leq x < \infty} \{(d+m+x)^{m-1} e^{-x/4}\}$.

Now

$$\frac{1}{n^{1/d}} \sim \frac{\alpha^{1/d}}{\log\frac{1}{\alpha}} \sim \frac{(\omega_d)^{1/d}}{d} \frac{r}{\log\frac{1}{r}} \quad \text{as} \quad r \to 0.$$

It follows that we may choose $0 < r_4 < r_3$, so that $\varrho = r - C_0 n^{-1/d} > 0$ for $r \leq r_4$, C_0 being the constant appearing in Theorem 2.1. Let $\alpha = \omega_d r^d$, $\beta = \omega_d \varrho^d$. In view of Theorem 2.1, if the balls of radius ϱ cover Sm times, then the balls of radius r cover Xm times. Hence $N_{rm} \leq N'_{\varrho m}$ and

$$P(N_{rm} > N) \leq P(N'_{\varrho m} > N) \tag{4.8}$$

Let $l(\alpha)=(1/\alpha)(\log(1/\alpha)+(d+m-1)\log\log(1/\alpha))$ and define y by the relation

$$N=l(\alpha)+\frac{x}{\alpha}=l(\beta)+\frac{y}{\alpha}\Leftrightarrow y=\beta(l(\alpha)-l(\alpha))+\frac{\beta}{\alpha}x. \tag{4.9}$$

We express β in the terms of r, and y in terms of r and x. We have

$$\beta=\omega_d\left(r-\frac{C_0}{n^{1/d}}\right)^d=\omega_d r^d+O\left(\frac{r^d}{\log 1/r}\right)=\alpha\left(1+O\left(\frac{1}{\log 1/r}\right)\right) \quad \text{as} \quad r\to 0. \tag{4.10}$$

We conclude from (4.10) by straightforward computations that

$$\begin{cases} \dfrac{1}{\beta}=\dfrac{1}{\alpha}\left(1+O\left(\dfrac{1}{\log 1/r}\right)\right) \\ \log\dfrac{1}{\beta}=\log\dfrac{1}{\alpha}+O\left(\dfrac{1}{\log 1/r}\right) \\ \log\log\dfrac{1}{\beta}=\log\log\dfrac{1}{\alpha}+O\left(\dfrac{1}{\log^2 1/r}\right), \end{cases} \tag{4.11}$$

which in turn yields

$$l(\beta)=l(\alpha)+O\left(\frac{1}{\beta}\right), \tag{4.12}$$

so that

$$y=O(1)+\left(1+O\left(\frac{1}{\log 1/r}\right)\right)x \quad \text{as} \quad r\to 0. \tag{4.13}$$

We may therefore choose $0<r_5<r_4$, $C_3>0$ so that

$$y\geqslant\frac{x}{2}-C_3, \tag{4.14}$$

provided $r\leqslant r_5$. If $x\geqslant 2C_3$, then $y\geqslant 0$ and we conclude from (4.7), (4.8), (4.14) that

$$P(N_{rm}>N)\leqslant C_2e^{-y/4}\leqslant C_2e^{C_3/4}e^{-x/8} \tag{4.15}$$

for $x\geqslant 2C_3$, $r\leqslant r_5$.

For $0\leqslant x\leqslant 2C_3$, $P(N_{rm}>N)\leqslant 1\leqslant e^{C_3/4}e^{-x/8}$. It follows that

$$P(N_{rm}>N)\leqslant C_4e^{-x/8}, \quad x\geqslant 0, \quad r\leqslant r_5, \tag{4.16}$$

where $C_4=e^{C_3/4}\operatorname{Max}(1, C_2)$. Renaming C_4, r_5 as C, r_1, we obtain (1.1). ∎

§ 5. Proof of inequality (1.2)

Our derivation will be based on the lower bound (5.1) for $P(N_{rm} > N)$. As mentioned in the introduction the measure space C_r can be identified with X so that Ω can be identified with $X \times X \times \ldots \times X \times \ldots$ The probability measure on Ω, denoted by $d\omega$, is then the product $dp \times dp \times \ldots \times dp \times \ldots$, where dp is the Riemannian volume measure on X. The points of Ω are denoted by ω.

THEOREM 5.1. *Let $\mu(\omega)$ = volume of the set of points of X not covered m times in the first N throws. Then*

$$P(N_{rm} > N) \geqslant E^2(\mu)/E(\mu^2) \tag{5.1}$$

$E(\mu)$, $E(\mu^2)$ denoting respectively the expectation of μ and μ^2.

Proof. We reproduce the proof found in [9]. Let

$$\phi(\omega) = \begin{cases} 1, & \text{if } X \text{ fails to be covered } m \text{ times in first } N \text{ throws} \\ 0, & \text{otherwise} \end{cases}$$

Observe that $\mu(\omega) = \mu(\omega)\phi(\omega)$. For if $\phi(\omega) = 1$, then this equation becomes $\mu(\omega) = \mu(\omega)$, and if $\phi(\omega) = 0$, then $\mu(\omega) = 0$ so that the equation becomes $0 = 0$. Applying Schwarz's inequality, we get $E^2(\mu) \leqslant E(\mu^2)E(\phi)$. But $E(\phi)$ = probability that X fails to be covered m times in first N throws $= P(N_{rm} > N)$, thus proving (5.1).

We derive next expressions for $E(\mu)$, $E(\mu^2)$.

THEOREM 5.2.

$$E(\mu) = \sum_{i=0}^{m-1} \binom{N}{i} \int_X (v(p,r))^i (1 - v(p,r))^{N-i}\, dp \tag{5.2}$$

Proof. Let $p \in X$, $\omega \in \Omega$ and define

$$\phi(p, \omega) = \begin{cases} 1, & \text{if } p \text{ is not covered } m \text{ times in first } N \text{ throws} \\ 0, & \text{otherwise} \end{cases}$$

Then $\mu(\omega) = \int_X \phi(p, \omega)\, dp$ and, by Fubini's theorem,

$$E(\mu) = \int_X \int_\Omega \phi(p, \omega)\, d\omega\, dp. \tag{5.3}$$

We have however

$$\int_\Omega \phi(p, \omega)\, d\omega = P(p \text{ is not covered } m \text{ times in first } N \text{ throws})$$

$$= \sum_{i=0}^{m-1} \binom{N}{i} (v(p,r))^i (1 - v(p,r))^{N-i}. \tag{5.4}$$

Substitution of (5.4) into (5.3) yields (5.2).

THEOREM 5.3 *Let* $v(p,q,r)=v[B(q,r)-B(p,r)], w(p,q,r)=v[B(p,r)\cap B(q,r)]$. *Then*

$$E(\mu^2)=\sum_{0\leq i,j\leq m-1}\frac{N!}{i!\,j!(N-i-j)!}\iint_{\delta(p,q)>2r}(v(p,r))^i(v(q,r))^j(1-v(p,r)-v(q,r))^{N-i-j}dq\,dp$$
$$+\sum_{0\leq i+k,\,j+k\leq m-1}\frac{N!}{i!\,j!\,k!(N-i-j-k)!}\iint_{\delta(p,q)\leq 2r}(v(p,q,r))^i(v(q,p,r))^j(w(p,q,r))^k$$
$$\times(1-v(p,r)-v(q,r)+w(p,q,r))^{N-i-j-k}dq\,dp \tag{5.5}$$

the indices i, j, k *in the second sum being* ≥ 0.

Proof. $\mu^2(\omega)=\int_X\int_X\phi(p,\omega)\phi(q,\omega)\,dp\,dq$ so that, by Fubini's Theorem

$$E(\mu^2)=\int_X\int_X\int_\Omega\phi(p,\omega)\phi(q,\omega)\,d\omega\,dp\,dq. \tag{5.6}$$

Now

$$\int_\Omega\phi(p,\omega)\phi(q,\omega)\,d\omega=P\text{ (Both } p \text{ and } q \text{ are not covered } m \text{ times in first } N \text{ throws).} \tag{5.7}$$

The above probability can be computed as follows. Suppose that of the first N balls thrown on X, i of the centers be in $B(q,r)-B(p,r)$, j in $B(p,r)-B(q,r)$, k in $B(p,r)\cap B(q,r)$ Then both p and q are not covered m times in first N throws iff $i+k,\ j+k\leq m-1$. It follows that

$$\int_\Omega\phi(p,\omega)\phi(q,\omega)\,d\omega=\sum_{0\leq i+k,\,j+k\leq m-1}\frac{N!}{i!\,j!\,k!(N-i-j-k)!}(v(p,q,r))^i(v(q,p,r))^j$$
$$\times(w(p,q,r))^k(1-v(p,r)-v(q,r)+w(p,q,r))^{N-i-j-k}. \tag{5.8}$$

Observe that for $\delta(p,q)>2r$, $v(p,q,r)=v(q,r)$, $w(p,q,r)=0$. Thus in this case, the sum in (5.8) simplifies to

$$\sum_{0\leq i,j\leq m-1}\frac{N!}{i!\,j!(N-i-j!}v(p,r))^i(v(q,r))^j(1-v(p,r)-v(q,r))^{N-i-j}. \tag{5.9}$$

Substituting (5.8), (5.9) into (5.6), we get (5.5). ■

We designate respectively the two sums appearing in the right side of (5.5) as $\sum_1$, $\sum_2$, so that $E(\mu^2)=\sum_1+\sum_2$. In the sequel, C denotes a generic positive constant depending only on m.

THEOREM 5.4 (i) $\exists \varepsilon > 0$ such that

$$\sum\nolimits_1 \leqslant E^2(\mu), \quad \text{for} \quad r \leqslant \varepsilon, \quad N\alpha \geqslant \frac{1}{\varepsilon}, \tag{5.10}$$

(ii) $\exists \varepsilon > 0$, $C > 0$ such that

$$\sum\nolimits_2 \leqslant \frac{C}{N}(N\alpha)^{m-d}\int_X (1-v(p,r))^N dp, \quad \text{for } r \leqslant \varepsilon, \quad N\alpha \geqslant 1. \tag{5.11}$$

Proof. (*i*) Squaring both sides of (5.2), we get

$$E^2(\mu) = \sum_{0 \leqslant i,j \leqslant m-1} \binom{N}{i}\binom{N}{j}\int_X\int_X (v(p,r))^i (v(q,r))^j (1-v(p,r))^{N-i}(1-v(q,r))^{N-j}\,dp\,dq. \tag{5.12}$$

We prove (5.10) by showing that each term of $\Sigma_1 \leqslant$ corresponding term of $E^2(\mu)$. This inequality follows from

$$\frac{N!}{i!\,j!(N-i-j)!} = \binom{N}{i}\binom{N-i}{j} \leqslant \binom{N}{i}\binom{N}{j} \tag{5.13}$$

and

$$(1-v(p,r)-v(q,r))^{N-i-j} \leqslant (1-v(p,r))^{N-i}(1-v(q,r))^{N-j} \quad \text{for } p,q \in X, \quad 0 < r \leqslant \varepsilon,$$
$$N\alpha \geqslant \frac{1}{\varepsilon}, \quad 0 < i,j \leqslant m-1. \tag{5.14}$$

(5.13) is obvious and we prove (5.14). Let $v(p,r) = \alpha + x$, $v(q,r) = \alpha + \pi$. Taking logarithms, (5.14) becomes

$$N[\log(1-\alpha-x) + \log(1-\alpha-y) - \log(1-2\alpha-x-y)]$$
$$+[(i+j)\log(1-2\alpha-x-y) - i\log(1-\alpha-x) - j\log(1>\alpha-y)] \geqslant 0 \tag{5.15}$$

According to Theorem 2.3, $\exists R > 0$, $C > 0$, such that $|x|$, $|y| \leqslant C\alpha^{1+1/d}$ for all $p, q \in X$ and $r < R$. Using the Taylor expansion for $\log(1-z)$, we obtain

$$\log(1-\alpha-x) = -\alpha - x - \frac{\alpha^2}{2} + O(\alpha^{2+1/d}) \quad \text{as} \quad \alpha \to 0 \tag{5.16}$$

with similar expressions for $\log(1-\alpha-y)$, $\log(1-2\alpha-x-y)$. It follows that the first bracketed term of (5.15) $= \alpha^2(1+O(\alpha^{1/d}))$, and the second bracketed term $= O(\alpha)$. Hence (5.15) may be rewritten as

$$N\alpha(1+O(\alpha^{1/d})) + O(1) \geqslant 0 \tag{5.17}$$

(5.17) clearly holds if $r \leqslant \varepsilon$, $N\alpha \geqslant 1/\varepsilon$, provided $\varepsilon \to 0$ is sufficiently small. Hence (5.14) holds and we have proven (5.10).

(ii) We shall estimate

$$I(p,r) = \int_{\delta(p,q)\leqslant 2r} (v(p,q,r))^i (v(q,p,r))^j (w(p,q,r))^k (1 - v(p,r) - v(p,q,r))^{N-i-j-k} dq \tag{5.18}$$

for small r, uniformly in p and in the indices i, j, k.

It follows from Lemma 2.2 and Theorems 2.3, 2.4 that $\exists r_1 > 0$ such that

$$v(p,r) \leqslant 2\alpha, \quad r \leqslant r_1, \quad \text{and} \quad p \text{ arbitrary} \tag{5.19}$$

$$\frac{\omega_{d-1}}{2^{d+1}} r^{d-1}\delta(p,q) \leqslant v(p,q,r) \leqslant 2\omega_{d-1} r^{d-1}\delta(p,q), \quad r \leqslant r_1, \quad \delta(p,q) \leqslant 2r. \tag{5.20}$$

Using (5.19), (5.20), and $w(p, q, r) \leqslant v(q, r)$, we obtain

$$I(p,r) \leqslant C r^{(i+j+k)d-(i+j)} \int_{\delta(p,q)\leqslant 2r} [\delta(p,q)]^{i+j} \left[1 - v(p,r) - \frac{\omega_{d-1} r^{d-1}}{2^{d+1}} \delta(p,q)\right]^N dq, \tag{5.21}$$

$r \leqslant r_1$ and p arbitrary.

An examination of the proof of Theorem 2.3 shows that $\exists 0 < r_2 < r_1$ such that the following holds:

(i) For each $p \in X$, a local normal coordinate η may be chosen at p valid for $\|\eta\| < r_2$

(ii) Let the element of volume $dq = k(p,\eta)d\eta$, $\|\eta\| < r_2$, where η is the chosen local normal coordinate at p. $k(p, n)$ is uniformly bounded for $p \in X$, $\|\eta\| < r_2$.

Hence the integral appearing in (5.21) is

$$\leqslant C \int_{\|\eta\|\leqslant 2r} \|\eta\|^{i+j} \left(1 - v(p,r) - \frac{\omega_{d-1} r^{d-1}}{2^{d+1}} \|\eta\|\right)^N d\eta \tag{5.22}$$

provided $r \leqslant r_2/2$.

Let $\varrho, \theta_1, \ldots, \theta_{d-1}$ be polar coordinates in the η-space ($\varrho = \|\eta\|$ and the θ_i's are respectively the radial and angular coordinates). The integral of (5.22) becomes

$$\int_S \int_0^r \varrho^{i+j+d-1} \left(1 - v(p,r) - \frac{\omega_{d-1} r^{d-1}}{2^{d+1}} \varrho\right)^N d\varrho \, d\sigma \tag{5.23}$$

where S is the unit d-dimensional sphere $\varrho=1$ and $d\sigma$ is the area element on S. Let $t=\frac{\omega_{d-1}r^{d-1}}{2^{d+1}}\varrho$. (5.21)–(5.23) yield

$$I(p,r)\leqslant C\alpha^{k-d+1}\int_0^{1-v}t^{i+j+d-1}(1-v-t)^N\,dt,\quad r<\frac{r_2}{2} \tag{5.24}$$

and p arbitrary where $v=v(p,r)$. Letting $t=(1-v)\lambda$, we get

$$I(p,r)\leqslant C\alpha^{k-d+1}(1-v)^N\int_0^1\lambda^{i+j+d-1}(1-\lambda)^N\,d\lambda. \tag{5.25}$$

The integral of (5.25) is recognized to be the beta function $B(i+j+d,N+1)=(i+j+d-1)!N!/(i+j+d+N)!$. Hence

$$I(p,r)\leqslant C\alpha^{k+d+1}(1-v)^N/N^{i+j+d},\quad r\leqslant r_2/2\quad\text{and}\quad p\text{ arbitrary}. \tag{5.26}$$

Substituting (5.26) into Σ_2, we get

$$\sum\nolimits_2\leqslant\frac{C}{N}\sum_{0\leqslant i+k,\,j+k\leqslant m-1}(N\alpha)^{k-d+1}\int_X(1-v(p,r))^N\,dp,\quad r\leqslant r_2. \tag{5.27}$$

Let $N\alpha\geqslant 1$. Since $k\leqslant m-1$, we have $(N\alpha)^{k-d+1}\leqslant(N\alpha)^{m-d}$. We conclude from (5.27) that

$$\sum\nolimits_2\leqslant\frac{C}{N}(N\alpha)^{m-d}\int_X(1-v(p,r))^N\,dp \tag{5.28}$$

for $r\leqslant r_2/2$, $N\alpha\geqslant 1$, thus proving (5.11) with the choice $\varepsilon=r_2/2$. ■

We can now provide the

Proof of inequality (1.2) Let $r_1=\text{Min}\,(1,\varepsilon)$. Theorems 5.1, 5.4 imply

$$P(N_{rm}>N)\geqslant\frac{E^2(\mu)}{E^2(\mu)+\frac{C}{N}(N\alpha)^{m-d}\int_X(1-v(p,r))^N\,dp},r\leqslant r_1\quad N\alpha\geqslant\frac{1}{r_1}. \tag{5.29}$$

so that

$$P(N_{rm}\leqslant N)\leqslant\frac{\frac{C}{N}(N\alpha)^{m-d}\int_X(1-v(p,r))^N\,dp}{E^2(\mu)},\quad r\leqslant r_1,\quad N\alpha\geqslant\frac{1}{r_1}. \tag{5.30}$$

From Theorems 2.3, 5.2, we conclude that $\exists\ 0 < r_2 < r_1$ such that

$$E(\mu) \geq C(N\alpha)^{m-1} \int_X (1 - v(p, r))^N dp, \quad r \leq r_2, \quad N\alpha \geq \frac{1}{r_2}. \tag{5.31}$$

Hence

$$P(N_{rm} \leq N) \leq \frac{C}{N(N\alpha)^{d+m-2} \int_X (1 - v(p, r))^N dp}, \quad r \leq r_2, \quad N\alpha \geq \frac{1}{r_2}. \tag{5.32}$$

We remark that (5.32) has been derived under the assumption that N is an integer. Since $P(N_{rm} \leq N) = P(N_{rm} \leq [N])$ and $N \geq [N]/2$ for $N \geq 1$, it is readily seen that (5.32) remains true for all $N \geq 1$, provided we replace r_2 by $r_2/2$ and change the constant C.

Let $N = (1/\alpha)(\log (1/\alpha) + (d + m - 1) \log\log (1/\alpha) + x)$, $x \leq 0$. Choose $0 < r_3 < r_2/2$ so that $\alpha \leq 1/e$ for $r \leq r_3$. We consider first the case $N \geq 1/(2\alpha) \log (1/\alpha)$ ($N \geq e/2$ so that (5.32) is applicable). Using the inequality $1 - z \geq e^{-z-z^2}$, $0 \leq z \leq \frac{1}{2}$, and Theorem 2.3, we choose $0 < r_4 < r_3/2$ so that

$$\int_X (1 - v(p, r))^N dp \geq \tfrac{1}{2} e^{-N\alpha}, \quad r \leq r_4 \tag{5.33}$$

(5.32), (5.33) yield

$$P(N_{rm} \leq N) \leq \frac{Ce^{\alpha N}}{N(N\alpha)^{d+m+1}} \leq 2^{d+m-1} Ce^x, \tag{5.34}$$

Suppose next that $N \leq (1/(2\alpha)) \log (1/\alpha)$. $P(N_{rm} \leq N) = 0$ for $N < 0$, in which case (1.2) is obviously true. We therefore assume $N \geq 0 \Leftrightarrow x \geq -\log (1/\alpha) - (d + m - 1) \log\log (1/\alpha)$. Let $n(r) = [(C_1/2r)^d]$ where $C_1 > 0$ is the constant appearing in Theorem 2.2. Choose $0 < r_5 < r_4$ so that $n = n(r) \geq 1$ for $r \leq r_5$. According to Theorem 2.2, there exist n points $p_1, \ldots, p_n$ so that the distance between any pair of these points $\geq C_1 n^{-1/d}$. Since $r \leq C_1 2n^{-1/d}$, we conclude that any open ball of radius r can cover at most one of the p_i's. Let N'_r = number of throws necessary to cover the p_i's. Since $N_{rm} \geq N'_r$, we have $P(N_{rm} \leq N) \leq P(N'_r \leq N)$. $P(N'_r \leq N)$ can be estimated by Theorem 3.1 (just change the terminology, replacing the phrase "ball falling into an urn" by "ball covering a point"). We obtain

$$P(N_{rm} \leq N) \leq P(N'_r \leq N) \leq \prod_{i=1}^{n} (1 - (1 - v_i)^N) \leq \exp\left(-\sum_{i=1}^{n} (1 - v_i)^N\right) \tag{5.35}$$

where $v_i = v(p_i, r)$.

Using the inequality $1-z \geqslant e^{-z-z^2}$, $0 \leqslant z \leqslant \frac{1}{2}$, and Theorem 2.3, we choose $0<r_6<r_5$ so that

$$\sum_{i=1}^{n}(1-v_i)^N \geqslant \frac{C}{\alpha}e^{-\alpha N}, \quad r \geqslant r_6. \tag{5.36}$$

Hence

$$P(N_{rm} \leqslant N) \leqslant \exp\left(-\frac{C}{\alpha}e^{-\alpha N}\right), \quad r \geqslant r_6. \tag{5.37}$$

For $N \leqslant (1/(2\alpha)) \log(1/\alpha)$, $-(C/\alpha)e^{-\alpha N} \leqslant -C\alpha^{-1/2}$. $-C\alpha^{-1/2} \leqslant -\log(1/\alpha)-(d+m-1) \times \log\log(1/\alpha) \leqslant x$ for r sufficiently small. We conclude from (5.37) that $\exists 0<r_7<r_6$ such that

$$P(N_{rm} \leqslant N) \leqslant e^x, \quad \text{provided } N \leqslant (1/(2\alpha))\log(1/\alpha) \quad \text{and} \quad r \leqslant r_7 \tag{5.38}$$

Renaming r_7 as r_1, (5.34) and (5.38) yield (1.2). ■

§6. Proof of Theorem (1.2)

Theorem 1.2 follows directly from the inequalities (1.1), (1.2). We have

$$P(N_{rm}) = \frac{1}{\alpha}\left(\log\frac{1}{\alpha} + (d+m-1)\log\log\frac{1}{\alpha} + E(X_{rm})\right). \tag{6.1}$$

We must show that $E(X_{rm}) = O(1)$ as $r \to 0$. Let $F_{rm}(x) = P(X_{rm} \leqslant x)$. Then $E(|X_{rm}|) = \int_{-\infty}^{\infty}|x|\,dF_{rm}(x)$. For $R>0$, we have

$$\int_0^R |x|\,dF_{rm}(x) = -\int_0^R x\,d(1-F_{rm}(x)) = -R(1-F_{rm}(R)) + \int_0^R (1-F_{rm}(x))\,dx \tag{6.2}$$

By (1.1), $1-F_{rm}(x) \leqslant Ce^{-x/8}$ for $0<r\leqslant r_1$, $x \geqslant 0$. Letting $R \to \infty$ in (6.2), we conclude that

$$\int_0^\infty |x|\,dF_{rm}(x) = \int_0^\infty (1-F_{rm}(x))\,dx \leqslant C\int_0^\infty e^{-x/8}\,dx = 8C, \quad 0<r\leqslant r_1. \tag{6.3}$$

Thus $\int_0^\infty |x|\,dF_{rm}(x)$ is uniformly bounded for $0<r\leqslant r_1$. A similar argument shows that $\int_{-\infty}^0 |x|\,dF_{rm}(x)$ is uniformly bounded for $0<r\leqslant r_1$. It follows that $E(X_{rm}) = O(1)$ as $r \to 0$.

We can also employ (1.1), (1.2) to prove the following generalization of Steutel's asymptotic formula for $E(N_{r1})$, in case X is the circle.

THEOREM 6.2. *Let X be the circle of unit circumference. Then*

$$E(N_{rm}) = \frac{1}{\alpha}\left(\log\frac{1}{\alpha} + \text{m}\log\log\frac{1}{\alpha} + \gamma_m + O(1)\right) \quad \text{as} \quad r \to 0, \tag{6.4}$$

where $\gamma_m = \gamma - \log(m-1)!$, γ being Euler's constant

Proof. We must prove that $\lim_{r\to 0} E(X_{rm}) = \lim_{r\to 0} \int_{-\infty}^{\infty} x\, df_{rm}(x) = \gamma_m$. As shown in [6], $\lim_{r\to 0} F_{rm}(x) = \exp(-e^{-x}/(m-1)!)$. Since $1 - F_{rm}(x) \leqslant Ce^{-x/8}$, $x \geqslant 0$, $r \leqslant r_1$, we conclude from (6.3) and the Dominated convergence theorem that

$$\lim_{r\to 0}\int_0^{\infty} x\, dF_{rm}(x) = \lim_{r\to 0}\int_0^{\infty}[1 - F_{rm}(x)]\, dx = \int_0^{\infty}\left[1 - \exp\left(-\frac{1}{(m-1)!}e^{-x}\right)\right]dx$$
$$= \int_0^{\infty} x\, d\left(\exp\left(-\frac{1}{(m-1)!}e^{-x}\right)\right). \tag{6.5}$$

A similar result holds for $\lim_{r\to 0}\int_{-\infty}$, so that $\lim_{r\to 0}\int_{-\infty}^{\infty} x\, dF_{rm}(x) = \int_{-\infty}^{\infty} x\, d(\exp(-e^{-x}/(m-1)!))$. Letting $t = (e^{-x})/(m-1)!$, we obtain

$$\int_{-\infty}^{\infty} x\, d\left(\exp\left(-\frac{1}{(m-1)!}e^{-x}\right)\right) = \frac{1}{(m-1)!}\int_{-\infty}^{\infty} x\, e^{-x}\exp\left(-\frac{1}{(m-1)!}e^{-x}\right)dx$$
$$= -\log(m-1)!\int_0^{\infty} e^{-t}\, dt - \int_0^{\infty}(\log t)\, e^{-t}\, dt. \tag{6.6}$$

We have $\int_0^{\infty} e^{-t}\, dt = 1$, $\int_0^{\infty}(\log t)\, e^{-t}\, dt = -\Gamma'(1) = -\gamma[1]$.

Hence $\lim_{r\to 0} E(X_{rm}) = \gamma - \log(m-1)!$

Finally, we obtain the analogs of Theorems 1.1, 1.2 for $\mathcal{C}_v$, the family of open balls of given volume v. In view of Theorem 2.3, we may choose $r_0 \leqslant R$(R is the number occuring in Theorem 2.3) so that $v(p, r) \geqslant \frac{1}{2}\omega_d r^d$ for $p \in X$, $0 \leqslant r \leqslant r_0$. Let $v_0 = \frac{1}{2}\omega_d r_0^d$. Thus for $0 < v = v_0$ and arbitrary p, $\exists B(p, r)$ of volume v. Hence $\mathcal{C}_v$ is well defined for $0 < v \leqslant v_0$. The balls of $\mathcal{C}_v$ are in $1-1$ correspondence with their centers. The probability measure P assigned to $\mathcal{C}_v$ is the volume measure on X via this correspondence. The random variables N_m are relabeled as N_{vm} and we define X_{vm} by $N_{vm} = (1/v)(\log(1/v) + (d+m-1)\log\log(1/v) + X_{vm})$. We have

THEOREM 6.3. *For each $m > 0$, $\exists v_1 > 0$ and $C > 0$, v_1 and C depending only on m, such that*

$$P(X_{vm} \geqslant x) \leqslant Ce^{-x/16}, \quad x \geqslant 0, \quad v \leqslant v_1. \tag{6.7}$$

$$P(X_{vm} \leqslant x) \leqslant Ce^{x/2}, \quad x \leqslant 0, \quad v \leqslant v_1. \tag{6.8}$$

THEOREM 6.4. *Let $E(N_{vm})$ be the expectation of N_{vm}. Then*

$$E(N_{vm}) = \frac{1}{v}\left(\log\frac{1}{v} + (d+m-1)\log\log\frac{1}{v} + O(1)\right) \quad as \quad v \to 0 \tag{6.9}$$

Proof. Theorem 6.4 is derived from (6.7), (6.8) in the same way that Theorem 1.2 is derived from (1.1), (1.2), so that we need only prove Theorem 6.3. Let $\tilde{B}(p, v)$, $p \in X$ and $v \leqslant v_0$, be the open ball of volume v centered at p. For given p, the function $v = v(p, r)$ is continuous and strictly increasing for $0 \leqslant r \leqslant r_0$, with $v(p, 0) = 0$, $v(p, r_0) \geqslant v_0$. It follows that the inverse function $r = r(p, v)$ is continuous and strictly increasing for $0 \leqslant v \leqslant v_0$, with $r(p, 0) = 0$, $r(p, v_0) \leqslant r_0$. Define $b(v) = \operatorname{Inf}_{p \in x} r(p, v)$, $C(v) = \sup_{p \in x} r(p, v)$, $\beta(v) = \omega_d b^d$, $\gamma(v) = \omega_d c^d$, $0 \leqslant v \leqslant v_0$.

According to Theorem 2.3 $\exists C > 0$ such that

$$|v - \omega_d[r(p, v)]^d| \leqslant C v^{(d+1)/d}, \quad p \in X, \quad v \leqslant v_0. \tag{6.10}$$

In (6.9) we may replace $\omega_d[r(p, v)]^d$ both by $\beta(v)$ and $\gamma(v)$. Hence

$$\beta(v) = v + O(v^{(d+1)/d}), \quad \gamma(v) = v + O(v^{(d+1)/d}) \quad \text{as} \quad v \to 0. \tag{6.11}$$

Now $B(p, b(v)) \subset \tilde{B}(p, v) \subset B(p, c(v))$, $v \leqslant v_0$. Hence $N_{cm} \leqslant N_{vm} \leqslant N_{bm}$, where $b = b(v)$, $c = c(v)$, so that

$$P(N_{vm} > N) \leqslant P(N_{bm} > N), \quad P(N_{vm} < N) \leqslant P(N_{cm} < N). \tag{6.12}$$

Let $N = (1/v)(\log(1/v) + (d+m-1)\log\log(1/v) + x)$
$= (1/\beta)(\log(1/\beta) + (d+m-1)\log\log(1/\beta) + y)$
$= (1/\gamma)(\log(1/\gamma) + (d+m-1)\log\log(1/\gamma) + z)$

It follows from (6.11) and an analysis similar to the one leading to formula (4.13) that

$$y = O\left(v^{1/d}\log\frac{1}{v}\right) + (1 + O(v^{1/d}))\,x, \quad z = O\left(v^{1/d}\log\frac{1}{v}\right) + (1 + O(v^{1/d}))\,x. \tag{6.13}$$

Hence $\exists 0 < v_1 \leqslant v_0$ such that $\beta(v), \gamma(v) \leqslant r_1$ for $v \leqslant v_1$ (r_1 is the number appearing in Theorem 1.1) and

$$\begin{aligned} &y \geqslant \frac{x}{2}, x \geqslant 1, \quad v \geqslant v_1 \\ &z \leqslant \frac{x}{2}, x \leqslant -1, \quad v \leqslant v_1 \end{aligned} \tag{6.14}$$

We conclude from (1.1), (1.2), (6.12), (6.14) that $\exists C > 0$ such that

$$P(N_{vm} > N) \leqslant Ce^{-x/16}, \quad x \geqslant 0, \quad v \leqslant v_1$$
$$P(N_{vm} < N) \leqslant Ce^{x/2}, \quad x \leqslant 0, \quad v \leqslant v_1 \tag{6.15}$$

(6.15) is identical with (6.6)–(6.7). ■

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