NEAREST NEIGHBOR BIRTH AND DEATH PROCESSES ON THE REAL LINE

BY

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0. Introduction

The purposes of this paper are to lay the foundations for infinite birth and death processes on R^ν, in general, and, in particular, to develop the theory of nearest neighbor birth and death processes on the real line. We are motivated to do this by the paper of F. Spitzer [11] which, among other things, contains the theory for time reversible, nearest neighbor birth and death processes on the integers. Many of our results were already anticipated in [11]. The only ingredient which Spitzer was lacking was sufficiently strong theory for the appropriate class of birth and death processes. These are processes in which an infinite number of individuals exist at each instant and the rate at which new individuals appear or old ones disappear depends on the instantaneous configuration of the existing individuals. In Spitzer's case, the place in which individuals can live is a lattice; for us it is a continuum. However, it is a simple matter to transfer our results to Spitzer's context (cf. remarks 3.14 and 5.10).

Section 1 is concerned with the basic results which we need in order to prove existence of the desired processes. In Section 2 we study birth and death processes in a bounded region. Here we rely on several of the ideas of Preston [10] in which he studied spatial birth and death processes for which the total population is always finite. Section 3 contains a theorem showing that under very general conditions the martingale problem for nearest neighbor birth and death processes on R is well posed. This is the only good uniqueness theorem for such martingale problems that we have been able to prove.

The last three sections are concerned with the existence of time reversible equilibrium

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states. In [11] Spitzer used a beautiful argument to find necessary and sufficient conditions for the existence of a time reversible equilibrium state for nearest neighbor birth and death processes on the integers. We prove the analogous results here. Our proof breaks naturally into three steps, which are carried out in Sections 4, 5, and 6. The results in Sections 4 and 6 rely heavily on the ideas of Spitzer [11]. Section 4 is also similar to some results of Ledrappier [7]; however, he has more stringent smoothness conditions than we do here and does not relate his results to a stochastic process.

The net result of Sections 4–6 is that the only possible time reversible equilibrium states for nearest neighbor birth and death processes on the real line are renewal measures for some density function with a finite first moment, and conversely every such renewal measure is a time reversible equilibrium state for some nearest neighbor birth and death process.

1. Foundations

We first want to describe the state space in which the birth and death process takes place. This space is very similar to the phase space for infinitely many classical particles used in statistical mechanics (see [6]), however there are two important differences. The first is that the particles have no momentum and the second is that we require certain half spaces to have infinitely many particles and no two particles to occupy the same point. This second difference changes the nature of the compact subsets of the state space considerably. Since the compact sets are important to us, we begin with a careful description of our state space.

Let $\tilde{E}$ be the set of all purely atomic locally finite measures, μ, on R^ν such that $\mu(\{y\}) \in \mathcal{N} = \{0, 1, 2, \ldots\}$ for all $y \in R^\nu$. We endow $\tilde{E}$ with the topology of weak convergence on compacts. That is $\mu_n \to \mu$ in $\tilde{E}$ if and only if $\int \varphi \, d\mu_n \to \int \varphi \, d\mu$ for all $\varphi \in C_0(R^\nu)$ (the space of continuous $\mathbf{C}$-valued functions on R^ν having compact support). It is easy to check that this topology makes $\tilde{E}$ into a Polish space (i.e. $\tilde{E}$ admits a metrization in which it is a complete separable metric space). Next, let E be the set of $\mu \in \tilde{E}$ such that $\mu(\{y\}) \in \{0, 1\}$ for all $y \in R^\nu$ and

$$\mu(\{x \in R^\nu : \langle x, \eta \rangle > 0\}) = \infty$$

for all $\eta \in (\{-1, 1\})^\nu$. Give E the relative topology it inherits as a subset of $\tilde{E}$. Obviously E is not closed in $\tilde{E}$. However, the next lemma shows that E is a G_δ subset of $\tilde{E}$, and therefore is again a Polish space (under a suitable metric).

1.1. Lemma. *The set E is a G_δ-subset of $\tilde{E}$. Moreover, a subset Γ of E is precompact in E if and only if each of the following is satisfied:*

(i) *there exist constants* C_N, $N \geqslant 1$, *such that* $\sup_{\mu \in \Gamma} \mu(Q_N) \leqslant C_N$ *for all* $N \geqslant 1$,

(ii) *for all* $\eta \in (\{-1, 1\})^\nu$

$$\lim_{N\to\infty} \inf_{\mu \in \Gamma} \mu\left(\left\{x \in \bar{Q}_N: \langle x, \eta\rangle \geqslant \frac{1}{N}\right\}\right) = \infty$$

(iii) *for each* $N \geqslant 1$ *there exists an* $n \geqslant 0$ *such that*

$$\sup_{\mu \in \Gamma} \sup_{Q \in H(N, n)} \mu(Q) \leqslant 1.$$

Here $Q_N = \{x \in R^\nu: |x_j| < N$ *for* $1 \leqslant j \leqslant \nu\}$ *and* $H(N, n)$ *is the set of open cubes* Q *having the form* $\{x \in R^\nu: (k_j - 1)N/n < x_j < (k_j + 1)N/n$ *with* $k_j \in \{-n+1, \ldots, n-1\}$ *for* $1 \leqslant j \leqslant \nu$.

Proof. To see that E is a G_δ set in $\tilde{E}$ note that

$$A_{M,N} \equiv \{\mu \in \tilde{E}: \mu(\{x \in Q_N: \langle x, \eta\rangle > 0\}) > M \quad \text{for all } \eta \in (\{-1, 1\})^d\}$$

and

$$B_{N,n} \equiv \{\mu \in \tilde{E}: \mu(\bar{Q}) < 2 \quad \text{for all } Q \in H(N, n)\}$$

are open sets in $\tilde{E}$ for all M, N and n. Since

$$E = (\bigcap_{M \geqslant 1} \bigcup_{N \geqslant 1} A_{M,N}) \cap (\bigcap_{N \geqslant 1} \bigcup_{n \geqslant 1} B_{N,n}),$$

we are done.

Next note that (i) is a necessary and sufficient condition for Γ to be precompact as a of $\tilde{E}$. Thus (i) is obviously necessary in order for Γ to be precompact in E. Now suppose that Γ is precompact in E and that (ii) fails. Then there is a sequence $\{\mu_N\}_1^\infty \subseteq \Gamma$ and some $\eta \in (\{-1, 1\})^\nu$ such that

$$\overline{\lim_{N\to\infty}} \mu_N\left(\left\{x \in \bar{Q}_N: \langle x, \eta\rangle \geqslant \frac{1}{N}\right\}\right) < \infty.$$

Since Γ is precompact in E, we may assume that $\mu_N \to \mu \in E$. But this means that for all $M \geqslant 1$:

$$\mu\left(\left\{x \in Q_M: \langle x, \eta\rangle > \frac{1}{M}\right\}\right) \leqslant \underline{\lim}_{N\to\infty} \mu_N\left(\left\{x \in Q_M: \langle x, \eta\rangle > \frac{1}{M}\right\}\right)$$

$$\leqslant \underline{\lim}_{N\to\infty} \mu_N\left(\left\{x \in \bar{Q}_N: \langle x, \eta\rangle \geqslant \frac{1}{N}\right\}\right),$$

and so $\mu(\{x: \langle x, \eta\rangle > 0\}) < \infty$, which is a contradiction. Next suppose that Γ is precompact in E and that (iii) fails. Then we can find an N such that for all $n \geqslant 1$ there exists a $\mu_n \in \Gamma$ and a $Q \in H(N, n)$ such that $\mu_n(Q) \geqslant 2$. Clearly we may assume that $\mu_n \to \mu \in E$. But then

$$\max_{Q\in H(N,m)} \mu(\bar{Q}) \geqslant \overline{\lim_{n\to\infty}} \max_{Q\in H(N,m)} \mu_n(\bar{Q})$$
$$\geqslant \overline{\lim_{n\to\infty}} \max_{Q\in H(N,n)} \mu_n(\bar{Q}) \geqslant 2$$

for all m, and so $\mu \notin E$. We have therefore established the necessity of (i), (ii), and (iii).

Now suppose that $\Gamma \subset E$ and that (i), (ii), and (iii) obtain. Given $\{\mu_n\} \subseteq \Gamma$, we may assume, because of (i), that $\mu_n \to \mu_0 \in \tilde{E}$. What remains to be shown is that $\mu_0 \in E$. But by (ii),

$$\lim_{N\to\infty} \mu_0\left(\left\{x \in \bar{Q}_N : \langle x, \eta\rangle \geqslant \frac{1}{N}\right\}\right) \geqslant \overline{\lim_{N\to\infty}} \lim_{n\to\infty} \mu_n\left(\left\{x \in \bar{Q}_N : \langle x, \eta\rangle \geqslant \frac{1}{N}\right\}\right)$$
$$\geqslant \lim_{N\to\infty} \inf_{\mu\in\Gamma} \mu\left(\left\{x \in \bar{Q}_N : \langle x, \eta\rangle \geqslant \frac{1}{N}\right\}\right) = \infty$$

for all $\eta \in (\{-1, 1\})^\nu$. Also by (iii), for each $N \geqslant 1$, there is an $m \geqslant 1$ such that

$$\max_{Q\in H(N,m)} \mu(Q) \leqslant \underline{\lim_{n\to\infty}} \max_{Q\in H(N,m)} \mu_n(Q) \leqslant 1.$$

Thus $\mu(\{y\}) \in \{0, 1\}$ for all $y \in Q_N$ and all $N \geqslant 1$. Q.E.D.

Define $C_b(E)$ and $B(E)$ to be, respectively, the set of all bounded continuous and bounded measurable functions on E into $\mathbf{C}$. Given $S \subseteq R^\nu$, define $C_b(E; S)$ $(B(E; S))$ to be the set of those $f \in C_b(E)$ $(B(E))$ such that $f(\mu) = f(\nu)$ for all $\mu, \nu \in E$ satisfying $\mu|_S = \nu|_S$. (Here, and throughout, $\mu|_S$ stands for the restriction of μ to S.) The following is a useful criterion for determining classes of functions on E.

1.2. Lemma. *Let S be a bounded measurable subset of R^ν and suppose $\mathcal{H} \subseteq B(E; S)$ is closed under bounded point-wise convergence. If for every finite partition $\mathcal{D}$ of S into measurable sets I, $\mathcal{H}$ contains linear combinations of functions having the form*

$$\mu \to \exp[i \sum_{I\in\mathcal{D}} \lambda_I \mu(I)],$$

where $\{\lambda_I : I \in \mathcal{D}\} \subseteq R$, then $\mathcal{H} = B(E; S)$. In particular, if $\mathcal{H} \subseteq B(E)$ is closed under bounded pointwise convergence and for every choice of $n \geqslant 1$ and bounded disjoint $\Gamma_1, \ldots, \Gamma_n \in \mathcal{B}_{R^\nu}$, $\mathcal{H}$ contains linear combinations of the functions

$$\mu \to \exp\left[i \sum_{j=1}^{n} \lambda_j \mu(\Gamma_j)\right],$$

where $\{\lambda_j\}_1^n \subseteq R$, then $\mathcal{H} = B(E)$.

Proof. The last assertion follows easily from the first, since, by the first part, the condition on $\mathcal{H}$ guarantees that $B(E; Q) \subseteq \mathcal{H}$ for all bounded cubes $Q \subseteq R^\nu$, and $\mathcal{H}$ is closed under bounded pointwise convergence.

To prove the first assertion it suffices to show that $C_b(E; S) \subseteq \mathcal{H}$. To this end, let $\{\mathcal{P}_n : n \geq 1\}$ be a sequence of finite partitions of S such that $\lim_{n\to\infty} \max_{I \in \mathcal{P}_n} \operatorname{diam}(I) = 0$. For each $I \in \bigcup_1^\infty \mathcal{P}_n$, let $a_I \in I$. Given $\mu \in E$, let N_μ be the smallest number such that $\mu(I) \in \{0, 1\}$ for all $I \in \bigcup_{N_\mu}^\infty \mathcal{P}_n$. For each $n \geq 1$, let $I_{1,n}, \ldots, I_{k_n,n}$ be an ordering of $\mathcal{P}_n$, and for $n \geq N_\mu$ define

$$\mu_n|_{S^c} = \mu \quad \text{and} \quad \mu_n|_S = \sum_{j=1}^{k_n} \mu(I_{j,n})\, \delta_{a_{I_{j,n}}}$$

Given $f \in C_b(E; S)$, there is a function $F_n : \mathcal{N}^{k_n} \to \mathbf{C}$ such that

$$F_n(\mu(I_{1,n}), \ldots, \mu(I_{k_n,n})) = \begin{cases} 0 & \text{if} \quad n < N_\mu \\ f(\mu_n) & \text{if} \quad n \geq N_\mu. \end{cases}$$

Since $\mu_n \to \mu$ as $n \to \infty$, we now see that our proof will be complete once we have shown that for any $n \geq 1$ and $F: \mathcal{N}^{k_n} \to \mathbf{C}$, the function $\mu \to F(\mu(I_{1,n}), \ldots, \mu(I_{k_n,n}))$ is in $\mathcal{H}$. To do this, it certainly suffices to treat the case when F vanishes off of a finite set, in which case F is the limit of linear combinations of functions having the form

$$z \to \exp\left[i \sum_{j=1}^{k_n} \lambda_j z_j\right]. \qquad \text{Q.E.D.}$$

For reasons which will become apparent very soon, we want to introduce yet another class of functions on E. Given $\mu \in E$ and $y \in R^\nu$, define

$$\mu\Delta\{y\} = \begin{cases} \mu + \delta_y & \text{if } \mu(\{y\}) = 0 \\ \mu - \delta_y & \text{if } \mu(\{y\}) = 1. \end{cases} \tag{1.3}$$

(We will often think of the elements of E as sets rather than measures. Thus the notation $\mu\Delta\{y\}$ for symmetric difference.) Clearly $\mu\Delta\{y\}$ is again in E. If $f \in B(E)$ and $y \in R^\nu$, define

$$\Delta_y f(\mu) = f_{,y}(\mu) = f(\mu\Delta\{y\}) - f(\mu). \tag{1.4}$$

It is obvious that $f_{,y}(\mu)$ is bounded and jointly measurable in y and μ. (In fact, by Lemma 1.2, it suffices to check this for f of the form $f(\mu) = \mu(\Gamma)$, where $\Gamma \in \mathcal{B}_{R^\nu}$. But then $f_{,y}(\mu) = \chi_\Gamma(y)(1 - 2\mu(\{y\}))$, and $\{(y, \mu): \mu(\{y\}) = 0\}$ is open.) Also note that if $f \in B(E; S)$, then $f_{,y}(\cdot) \in B(E; S)$ for all $y \in R^\nu$. Given a bounded measurable set S in R^ν, let $\mathcal{D}(E; S)$ be the set of $f \in C_b(E; S)$ such that

$$\sup_{\mu \in E} \int |f_{,y}(\mu)|\, \mu(dy) < \infty;$$

and let $\mathcal{D}(E)$ stand for the set of all $f: E \to \mathbf{C}$ such that $f \in \mathcal{D}(E; S)$ for some bounded measurable S in R^ν.

1.5. LEMMA. *For each $\varphi \in C_0^+(R^\nu)$ (the space of non-negative continuous functions on R^ν having compact support), $\lambda \in \mathbf{C}$ and $C>0$, the function f given by $f(\mu)=\exp[\lambda(\int \varphi(y)\mu(dy) \wedge C)]$ is an element of $\mathcal{D}(E;\operatorname{supp}(\varphi))$. In particular, for any bounded open set G in R^ν, the smallest class of functions $f\colon E\to\mathbf{C}$ which contains $\mathcal{D}(E;G)$ and is closed under bounded pointwise convergence is $B(E;G)$.*

Proof. Let $\varphi \in C_0^+(R^\nu)$, $\lambda\in\mathbf{C}$, and $C<\infty$ be given and define f accordingly. Certainly $f\in C_b(E;\operatorname{supp}\varphi)$. Moreover, if $a\in R^\nu$ and $\mu(\{a\})\neq 0$, then

$$f_{,a}(\mu)=\begin{cases} 0 & \text{if } \displaystyle\int\varphi(y)\,\mu(dy)-\varphi(a)\geqslant C \\ (e^{-\lambda\varphi(a)}-1)\,f(\mu) & \text{if } \displaystyle\int\varphi(y)\,\mu(dy)\leqslant C \\ e^{\lambda\int\varphi(y)\,\mu(dy)-\varphi(a)}-e^{\lambda C} & \text{if } \displaystyle\int\varphi(y)\,\mu(dy)-\varphi(a)<C<\int\varphi(y)\,\mu(dy). \end{cases}$$

Choose M so that for $|x|\leqslant\|\varphi\|$, $|e^{\lambda x}-1|\leqslant M|x|$. If $\int\varphi(y)\mu(dy)-\|\varphi\|\geqslant C$, then $\int|f_{,a}(\mu)|\mu(da)=0$. If $\int\varphi(y)\mu(dy)\leqslant C$, then

$$\int|f_{,a}(\mu)|\,\mu(da)\leqslant Me^{|\lambda|C}\int\varphi(a)\mu(da)\leqslant MCe^{|\lambda|C}.$$

If $\int\varphi(y)\mu(dy)-\|\varphi\|<C<\int\varphi(y)\mu(dy)$, then either $f_{,a}(y)=0$ or $|f_{,a}(\mu)|\leqslant Me^{|\lambda|C}|\varphi(a)|$, and so

$$\int|f_{,a}(\mu)|\,\mu(da)\leqslant Me^{|\lambda|C}\int\varphi(a)\,\mu(da)$$
$$\leqslant Me^{|\lambda|C}(C+\|\varphi\|).$$

Thus $f\in\mathcal{D}(E,\operatorname{supp}(\varphi))$.

To prove the second part, it is sufficient, in view of Lemma 1.2 and the fact that $\mathcal{D}(E,G)$ is an algebra, to prove that for any measurable $\Gamma\subseteq G$ and $\lambda\in R$ the map $\mu\to\exp[i\lambda\mu(\Gamma)]$ is a member of the smallest class containing $\mathcal{D}(E;G)$ and closed under bounded pointwise convergence. But χ_Γ is in the smallest class which contains $C_0(G)$ and is closed under bounded point-wise convergence. Q.E.D.

Define $\Omega=D([0,\infty),E)$, the space of right-continuous trajectories on $[0,\infty)$ into E having left limits. Given $\omega\in\Omega$, let $\mu_t(\omega)$ be the position of ω at time t and define $\mathcal{M}_t=\mathcal{B}[\mu_s\colon 0\leqslant s\leqslant t]$, $t\geqslant 0$. We give Ω the Skorohod topology, and denote by $\mathcal{M}$ the Borel field of subsets of Ω. Note that Ω is a Polish space and $\mathcal{M}=\sigma(\bigcup_{t\geqslant 0}\mathcal{M}_t)$. Given non-anticipating functions $b=b(t,y;\omega)$ and $d=d(t,y;\omega)$ on $[0,\infty)\times\Omega$ into the set of non-negative bounded measurable functions on R^ν, we define the non-anticipating operator $\mathcal{L}_t$ on $\mathcal{D}(E)$ by

$$\mathcal{L}_t f(\mu)=\int b(t,y)\,f_{,y}(\mu)\,dy+\int d(t,y)\,f_{,y}(\mu)\,\mu(dy). \tag{1.6}$$

1.7. THEOREM. *Let b, d, and $\mathfrak{L}_t$ be given as in the proceeding and assume that b and d are uniformly bounded by numbers B and D, respectively. Let α: $[0, \infty) \times \Omega \to E$ be a right continous, non-anticipating function having left limits and P a probability measure on $(\Omega, \mathcal{M})$ such that $P(\alpha_0 = \mu^0) = 1$ for some $\mu^0 \in E$. Then the following are equivalent:*

(i) *$f(\alpha_t) - \int_0^t \mathfrak{L}_s f(\alpha_s) ds$ is a P-martingale for all $f \in \mathcal{D}(E)$,*

(ii) *$f(t \wedge T, \alpha_{t\wedge T}) - \int_0^{t\wedge T} (\partial/\partial s + \mathfrak{L}_s) f(s, \alpha_s) ds$ is a P-martingale for all bounded measurable f: $[0, T] \times E \to \mathbf{C}$ such that $f(\cdot, \mu) \in C^1([0, T])$ for all $\mu \in E$, and there is a bounded open cube Q for which $f(t, \cdot) \in B(E; Q)$, $t \in [0, T]$ and $|\partial f/\partial s(t, \mu)| \leq C\mu(Q)$, $(t,\mu) \in [0, T] \times E$, for some $C < \infty$,*

(iii) *$f(\alpha_t) \exp[-\int_0^t (\mathfrak{L}_s f/f)(\alpha_s) ds]$ is a P-martingale for all f: $E \to \mathbf{C}$ such that $|f|$ is uniformly positive and $f \in B(E; Q)$ for some bounded open cube Q,*

(iv) $$\exp\left[\sum_{j=1}^n \lambda_j \alpha_t(s_j) - \sum_{j=1}^n \int_0^t ds \int_{S_j} (e^{\lambda_j} - 1) b(s, y)\, dy - \sum_{j=1}^n \int_0^t ds \int_{S_j} (e^{-\lambda_j} - 1) d(s,y)\, \alpha_s(dy)\right]$$

is a P-martingale for all $n \geq 1$, $\{\lambda_j\}_1^n \subseteq \mathbf{C}^n$, and mutually disjoint bounded sets $S_1, \ldots, S_n \in \mathcal{B}_{R^\nu}$,

(v) $$\exp\left[i \sum_{j=1}^n \lambda_j \alpha_t(S_j) - \sum_{j=1}^n \int_0^t ds \int_{S_j} (e^{i\lambda_j} - 1) b(s, y)\, dy - \sum_{j=1}^n \int_0^t ds \int_{S_j} (e^{-i\lambda_j} - 1) d(s, y)\, \alpha_s(dy)\right]$$

is a P-martingale for all $n \geq 1$, $\{\lambda_j\}_1^n \subseteq R^n$, and mutually disjoint bounded sets $S_1, \ldots, S_n \in \mathcal{B}_{R^\nu}$.

Moreover, if P satisfies one of these equivalent conditions, then for any bounded open cube Q and $T > 0$

$$P(\sup_{0\leq t\leq T} \alpha_t(\bar{Q}) - \mu^0(\bar{Q}) \geq N) \leq \left(\frac{eB|Q|T}{N}\right)^N \tag{1.8}$$

for $N > eB|Q|T$, where $|Q|$ denotes the Lebesgue measure of Q.

Proof. We will first show that (i) implies the estimate in (1.8). Given Q, choose $\varphi \in C_0(R^\nu)$ so that $\chi_{\bar{Q}} \leq \varphi \leq 1$. Given $T > 0$ and $N > eB|Q|T$, define f_λ for $\lambda > 0$ by

$$f_\lambda(\mu) = \exp\left[\lambda\left(\left(\int \varphi(y)(\mu(dy) - \mu^0(dy))\right) \wedge N\right)\right]$$

By Lemma 1.5, $f_\lambda \in \mathcal{D}(E)$ for all $\lambda > 0$, and so, by (i), $f_\lambda(\alpha\ ) - \int_0^t \mathfrak{L}_s f_\lambda(\alpha_s) ds$ is a P-martingale. Since f_λ is uniformly positive, we can apply Lemma 2.1 of [14] to conclude that

$$f_\lambda(\alpha_t) \exp\left[-\int_0^t \frac{\mathfrak{L}_s f_\lambda}{f_\lambda}(\alpha_s)\, ds\right]$$

is a P-martingale. Let $\tau = \inf\{t \geq 0: \int \varphi(y)\,\alpha_t(dy) - \int \varphi(y)\mu^0(dy) \geq N\}$. Then, by Doob's stopping time theorem and an easy calculation, we see that

$$X_\lambda(t) \equiv \exp\left[\lambda\left(\int \varphi(y)(\alpha_{t\wedge T}(dy) - \mu^0(dy)) \wedge N\right) - \int_0^{t\wedge\tau} ds \int (e^{\lambda\varphi(y)} - 1)\, b(s,y)\, dy\right]$$

is a P-supermartingale for all $\lambda > 0$. In particular,

$$e^{\lambda N} P(\tau \leq T) \leq E^P\left[\exp\left[\lambda\left(\int \varphi(y)(\alpha_{T\wedge\tau}(dy) - \mu^0(dy)) \wedge N\right)\right]\right]$$
$$\leq \exp[(e^\lambda - 1) B |\operatorname{supp}\varphi| T]\, E^P[X_\lambda(T)] \leq \exp[(e^\lambda - 1) B |\operatorname{supp}\varphi| T].$$

Taking a sequence of φ's which decrease to $\chi_{\bar{Q}}$, we now have

$$P(\sup_{0\leq t\leq T} \alpha_t(\bar{Q}) - \mu^0(\bar{Q}) \geq N) \leq \exp\{-\lambda N + (e^\lambda - 1) B |Q| T\}$$

for all $\lambda > 0$. Since $N > eB|Q|T$, we can choose λ so that $e^{-\lambda} = (B|Q|T)/N$, and thereby obtain (1.8).

Once (1.8) has been obtained from (i), it is simple to show that for any bounded open cube Q in R^ν and $f \in B(E; Q)$, $f(\alpha_t) - \int_0^t \mathfrak{L}_s f(\alpha_s)\,ds$ is a P-martingale. Indeed, by (1.8), $E^P[\alpha_t(Q)]$ is finite for all $t \geq 0$, and one can use this to show that the set of $f \in B(E; Q)$ such that $f(\alpha_t) - \int_0^t \mathfrak{L}_s f(\alpha_s)\,ds$ is a martingale is closed under bounded point-wise convergence. One now simply applies Lemma 1.5. The case when f depends on t as well as μ is now easy (cf. Theorem 2.1 in [14]).

Next assume that (ii) holds. Then since (ii) implies (i), the estimate (1.8) obtains, and we can use Lemma 2.1 in [14] together with (ii) and (1.8) to arrive at (iii).

Clearly (iv) is a special case of (iii) and (v) of (iv). Finally assume that (v) holds and let Q be a bounded open cube in R^ν. Given $T > 0$ and $N > eB|Q|T$, define $\tau = \inf\{t \geq 0: \alpha_t(\bar{Q}) - \mu^0(\bar{Q}) \geq N\}$, $b_N(t,y) = \chi_{(-\infty,N)}(\alpha_{t\wedge\tau}(\bar{Q}) - \mu^0(\bar{Q}) b(t,y)$, and $d_N(t,y) = \chi_{(-\infty,N)}(\alpha_{t\wedge\tau}(\bar{Q}) - \mu^0(\bar{Q}))\, d(t,y)$. Then if $\mathfrak{P}$ is a finite partition of Q into measurable sets I and $\{\lambda_I: I \in \mathfrak{P}\} \subseteq R$:

$$\exp\left[i \sum_{I\in\mathfrak{P}} \lambda_I \alpha_{t\wedge\tau}(I) - \sum_{I\in\mathfrak{P}} \int_0^t \left(\int_I b_N(s,y)(e^{i\lambda_I} - 1)\, dy + \int_I d_N(s,y)(e^{-i\lambda_I} - 1)\, \alpha_s(dy)\right) ds\right]$$

is a P-martingale. Since $\int_{\bar{Q}} b_N(s,y)\,dy \leq B|Q|$ and $\int_a d_N(s,y)\,\alpha_s(dy) \leq \mu^0(\bar{Q}) + N$, it follows, after an application of Lemma 2.1 in [14] that

$$f(\alpha_{t\wedge\tau}) - \int_0^t \mathfrak{L}_s^{(N)} f(\alpha_{s\wedge\tau})\, ds$$

is a P-martingale for $f \in B(E, Q)$ of the form

$$f(\mu) = \exp[i \sum_{I \in p} \lambda_i \mu(I)].$$

Here $\mathfrak{L}_t^{(N)}$ is defined in terms of b_N and d_N as in (1.6). Since $\mathfrak{L}_s^{(N)}$ is bounded on $B(E, Q)$, we can now apply Lemma 1.2 to conclude that

$$f(\alpha_{t \wedge \tau}) - \int_0^t \mathfrak{L}_s^{(N)} f(\alpha_{s \wedge \tau})\, ds$$

is a P-martingale for all $f \in B(E, Q)$. It is now easy to derive the estimate (1.8) for this Q, T, and N. Since Q, T, and N are arbitrary, it follows that (v) implies (1.8); and the rest of the proof is now easy. Q.E.D.

The next result is proved in exactly the same way as Theorem 3.1 in [13].

1.9. THEOREM. *Let b, d, $\mathfrak{L}_t$ and α_t be as in Theorem 1.7 and suppose P on $(\Omega, \mathfrak{M})$ satisfies $P(\alpha_0 = \mu^0) = 1$ for some $\mu^0 \in E$ and P satisfies one of the equivalent conditions there. Given a stopping time τ, let $\omega \to P_\omega$ be a regular conditional probability distribution* (r.c.p.d.) *of $P \mid \mathfrak{M}_\tau$. Then there is a P null set $N \in \mathfrak{M}_\tau$ such that for all $\omega \notin N$ and all $f \in \mathfrak{D}(E)$*

$$f(\alpha_t) - f(\alpha_{t \wedge \tau(\omega)}) - \int_{t \wedge \tau(\omega)}^t \mathfrak{L}_s f(\alpha_s)\, ds$$

is a P_ω-martingale.

1.10. THEOREM. *Let b, d, $\mathfrak{L}_t$, and α_t be as in Theorem 1.7 and suppose P on $(\Omega, \mathfrak{M})$ satisfies $P(\alpha_0 = \mu^0) = 1$ for some $\mu^0 \in E$ and one of the equivalent conditions* (i)–(v). *Given $t \geq 0$ and a bounded set $\Gamma \in \mathcal{B}_{R^\nu}$, let $\eta_t^+(\Gamma)\,(\eta_t^-(\Gamma))$ equal the number of $s \in [0, t]$ such that $(\alpha_s - \alpha_{s-})\mid_\Gamma = \delta_y$ $((\alpha_s - \alpha_{s-})\mid_\Gamma = -\delta_y)$ for some $y \in \Gamma \setminus \operatorname{supp}(\alpha_{s-})\,(y \in \Gamma \cap \operatorname{supp}(\alpha_{s-}))$. Then for any $\lambda \in \mathbf{C}$, bounded measurable g_+ and g_- on $R^\nu \to \mathbf{C}$ having compact support:*

$$\exp\Bigg[\lambda(\alpha_t(\Gamma) - \mu^0(\Gamma)) + \int g_+(y)\, \eta_t^+(dy) + \int g_-(y)\, \eta_t^-(dy) - \int_0^t \left(\int (e^{\lambda \chi_\Gamma(y) + g_+(y)} - 1)\, b(u, y)\, dy\right) du$$
$$- \int_0^t \left(\int (e^{-\lambda \chi_\Gamma(y) + g_-(y)} - 1)\, d(u, y)\, \alpha_u(dy)\right) du\Bigg]$$

is a P-martingale.

Proof. Let $n \geq 1$ be temporarily fixed, and for $k \geq 0$ let $\omega \to P_\omega^k$ be a r.c.p.d. of $P \mid \mathfrak{M}_{k/n}$. By Theorems 1.7 and 1.9, there is a P-null set $N \in \mathfrak{M}_{k/n}$ such that if $\omega \to f_\omega$ in an $\mathfrak{M}_{k/n}$-measurable map of Ω into $B(E, Q)$ (for some fixed open cube Q in R^ν) then $0 < \inf_{\omega, \mu} |f_\omega(\mu)| \leq \sup_{\omega, \mu} |f_\omega(\mu)| < \infty$ implies

$$\frac{f_\omega(\alpha_{t \vee (k/n)})}{f_\omega(\alpha_{k/n})} \exp\left[-\int_{k/n}^{(k/n) \vee t} \frac{\mathfrak{L}_u f_\omega}{f_\omega}(\alpha_u)\, du\right]$$

is a P_ω^k martingale for all $\omega \notin N$. In particular, if

$$f_\omega(\mu) = \exp\left[\lambda\mu(\Gamma) + \gamma(\mu, \alpha_{k/n}(\omega))\right]$$

where

$$\gamma(\mu, \nu) = \begin{cases} g_+(y) & \text{if} \quad (\mu-\nu)|_{\operatorname{supp}(g_+)\cup\operatorname{supp}(g_-)} = \delta_y \\ g_-(y) & \text{if} \quad (\mu-\nu)|_{\operatorname{supp}(g_+)\cup\operatorname{supp}(g_-)} = -\delta_y \\ 0 & \text{otherwise,} \end{cases}$$

then

$$\begin{aligned}\exp\Bigg[&\lambda(\alpha_{t\vee(k/n)}(\Gamma) - \alpha_{k/n}(\omega;\Gamma)) + \gamma(\alpha_{t\vee(k/n)}, \alpha_{k/n}(\omega)) \\ &- \int_{k/n}^{(k/n)\vee t}\left(\int(\exp\{\lambda\chi_\Gamma(y) + \gamma(\alpha_u+\delta_y, \alpha_{k/n}(\omega)) - \gamma(\alpha_u, \alpha_{k/n}(\omega))\} - 1)\, b(u,y)\,dy\right)du \\ &- \int_{k/n}^{(k/n)\vee t}\left(\int(\exp\{-\lambda\chi_\Gamma(y) + \gamma(\alpha_u-\delta_y, \alpha_{k/n}(\omega)) - \gamma(\alpha_u, \alpha_{k/n}(\omega))\} - 1)\, d(u,y)\,\alpha_u(dy)\right)du\Bigg]\end{aligned}$$

is a P_ω^k-martingale for all $\omega \notin N$. Here we can take Q to be any bounded open cube containing $\Gamma \cup \operatorname{supp}(g_+) \cup \operatorname{supp}(g_-)$. Thus,

$$\begin{aligned}X_{\lambda,g\pm}^{(n)}(t) \equiv \exp\Bigg[&\lambda(\alpha_t(\Gamma) - \mu^0(\Gamma)) + \sum_k \gamma(\alpha_{t\wedge(k+1)/n)}, \alpha_{t\wedge(k/n)}) \\ &- \int_0^t\left(\int(\exp\{\lambda\chi_\Gamma(y) + \gamma(\alpha_u+\delta_y, \alpha_{u_n}) - \gamma(\alpha_u, \alpha_{u_n})\} - 1)\, b(u,y)\,dy\right)du \\ &- \int_0^t\left(\int(\exp\{-\lambda\chi_\Gamma(y) + \gamma(\alpha_u-\delta_y, \alpha_{u_n}) - \gamma(\alpha_u, \alpha_{u_n})\} - 1)\, d(u,y)\,\alpha_u(dy)\right)du\Bigg]\end{aligned}$$

is a P-martingale, where $u_n = [nu]/n$.

Clearly the exponent of $X_{\lambda,g\pm}^{(n)}(t)$ tends to

$$\begin{aligned}\lambda(\alpha_t(\Gamma) - \mu^0(\Gamma)) + \int g_+(y)\,\eta_t^+(dy) + \int g_-(y)\,\eta_t^-(dy) - \int_0^t\left(\int(e^{\lambda\chi_\Gamma(y)+g_+(y)} - 1)\, b(u,y)\,dy\right)du \\ - \int_0^t\left(\int(e^{-\lambda\chi_\Gamma(y)+g_-(y)} - 1)\, d(u,y)\,\alpha_u(dy)\right)du\end{aligned}$$

as $n \to \infty$. Thus the proof will be complete once we show that $\{X_{\lambda,g\pm}^{(n)}(t) \colon n \geqslant 1\}$ is uniformly P-integrable for all $t \geqslant 0$. Note that

$$|X_{\lambda,g\pm}^{(n)}(t)| \leqslant \left|\exp\left[\sum_k \gamma(\alpha_{((k+1)/n)\wedge t}, \alpha_{(k/n)\wedge t})\right]\right| \exp\left[At(|Q| + \sup_{0\leqslant u\leqslant t}\alpha_u(Q))\right],$$

where

$$A = \left(\sup_{y,\mu,\nu}|e^{\lambda\chi_\Gamma(y)+\gamma(\mu+\delta_y,\nu)-\gamma(\mu,\nu)} - 1|\,B\right) \vee \left(\sup_{y,\mu,\nu}|e^{-\lambda\chi_\Gamma(y)+\gamma(\mu+\delta_y,\nu)-\gamma(\mu,\nu)} - 1|\,D\right),$$

and; by (1.8), $\exp[\beta \sup_{0\leqslant u\leqslant t} \alpha_u(Q)]$ is P-integrable for all $\beta>0$. Thus we need only show that

$$\sup_n E^P[|\exp[\sum_k \gamma(\alpha_{((k+1)/n)\wedge t}, \alpha_{(k/n)\wedge t})]|^r]<\infty$$

for some $r>1$. For this purpose, we can, and will, assume that g_+ and g_- are real valued. Then

$$\begin{aligned}|\exp[\sum_k \gamma(\alpha_{((k+0)/n)\wedge t}, \alpha_{(k/n)\wedge t})]|^2 &= \exp[2\sum_k \gamma(\alpha_{((k+1)/n)\wedge t}, \alpha_{(k/n)\wedge t})]\\ &\leqslant (X^{(n)}_{0,4g^\pm}(t))^{1/2}\exp[A's(|Q|+\sup_{0\leqslant u\leqslant t}\alpha_u(Q))],\end{aligned}$$

where A' is determined in the same sort of way as A above. Since

$$E^P[X^{(n)}_{0,4g^\pm}(t)]=1,$$

this completes the proof. Q.E.D.

1.11. Corollary. *Let everything be as in Theorem 1.10. Then* $\alpha_. = \mu^0+\eta_.^+ - \eta_.^-$ (a.s., P). *In particular, if* Γ *is a bounded element of* $\mathcal{B}_{R^\nu}$ *and* α_t^Γ *denotes the restriction of* α_t *to* Γ, *then* $s\to\alpha_s^\Gamma$, $0\leqslant s\leqslant t$, *consists, P-almost surely, of a finite number of jumps, each of which entails the addition or deletion of exactly one atom. Finally, if* $\Gamma\in\mathcal{B}_{R^\nu}$ *is a set of Lebesgue measure zero, then* $\alpha_.^\Gamma = \mu^0|_\Gamma - \eta_.^-|_\Gamma$ (a.s., P).

Proof. Given a bounded $\Gamma\in\mathcal{B}_{R^\nu}$, take $\lambda=i\theta$ and $g_\pm=\mp i\theta\chi_\Gamma$ in Theorem 1.10 for some $\theta\in R$. Then

$$E[\exp[i\theta(\alpha_t(\Gamma)-\mu^0(\Gamma)-\eta_t^+(\Gamma)+\eta_t^-(\Gamma))]]=1$$

for all $t\geqslant 0$. Since $\mathcal{B}_{R^\nu}$ is countably generated, this proves that $\alpha_t=\mu^0+\eta_t^+-\eta_t^-$ for each $t\geqslant 0$. Since both sides of the preceding equation are right continuous in t, the first assertion is now proved. The second assertion is an immediate consequence of the first. Finally, to prove the last assertion, we must check that $\eta_t^+(\Gamma)=0$ (a.s., P) if $|\Gamma|=0$. But if $\lambda=0$, $g_+=i\theta\chi_{\Gamma\cap B(0,N)}$ ($B(0,N)$ is the ball with center 0 and radius N), and $g_-=0$ in Theorem 1.10, then

$$E^P[e^{i\theta\eta_t^+(\Gamma\cap B(0,N))}]=1,$$

and so the desired conclusion follows immediately. Q.E.D.

1.12. Lemma. *Let everything be as in Theorem 1.10. If* $T>0$ *and* $S\subseteq R^\nu$ *is a finite set, then there is an* $0<\varrho<1$ *depending only on* DT *such that:*

$$P\left(\inf_{0\leqslant t\leqslant T}\alpha_t(S)\leqslant\frac{\mu^0(S)}{2}e^{-DT}\right)\leqslant\varrho^{\mu^0(S)}.$$

Proof. First note that, by Corollary 1.11, $\alpha_.(S)$ is P-almost surely non-increasing. Assume that $\mu^0(S)=N\geqslant 1$. We must show that $P(\alpha_T(S)\leqslant\frac{1}{2}Ne^{-DT})\leqslant\varrho^N$ for some $0\leqslant\varrho<1$ depending only on DT.

Set

$$f_n(t,\mu)=\begin{cases}\sum_{k=n}^{\mu(S)}\binom{\mu(S)}{k}e^{-kDt}(1-e^{-Dt})^{\mu(S)-k} & \text{if} \quad \mu(S)\geqslant n\\ 0 \quad \text{if} \quad \mu(S)<n\end{cases}$$

for $n\geqslant 0$, $t\geqslant 0$, and $\mu\in E$. It is easy to check that

$$\frac{\partial f_n}{\partial t}=D\int(f_n(\mu-\delta_y)-f_n(\mu))\,\mu(dy)$$

and $f_n(t,\mu+\delta_y)\geqslant f_n(t,\mu)$ for all $y\in R^\nu$ and $\mu\in E$. From these observations it is easy to see that $f_n(T-(t\wedge T),\alpha_{t\wedge T})$ is a P-submartingale. Hence

$$P(\alpha_T(S)\geqslant n)=E^P[f_n(0,\alpha_T)]\geqslant E^P[f_n(T,\alpha_0)]=f_n(T,\mu^0),$$

and so

$$P(\alpha_T(S)<n)\leqslant\sum_{k=0}^{n-1}\binom{N}{k}e^{-kDT}(1-e^{-DT})^{N-k}.$$

The desired conclusion is immediate from this. Q.E.D.

1.13. LEMMA. *Let everything be as in Theorem 1.10. Given $L>0$, let $Q=\{x\in R^\nu\colon -L<x_j<L,$ $1\leqslant j\leqslant\nu\}$; and for $n\geqslant 1$ and $\mathbf{k}\in(\{-n+1,\ldots,n-1\})^\nu$, set*

$$Q_{\mathbf{k}}^{(n)}=\left\{x\in Q\colon \frac{(k_j-1)L}{n}<x_j<\frac{(k_j+1)L}{n},\ 1\leqslant j\leqslant\nu\right\}.$$

Then for each $\varepsilon>0$ there is an $N\geqslant 1$, depending only on BLT and $\mu^0(Q)$, such that if $n\geqslant N$ and $\max_{\mathbf{k}}\mu^0(Q_{\mathbf{k}}^{(n)})\leqslant 1$, then

$$P(\max_{\mathbf{k}}\sup_{0\leqslant t\leqslant T}\alpha_t(Q_{\mathbf{k}}^{(n)})>1)<\varepsilon.$$

Proof. Let $\alpha_t^+=\mu^0+\eta_t^+$. Clearly $\alpha_.^+$ is non-decreasing and $\alpha_.\leqslant\alpha_.^+$. Thus, it suffices for us to prove that for each $\varepsilon>0$ a suitable choice of N can be made so that $n\geqslant N$ and $\max\mu^0(Q_{\mathbf{k}}^{(n)})\leqslant 1$ implies

$$P(\max_{\mathbf{k}}\alpha_T^+(Q_{\mathbf{k}}^{(n)})>1)<\varepsilon.$$

We do this first under the assumption that $b\equiv B$, and for convenience we will assume $\nu=1$.

Let n be given, and assume that $\mu^0(Q_k^{(n)})\leqslant 1$ for $-n+1\leqslant k\leqslant n-1$. Denote by $\mathfrak{J}^{(n)}$ the set of subsets, S, of $\{-n+1, ..., n\}$ such that $\{k\colon -n+1\leqslant k\leqslant n$ and $\mu^0(((k-2)L/n, (k+1)L/n]\cap(-L, L))\geqslant 1\}\subseteq S^c$ and for any $k\in S\cap\{-n+2, ..., n-1\}$ neither $k-1$ nor $k+1$ is in S. Then

$$P\left(\max_{|k|<n}\alpha_T^+\left(\left(\frac{(k-1)L}{n}, \frac{(k+1)L}{n}\right]\right)\leqslant 1\right)$$

$$\geqslant\sum_{S\in\mathfrak{J}^{(n)}}P\left(\eta_T^+\left(\left(\frac{(k-1)L}{n}, \frac{kL}{n}\right]\right)=1,\ k\in S\ \text{and}\ \eta_T^+\left(\left(\frac{(k-1)L}{n}, \frac{kL}{n}\right]\right)=0,\ k\in S^c\right).$$

Notice that, since $b\equiv B$, $\{\eta_T^+(((k-1)L/n, kL/n])\colon -n+1\leqslant k\leqslant n\}$ are mutually independent $\mathcal{N}$-valued random variables such that

$$P\left(\eta_T^+\left(\left(\frac{(k-1)L}{n}, \frac{kL}{n}\right]\right)=m\right)=e^{-BLT/n}\left(\frac{BLT}{n}\right)^m\Big/m!$$

(This observation is immediate from Theorem 1.10.) Thus

$$P\left(\max_{|k|<n}\alpha_T^+\left(\left(\frac{(k-1)L}{n}, \frac{(k+1)L}{n}\right]\right)\leqslant 1\right)\geqslant e^{-2BLT}\sum_{S\in\mathfrak{J}^{(n)}}\left(\frac{BLT}{n}\right)^{|S|}$$

$$=e^{-2BLT}\sum_{j=0}^{2n}|\mathfrak{J}_j^{(n)}|\left(\frac{BLT}{n}\right)^j,$$

where $\mathfrak{J}_j^{(n)}=\{S\in\mathfrak{J}^{(n)}\colon |S|=j\}$. It is easy to check that $|\mathfrak{J}_j^{(n)}|\leqslant\binom{2n}{j}$ and that for each j

$$\lim_{n\to\infty}|\mathfrak{J}_j^{(n)}|\Big/\binom{2n}{j}=1$$

at a rate that depends only on $\mu^0(Q)$. This completes the proof in the case when $b\equiv B$.

In general, we proceed as follows. Given n, we can use the reasoning just given if we can show that the vector

$$\left(\eta_T^+\left(\left(-L, \frac{(-n+1)L}{n}\right]\right), \dots, \eta_T^+\left(\left(\frac{(n-1)L}{n}, L\right]\right)\right)$$

is stochastically smaller than in the case $b\equiv B$. To this end, define

$$\Delta_k(\omega)=\frac{BLT}{n}-\int_0^T\left(\int_{((k-1)L)/n}^{kL/n}b(u, y;\omega)\,dy\right)du$$

for $-n+1\leqslant k\leqslant n$. For $\omega\in\Omega$, let μ_ω be the probability measure on $\mathcal{N}^{2n}$ given by:

$$\eta_\omega(\{(m_{-n+1}, \dots, m_n)\})=\prod_{k=-n+1}^{n}e^{-\Delta_k(\omega)}\frac{(\Delta_k(\omega))^{m_k}}{m_k!};$$

and define $\tilde{P}$ on $\Omega \times \mathcal{N}^{2n}$ so that

$$\tilde{P}(A \times \{(m_{-n+1}, \ldots, m_n)\}) = E^P[\mu.(\{(m_{-n+1}, \ldots, m_n)\}), A].$$

Then an easy computation shows that if

$$X_k = \eta_T^+\left(\left(\frac{(k-1)L}{n}, \frac{kL}{n}\right]\right) + m_k, \quad -n+1 \leqslant k \leqslant n,$$

then

$$E^{\tilde{P}}\left[\exp\left[i \sum_{-n+1}^{n} \lambda_k X_k\right]\right] = \exp\left[\frac{BLT}{n} \sum_{-n+1}^{n} (e^{i\lambda_k} - 1)\right],$$

and so

$$\tilde{P}(X_k = l_k, \ -n+1 \leqslant k \leqslant n) = e^{-2BLT} \prod_{-n+1}^{n} \left(\frac{BLT}{n}\right)^{l_k} \Big/ l_k! \qquad \text{Q.E.D.}$$

1.14. Lemma. *Let everything be as in Theorem 1.10. Given $s \geqslant 0$ and a bounded open cube Q in R^ν there is a constant C depending on s, B, D, $|Q|$, and $\mu^0(Q)$ such that:*

$$P((\exists t \in [s, s+\varepsilon))\, \alpha_t^Q \neq \alpha_s^Q) \leqslant C\varepsilon.$$

Proof. Let $\tau = \inf\{t \geqslant s: \alpha_t^Q \neq \alpha_s^Q\}$. Let $\omega \to P_\omega$ be a r.c.p.d. of $P \,|\, \mathcal{M}_s$ and $\eta_t = \eta_t^+ + \eta_t^-$. Then, by Theorem 1.9

$$P_\omega(\tau \leqslant t) = E^{P_\omega}[\eta_{t\wedge\tau}(Q) - \eta_s(Q)] = E^{P_\omega}\left[\int_s^{t\wedge T}\left(\int_Q b(u,y)\,dy + \int_Q d(u,y)\,\alpha_u(dy)\right)du\right]$$
$$\leqslant (t-s)(B|Q| + D\alpha_s(Q)).$$

The result now follows from an application of (1.8). Q.E.D.

1.15. Theorem. *Let I be an index set and for each $\alpha \in I$ let b_α and d_α be non-anticipating functions on $[0, \infty) \times \Omega$ into $B^+(R^\nu)$ such that $B = \sup_\alpha \|b_\alpha\| < \infty$ and $D = \sup_\alpha \|d_\alpha\| < \infty$. Let $\{\mu^\alpha: \alpha \in I\}$ be a precompact subset of E. For $\alpha \in I$, suppose P_α is a probability measure on $(\Omega, \mathcal{M})$ such that $P_\alpha(\mu_0 = \mu^\alpha) = 1$ and $f(\mu_t) - \int_0^t \mathcal{L}_u^\alpha f(\mu_u)\,du$ is a P_α-martingale for all $f \in \mathcal{D}(E)$, where $\mathcal{L}_t^\alpha$ is defined in terms of b_α and d_α as in (1.6). Then $\{P_\alpha: \alpha \in I\}$ is precompact in the weak topology on probability measures on Ω.*

Proof. Using Lemma 1.14, one can show, by standard methods that the Skorohod modulus of continuity on finite time intervals can be controlled independent of $\alpha \in I$. Thus it suffices to show that for any $\varepsilon > 0$ and $T > 0$ there is a compact $K \subseteq E$ such that

$$\sup_\alpha P_\alpha(\sup_{0 \leqslant t \leqslant T} \mu_t \notin K) \leqslant \varepsilon.$$

This can be done by using the characterization of compacts given in Lemma 1.1 and then using Lemmas 1.12 and 1.13 together with estimate (1.8). Q.E.D.

2. The martingale problem; birth and death in a bounded region

Let b: $R^\nu \times E \to [0, \infty)$ and d: $R^\nu \times E \to [0, \infty)$ be bounded measurable functions and define $\mathcal{L}$ on $\mathcal{D}(E)$ by

$$\mathcal{L}f(\mu) = \int b(y, \mu) f_{,y}(\mu)\, dy + \int d(y, \mu) f_{,y}(\mu)\, \mu(dy). \tag{2.1}$$

A probability measure P on $(\Omega, \mathcal{M})$ is said to solve *the martingale problem for $\mathcal{L}$ starting from* $\mu \in E$ if $P(\mu_0 = \mu) = 1$ and $f(\mu_t) - \int_0^t \mathcal{L}f(\mu_s)ds$ is a P-martingale for all $f \in \mathcal{D}(E)$.

2.2. THEOREM. *Let b, d, and $\mathcal{L}$ be as above. If there exists a bounded open cube Q in R^ν such that $b(y, \cdot) = d(y, \cdot) \equiv 0$ for $y \notin Q$, then for each $\mu \in E$ there is exactly one solution P_μ to the martingale problem for $\mathcal{L}$ starting from μ. Moreover, $\mu \to P_\mu(A)$ is measurable for all $A \in \mathcal{M}$ and the family $\{P_\mu: \mu \in E\}$ is strong Markov. Finally if $\tau_0 \equiv 0$ and $\tau_n = \inf\{t \geq \tau_{n-1}: \mu_t \neq \mu_{\tau_{n-1}}\}$, $n \geq 1$, then for all $t > 0$ and $n \geq 1$:*

(i) $P_\mu(\tau_n \leq t$ *and* $\eta_{t_n} - \eta_{\tau_{n-1}} = \delta_y$ *for some* $y \in \Gamma \mid \mathcal{M}_{\tau_{n-1}}) = (1 - \exp[-(t - (t \wedge \tau_{n-1})) \times M(\eta_{\tau_{n-1}})])[b(\Gamma, \mu_{\tau_{n-1}})]/[M(\mu_{\tau_{n-1}})]$ *(a.s., P_μ) on* $\{\tau_{n-1} < \infty$ *and* $M(\mu_{\tau_{n-1}}) > 0\}$ *and equals* 0 *elsewhere.*

(ii) $P_\mu(\tau_n \leq t$ *and* $\mu_{\tau_n} - \mu_{\tau_{n-1}} = -\delta_y$ *for some* $y \in \Gamma \mid \mathcal{M}_{\tau_{n-1}}) = (1 - \exp[-(t - (t \wedge \tau_{n-1})) \times M(\mu_{\tau_{n-1}})])[d(\Gamma, \mu_{\tau_{n-1}})]/[M(\mu_{\tau_{n-1}})]$ *(a.s., P_μ) on* $\{\tau_{n-1} < \infty$ *and* $M(\mu_{\tau_{n-1}}) > 0\}$ *and equals* 0 *elsewhere.*

(iii) $P_\mu(\tau_n \leq t) \searrow 0$ *as* $n \to \infty$, *where*

$$b(\Gamma, \mu) = \int_\Gamma b(y, \mu)\, dy,$$

$$d(\Gamma, \mu) = \int_\Gamma d(y, \mu)\, \mu(dy),$$

and

$$M(\mu) = b(Q, \mu) + d(Q, \mu).$$

Proof. Clearly uniqueness of solutions will be established once we show that any solution must satisfy (i), (ii), and (iii). Let P be a solution to the martingale problem for $\mathcal{L}$ starting from μ^0 and define $\xi_N = \inf\{t \geq 0: \mu_t(Q) - \mu^0(Q) \geq N\}$. By estimate (1.8), $P(\xi_N \leq t) \searrow 0$ as $N \to \infty$. Also, if we can prove (i) and (ii) then

$$P(\tau_n \leq t, \xi_N > \tau_{n-1}) \leq E^P[1 - \exp[-(t - (t \wedge \tau_{n-1}))(\|b\|\,|Q| + \|d\|(\mu^0(Q) + N)], \xi_N > \tau_{n-1}],$$

from which it is easy to conclude that:

$$P(\tau_n \leqslant t < \xi_N) \searrow 0$$

as $n \to \infty$ for each $N \geqslant 1$. Thus (i) and (ii) imply (iii).

We now prove (i) and (ii). Using the notation and results of Theorems 1.9 and 1.10 we have

$$\begin{aligned}
(2.3) \qquad & P(\tau_n \leqslant t \text{ and } \mu_{\tau_n} - \mu_{\tau_{n-1}} = \delta_y \text{ for some } y \in \Gamma \,|\, \mathcal{M}_{\tau_{n-1}}) \\
&= P_\omega^{n-1}(\tau_n \leqslant t \text{ and } \mu_{\tau_n} - \mu_{\tau_{n-1}} = \delta_y \text{ for some } y \in \Gamma) \\
&= E^{P_\omega^{n-1}}[\eta^+_{t \wedge \tau_n}(\Gamma) - \eta^+_{t \wedge \tau_{n-1}}(\Gamma)] \\
&= E^{P_\omega^{n-1}}\left[\int_{t \wedge \tau_{n-1}(\omega)}^{t \wedge \tau_n(\omega)} \int_\Gamma b(y, \mu_s)\, dy\, ds\right] \\
&= b(\Gamma, \mu_{\tau_{n-1}}) \int_{t \wedge \tau_{n-1}(\omega)}^{t} P_\omega^{n-1}(\tau_n > s)\, ds,
\end{aligned}$$

where $\omega \to P_\omega^{n-1}$ is a r.c.p.d. of $P \,|\, \mathcal{M}_{\tau_{n-1}}$. Similarly,

$$\begin{aligned}
(2.4) \qquad & P(\tau_n \leqslant t \text{ and } \mu_{\tau_n} - \mu_{\tau_{n-1}} = -\delta_y \text{ for some } y \in \Gamma \,|\, \mathcal{M}_{\tau_{n-1}}) \\
&= P_\omega^{n-1}(\tau_n \leqslant t \text{ and } \mu_{\tau_n} - \mu_{\tau_{n-1}} = -\delta_y \text{ for some } y \in \Gamma) \\
&= d(\Gamma, \mu_{\tau_{n-1}}) \int_{t \wedge \tau_{n-1}(\omega)}^{t} P_\omega^{n-1}(\tau_n > s)\, ds.
\end{aligned}$$

Taking $\Gamma = Q$ in these two equations and adding we get:

$$P_\omega^{n-1}(\tau_n \leqslant t) = M(\mu_{\tau_{n-1}}) \int_{t \wedge \tau_{n-1}(\omega)}^{t} P_\omega^{n-1}(\tau_n > s)\, ds,$$

which implies immediately that:

$$P_\omega^{n-1}(\tau_n \leqslant t) = \begin{cases} 0 & \text{if } \tau_{n-1}(\omega) = \infty \text{ or } \tau_{n-1}(\omega) < \infty \text{ and } M(\mu_{\tau_{n-1}(\omega)}(\omega)) = 0 \\ 1 - \exp[-(t - (t \wedge \tau_{n-1}(\omega))\, M(\mu_{\tau_{n-1}(\omega)}(\omega))] & \text{otherwise.} \end{cases}$$

Plugging this back into (2.3) and (2.4), one arrives at (i) and (ii) respectively. Thus (i), (ii), and (iii) have been established for any solution to the martingale problem for $\mathfrak{L}$ starting from μ, and so uniqueness of such solutions is obvious.

Next assume that b and d have the property that $b(\cdot, \mu) = d(\cdot, \mu) \equiv 0$ for μ such that $\mu(Q) \geqslant N$. In this case $\mathfrak{L}$ is bounded, and, therefore, there are several ways of establishing the existence of a measurable Markov family of solutions $\{P_\mu : \mu \in E\}$ (cf. Theorem 2.1 in [12] for instance). In general, let $b_N(y, \mu) = \chi_{[0,N)}(\mu(Q))\, b(y, \mu)$ and $d_N(y, \mu) = \chi_{[0,N)}(\mu(Q))\, d(y, \mu)$, and define $\mathfrak{L}^{(N)}$ accordingly. For each $N \geqslant 1$, a measurable Markov family of solutions

$\{P_\mu^N: \mu \in E\}$ exists. Note that if $\sigma_N = \inf\{t \geq 0: \mu_t(Q) \geq N\}$, then, for any $\mu \in E$,

$$f(\mu_{t\wedge\sigma_N}) - \int_0^t \mathcal{L}^{(N)} f(\mu_{s\wedge\sigma_N})\, ds$$

is a P^{N+1}-martingale for all $f \in \mathcal{D}(E)$. Thus, by uniqueness P_μ^{N+1} equals P_μ^N on $\mathcal{M}_{\sigma_N}$ for all N. Moreover, by estimate (1.8), $P_\mu^N(\sigma_N \leq t) \to 0$ as $N \to \infty$ for all $t > 0$ and $\mu \in E$. Hence by standard extension theorems, there is, for each $\mu \in E$, a unique P_μ on $(\Omega, \mathcal{M})$ such that P_μ equals P_μ^N on $\mathcal{M}_{\sigma_N}$ for all $N \geq 1$. Clearly P_μ solves the martingale problem for $\mathcal{L}$ starting from μ and $\mu \to P_\mu(A)$ is measurable for all $A \in \mathcal{M}$. Finally, because of uniqueness, it is easy to derive from these facts that $\{P_\mu: \mu \in E\}$ is a strong Markov family (cf. [13]). Q.E.D.

2.5. THEOREM. *Let everything be as in Theorem 2.2 and let $\{P_\mu: \mu \in E\}$ be the unique family of solutions constructed there. Given $f \in B(E)$, set $u_f(t, \mu) = E^{P_\mu}[f(\mu_t)]$. Then $u_f(t, \mu)$ is the unique bounded measurable function on $[0, \infty) \times E$ into $\mathbf{C}$ such that $u(0, \cdot) = f(\cdot)$, $u(\cdot, \mu) \in C^1([0, \infty))$ for each $\mu \in E$ and $\partial u/\partial t = \mathcal{L} u$ on $[0, \infty) \times E$. Furthermore, if there is a bounded open cube Q' such that $b(y, \cdot)$ and $d(y, \cdot)$ are in $B(E; Q')$ for all $y \in R^\nu$, then $u_f(t, \cdot) \in B(E; Q')$ for all $t \geq 0$ if $f \in B(E; Q')$.*

Proof. Let $T > 0$ and f: $[0, T] \times E \to \mathbf{C}$ be given such that f is bounded and measurable, $f(\cdot, \mu) \in C^1([0, T])$ for all $\mu \in E$, and $|\partial f/\partial t(t, \mu)| \leq C\mu(Q)$, $(t, \mu) \in [0, T] \times E$, for some $C < \infty$. Given $\mu^0 \in E$, let $\tilde{f}$ be defined by $\tilde{f}(t, \mu) = f(t, \mu|_Q + \mu^0|_{Q^c})$ on $[0, T] \times E$. Then $\tilde{f}$ satisfies the conditions given in (ii) of Theorem 1.7, and so

$$\tilde{f}(t \wedge T, \mu_{t\wedge T}) - \int_0^{t\wedge T} \left(\frac{\partial}{\partial s} + \mathcal{L}\right) \tilde{f}(s, \mu_s)\, ds$$

is a P_{μ^0}-martingale. Since $\mu.|_{Q^c} \equiv \mu^0|_{Q^c}$ (a.s. P_{μ^0}), this shows that

$$f(t \wedge T, \mu_{t\wedge T}) - \int_0^{t\wedge T} \left(\frac{\partial}{\partial s} + \mathcal{L}\right) f(s, \mu_s)\, ds$$

is a P_{μ^0}-martingale. In particular, if $f \in B(E)$ and u: $[0, \infty) \times E \to \mathbf{C}$ is bounded measurable and satisfies $u(0, \cdot) = f(\cdot)$, $u(\cdot, \mu) \in C^1([0, \infty))$, and $\partial u/\partial t = \mathcal{L} u$ on $[0, \infty) \times E$, then, for each $\mu \in E$, $u(T - (t \wedge T), \mu_{t\wedge T})$ is a P_μ-martingale; and so

$$u(T, \mu) = E^{P_\mu}[f(\mu_T)] = u_f(t, \mu).$$

We next want to show that u_f satisfies the asserted properties. First note that, by the preceding paragraph,

$$u_f(t, \mu) = E^{P_\mu}[f(\mu_t)] = f(\mu) + E^{P_\mu}\left[\int_0^t \mathcal{L} f(\mu_s)\, ds\right].$$

Next note that if $0 \leqslant s < t$, then

$$|E^{P_\mu}[\mathfrak{L}f(\mu_t)] - E^{P_\mu}[\mathfrak{L}f(\mu_s)]| \leqslant E^{P_\mu}[|\mathfrak{L}f(\eta_t) - \mathfrak{L}f(\mu_s)|, \mu_t|_Q \neq \mu_s|_Q]$$
$$\leqslant CE^{P_\mu}[\sup_{0\leqslant u \leqslant t} (\mu_u(Q))^2]^{1/2}(P_\mu(\mu_t|_Q \neq \mu_s|_Q))^{1/2}$$

and the last line tends to zero as $t-s \to 0$ by (1.8) and Lemma 1.14. Thus $u_f(\cdot, \mu) \in C^1([0, \infty))$ for all $\mu \in E$. Next define, for $N \geqslant 1$, $\mathfrak{L}^{(N)}$ and $\{P_\mu^N: \mu \in E\}$ as in the proof of Theorem 2.2. Then $\mathfrak{L}^{(N)}$ is bounded and so $u^{(N)}(t, \mu) = e^{t\mathfrak{L}^{(N)}} f(\mu)$ is well defined. By the preceding paragraph,

$$u^{(N)}(t, \mu) = E^{P_\mu^N}[f(\mu_t)],$$

and, by the proof of Theorem 2.2, $E^{P_\mu^N}[f(\mu_t)] \to E^{P_\mu}[f(\mu_t)] = u_f(t, \mu)$. Also, for each N,

$$u^{(N)}(t, \mu) = f(\mu) + \int_0^t \mathfrak{L}^{(N)} u^{(N)}(s, \mu)\, ds.$$

Thus, by letting $N \nearrow \infty$, we see that

$$u_f(t, \mu) = f(\mu) + \int_0^t \mathfrak{L} u_f(s, \mu)\, ds.$$

This completes the proof that $\partial u_f / \partial t = \mathfrak{L} u_f$ on $[0, \infty) \times E$.

Finally, if $b(y, \cdot)$, $d(y, \cdot) \in B(E, Q')$ for all $y \in R^\nu$ and μ^0, $\nu^0 \in E$ have the property that $\mu^0|_{Q'} = \nu^0|_{Q'}$, define $\Phi: \Omega \to \Omega$ so that $\mu_t(\Phi(\omega)) = \mu_t|_{Q'} + \mu^0|_{Q'^c}$ for all $t \geqslant 0$. Then, by uniqueness, $P_{\mu^0} = P_{\nu^0} \circ \Phi^{-1}$. Since

$$E^{P_{\mu^0}}[f(\mu_t)] = E^{P_{\mu^0} \circ \Phi^{-1}}[f(\mu_t)]$$

for $f \in B(E; Q')$, the proof of the theorem is complete. Q.E.D.

2.6. Theorem. *Let everything be as in Theorem 2.2 and let $\nu \in E$ such that $\nu(Q) = 0$. Set $E_\nu(Q) = \{\mu \in E: \mu|_{Q^c} = \nu|_{Q^c}\}$ and assume that there is a $\delta > 0$ such that $d(y, \mu) \geqslant \delta$ for all $y \in Q$ and $\mu \in E_\nu(Q)$. Then for each $N \geqslant 0$ there is a $C_N < \infty$ such that $E^{P_\mu}[\tau] \leqslant C_N$ for all $\mu \in E_\nu(Q)$ satisfying $\mu(Q) = N$, where $\tau = \inf\{t \geqslant 0: \mu_t(Q) = 0\}$. Moreover, if $\int b(y, \nu)\, dy > 0$, then $0 < E^{P_\nu}[\tau'] < \infty$, where $\tau' = \inf\{t \geqslant 0: \mu_t(Q) = 0$ and $(\exists s \in [0, t))\ \mu_s(Q) \neq 0\}$.*

Proof. We learned the basic idea behind this proof from Preston [10].

Choose $B > 0$ so that $\int b(y, \mu)\, dy \leqslant B$ for all $\mu \in E$. Set $\gamma = B/\delta$ and define $\psi: \mathfrak{N} \to [0, \infty)$ so that $\psi(0) = 0$ and $\psi(n) - \psi(n-1) = (n-1)!\, \gamma^{n-1} \sum_{k=n}^\infty \gamma^{-k}/k!$, $n \geqslant 1$. Then it is easy to check that

$$B(\psi(n+1) - \psi(n)) + \delta n(\psi(n-1) - \psi(n)) = -B$$

for all $n \geq 1$. Given $N \geq 2$, define $f_N(\mu) = \psi(\mu(Q) \wedge N)$. Then if $\mu \in E_\nu(Q)$ and $0 < \mu(Q) < N$:

$$\mathfrak{L} f_N(\mu) = \int b(y, \mu)(\psi(\mu(Q)+1) - \psi(\mu(Q)))\, dy + \int d(y, \mu)(\psi(\mu(Q)-1) - \psi(\mu(Q)))\, \mu(dy)$$
$$\leqslant B(\psi(\mu(Q)+1) - \psi(\mu(Q)) - \delta\mu(Q)(\psi(\mu(Q)-1) - \psi(\mu(Q)))) = -B.$$

Given $\mu \in E_\nu(Q)$,

$$f_N(\mu_t) - \int_0^t \mathfrak{L} f_N(\mu_s)\, ds$$

is a P_μ-martingale. Thus if $\xi_N = \inf\{t \geq 0: \mu_t(Q) \geq N\}$, then $f_N(\mu_{t\wedge\xi_N\wedge\tau}) - \int_0^{t\wedge\xi_N\wedge\tau} \int f_N(\mu_s)\, ds$ is a P_μ-martingale. Since $P_\mu(\mu_t \in E_\nu(Q)$ for all $t \geq 0) = 1$, we now have that $f_N(\mu_{t\wedge\xi_N\wedge\tau}) + B(t \wedge \xi_N \wedge \tau)$ is a P_μ-supermartingale. Thus

$$\psi(\mu(Q)) \geqslant f_N(\mu) \geqslant E^{P_\mu}[f_N(\mu_{t\wedge\xi_N\wedge\tau})] + BE^{P_\mu}[t \wedge \xi_N \wedge \tau] \geqslant BE^{P_\mu}[t \wedge \xi_N \wedge \tau].$$

Since $\xi_N \nearrow \infty$ (a.s., P_μ), this proves that

$$E^{P_\mu}[\tau] \leqslant \psi(\mu(Q))/B.$$

Finally, note that $\tau' > 0$ (a.s., P_ν) because $\mu_t(Q)$ cannot change from 0 to something not 0 and then back to 0 instantaneously. Thus $E^{P_\nu}[\tau'] > 0$. On the other hand, if $\sigma = \inf\{t \geq 0: \mu_t(Q) \neq 0\}$, then, by Theorem 2.2, $E^{P_\nu}[\sigma] = 1/\int b(y, \nu)\, dy$ and $P_\nu(\mu_\sigma(Q) = 1) = 1$. Thus

$$E^{P_\nu}[\tau'] = E^{P_\nu}[\sigma] + E^{P_\nu}[E^{P_{\mu(\sigma)}}[\tau]] \leqslant \frac{1}{\int b(y, \nu)\, dy} + C_1. \qquad \text{Q.E.D.}$$

2.7. THEOREM. *Let everything be as in Theorem 2.2. Given $\nu \in E$ such that $\nu(Q) = 0$, $\{P_\mu: \mu \in E_\nu(Q)\}$ is a measurable, strong Markov family. Moreover, if there is a $\delta > 0$ such that $d(y, \mu) \geqslant \delta$ for all $y \in Q$ and $\mu \in E_\nu(Q)$, then for all $\mu \in E_\nu(Q)$ and $f \in B(E)$:*

$$\lim_{t\to\infty} E^{P_\mu}[f(\mu_t)] = \int f\, dm^\nu,$$

where

$$\int f\, dm^\nu = \begin{cases} f(\nu) & \text{if } \int b(y, \nu)\, dy = 0 \\ E^{P_\nu}\left[\int_0^{\tau'} f(\mu_t)\, dt\right] \Big/ E^{P_\nu}[\tau'] & \text{if } \int b(y, \nu)\, dy > 0. \end{cases}$$

(Here τ' is as in the last part of Theorem 2.6.) In particular, $\{P_\mu: \mu \in E_\nu(Q)\}$ is an ergodic

Markov family and its unique invariant probability measure m^ν has the property that

$$m^\nu(\{\nu\}) = \begin{cases} 1 & \text{if} \quad \int b(y,\nu)\,dy = 0 \\ \left(\int b(y,\nu)\,dy\, E^{P_\nu}[\tau']\right)^{-1} & \text{if} \quad \int b(y,\nu)\,dy > 0. \end{cases}$$

Proof. The first assertion is immediate from Theorem 2.2. To prove the second assertion we show first that

$$\lim_{t\to\infty} E^{P_\nu}[f(\mu_t)] \tag{2.8}$$

exists for all $f \in B(E)$. Indeed, this is obvious when $\int b(y,\nu)\,dy = 0$, since, in that case, $P_\nu(\mu_t = \nu$ for all $t \geq 0) = 1$. When $\int b(y,\nu)\,dy > 0$, $E^{P_\nu}[\tau'] < \infty$, and clearly τ' is a renewal time for the process P_ν. Thus the limit in (2.8) exists by Section 9.8 in [1]. The idea of using the renewal theorem here is due to Preston [10].

Next note that if $\mu \in E_\nu(Q)$, then

$$E^{P_\mu}[f(\mu_t]) = E^{P_\mu}[u_f[t-\tau,\nu),\ \tau \leq t] + E^{P_\mu}[f(\mu_t),\ \tau > t] \to \lim_{t\to\infty} u_f(t,\nu),$$

as $t \to \infty$, where τ is as in Theorem 2.6 and $u_f(t,\mu) = E^{P_\mu}[f(\mu_t)]$. Thus our proof will be complete once we identify the limit in (2.8) as $\int f\,dm^\nu$. If $\int b(y,\nu)\,dy = 0$, there is nothing more to do. If $\int b(y,\nu)\,dy > 0$,

$$\int_0^\infty e^{-\lambda t} E^{P_\nu}[f(\mu_t)]\,dt = E^{P_\nu}\left[\int_0^{\tau'} e^{-\lambda t} f(\mu_t)\,dt\right] + E^{P_\nu}[e^{-\lambda\tau'}]\int_0^\infty e^{-\lambda t} E^{P_\nu}[f(\mu_t)]\,dt.$$

Hence
$$\int_0^\infty e^{-\lambda t} E^{P_\nu}[f(\mu_t)]\,dt = E^{P_\nu}\left[\int_0^{\tau'} e^{-\lambda t} f(\mu_t)\,dt\right] \Big/ (1 - E^{P_\nu}[e^{-\lambda\tau'}]).$$

Thus since the limit in (2.8) exists it must be equal to

$$\lim_{\lambda\to 0} \lambda \int_0^\infty e^{-\lambda t} E^{P_\nu}[f(\mu_t)]\,dt = E^{P_\nu}\left[\int_0^{\tau'} f(\mu_t)\,dt\right] \Big/ E^{P_\nu}[\tau'].$$

Finally, it is obvious that $m^\nu(\{\nu\}) = 1$ if $\int b(y,\nu)\,dy = 0$; and if $\int b(y,\nu)\,dy > 0$, then $E^{P_\nu}[\tau'] < \infty$ and $E^{P_\nu}[\int_0^{\tau'} \chi_{\{\nu\}}(\mu_t)\,dt] = E^{P_\nu}[\sigma] = (\int b(y,\nu)\,dy)^{-1}$, where $\sigma = \inf\{t \geq 0 : \mu_t(Q) \neq 0\}$.

Q.E.D.

2.9. Lemma. *Let b and d be bounded measurable functions on $R^\nu \times E$ into $[0,\infty)$ and define $\mathcal{L}$ accordingly. Suppose m is a probability measure on $(E, \mathcal{B}_E)$ and Q is a bounded open cube in R^ν such that $\int \mathcal{L} f\,dm = 0$ for all $f \in \mathcal{D}(E;Q)$. Then*

$$\int\left(\int_Q d(y,\mu)\,\mu(dy)\right)m(d\mu) \leqslant \|b\|\,|Q|.$$

In particular, for all $f \in B(E, Q)$, $\mathcal{L}f \in L^1(m)$ *and* $\int \mathcal{L}f\,dm = 0$.

Proof. Given $N \geqslant 0$, choose $\psi \in C_0^\infty(R)$ such that $\psi \equiv 1$ on $[0, N]$ and $\psi \equiv 0$ off $(-1, N+1)$. Let $\{\varphi_n\}_1^\infty \subseteq C_0^\infty(Q)$ such that $0 \leqslant \varphi_n \nearrow \chi_Q$ and set $f_n(\mu) = \psi(\int \varphi_n(y)\mu(dy))$. Then it is easy to check that $f_n \in \mathcal{D}(E; Q)$ for all n, $\sup_n \sup_\mu \int |f_{n,y}(\mu)|\,\mu(dy) < \infty$, and $f_n(\mu) \to \chi_{[0,N]}(\mu(Q))$ boundedly and pointwise for all $\mu \in E$. Hence, if $f(\mu) = \chi_{[0,N]}(\mu(Q))$, $\mu \in E$, then $\mathcal{L}f_n \to \mathcal{L}f$ boundedly and pointwise for $\mu \in E$ and so $\int \mathcal{L}f\,dm = 0$. But

$$\mathcal{L}f(\mu) = -\chi_{\{N\}}(\mu(Q))\int_Q b(y,\mu)\,dy + \chi_{\{N+1\}}(\mu(Q))\int_Q d(y,\mu)\,\mu(dy),$$

and therefore

$$\int_{\{\mu:\,\mu(Q)=N\}}\left(\int_Q b(y,\mu)\,dy\right)m(d\mu) = \int_{\{\mu:\,\mu(Q)=N+1\}}\left(\int_Q d(y,\mu)\,\mu(dy)\right)m(d\mu).$$

Summing over $N \geqslant 0$, we arrive at the desired estimate. The estimate immediately implies that $\mathcal{L}f \in L^1(m)$ for all $f \in B(E; Q)$. Finally, consider the class $\mathcal{H}$ of $f \in B(E; Q)$ such that $\int \mathcal{L}f\,dm = 0$. Using the preceding estimate, it is easy to see that $\mathcal{H}$ is closed under bounded pointwise limits. Since $\mathcal{D}(E, Q) \subseteq \mathcal{H}$, it follows from Lemma 1.5 that $\mathcal{H} = B(E; Q)$. Q.E.D.

2.10. Theorem. *Let b and d be bounded measurable functions on $R^\nu \times E$ into $[0, \infty)$ and Q a bounded open cube in R^ν such that $b(y, \cdot) = d(y, \cdot) \equiv 0$ for $y \notin Q$. Define $\mathcal{L}$ accordingly and let $\{P_\mu : \mu \in E\}$ be the associated Markov family given in Theorem 2.2. If m is a probability measure on $(E, \mathcal{B}_E)$ such that $\int \mathcal{L}f\,dm = 0$ ($\int f\mathcal{L}g\,dm = \int g\mathcal{L}f\,dm$) for all $f \in \mathcal{D}(E)$ ($f, g \in \mathcal{D}(E)$), then m satisfies $\int f\,dm = \int E^{P\cdot}[f(\mu_t)]\,dm$ ($\int g E^{P\cdot}[f(\mu_t)]\,dm = \int f E^{P\cdot}[g(\mu_t)]\,dm$) for all $f \in B(E)$ ($f, g \in B(E)$).*

Proof. By Lemma 1.5 and Lemma 2.9, it is easy to see that $\mathcal{L}h \in L^1(m)$ for all $h \in B(E)$ and $\int \mathcal{L}f\,dm = 0$ ($\int f\mathcal{L}g\,dm = \int g\mathcal{L}f\,dm$) for all $f \in B(E)$ ($f, g \in B(E)$). (In fact, the $L^1(m)$-norm of $\mathcal{L}h$ is bounded by $2\|b\|\,|Q|\,\|h\|$.) Thus if $f \in B(E)$ ($f, g \in B(E)$) and $u_f(t, \cdot)$ ($u_f(t, \cdot)$ and $u_g(t, \cdot)$) are defined as in Theorem 2.5, then for $t \geqslant 0$: $\int \mathcal{L}u_f(s, \cdot)\,dm = 0$ ($\int u_f(s, \cdot)\mathcal{L}u_g(t-s, \cdot)\,dm = \int u_g(t-s, \cdot)\mathcal{L}u_f(s, \cdot)\,dm$), $0 \leqslant s \leqslant t$. Since $d/ds \int u_f(s, \cdot)\,dm = \int \mathcal{L}u_f(s, \cdot)\,dm$ ($d/ds \int u_f(s, \cdot) \times u_g(t-s, \cdot)\,dm = -\int u_f(s, \cdot)\mathcal{L}u_g(t-s, \cdot)\,dm + \int u_g(t-s, \cdot)\mathcal{L}u_f(s, \cdot)\,dm = 0$), we now have the desired result. Q.E.D.

2.11. Theorem. *Let b and d be bounded measurable functions on $R^\nu \times E$ into $[0, \infty)$ and Q a bounded open cube in R^ν such that $b(y, \cdot) = d(y, \cdot) \equiv 0$ for $y \notin Q$ and $b(y, \cdot)$, $d(y, \cdot) \in B(E; Q)$*

for all $y \in Q$. *Define* $\mathcal{L}$ *accordingly and let* $\{P_\mu: \mu \in E\}$ *be the associated Markov family given in Theorem 2.2. If* m *is a probability measure on* $(E, \mathcal{B}_E)$ *such that* $\int \mathcal{L} f\, dm = 0$ $(\int f \mathcal{L} g\, dm = \int g \mathcal{L} f\, dm)$ *for all* $f \in \mathcal{D}(E; Q)$ $(f, g \in \mathcal{D}(E; Q))$, *then* m *satisfies* $\int f\, dm = \int E^{P}\cdot[f(\mu_t)]\, dm$ $(\int g E^{P}\cdot[f(\mu_t)]\, dm = \int f E^{P}\cdot[g(\mu_t)]\, dm)$ *for all* $f \in B(E; Q)$ $(f, g \in B(E; Q))$.

Proof. The proof follows that of Theorem 2.10. The only extra ingredient needed is the last part of Theorem 2.5. Q.E.D.

2.12. COROLLARY. *Let* b *and* d *be bounded measurable functions on* $R^\nu \times E$ *into* $[0, \infty)$ *and* Q *a bounded open cube in* R^ν *such that* $b(y, \cdot) = d(y, \cdot) \equiv 0$ *for* $y \notin Q$. *Let* $\nu \in E$ *have the property that* $\nu(Q) = 0$ *and* $d(y, \mu) \geqslant \delta > 0$ *for all* $y \in Q$ *and* $\mu \in E_\nu(Q)$. *If* m *is a probability measure on* $(E, \mathcal{B}_E)$ *such that* $m(E_\nu(Q)) = 1$ *and* $\int \mathcal{L} f\, dm = 0$ *for all* $f \in \mathcal{D}(E; Q)$, *then* m *is the measure* m^ν *described in Theorem 2.7. In particular, there is exactly one such measure* m.

Proof. In view of Theorem 2.7, we need only show that m is a stationary measure for $\{P_\mu: \mu \in E_\nu(Q)\}$. To this end, define $b_\nu(y, \mu) = b(y, \mu|_Q + \nu|_{Q^c})$ and $d_\nu(y, \mu) = d(y, \mu|_Q + \nu|_{Q^c})$ for $y \in R^\nu$ and $\mu \in E$. If $\mathcal{L}^{(\nu)}$ is the associated operator, then $\int \mathcal{L}^{(\nu)} f\, dm = 0$ for all $f \in B(E; Q)$, since $m(E_\nu(Q)) = 1$. Thus, by Theorem 2.11, m is stationary for the corresponding $\{P^\nu_\mu: \mu \in E\}$. But $P^\nu_\mu = P_\mu$ for $\mu \in E_\nu(Q)$ and $m(E_\nu(Q)) = 1$. Thus m is stationary for $\{P_\mu: \mu \in E_\nu(Q)\}$. Q.E.D.

3. The martingale problem, a special case

Suppose that we are given b and d, as the beginning of Section 2, and define $\mathcal{L}$ on $\mathcal{D}(E)$ accordingly. The purpose of this section is to find out what can be said about the martingale problem when we drop the assumption that $b(y, \cdot) = d(y, \cdot) \equiv 0$ for y outside some bounded set. Our results here are rather incomplete and are really satisfactory only in a very special case to be described below. Before getting into that, we present the next general theorem.

3.1. THEOREM. *Assume that* b *and* d *have the following additional properties:*

(i) *for all* $\mu \in E$ *and* $\{\mu_n\}_1^\infty \subseteq E$ *such that* $\mu_n \to \mu$, $b(\cdot, \mu_n) \to b(\cdot, \mu)$ *in (Lebesgue) measure,*

(ii) *for all* $(y, \mu) \in R^\nu \times E$ *and* $\{(y_n, \mu_n)\}_1^\infty \subseteq R^\nu \times E$ *such that* $\mu_n(\{y_n\}) = 1$ *and* $(y_n, \mu_n) \to (y, \mu)$, $d(y_n, \mu_n) \to d(y, \mu)$.

Then, for all $f \in \mathcal{D}(E)$, $\mathcal{L} f$ *is a bounded continuous function. Moreover, for each* $\mu \in E$, *there is a solution* P_μ *to the martingale problem for* $\mathcal{L}$ *starting from* μ. *Finally, there is a choice of* $\mu \to P_\mu$ *such that* $\{P_\mu: \mu \in E\}$ *is a measurable, strong Markov family.*

Proof. The proof that $\mathcal{L} f$ is bounded and continuous for $f \in \mathcal{D}(E)$ is left to the reader. As for existence of solutions, let $Q_N = \{x \in R^\nu: |x_j| < N \text{ for } 1 \leqslant j \leqslant \nu\}$ and define $b_N(y, \cdot) =$

$\chi_{Q_N}(y)b(y, \cdot)$, $d_N(y, \cdot)=\chi_{Q_N}(y)d(y, \cdot)$, and $\mathfrak{L}^{(N)}$ accordingly. Given $\mu\in E$, let P_μ^N be the solution to the martingale problem for $\mathfrak{L}^{(N)}$ starting from μ. By Theorem 1.15 $\{P_\mu^N: N\geq 1\}$ is weakly compact on Ω. Let $\{P_\mu^{(N')}\}$ be a convergent subsequence and set $P=\lim_{N'\to\infty}P_\mu^{(N')}$. Clearly $P(\mu_0=\mu)=1$. Moreover, for any $f\in\mathfrak{D}(E)$, $\mathfrak{L}^{(N')}f=\mathfrak{L}f$ whenever N' is sufficiently large. Since $\mathfrak{L}f$ is continuous, it is now easy to check that P solves the martingale problem for $\mathfrak{L}$ starting from μ. Finally, the assertion about the possibility of selecting $\mu\to P_\mu$ so that $\{P_\mu: \mu\in E\}$ is measurable and strongly Markovian is easily derived by an obvious adaptation of the argument given by Krylov [5]. Q.E.D.

Unfortunately, Theorem 3.1 is not very useful. In particular, the relationship between $\mathfrak{L}$ and $\{P_\mu: \mu\in E\}$ is too weak to draw any important conclusions about the properties of $\{P_\mu: \mu\in E\}$ from facts about $\mathfrak{L}$. For instance, it is impossible to show, from this theorem, that $\int f\mathfrak{L}g\,dm=\int g\mathfrak{L}f\,dm$ for all $f, g\in\mathfrak{D}$ (and some probability measure m on $(E, \mathcal{B}_E)$) implies $\int gE^{P}\cdot[f(\mu_t)]dm=\int fE^{P}\cdot[g(\mu_t)]dm$. In fact, similar implications in other contexts are well-known to be false (cf. [4] for example).

We now turn our attention to a very special situation in which it is possible to prove much more refined and useful conclusions. In the first place, we will restrict ourselves to one dimension. Secondly, we will assume that our coefficients b and d depend only on "nearest neighbors". That is, we assume that there are bounded measurable functions $\beta: \Sigma_3\to[0, \infty)$ and $\delta: \Sigma_3\to[0, \infty)$, where $\Sigma_3=\{(l, y, r)\in R^3: l<y<r\}$, such that:

$$b(y, \mu)=\beta(l_\mu(y), y, r_\mu(y)) \tag{3.2}$$

and

$$d(y, \mu)=\delta(l_\mu(y), y, r_\mu(y)), \tag{3.3}$$

where

$$l_\mu(y)=\sup\{l: l<y \quad\text{and } \mu(\{l\})=1\}$$

and

$$r_\mu(y)=\inf\{r: r>y \quad\text{and } \mu(\{r\})=1\}.$$

If one considers the analogue of our set-up on the integers, then existence and uniqueness can be proved for many reasonable choices of b and d by adapting Liggett's [8] techniques in the same way as we did in [3]. Moreover, L. Gray has developed a very powerful method of proving uniqueness for spin-flip type processes and he has used his technique to prove the first part of our Theorem 3.13 in the integer context. In fact, he discovered his technique before we learned how to get away from our original assumption that d be independent of μ. It was only after we had learned of his work that we found a way of proving Theorem 3.13 in the form that it now appears. Although we believe that Gray's ideas can be adapted to cover the real-line case, we prefer the approach that we give below because it seems to us more direct and shows that not only do the finite dimensional marginals of the approximants converge strongly but the approximating processes themselves converge strongly. This latter fact is a peculiarity of the nearest neighbor assumption and may prove useful in the future.

It what follows, we will make frequent use of the following construction. Given a probability measure P on $(\Omega, \mathcal{M})$, an $\omega \in \Omega$ and a $T>0$, let $\delta_\omega \otimes_T^0 P$ denote the unique probability measure on $(\Omega, \mathcal{M})$ such that

$$E^{\delta_\omega \otimes_T^0 P}[f_1(\mu_{s_1}) \dots f_m(\mu_{s_m}) g_1(\mu_{t_1}) \dots g_n(\mu_{t_n})] = f_1(\mu_{s_1}(\omega)) \dots f_m(\mu_{s_m}(\omega)) E^P[g_1(\mu_{t_1-T}) \dots g_n(\mu_{t_n-T})]$$

for all $m, n \geq 1$, $0 \leq s_1 < \dots < s_m < T \leq t_1 < \dots < t_n$ and $f_1, \dots, f_m, g_1, \dots, g_n \in B(E)$. For more details see [13].

For $t \geq 0$ and Λ an interval, let $\mathcal{M}_t^\Lambda$ be the smallest σ-algebra of subsets of Ω with respect to which

$$\omega \to \int_\Lambda \varphi(y)\, \mu_s(\omega; dy)$$

is measurable for all $0 \leq s \leq t$ and $\varphi \in C_b(R)$; and set $\mathcal{M}^\Lambda = \sigma(\bigcup_{t \geq 0} \mathcal{M}_t^\Lambda)$.

3.4. LEMMA. *Suppose that b: $R \times E \to [0, \infty)$ and d: $R \times E \to [0, \infty)$ are bounded measurable functions and that Λ is an interval (open or closed) such that $b(y, \cdot)$, $d(y, \cdot) \in \mathcal{B}(E, \Lambda)$ for all $y \in \Lambda$. Let $\mathcal{L}$ be the operator associated with b and d. Define $b^\Lambda(y, \cdot) = I_\Lambda(y) b(y, \cdot)$ and $d^\Lambda(y, \cdot) = I_\Lambda(y) d(y, \cdot)$, and $\mathcal{L}^\Lambda$ accordingly for b^Λ and d^Λ. Denote by $\{P_\mu^\Lambda; \mu \in E\}$ the family of solutions to the martingale problem for $\mathcal{L}^\Lambda$. Given $T>0$ and a right-continuous function η: $[0, T] \to E$, let $P^{\Lambda, \eta} = \delta_\eta \otimes_T^0 P_{\eta(T)}^\Lambda$. If P is a probability measure on $(\Omega, \mathcal{M}^\Lambda)$ such that $P(\mu_t|_\Lambda = \eta(t)|_\Lambda, 0 \leq t \leq T) = 1$ and for all $f \in \mathcal{B}(E, N)$ $(f(\mu_t) - \int_T^t \mathcal{L}^\Lambda f(\mu_s) ds, \mathcal{M}_t^\Lambda, P)$ is a martingale after time T, then P equals $P^{\Lambda, \eta}$ on $\mathcal{M}^\Lambda$.*

Proof. Define $\tilde{\mu}_t(\omega) = \mu_t(\omega)|_\Lambda + \eta(t \wedge T)|_{\Lambda^c}$. It is then easy to check that

$$P(\tilde{\mu}_t = \eta(t), 0 \leq t \leq T) = 1$$

and that $f(\tilde{\mu}_t) - \int_T^t \mathcal{L}^\Lambda f(\tilde{\mu}_s) ds$ is a P-martingale after time T for all $f \in \mathcal{B}(E, \Lambda)$. Thus the distribution $\tilde{P}$ of $\tilde{\mu}$ under P must be $P^{\Lambda, \eta}$. Since $\tilde{\mu}.|_\Lambda = \mu.|_\Lambda$, this completes the proof.

Q.E.D.

3.5. LEMMA. *Let b, d, and $\mathcal{L}$ be as in Lemma 3.4. Suppose that P is a probability measure on $(\Omega, \mathcal{M})$ such that $f(\mu_t) - \int_0^t \mathcal{L} f(\mu_s) ds$ is a martingale for all $f \in \mathcal{D}$, and let $\{P_\omega\}$ be a* r.c.p.d. *of $P|\mathcal{M}^\Lambda$. Then there is a P-null set $N \in \mathcal{M}^\Lambda$ such that for all $\omega \notin N$ and all $f \in \mathcal{D}(\Lambda^c)$, $f(\mu_t) - \int_0^t \mathcal{L} f(\mu_s) ds$ is a P_ω-martingale.*

Proof. Let $n\geq 1$, $\theta_1, \ldots, \theta_n \in R$, and $\Gamma_1, \ldots, \Gamma_n$ be mutually disjoint bounded measurable subsets of $R\setminus\Lambda$. We must show that

$$X(t)=\exp\left[\sum_{j=1}^{n}\theta_j\mu_t(\Gamma_j)-\sum_{j=1}^{n}\int_0^t ds\int_{\Gamma_j}(e^{\theta_j}-1)\,b(y,\mu_s)ds-\sum_{j=1}^{n}\int_0^t ds\int_{\Gamma_j}e^{-\theta_j}-1)\,d(y,\mu_s)\,\mu_s(dy)\right]$$

is a P_ω-martingale for P-almost all ω. Given $0\leq t_1<t_2$, set

$$X^{t_1}(t_2)=X(t_2)/X(t_1).$$

Let $\{P_\omega^{t_1}\}$ be a r.c.p.d. of $P\,|\,\mathcal{M}_{t_1}$. Then, by Lemma 3.4 and Theorem 1.9 there is a P-null set $N_{t_1}\in\mathcal{M}_{t_1}$ such that $P_\omega^{t_1}$ equals $\delta_\omega\otimes_{t_1}^{0}P^{\Lambda}_{\mu_{t_1}(\omega)}$ on $\mathcal{M}^{\Lambda}$ if $\omega\notin N_{t_1}$. For $\omega\notin N_{t_1}$, define Q_ω on $\mathcal{M}^{\Lambda}$ so that

$$\frac{dQ_\omega}{dP_\omega^{t_1}}=E^{P_\omega^{t_1}}[X^{t_1}(t_2)\,|\,\mathcal{M}^{\Lambda}].$$

Proceeding as in Theorem 3.2 of [12], we see that for any $m\geq 1$, $\lambda_1, \ldots, \lambda_m\in R^1$ and mutually disjoint Borel sets $\Lambda_1, \ldots, \Lambda_m$ in Λ, $(Y(t), \mathcal{M}_t^{\Lambda}, Q_\omega)$ is a martingale after time t_1, where

$$Y(t)=\exp\left[\sum_{j=1}^{m}\lambda_j\mu_t(\Lambda_j)-\sum_{j=1}^{m}\int_{t_1}^t ds\int_{\Lambda_j}(e^{\lambda_j}-1)\,b(y,\mu_s)dy-\sum_{j=1}^{n}\int_{t_1}^t ds\int_{\Lambda_j}(e^{-\lambda_j}-1)\,d(y,\mu_s)\,\mu_s(dy)\right].$$

Hence, for $\omega\notin N_{t_1}$ and $f\in\mathcal{B}(E,\Lambda)$; $(f(\mu_t)-\int_0^t\mathcal{L}f(\mu_s)ds,\ \mathcal{M}_t^{\Lambda}, Q_\omega)$ is a martingale after time t_1. Since $Q_\omega(\mu_t|_\Lambda=\mu_t(\omega)|_\Lambda,\ 0\leq t\leq t_1)=1$ this shows that $Q_\omega=\delta_\omega\otimes_{t_1}^{0}P^{\Lambda}_{\mu_{t_1}}$ on $\mathcal{M}^{\Lambda}$ for $\omega\notin N_{t_1}$. Thus

$$E^{P_\omega^{t_1}}[X^{t_1}(t_2)\,|\,\mathcal{M}^{\Lambda}]=1\quad(\text{a.s., } P_\omega^{t_1})$$

for all $\omega\notin N_{t_1}$. If $A\in\mathcal{M}^{\Lambda}$ and $B\in\mathcal{M}_{t_1}$, we now have:

$$\begin{aligned}E^P[E^{P_\cdot}[X(t_2),B],A]=E^P[X(t_2),\ A\cap B]&=E^P[X(t_1)\,E^{P_\cdot^{t_1}}[X^{t_1}(t_2),A),B]\\&=E^P[X(t_1)\,P_\cdot^{t_1}(A),B]\\&=E^P[X(t_1),\ A\cap B]=E^P[E^{P_\cdot}[X(t_1),B],A].\end{aligned}$$

That is, for each $0\leq t_1<t_2$ and $B\in\mathcal{M}_{t_1}$

$$E^{P_\omega}[X(t_2),B]=E^{P_\omega}[X(t_1),B]\tag{3.6}$$

for P-almost all ω. It is now clear that we can choose one P-null set $N\in\mathcal{M}^{\Lambda}$ so that (3.6) holds for all $\omega\notin N$, $0\leq t_1<t_2$, and $B\in\mathcal{M}_{t_1}$. This set N still depends on n, $\theta_1, \ldots, \theta_n$, and $\Gamma_1, \ldots, \Gamma_n$. However, by an obvious procedure, it is possible to choose one P-null set from $\mathcal{M}^{\Lambda}$ which works for all n, $\theta_1, \ldots, \theta_n$ and $\Gamma_1, \ldots, \Gamma_n$. Q.E.D.

3.7. LEMMA. *Let b and d be coefficients satisfying (3.2) and (3.3) and assume that $d(\cdot, \cdot) \leqslant D$. Let $\Lambda = (\alpha, \beta)$ and set*

$$\bar{b}(y, \mu) = I^{\Lambda}(y) b(y, \mu \cup \{\alpha, \beta\})$$

$$\bar{d}(y, \mu) = \begin{cases} D & \text{if } y \in \partial\Lambda \\ I_{\Lambda}(y) d(y, \mu \cup \{\alpha, \beta\}) & \text{otherwise.} \end{cases}$$

Given a P on $(\Omega, \mathcal{M})$ such that: $P(\mu_0|_{\bar{\Lambda}} = \mu^0|_{\bar{\Lambda}})$, where $\mu^0(\{\alpha\}) = \mu^0(\{\beta\}) = 1$, and $f(\mu_t) - \int_0^t \mathcal{L}f(\mu_s)\, ds$ is a martingale for all $f \in \mathcal{B}(E, \bar{\Lambda})$, there is a $\tilde{P}$ such that: $P = \tilde{P}$ on $\mathcal{M}^{\Lambda}$ and $\tilde{P} \otimes_{\tau}^{0} \bar{P}_{\mu_\tau} = \bar{P}_{\mu^0}$, where $\tau = \inf\{t \geqslant 0: \mu_t(\{\alpha\})\mu_t(\{\beta\}) = 0\}$, $\bar{P}_{\mu}$ is the solution to the martingale problem for $\bar{b}$ and $\bar{d}$ starting from μ and $\tilde{P} \otimes_{\tau}^{0} \bar{P}_{\mu_\tau}(A) = E^{\tilde{P}}[\delta_{\cdot} \otimes_{\tau(\cdot)}^{0} \bar{P}_{\mu_{\tau(\cdot)}}(A)]$.

Proof. Let ν be the probability measure on $([0, \infty)^2, \mathcal{B}_{[0,\infty)^2})$ satisfying $\nu([x, \infty), [y, \infty)) = e^{-(x+y)}$. Consider the measurable space $(\Omega \times [0, \infty)^2, \mathcal{M} \times \mathcal{B}_{[0,\infty)^2})$. We can think of $\mu_\cdot$ as defined on $\Omega \times [0, \infty)^2$ by $\mu_t((\omega, x, y)) = \mu_t(\omega)$. We also define $X(\omega, x, y) = x$ and $Y(\omega, x, y) = y$. Define

$$\sigma_\alpha = \left(\inf\left\{t \geqslant 0: \int_0^{t \wedge \tau_\alpha} (D - d(\alpha, \mu_s))\, ds \geqslant X\right\}\right) \wedge \tau_\alpha,$$

$$\sigma_\beta = \left(\inf\left\{t \geqslant 0: \int_0^{t \wedge \tau_\beta} (D - d(\beta, \mu_s))\, ds \geqslant Y\right\}\right) \wedge \tau_\beta,$$

and

$$\zeta = \sigma_\alpha \wedge \sigma_\beta,$$

where $\tau_\alpha = \inf\{t \geqslant 0: \mu_t(\{\alpha\}) = 0\}$ and $\tau_\beta = \inf\{t \geqslant 0: \mu_t(\{\beta\}) = 0\}$. Define

$$\tilde{\mu}_t = \mu^0|_{\bar{\Lambda} - c} + I_{(t,\infty)}(\sigma_\alpha)\,\delta_{\{\alpha\}} + I_{(t,\infty)}(\sigma_\beta)\,\delta_\beta + \mu_t|_{\Lambda}.$$

We want to show that if $\tilde{P}$ is the distribution of $\tilde{\mu}$ under $P \times \nu$, then $\tilde{P}$ has the desired properties. It is easy to see that this comes down to checking that if $f \in \mathcal{B}(E, \bar{\Lambda})$, then

$$\left(f(\tilde{\mu}_{t \wedge \zeta}) - \int_0^{t \wedge \zeta} \bar{\mathcal{L}}f(\tilde{\mu}_s)\, ds,\ \tilde{\mathcal{M}}_t^{\bar{\Lambda}},\ P \times \nu\right)$$

is a martingale, where $\tilde{\mathcal{M}}_t^{\bar{\Lambda}} = \sigma(\tilde{\mu}_s|_{\bar{\Lambda}}: 0 \leqslant s \leqslant t)$. But this is equivalent to showing that

$$\left(f(\tilde{\mu}_{t \wedge \zeta}) - \int_0^{t \wedge \zeta} \bar{\mathcal{L}}f(\tilde{\mu}_s)\, ds,\ \tilde{\mathcal{M}}_{t \wedge \zeta}^{\bar{\Lambda}},\ P \times \nu\right)$$

is a martingale. Since $\mathcal{M}_{t \wedge \zeta}^{\bar{\Lambda}} \subseteq \mathcal{M}_t \times \mathcal{B}_{[0,\infty)^2}$ for all $t \geqslant 0$, we can afford to replace P by any P' which equals P on $\mathcal{M}_\tau$. In particular, take $P' = P \otimes_{\tau}^{0} \bar{P}_{\mu_\tau}$. If we can show that for all

$f \in \mathcal{B}(E, \bar{\Lambda})$: $(f(\tilde{\mu}_t) - \int_0^t \bar{\mathfrak{L}} f(\tilde{\mu}_s)\,ds, \tilde{\mathcal{M}}_t^{\bar{\Lambda}}, P' \times \nu)$ is a martingale, then we will be done. Surely this is true for $f \in \mathcal{B}(E, \Lambda)$; and so we will be done if we show that

$$P' \times \nu(\sigma_\alpha > s, \sigma_\beta > t \,|\, \sigma(\mu_t|_\Lambda : t \geq 0)) = e^{-D(s+t)} \quad \text{(a.s. } P' \times \nu).$$

To this end, let $A \in \sigma(\mu_t|_\Lambda : t \geq 0)$ and $s, t \geq 0$ be given. Then

$$\begin{aligned} &P' \times \nu(\{\sigma_\alpha > s\} \cap \{\sigma_\beta > t\} \cap A) \\ &\quad = P' \times \nu\left(\left\{\tau_\alpha > s,\ X > \int_0^s (D - d(\alpha, \mu_u))\,du\right\} \cap \left\{\tau_\beta > t,\ Y > \int_0^t (D - d(\beta, \mu_u))\,d\mu\right\} \cap A\right) \\ &\quad = E^{P'}\left[\mu_{s \wedge \tau_\alpha}(\{\alpha\}) \exp\left(-\int_0^s (D - d(\alpha, \mu_u))\,du\right)\right. \\ &\qquad\qquad \left. \times\, \mu_{t \wedge \tau_\beta}(\{\beta\}) \exp\left(-\int_0^t (D - d(\beta, \mu_u))\,du\right),\ A\right]. \end{aligned}$$

Note that, by Lemma 3.5, if $\{P'_\omega\}$ is a r.c.p.d. of $P' \,|\, \mathcal{M}^\Lambda$, then for P'-almost all ω:

$$\mu_t(\{\alpha\}) + \int_0^t d(\alpha, \mu_u)\,\mu_u(\{\alpha\})\,du$$

$$\mu_t(\{\beta\}) + \int_0^t d(\beta, \mu_u)\,\mu_u(\{\beta\})\,du$$

and

$$\mu_t(\{\alpha\})\,\mu_t(\{\beta\}) + \int_0^t (d(\alpha, \mu_u)\,\mu_u(\{\alpha\}) + d(\beta, \mu_u)\,\mu_u(\{\beta\}))\,du$$

are all P'_ω-martingales. Thus, so are

$$X(t) \equiv \mu_t(\{\alpha\}) \exp\left[\int_0^t d(\alpha, \mu_u)\,du\right]$$

$$Y(t) \equiv \mu_t(\{\beta\}) \exp\left[\int_0^t d(\beta, \mu_u)\,du\right]$$

and

$$Z(t) \equiv X(t)\,Y(t).$$

Hence for P'-almost all ω:

$$E^{P'_\omega}[X(s \wedge \tau_\alpha)\,Y(t \wedge \tau_\beta)] = E^{P'_\omega}[Z(s \wedge t \wedge \tau_\alpha \wedge \tau_\beta)] = E^{P'_\omega}[Z(0)]).$$

Therefore,

$$E^{P'}\left[\mu_{s\wedge\tau_\alpha}(\{\alpha\})\exp\left(-\int_0^s (D-d(\alpha,\mu_u))\,du\right)\mu_{t\wedge\tau_\beta}(\{\beta\})\exp\left(-\int_0^t (D-d(\beta,\mu_u))\,du\right),\ A\right]$$

$$=e^{-D(t+s)}E^{P'}[X(s\wedge\tau_\alpha)\,Y(t\wedge\tau_\beta),\ A]$$

$$=e^{-D(t+s)}E^{P'}[Z(0),\ A]=e^{-D(t+s)}P'(A),$$

Since $P'(Z(0)=1)=1$. Q.E.D.

Let $\Lambda=(\alpha,\beta)$ and define $\tilde{b}$ and $\tilde{d}$ by

$$\tilde{b}(y,\mu)=I_{(r_\mu(\alpha),\,l_\mu(\beta))}(y)\,b(y,\mu)$$

$$\tilde{d}(y,\mu)=\begin{cases} d(y,\mu) & \text{if } y\in(r_\mu(\alpha),\,l_\mu(\beta)) \\ D & \text{if } y\in\{r_\mu(\alpha),\,l_\mu(\beta)\} \\ 0 & \text{otherwise.}\end{cases}$$

Let $\{\tilde{P}_\mu:\mu\in E\}$ be the associated family of solutions to the martingale problem.

3.8. THEOREM. *Let $\mu^0\in E$ be such that $\mu^0(\Lambda)\geqslant 2$ and set $\alpha_0=r_{\mu^0}(\alpha)$, $\beta_0=l_{\mu^0}(\beta)$, and $\Lambda_0=(\alpha_0,\beta_0)$. Suppose P on $(\Omega,\mathcal{M})$ is such that*

$$P(\mu_0|_{\bar{\Lambda}_0}=\mu^0|_{\bar{\Lambda}_0})\ \textit{and}$$

$f(\mu_t)-\int_0^t \mathfrak{L}f(\mu_s)$ is a martingale for all $f\in\mathcal{B}(E,\Lambda)$. Then there is a $\tilde{P}$ such that:

$$P=\tilde{P}\quad\text{on } \mathcal{M}^{\Lambda_0}\quad\text{and } \tilde{P}_{\mu_0}=\tilde{P}\otimes_\tau^0\tilde{P}_{\mu_\tau},$$

where $\tau=\inf\{t\geqslant 0:\mu_t(\{\alpha_0\})\mu_t(\{\beta_0\})=0\}$.

Proof. Let $\tilde{P}$ be as in the preceeding lemma relative to P and Λ_0. Q.E.D.

3.9. LEMMA. *Let b and d be bounded measurable birth and death coefficients with $d(\cdot,\cdot)\leqslant D$ and define $\mathfrak{L}$ accordingly. Let $I_1=(l_1,r_1)$ and $I_2=(l_2,r_2)$ and suppose that $r_1<l_2$. Let $\mu^0\in E$ be such that $\mu^0(\{l_1\})=\mu^0(\{r_2\})=1$ and $\mu^0(I_1)\wedge\mu^0(I_2)\geqslant N-1$. Let P be any measure on $(\Omega,\mathcal{M})$ such that $P(\mu_0|_{[l_1,r_2]}=\mu^0|_{[l_1,r_2]})=1$ and $f(\mu_t)-\int_0^t\mathfrak{L}f(\mu_s)ds$ is a P-martingale for all $f\in\mathcal{B}(E;[l_1,r_2])$. Define*

$$f_N(s,\mu)=\begin{cases}(1-\exp[-2D(T-s)])^{\mu(I_1)\wedge\mu(I_2)\wedge(N-1)} & \text{for}\quad s\leqslant T\\ 0 & \text{for}\quad s>T.\end{cases}$$

Then

$$E^P[f_N(\tau,\mu_\tau)]\leqslant(1-\exp[-2DT])^N, \tag{3.10}$$

where $\tau=\inf\{t\geqslant 0:\mu_t(\{l_1\})\mu_t(\{r_2\})=0\}$.

Proof. Let $x_0=l_1$ and $y_0=r_2$ and $x_1, \ldots, x_{N-1}\in I_1$, $y_1, \ldots, y_{N-1}\in I_2$ be such that $\mu^0(\{x_i\})=\mu^0(\{y^0\})=1$ for $i=1, 2, \ldots, N-1$. Define

$$h(s,\mu)=\begin{cases}(1-\exp[-2D(T-s)])^{\sum_{i=0}^{N-1}\mu(\{x_i\})\mu(\{y_i\})} & \text{if} \quad s\leqslant T\\ 0 & \text{if} \quad s>T.\end{cases}$$

Since $\mu_\tau(\{x_0\})\mu_\tau(\{y_0\})=0$ and $\sum_{i=1}^{N-1}\mu(\{x_i\})\mu(\{y_i\})\leqslant\mu(I_1)\wedge\mu(I_2)$, it follows that $P(f_N(\tau,\mu_\tau)\leqslant h(\tau,\mu_\tau)=1$. Thus it suffices to show that

$$E^P[h(\tau,\mu_\tau)]\leqslant(1-\exp[-2DT])^N.$$

Now $(\partial/\partial s)\,h(s,\mu)=-D\int(h(s,\mu\diagdown\delta_y)-h(s,\mu))\mu(dy)$, and therefore, since $d\leqslant D$ and $h(s,\mu)\leqslant h(s,\mu')$ if $\mu\supseteq\mu'$ it follows that

$$\mathfrak{L}h(s,\mu)\leqslant-\frac{\partial}{\partial s}h(s,\mu).$$

Since $h(s\wedge T,\mu_{s\wedge T})-\int_0^{s\wedge T}(\partial/\partial r+\mathfrak{L})h(r,\mu_r)dr$ is a martingale, we have

$$\begin{aligned}E^P[h(\tau,\mu_\tau)]&=E^P[h(\tau\wedge T,\mu_{\tau\wedge T})]\\&=E^P\left[h(\tau\wedge T,\mu_{\tau\wedge T})-\int_0^{\tau\wedge T}\left(\frac{\partial}{\partial r}+\mathfrak{L}\right)h(r,\mu_r)\,dr\right]+E^P\left[\int_0^{\tau\wedge T}\left(\frac{\partial}{\partial r}+\mathfrak{L}\right)h(r,\mu_r)\,dr\right]\\&\leqslant h(0,\mu_0)=(1-\exp[-2DT])^N.\end{aligned}$$

Q.E.D.

3.11. THEOREM. *Let $[c,d]=I\subset\Lambda=(\alpha,\beta)$ and let P be any measure on $(\Omega,\mathfrak{M})$ such that $f(\mu_t)-\int_0^t\mathfrak{L}f(\mu_s)ds$ is a P-martingale for all $f\in\mathcal{B}(E;\Lambda)$ and such that $P(\mu_0=\mu^0)=1$ for some $\mu^0\in E$. Let $\hat\mu\in E$ and let $\Lambda^{\hat\mu}=[r_{\hat\mu}(\alpha)\,l_{\hat\mu}(\beta)]$. Assume that $\hat\mu|_{\Lambda^{\hat\mu}}=\mu^0|_{\Lambda^{\hat\mu}}$ and that $\hat\mu((\alpha,c))\wedge\hat\mu((d,\beta))\geqslant n$. Then*

$$\tfrac{1}{2}\|P-\tilde P_{\hat\mu}\|_{\mathrm{Var}\,\mathfrak{M}_t^I}\leqslant(1-\exp[-2Dt])^n,$$

where $\{\tilde P_\mu:\mu\in E\}$ is as before Theorem 3.8.

Proof. The proof is by induction on n. Let $A\in\mathfrak{M}_t^I$ be fixed. It suffices to show that

$$|P(A)-\tilde P_{\hat\mu}(A)|\leqslant(1-\exp[-2Dt])^n.$$

Let $\tilde P$ and τ be as in Theorem 3.8; and let $\omega\to\tilde P^\omega$ be a r.c.p.d. of $\tilde P|\mathfrak{M}_\tau$. Then by Theorem 3.8,

$$\begin{aligned}(3.12)\qquad |P(A)-\tilde P_{\hat\mu}(A)|&=|E^{\tilde P}[\tilde P^\omega(A)-\delta_\omega\otimes^0_{\tau(\omega)}\tilde P_{\mu_{\tau(\omega)}}(A)]|\\&\leqslant E^{\tilde P}[|\tilde P^\omega(A)-\delta_\omega\otimes^0_{\tau(\omega)}(A)|].\end{aligned}$$

Suppose now that $n=1$. If $\tau\geqslant t$ then $\tilde P^\omega(A)-\delta_\omega\otimes^0_{\tau(\omega)}\tilde P_{\mu_{\tau(\omega)}}(A)=0$ (a.s. $\tilde P$), and hence

$$|P(A)-\tilde P_{\hat\mu}(A)|\leqslant\tilde P(\tau\leqslant t)=1-\exp[-2Dt].$$

Assume next that the theorem is true for $n \leqslant N-1$ and that $n \geqslant N$. Since $\tilde{P}^{\omega}$ is concentrated on the atom $[\omega]_{\tau(\omega)}$ and $f(\mu_s) - \int_0^s \mathcal{L}f(\mu_s)ds$ is a $\tilde{P}^{\omega}$ martingale after time $\tau(\omega)$ for all $f \in \mathcal{B}(E, \Lambda)$, the hypotheses of the theorem apply to $\tilde{P}^{\omega}$ beginning at time $\tau(\omega)$ and hence, by the inductive hypothesis,

$$|\tilde{P}^{\omega}(A) - \delta_{\omega} \otimes^0_{\tau(\omega)} \tilde{P}_{\mu_{\tau(\omega)}}(A)| \leqslant f_N(\tau(\omega), \mu_{\tau_{\omega}}),$$

where f_N is defined as in Lemma 3.9 with $T=t$ and $I_1 = (r_{\hat{\mu}}(\alpha), c)$ and $I_2 = (d, l_{\hat{\mu}}(\beta))$. The proof is now completed by substituting this bound into (3.12) and applying Lemma 3.9.

Q.E.D.

3.13. THEOREM. *Let b and d be bounded functions of the form (3.2) and (3.3). Then for each $\mu \in E$ there is exactly one solution P_{μ} to the martingale problem for $\mathcal{L}$ starting from μ. Moreover, the family $\{P_{\mu}: \mu \in E\}$ is measurable and strongly Markovian. Finally, for each $\varepsilon > 0$ and $T > 0$ there is an $N = N(\varepsilon, T)$ such that if $\Lambda = (a, b)$, $I = [c, f]$, and $\mu^0 \in E$ satisfy $a < c < d < b$ and $\mu^0((a, c)) \wedge \mu^0((d, b)) \geqslant N$, then for any probability measure P on $(\Omega, \mathcal{M})$ satisfying $P(\mu_0 = \mu) = 1$ for some $\mu \in E$ with $\mu|_{\Lambda} = \mu^0|_{\Lambda}$ plus $f(\mu_t) - \int_0^t \mathcal{L}f(\mu_s)ds$ is a P-martingale for all $f \in \mathcal{D}(E; \Lambda)$ we have*

$$\| P - P_{\mu^0} \|_{\mathrm{Var}\, \mathcal{M}_T^I} \leqslant \varepsilon.$$

Proof. The last assertion is an immediate consequence of Theorem 3.11 and therefore the uniqueness of P_{μ^0} is obvious. Moreover, once we prove the existence of a measurable family of solutions, the fact that the family enjoys the strong Markov property is an easy consequence of Theorem 1.10 and uniqueness (cf. [13]). Thus it remains only to establish the existence of a measurable family. To this end, let $\Lambda_n = (-n, n)$, $b_n(y, \mu) = \chi_{\Lambda_n}(y) b(y, \mu)$, and $d_n(y, \mu) = \chi_{\Lambda_n}(y) d(y, \mu)$. Denote by $\{P_{\mu}^n: \mu \in E\}$ the family of solutions associated with the corresponding operators $\mathcal{L}_n$. By Theorem 1.15, for each $\mu \in E$ the sequence $\{P_{\mu}^n: n \geqslant 1\}$ is weakly precompact; and by Theorem 3.11 for any bounded interval I, $T > 0$ and bounded $\mathcal{M}_T^I$-measurable $\Phi: \Omega \to \mathbf{C}$, $E^{P_{\mu}^n}[\Phi] \to E^{P_{\mu}}[\Phi]$. We will now use this to show that P_{μ} solves the martingale problem for $\mathcal{L}$ starting from μ. Let $0 \leqslant t_1 < t_2$, $I = [c, d]$, and $A \in \mathcal{M}_{t_1}^I$ be given. If $f \in \mathcal{D}(E; I)$, then for n satisfying $I \subseteq (-n, n)$:

$$E^{P_{\mu}^n}[f(\mu_{t_2}) - f(\mu_{t_1}), A] = E^{P_{\mu}^n}\left[\int_{t_1}^{t_2} \mathcal{L}f(\mu_s)\, ds, A\right].$$

Since the term on the left tends to $E^{P_{\mu}}[f(\mu_{t_2}) - f(\mu_{t_1}), A]$ as $n \to \infty$, it suffices for us to check that

$$E^{P_{\mu}}[\mathcal{L}f(\mu_s), A] = \lim_{n \to \infty} E^{P_{\mu}^n}[\mathcal{L}f(\mu_s), A]$$

for each $t_1 \leqslant s \leqslant t_2$. But, for all $k \geqslant |c| \vee |d|$:

$$E^{P_\mu}[\mathfrak{L}f(\mu_s),\ A \cap \{l_{\mu_s}(c) \geqslant -k \text{ and } r_{\mu_s}(d) \leqslant k\}]$$
$$= \lim_{n\to\infty} E^{P^n_\mu}[\mathfrak{L}f(\mu_s),\ A \cap \{l_{\mu_s}(c) \geqslant -k \text{ and } r_{\mu_s}(d) \leqslant k\}];$$

and, by Lemma 1.12, it is easy to check that

$$\lim_{k\to\infty} \sup_{1\leqslant n\leqslant\infty} P^n_\mu(l_{\mu_s}(c) < -k \text{ or } r_{\mu_s}(d) > k) = 0,$$

where $P^\infty_\mu = P_\mu$. Q.E.D.

3.14. *Remark.* The techniques used to prove Theorem 3.13 are very special. Unfortunately, they do not seem to lend themselves to easy generalization beyond obvious variations on what we have done here. In fact for second nearest neighbor interactions with bounded birth and death rates it is easy to construct examples of non-uniqueness. One possible way to generalize what we have done here would be to replace Lebesgue measure in the birth term of $\mathfrak{L}$ with some other locally finite measure. Everything goes through as above with the obvious modifications. If we use counting measure on the integers instead of Lebesgue measure then an obvious modification of the proof of Theorem 3.13 yields existence and uniqueness for the birth and death processes on the integers in [11].

In more than one dimension the situation is more complicated. We do not see how to handle the question of uniqueness for the martingale problem in more than one dimension. Of course, there are a few special cases which are amenable to known techniques; for instance, one can use methods familiar in the study of spin-flip models to treat the case in which a "hard core" exists (i.e. when there is an $\varepsilon > 0$ such that $\mu(\{a\}) = 1$ implies $b(y,\mu) = 0$ for $|y-a| < \varepsilon$). However, a satisfactory general theory appears to be difficult.

4. Reversible equilibrium states—necessary conditions

Let β and δ be positive bounded measurable functions on $[0, \infty)^2$, and suppose that

$$b(y, \mu) = \beta(y - l_\mu(y), r_\mu(y) - y)$$

and

$$d(y, \mu) = \delta(y - l_\mu(y), r_\mu(y) - y).$$

Let $\mathfrak{L}$ be the operator on $\mathcal{D}(E)$ determined by b and d. We say that a probability measure, m, on E is a time reversible equilibrium state for $\mathfrak{L}$ if

$$\int \varphi \mathfrak{L}\psi\, dm = \int \psi \mathfrak{L}\varphi\, dm \quad \text{for all } \varphi, \psi \in \mathcal{D}. \tag{4.1}$$

Let $g(l, r) = \beta(l, r)/\delta(l, r)$. The goal of this section is to find necessary conditions on g in order for (4.1) to hold for some m. The results and methods in this section are due to

Spitzer [11] in the context of birth and death processes on the integers. Our contribution here is merely to make the modifications necessary to fit the present situation.

We need the following additional assumption:

(4.2) β is uniformly positive on compact subsets of $(0, \infty)^2$ and δ is uniformly positive on compact subsets of $[0, \infty)^2$.

These of course imply that

(4.3) g is locally bounded on $[0, \infty)^2$ and uniformly positive on compact subsets of $(0, \infty)^2$.

For the rest of this section let m be a fixed probability measure on E satisfying (4.1).

If Λ is a finite interval, we denote by $\tilde{\mathcal{B}}^\Lambda$ the smallest σ-algebra for which every element in $\{\varphi: \varphi \in B(E; \Gamma)$ for some Γ with $\Gamma \cap \Lambda = \varnothing\}$ is measurable, and by $m^{(\Lambda,\mu)}$ the r.c.p.d. of $m \mid \tilde{\mathcal{B}}^\Lambda$ evaluated at μ. Of course $m^{(\Lambda,\mu)}$ is a probability measure on $E_\mu(\Lambda)$. ($E_\mu(\Lambda)$ is defined in Theorem 2.6.)

The key to our analysis is the following observation. If $\varphi, \psi \in \mathcal{D}(E; \Lambda)$ and $\gamma \in \mathcal{D}(E, \Gamma)$ with $\Lambda \cap \Gamma = \varnothing$, then since $\|\Delta_y \varphi\| \, \|\Delta_y \gamma\| \equiv 0$ it follows that

$$\mathfrak{L}(\varphi\gamma) = \gamma \mathfrak{L}\varphi + \varphi \mathfrak{L}\gamma,$$

and thus

$$0 = \int \varphi\gamma \mathfrak{L}1 \, dm = \int \mathfrak{L}(\varphi\gamma) \, dm \tag{4.4}$$

$$= \int (\gamma \mathfrak{L}\varphi + \varphi \mathfrak{L}\gamma) \, dm = 2 \int \gamma \mathfrak{L}\varphi \, dm = 2 \int \varphi \mathfrak{L}\gamma \, dm.$$

Thus

$$\int \gamma(\varphi \mathfrak{L}\psi) \, dm = \int \psi \mathfrak{L}(\gamma\varphi) \, dm = \int \psi(\gamma \mathfrak{L}\varphi + \varphi \mathfrak{L}\gamma) \, dm = \int \gamma(\psi \mathfrak{L}\varphi) \, dm. \tag{4.5}$$

Equation (4.5) holds for all $\varphi, \psi \in \mathcal{D}(E; \Lambda)$ and all $\gamma \in \mathcal{D}(E; \Gamma)$ provided $\Gamma \cap \Lambda = \varnothing$. Therefore for each pair $\varphi, \psi \in \mathcal{D}(E; \Lambda)$ we have

$$\int \varphi \mathfrak{L}\psi \, dm^{(\Lambda,\mu)} = \int \psi \mathfrak{L}\varphi \, dm^{(\Lambda,\mu)} \quad \text{(a.e., } m). \tag{4.6}$$

From Lemma 2.9 it follows that for a.e. (m) μ

$$\int \left(\int_\Lambda d(y, \mu') \mu'(dy) \right) m^{(\Lambda,\mu)}(d\mu') < \infty.$$

Therefore for a.e. (m) μ the set of pairs for which (4.6) holds is closed under bounded pointwise convergence, and it is clearly closed under finite linear combinations. Let $\{h_j, j\geq 1\}$ be a countable dense (in the uniform norm) set in $C_0^+(\Lambda)$ and let $\mathfrak{J}\subset \mathcal{D}(E;\Lambda)$ be the set of functions of the form

$$\exp\left[i\sum_{j=1}^{k}\lambda_j\left(\left(\int h_j(y)\,\mu(dy)\wedge n\right)\right)\right]$$

for positive integral n and rational λ_j. By Lemma 1.5 $\mathfrak{J}\subset \mathcal{D}(E;\Lambda)$. Since $\mathfrak{J}$ is countable there is a measurable set $N\in\mathcal{B}_E$ with $m(N)=0$ such that if $\mu\notin N$ then (4.6) holds for all φ, $\psi\in\mathfrak{J}$. By Lemma 1.2 the set of functions which is closed under bounded pointwise limits and finite linear combinations and contains $\mathfrak{J}$ is $B(E;\Lambda)$. Thus, except for a set of m measure zero, (4.6) holds for all φ, $\psi\in B(E;\Lambda)$. Now repeating the argument in (4.4) we see that, a.s. (m), if $\varphi\in B(E;\Lambda_1)$ and $\psi\in B(E;\Lambda_2)$ with $\Lambda_1\cap\Lambda_2=\emptyset$ and $\Lambda_1\cup\Lambda_2\subset\Lambda$, then

$$\int\psi(\mu')\,\mathfrak{L}\varphi(\mu')\,m^{(\Lambda,\mu)}(d\mu')=0. \tag{4.7}$$

Let λ_k denote Lebesgue measure on R^k. We first show that g satisfies a certain equation a.e. (λ_3).

4.8. LEMMA. *Let $a<c<b$. Then for a.e. (m) μ we have*

$$g(x-l_\mu(a),\,y-x)\,g(y-l_\mu(a),\,r_\mu(b)-y)=g(y-x,\,r_\mu(b)-y)\,g(x-l_\mu(a),\,r_\mu(b)-x) \tag{4.9}$$

for a.e. (λ_2) $(x,y)\in(a,c)\times(c,b)$.

Proof. Let $\Lambda=(a,b)$, and if A_1, A_2 are Borel subsets of Λ let $S(A_1)$ be the event $\mu(A_1)=0$ and $T(A_1)$ $(T(A_1,A_2))$ be the event $\mu(A_1)=1$ $(\mu(A_1)=\mu(A_2)=1)$.

Let $\mu\setminus\Lambda$ be the element of E defined by $(\mu\setminus\Lambda)(A)=\mu(A)-\mu(A\cap\Lambda)$ and set $b_\Lambda(y,\mu)=b(y,\mu\setminus\Lambda)$ and $d_\Lambda(y,\mu)=d(y,(\mu\setminus\Lambda)+\delta_y)$.

Now given $A\subset\Lambda$ let $\varphi(\mu)=\mu(A)\wedge 1$ and $\psi(\mu)=\chi_{S(\Lambda\setminus A)}(\mu)$, the indicator function of the event $S(\Lambda\setminus A)$.

$$\mathfrak{L}\varphi(\mu)=\int_A b(y,\mu)\,dy\,\chi_{S(A)}(\mu)-d(y_\mu^A,\mu)\,\chi_{T(A)}(\mu),$$

where on $T(A)$, y_μ^A is defined to be the unique $y\in A$ for which $\mu(\{y\})=1$. From (4.7) we have, for a.e. (m) μ

$$0=\int\left[\int_A b(y,\mu')\,dy\,\chi_{S(A)}(\mu')\,\chi_{S(\Lambda\setminus A)}(\mu')-d(y_{\mu'}^A,\mu')\,\chi_{T(A)}(\mu')\,\chi_{S(\Lambda\setminus A)}(\mu')\right]m^{(\Lambda,\mu)}(d\mu').$$

Thus

$$\int_{S(\Lambda)}\left[\int_A b(y,\mu')\,dy\right] m^{(\Lambda,\mu)}(d\mu') = \int_{T(A)\cap S(\Lambda\setminus A)} d(y^A_{\mu'}, u')\, m^{(\Lambda,\mu)}(d\mu').$$

On $S(\Lambda)$, $b(y,\mu') = b^\Lambda(y,\mu)$ a.e. $(m^{(\Lambda,\mu)})$, and $T(A)\cap S(\Lambda\setminus A) = T(A,\Lambda)$. Thus there is a set N with $m(N)=0$ such that if $\mu\notin N$

$$m^{(\Lambda,\mu)}(S(\Lambda))\int_A b_\Lambda(y,\mu)\,dy = \int_{T(A,\Lambda)} d(y^A_{\mu'},\mu')\, m^{(\Lambda,\mu)}(d\mu') \tag{4.10}$$

for all $A\subset\Lambda$. If $\mu\notin N$ (4.10) implies that the measure $A\to m^{(\Lambda,\mu)}(T(A,\Lambda))$ is absolutely continuous with respect to λ_1 and has density

$$f^{(\Lambda,\mu)}(y) = \frac{b_\Lambda(y,\mu)}{d_\Lambda(y,\mu)} m^{(\Lambda,\mu)}(S(\Lambda)).$$

Now let $A\subset(a,c)$ and $B\subset(c,b)$. Let φ be as before and set $\psi = \chi_{T(B)\cap S(\Lambda\setminus(A\cup B))}$. Using these φ and ψ in (4.7) gives, for $\mu\notin N$,

$$\begin{aligned}\int_{T(B,\Lambda)}\left[\int_A b(x,\mu')\,dx\right] m^{(\Lambda,\mu)}(d\mu') &= \int_{T(A,B)\cap S(\Lambda\setminus(A\cup B))} d(y^A_{\mu'},\mu')\, m^{(\Lambda,\mu)}(d\mu') \\ &= \int_B\left[\int_A b(x,(\mu\setminus\Lambda)+\delta_y)\,dx\right] f^{(\Lambda,\mu)}(y)\,dy.\end{aligned} \tag{4.11}$$

Thus the measure on $(a,c)\times(c,b)$ determined by

$$A\times B\to m^{(\Lambda,\mu)}(T(A,B)\cap S(\Lambda\setminus(A\cup B))) \tag{4.12}$$

is absolutely continuous with respect to λ_2 and has density

$$h_1^{(\Lambda,\mu)}(x,y) = \frac{b_\Lambda(y,\mu)}{d_\Lambda(y,\mu)}\,\frac{b(x,(\mu\setminus\Lambda)+\delta_y)}{d(x,(\mu\setminus\Lambda)+\delta_x+\delta_y)} m^{(\Lambda,\mu)}(S(\Lambda)). \tag{4.13}$$

If we interchange the roles of A and B in the above derivation we get

$$h_2^{(\Lambda,\mu)}(x,y) = \frac{b_\Lambda(x,\mu)}{d_\Lambda(x,\mu)}\,\frac{b(y,(\mu\setminus\Lambda)+\delta_x)}{d(y,(\mu\setminus\Lambda)+\delta_x+\delta_y)} m^{(\Lambda,\mu)}(S(\Lambda))$$

as the density of the measure given by (4.12). Thus for a.e. (m) μ we have $h_1^{(\Lambda,\mu)}(x,y) = h_2^{(\Lambda,\mu)}(x,y)$ for a.e. (λ_2) $(x,y)\in(a,c)\times(c,b)$. Since $m^{(\Lambda,\mu)}(S(\Lambda))>0$ a.s. (m) (see Corollary 2.11), the conclusion follows immediately upon substituting the definitions of b and d into the equation $h_1^{(\Lambda,\mu)} = h_2^{(\Lambda,\mu)}$. Q.E.D.

4.14. Lemma. *Let $a<b$. Then λ_2 on $(-\infty,a)\times(b,\infty)$ is equivalent to the joint distribution of $(l_\mu(a), r_\mu(b))$ under m.*

Proof. We first show that λ_2 is absolutely continuous with respect to the distribution of $(l_\mu(a), r_\mu(b))$ under m. To do this it suffices to show that for each $\alpha < a$ and $\gamma > b$ and $\Gamma \subset (\alpha, a) \times (b, \gamma)$ with $m((l_\mu(a), r_\mu(b)) \in \Gamma) = 0$ we must have $\lambda_2(\Gamma) = 0$. This will follow if, setting $\Lambda = (\alpha, \gamma)$, $m^{(\Lambda,\mu)}((l_{\mu'}(a), r_{\mu'}(b) \in \Gamma) = 0$ implies that $\lambda_2(\Gamma) = 0$ for a.e. $(m)\,\mu$. Fix $c \in (a, b)$ and let $h_1^{(\Lambda,\mu)}(x, y)$ be the density with respect to λ_2 for the measure on $(\alpha, c) \times (c, \gamma)$ determined by (4.12). $h_1^{(\Lambda,\mu)}$ is given by (4.13) and is strictly positive in x and y for a.e. $(m)\,\mu$. Note that $T(A, B) \cap S(\Lambda \setminus (A \cup B)) = T(A, B) \cap T((\alpha, c), [c, \gamma))$. If $\mu \in T((\alpha, c), [c, \gamma))$, let x_μ^l (x_μ^r) denote the unique $x \in (\alpha, c)$ $(x \in [c, \gamma))$ such that $\mu(x) = 1$. Then if $\Gamma \subset (\alpha, c) \times [c, \gamma)$

$$m^{(\Lambda,\mu)}((x_{\mu'}^l, x_{\mu'}^r) \in \Gamma \quad \text{and} \quad T((\alpha, c), [c, \gamma))) = \int_\Gamma h_1^{(\Lambda,\mu)}(x, y)\, dx\, dy$$

for a.e. $(m)\,\mu$. The first half of the lemma now follows from the positivity of $h_1^{(\Lambda,\mu)}$ and

$$m^{(\Lambda,\mu)}((l_{\mu'}(a), r_{\mu'}(b)) \in \Gamma) \geqslant m^{(\Lambda,\mu)}((x_{\mu'}^l, x_{\mu'}^r) \in \Gamma \quad \text{and} \quad T((\alpha, c), [c, \gamma)))$$
$$= \int_\Gamma h_1^{(\Lambda,\mu)}(x, y)\, dx\, dy.$$

Conversely to show that the distribution of $(l_\mu(a), r_\mu(b))$ under m is absolutely continuous with respect to Lebesgue measure it suffices to show that for each $\Lambda = (\alpha, \gamma) \supset (a, b)$ the distribution of $(l_{\mu'}(a), r_{\mu'}(b))$ on $(\alpha, a) \times (b, \gamma)$ under $m^{(\Lambda,\mu)}$ is absolutely continuous with respect to λ_2 for a.e. $(m)\mu$. From (4.7) and Lemma 2.9 we see that if Q is an interval contained in Λ then

$$\delta\, m^{(\Lambda,\mu)}(\mu'(Q) \geqslant 1) \leqslant \int \left[\int_Q d(y, \mu')\, \mu'(dy)\right] m^{(\Lambda,\mu)}(d\mu') \leqslant \|b\|\, \lambda_1(Q),$$

where $\delta = \inf\{d(y, \mu') \colon y \in \Lambda \text{ and } \mu' \in E_\mu(\Lambda)\}$. By (4.2), $\delta > 0$. Now let I be an interval in Λ disjoint from Q and set $\varphi_N(\mu) = \mu(I) \wedge N$ and $\psi(\mu) = \mu(Q) \wedge 1$. By (4.7) we have for a.e. $(m)\,\mu$

$$0 = \int \psi \mathfrak{L} \varphi_N(\mu')\, m^{(\Lambda,\mu)}(d\mu')$$
$$= \int_{\{\mu'(Q) \geqslant 1,\, \mu'(I) < N\}} \left[\int_I b(y, \mu')\, dy\right] m^{(\Lambda,\mu)}(d\mu') - \int_{\{\mu'(Q) \geqslant 1,\, \mu'(I) \leqslant N\}} \left[\int_I d(y, \mu')\, \mu'(dy)\right] m^{(\Lambda,\mu)}(d\mu').$$

Thus

$$\|b\|\, \lambda_1(I)\, m^{(\Lambda,\mu)}(\mu'(Q) \geqslant 1, \mu'(I) < N) \geqslant \delta\, m^{(\Lambda,\mu)}(\mu'(Q) \geqslant 1, 1 \leqslant \mu'(I) \leqslant N).$$

Letting N go to infinity and combining this with the previous inequality we have

$$\|b\|^2 \lambda_1(I)\, \lambda_1(Q)/\delta^2 \geqslant m^{(\Lambda,\mu)}(\mu'(Q) \geqslant 1, \mu'(I) \geqslant 1).$$

Now if $Q \subset (\alpha, a)$ and $I \subset (b, \gamma)$

$$m^{(\Lambda,\mu)}(\mu'(Q) \geqslant 1, \mu'(I) \geqslant 1) \geqslant m^{(\Lambda,\mu)}((l_{\mu'}(a), r_{\mu'}(b)) \in Q \times I).$$

Thus

$$\|b\|^2 \lambda \xi(Q \times I)/\delta^2 \geqslant m^{(\Lambda,\mu)}((l_{\mu'}(a), r_{\mu'}(b)) \in Q \times I),$$

which implies the desired result and completes the proof. Q.E.D.

4.15. LEMMA. *For a.e.* (λ_3), $(x, y, z) \in [0, \infty)^3$

$$g(x, y)g(x+y, z) = g(y, z)g(x, y+z). \tag{4.16}$$

Proof. Fix $a<c<b$. From Lemmas 4.8 and 4.14 we have for a.e. (λ_4) $(l, x, y, r) \in (-\infty, a) \times (a, c) \times (c, b) \times (b, \infty)$

$$g(x-l, y-x)g(y-l, r-y) = g(y-x, r-y)g(x-l, r-x). \tag{4.17}$$

Since (4.17) holds a.e. (λ_4) for all rational $a<c<b$ it follows that (4.17) holds a.e. (λ_4) on $A = \{(l, x, y, r): l \leqslant x \leqslant y \leqslant r\}$. Consider the transformation $\Phi: A \to [0, \infty)^3$ given by

$$\Phi(l, x, y, r) = (x-l, y-x, r-y).$$

Let B be the subset of A on which (4.17) holds. It suffices to show that $\lambda_3([0, \infty)^3 \setminus \Phi(B)) = 0$. Let $\Psi: A \to R \times [0, \infty)^3$ be the transformation $\Psi(l, x, y, r) = (l, \Phi(l, x, y, r))$. The Jacobian of Ψ is identically one. Therefore since $\lambda_4(A \setminus B) = 0$, it follows that $\lambda_4(\Psi(A) \setminus \Psi(B)) = 0$. Thus for a.e. (λ_1) l the section of $\Psi(A) \setminus \Psi(B)$ at l has λ_3 measure 0. Let l_0 be such an l and, denoting the l_0 section by subscripts, we have

$$\begin{aligned} 0 &= \lambda_3((\Psi(A) \setminus \Psi(B))_{l_0}) = \lambda_3([0, \infty)^3 \setminus (\Psi(B))_{l_0}) \\ &= \lambda_3([0, \infty)^3 \setminus \Phi(B_{l_0})) \geqslant \lambda_3([0, \infty)^3 \setminus \Phi(B)). \end{aligned}$$

Q.E.D.

Our next goal is to show that if g satisfies (4.3) and the conclusions of Lemma 4.15 then there is a measurable function f such that

$$g(x, y) = f(x)f(y)/f(x+y) \quad \text{a.e. } (\lambda_2) \quad \text{on } [0, \infty)^2. \tag{4.18}$$

This could be done by writing down what f should be (see (4.26)) and then checking that it satisfies (4.18). However, the computations involved in this approach are extremely tedious, so we first prove the existence of such an f. The question of the existence of such an f is very similar to the analogous question in the theory of cocycles (see [9]) and we learned the proof of Lemma 4.20 below from [9]; however, to the best of our knowledge, expression (4.27) is new.

4.19. LEMMA. *If g satisfies (4.16) a.e. (λ_3) and $h=g$ a.e. (λ_2) then h satisfies (4.16) a.e. (λ_3).*

Proof. It clearly suffices to show that the corresponding factors are equal a.e. (λ_3). For the factors involving only two of the variables this is obvious. Thus we show that $g(x+y, z)=h(x+y, z)$ $(g(x, y+z)=h(x, y+z))$ a.e. (λ_3). But for each fixed y, $g(x+y, z)=h(x+y, z)$ $(g(x, y+z)=h(x, y+z))$ for a.e. (λ_2) (x, z). The desired conclusion follows from Fubini's theorem. Q.E.D.

4.20. LEMMA. *Suppose g satisfies (4.3) and (4.16) holds a.e. (λ_3). Then there is a positive measurable function f and a positive C^∞ function G which satisfies* (4.16) *pointwise and such that*

$$g(x,y)=G(x,y)\frac{f(x)f(y)}{f(x+y)} \quad \text{a.e. } (\lambda_2) \tag{4.21}$$

Moreover $G(0, y)=G(x, 0)\equiv 1$.

Proof. It suffices to prove (4.21) for $x, y>0$. Thus by (4.3) we may assume that all of the integrals which follow are convergent. Let $\tilde{g}(x, y)=\ln(g(x,y))$ and let $a(x)\geqslant 0$ be a C^∞ function with compact support in $(0, \infty)$ and $\int_0^\infty a(x)\,dx=1$. Set

$$\tilde{f}(x)=\int_0^\infty\int_0^\infty[\tilde{g}(x,t)+\tilde{g}(s,t+x)-\tilde{g}(s,t)]\,a(s)\,a(t)\,ds\,dt.$$

and $f(x)=\exp[\tilde{f}(x)]$. Let

$$\tilde{G}_0(x, y) = \tilde{g}(x, y)-\tilde{f}(x)-\tilde{f}(y)+\tilde{f}(x+y)$$

and $G_0(x, y)=\exp[\tilde{G}_0(x, y)]=g(x, y)[f^{-1}(x)f^{-1}(y)/f^{-1}(x+y)]$. G_0 clearly satisfies (4.16) a.e. (λ_3). Now

$$\begin{aligned}\tilde{G}_0(x,y)=\int_0^\infty\int_0^\infty[&\tilde{g}(x,y)-\tilde{g}(s,x)-\tilde{g}(s,y)+\tilde{g}(s,x+y)-\tilde{g}(x,t)+\tilde{g}(s,x)+\tilde{g}(s,t)-\tilde{g}(s,t+x)\\ &-\tilde{g}(y,t)+\tilde{g}(s,y)+\tilde{g}(s,t)-\tilde{g}(s,t+y)\\ &+\tilde{g}(x+y,t)-\tilde{g}(s,x+y)-\tilde{g}(s,t)+\tilde{g}(s,x+y+t)]\,a(s)\,a(t)\,ds\,dt.\end{aligned}$$

Using the identity

$$\tilde{g}(x, y)-\tilde{g}(z, x)+\tilde{g}(z, x+y) = \tilde{g}(z+x, y) \quad \text{a.e. } (\lambda_3)$$

four times together with the observation that if $\lambda_1(N)=0$ and $M=\{(x, y)\colon x+y\in N\}$ then $\lambda_2(M)=0$ we see that for a.e. (λ_2) (x, y)

$$\begin{aligned}\tilde{G}_0(x,y)=\int_0^\infty\int_0^\infty&[(\tilde{g}(s+x,y)-\tilde{g}(s,y))-(\tilde{g}(x+s,t)-\tilde{g}(s,t))\\ &-(\tilde{g}(y+s,t)-\tilde{g}(s,t))+(\tilde{g}(x+y+s,t)-\tilde{g}(s,t))]\,a(s)\,a(t)\,ds\,dt.\end{aligned} \tag{4.22}$$

Similarly $\tilde{g}(s+x, y)+\tilde{g}(s+x+y, t)=\tilde{g}(y, t)+\tilde{g}(s+x, y+t)$ a.e. (λ_4) and $\tilde{g}(y, t)-\tilde{g}(s, y)=\tilde{g}(s+y, t)-\tilde{g}(s, y+t)$ a.e. (λ_3). Successively substituting these equations into (4.22) we see that for a.e. $(\lambda_2)\,(x, y)$

$$\text{(4.23)}\quad \tilde{G}_0(x, y)=\int_0^\infty\int_0^\infty [\tilde{g}(s+x, t+y)+\tilde{g}(s, t)-\tilde{g}(s, t+y)-\tilde{g}(s+x, t)]\,a(s)\,a(t)\,ds\,dt$$
$$=\int_0^\infty\int_0^\infty \tilde{g}(s, t)[a(s-x)\,a(t-y)+a(s)\,a(t)-a(s)\,a(t-y)-a(s-x)\,a(t)]\,ds\,dt.$$

Denoting the right side of (4.23) by $\tilde{G}(x, y)$ we see that $G(x, y)=\exp[\tilde{G}(x, y)]$ is a positive C^∞ function which, by Lemma 4.19 satisfies (4.16) a.e. (λ_3), and hence satisfies (4.16) everywhere. Moreover (4.21) holds and, since $\tilde{G}(0, y)=\tilde{G}(x, 0)\equiv 0$, we have $G(0, y)=G(x, 0)\equiv 1$. Q.E.D.

4.24. LEMMA. *Let G be a positive C^∞ function on $[0, \infty)^2$ satisfying (4.16) and with $G(0, y)=G(x, 0)\equiv 1$. Then there is a C^∞ function $f>0$ such that*

$$G(x, y)=\frac{f(x)\,f(y)}{f(x+y)}.$$

Proof. Let $\tilde{G}(x,y)=\ln(G(x,y))$ and $\tilde{G}_1(x,y)=(\partial/\partial x)\tilde{G}(x,y)$. Set $f(t)=\exp[\int_0^t \tilde{G}_1(0,u)\,du]$. f is clearly positive and C^∞. Also

$$f(t)\,f(s)/f(t+s)=\exp\left[\int_0^{t+s}\tilde{G}_1(0, u)\,du-\int_0^t \tilde{G}_1(0, u)\,du-\int_0^s \tilde{G}_1(0, u)\,du\right]$$
$$=\exp\left[\int_0^t [\tilde{G}_1(0, u+s)-\tilde{G}_1(0, u)]\,du\right].$$

From (4.16) one easily finds that $\tilde{G}_1(0, u+s)-\tilde{G}_1(0, u)=\tilde{G}_1(u, s)$. Thus

$$f(t)\,f(s)/f(t+s)=\exp\left[\int_0^t \tilde{G}_1(u, s)\,du\right]=\exp[\tilde{G}(t, s)]=G(t, s). \qquad \text{Q.E.D.}$$

4.25. THEOREM. *Let g be a measurable function which satisfies (4.3) and (4.16) a.e. (λ_3). Then there is a positive measurable function f on $[0, \infty)$ such that $g(x, y)=f(x)f(y)/f(x+y)$ a.e. (λ_2).*

Proof. This follows immediately from Lemmas 4.20 and 4.24. Q.E.D.

4.26. *Remark.* The function f in (4.18) can be taken to be

$$\text{(4.27)}\qquad f(t)=\begin{cases}\exp\left[\int_0^1 \tilde{g}(s+1, t-1)\,ds-\int_1^t \tilde{g}(1, s)\,ds-\tilde{g}(1, t-1)\right] & \text{if } t\geqslant 1\\[1ex] \exp\left[\int_t^1 \tilde{g}(t, s)\,ds-\int_0^t \tilde{g}(1-t, t+s)\,ds+t\tilde{g}(1-t, t)\right] & \text{if } t\leqslant 1.\end{cases}$$

To see this let $\bar{f}$ be as in (4.18). (We know such an $\bar{f}$ exists by Theorem 4.25.) Substituting the right side of (4.18) for g in (4.27) we see that for a.e. (λ_1) t

$$f(t)=\bar{f}(t)(\bar{f}(1))^{-1},$$

and thus $f(t)$ also satisfies (4.18). If g is continuous, $f(t)$ given by (4.27) is the unique continuous function satisfying (4.18) and having $f(1)=1$.

5. Reversible equilibrium states–sufficient conditions

Let b and d be as in Section 4 and satisfy (4.2). Again we denote $\beta(l,r)/\delta(l,r)$ by $g(l,r)$. From Theorem 4.25 and Lemma 4.15 we know that if a time reversible equilibrium state is going to exist then $g(l,r)$ must be equal a.s. (λ_2) to $f(l)f(r)/f(l+r)$, where f is given by (4.27). In particular since g is locally bounded on $[0,\infty)^2$, f is locally bounded on $[0,\infty)$. Let $M(t)=\sup_{0\leqslant s\leqslant t} f(s)$. If $l\leqslant a<b\leqslant r$ define

$$u(l,a,b,r)=f(r-l)+\sum_{k=1}^{\infty}\int_{a\leqslant x_1<x_2<\dots<x_k\leqslant b}\int f(x_1-l)\prod_{j=2}^{k}f(x_j-x_{j-1})f(r-x_k)\,dx_1\dots dx_k$$

where $\prod_{j=2}^{1}$ is taken to be one. One easily checks that

$$u(l,a,b,r)\leqslant M(r-l)\exp\left[(r-l)M(r-l)\right]<\infty.$$

Denote the interval (a,b) by Λ. If $\nu\in E$ is such that $l_\nu(a)=l$ and $r_\nu(b)=r$ define $m^{(\Lambda,\nu)}$ to be the measure on $E_\nu(\Lambda)$ for which

$$\text{(5.1)}\quad u(l,a,b,r)\int\varphi(\mu)\,m^{(\Lambda,\nu)}(d\mu)$$
$$=\varphi(\phi)f(r-l)+\int_a^b\varphi(\{x\})f(x-l)f(r-x)\,dx$$
$$+\sum_{k=2}^{\infty}\int_{a\leqslant x_1<\dots<x_k\leqslant b}\int\varphi(\{x_1,\dots,x_k\})f(x_1-l)\prod_{j=2}^{k}f(x_j-x_{j-1})f(r-x_k)\,dx_1\dots dx_k$$

for all $\varphi\in B(E,\Lambda)$. Here $\{x_1\dots x_k\}$ denotes any element of E whose restriction to Λ is $\sum_{i=1}^{k}\delta_{x_i}$.

Let $\mathcal{G}_f$ be the set of probability measures, m, on $(E,\mathcal{B}_E)$ such that for every finite interval, Λ, the r.c.p.d. of $m\,|\,\tilde{\mathcal{B}}^{\Lambda}$ evaluated at ν is the measure on $E_\nu(\Lambda)$ given by (5.1). This definition of $\mathcal{G}_f$ in terms of conditional probabilities involving f is analogous to the usual definition of Gibbs states in terms of conditional probabilities involving a potential. We will see in the next section that when f is a probability density with finite first moment, then $\mathcal{G}_f$ has the renewal measure determined by f as its only element. Moreover this is essentially the only case in which $\mathcal{G}_f$ is not empty. In this case (5.1) describes the renewal measure in terms of its conditional probabilities and (6.2) describes it in terms of its marginals.

5.2. LEMMA. *Let b and d be nearest neighbor birth and death rates as above and assume that $g(l, r) = f(l)f(r)/f(l+r)$ a.e. (λ_2) for some positive locally bounded measurable function, f. Let $m^{(\Lambda,\nu)}$ be as in (5.1). Then for all finite intervals $\Lambda = (a, b)$ there is a set $N \subset (-\infty, a) \times (b, \infty)$ with $\lambda_2(N) = 0$ such that for all $\nu \in E$ with $(l_\nu(a), r_\nu(b)) \notin N$*

$$\int \varphi \mathfrak{L} \psi \, dm^{(\Lambda,\nu)} = \int \psi \mathfrak{L} \varphi \, dm^{(\Lambda,\nu)} \tag{5.3}$$

for all $\varphi, \psi \in B(E; \Lambda)$.

Proof. Assume first that $g(l, r) = f(l)f(r)/f(l+r)$ everywhere. Then

$$\varphi(\mu)\,\mathfrak{L}\psi(\mu) - \psi(\mu)\,\mathfrak{L}\varphi(\mu) = \int_a^b b(y,\mu)[\varphi(\mu)\psi(\mu+\delta_y) - \psi(\mu)\varphi(\mu+\delta_y)]\,dy$$
$$+ \int_a^b d(y,\mu)[\varphi(\mu)\psi(\mu-\delta_y) - \psi(\mu)\varphi(\mu-\delta_y)]\,\mu(dy).$$

Denote $l_\nu(a)$ by l and $r_\nu(b)$ by r, and let $M_k = \{\mu \in E_\nu(\Lambda): \mu(\Lambda) = k\}$. Then

$$\begin{aligned}
\int (\varphi\mathfrak{L}\psi - \psi\mathfrak{L}\varphi)\,dm^{(\Lambda,\nu)} &= \int \Bigg[\int_a^b b(y,\mu)[\varphi(\mu)\psi(\mu+\delta_y) - \psi(\mu)\varphi(\mu+\delta_y)]\,dy \\
&\quad + \int_a^b d(y,\mu)[\varphi(\mu)\psi(\mu-\delta_y) - \psi(\mu)\varphi(\mu-\delta_y)]\,\mu(dy)\Bigg]\, m^{(\Lambda,\nu)}(d\mu) \\
&= \sum_{k=0}^{\infty} \Bigg\{\int_{M_k}\Bigg[\int_a^b b(y,\mu)\varphi(\mu)\psi(\mu+\delta_y)\,dy\Bigg]\, m^{(\Lambda,\nu)}(d\mu) \\
&\quad - \int_{M_{k+1}}\Bigg[\int_a^b d(y,\mu)\psi(\mu)\varphi(\mu-\delta_y)\,\mu(dy)\Bigg]\, m^{(\Lambda,\nu)}(d\mu)\Bigg\} \\
&\quad - \sum_{k=0}^{\infty} \Bigg\{\int_{M_k}\Bigg[\int_a^b b(y,\mu)\psi(\mu)\varphi(\mu+\delta_y)\,dy\Bigg]\, m^{(\Lambda,\nu)}(d\mu) \\
&\quad - \int_{M_{k+1}}\Bigg[\int_a^b d(y,\mu)\varphi(\mu)\psi(\mu-\delta_y)\,\mu(dy)\Bigg]\, m^{(\Lambda,\nu)}(d\mu)\Bigg\}.
\end{aligned}$$

It suffices to show that each of the terms in the two summations is zero. The terms $k=0$ and $k=1$ require slightly different notation, but the idea is exactly the same as for the general term with which we deal. We consider only the terms of the first series. The results for the second series follow by interchanging φ and ψ. Fix k and set $z_0 = x_0 = l$ and $z_{k+1} = x_{k+1} = r$. Using the identify $\delta(l, r) = \beta(l, r)f(l+r)/f(l)f(r)$ we have

$$\int_{M_k}\left[\int_a^b b(y,\mu)\varphi(\mu)\psi(\mu+\delta_y)\,dy\right]m^{(\Lambda,\nu)}(d\mu)-\int_{M_{k+1}}\left[\int_a^b d(y,\mu)\psi(\mu)\varphi(\mu-\delta_y)\mu(dy)\right]m^{(\Lambda,\nu)}(d\mu)$$

$$=\sum_{n=0}^{k}\int_{a\leqslant x_1<\ldots<x_k\leqslant b}\int\prod_{j=1}^{k+1}f(x_j-x_{j-1})$$

$$\times\left[\int_{x_n\vee a}^{x_{n+1}\wedge b}\beta(y-x_n,x_{n+1}-y)\,\varphi(\{x_1,\ldots,x_k\})\,\psi(\{x_1,\ldots,x_k,y\})\,dy\right]dx_1\ldots dx_k\,u^{-1}(l,a,b,r)$$

$$-\sum_{n=1}^{k+1}\int_{a\leqslant z_1<\ldots<z_{k+1}\leqslant b}\int\sum_{j=1}^{k+2}f(z_j-z_{j-1})\,\beta(z_n-z_{n-1},z_{n+1}-z_n)\left[\frac{f(z_{n+1}-z_{n-1})}{f(z_{n+1}-z_n)\,f(z_n-z_{n-1})}\right]$$

$$\times\varphi(\{z_1,\ldots,z_{n-1},z_{n+1},\ldots,z_{k+1}\})\,\psi(\{z_1,\ldots,z_{k+1}\})\,dz_1\ldots dz_{k+1}\,u^{-1}(l,a,b,r)=0.$$

One obtains the last equality by comparing the term $n=m$ in the first series with the term $n=m+1$ in the second series.

If the equality $g(l,r)=f(l)f(r)/f(l+r)$ only holds a.e. (λ_2) then the above computation still yields zero for $k\geqslant 1$. When $k=0$ we have

$$f(r-l)\int_a^b\beta(y-l,r-y)\,\varphi(\varnothing)\,\psi(\{y\})\,dy\,u^{-1}(l,a,b,r)$$

$$-\int_a^b f(x-l)\,f(r-x)\,\delta(x-l,r-x)\,\varphi(\varnothing)\,\psi(\{x\})\,dx\,u^{-1}(l,a,b,r),$$

which is zero for a.e. (λ_2) $(l,r)\in(-\infty,a]\times[b,\infty)$. Q.E.D.

5.4. THEOREM. *Let b and d be nearest neighbor birth and death rates as above and satisfying (4.2). Assume that $g(l,r)=f(l)f(r)/f(l+r)$ a.e. (λ_2) for some positive measurable locally bounded function f. Then every probability measure $m\in\mathcal{G}_f$ is a time reversible equilibrium state for $\mathfrak{L}$. That is*

$$\int\varphi\mathfrak{L}\psi\,dm=\int\psi\mathfrak{L}\varphi\,dm,\quad\psi,\varphi\in\mathcal{G}(E).\tag{5.5}$$

Proof. From the definition of $\mathcal{G}_f$ and Lemma 5.2 it suffices to show that if $a<b$ then the joint distribution of $(l_\mu(a),r_\mu(b))$ under m is absolutely continuous with respect to λ_2. Fix $a<b$. If $\tilde{a}<a$ and $\tilde{b}>b$ then the conditional distribution of $(l_\mu(a),r_\mu(b))$ on $(\tilde{a},a]\times[b,\tilde{b})$ given $\mathfrak{B}^{(\tilde{a},\tilde{b})}$ is easily seen to be absolutely continuous with respect to λ_2. Since this is true for each $\tilde{a}<a$ and $\tilde{b}>b$, we have the desired result. Q.E.D.

5.6. THEOREM. *Let b and d be nearest neighbor birth and death rates as above and satisfying (4.2). Then necessary and sufficient conditions for the existence of a time reversible equilibrium state, m, are that for some positive measurable locally bounded function f*

(a) $g(l, r) = f(l)f(r)/f(l+r)$ *a.e.* (λ_2)

and

(b) $m \in \mathcal{G}_f$.

Proof. The only thing not yet proved is the necessity of (b). Fix $\Lambda = (a, b)$. We must show that, if m is a time reversible equilibrium state, then $m^{(\Lambda,\nu)}$, the r.c.p.d. of $m|\tilde{\mathcal{B}}^{\Lambda}$ evaluated at ν, is given by (5.1). From (4.6) we know that

$$\int \mathfrak{L}\varphi(\mu)\, m^{(\Lambda,\nu)}(d\mu) = 0 \quad \text{a.s. } (m)$$

for all $\varphi \in \mathcal{D}(E; \Lambda)$. If we set $b_\Lambda(y, \mu) = \chi_\Lambda(y) b(y, \mu)$, $d_\Lambda(y, \mu) = \chi_\Lambda(y) d(y, \mu)$ and define $\mathfrak{L}^{(\Lambda)}$ accordingly then $\mathfrak{L}\varphi(\mu) = \mathfrak{L}^{(\Lambda)}\varphi(\mu)$ for $\varphi \in \mathcal{D}(E; \Lambda)$, and hence for all $\varphi \in \mathcal{D}(E; \Lambda)$

$$\int \mathfrak{L}^{(\Lambda)}\varphi(\mu)\, m^{(\Lambda,\nu)}(d\mu) = 0 \quad \text{a.s. } (m).$$

The proof is completed by using Lemma 5.2, Corollary 2.12 and Lemma 4.14. Q.E.D.

The reason for calling a measure which satisfies (4.1) time reversible for $\mathfrak{L}$ is made clear by the following theorem.

5.7. THEOREM. *Let b and d be non-negative bounded functions satisfying (3.2) and (3.3) and define $\mathfrak{L}$ accordingly. Let m be a time reversible equilibrium state for $\mathfrak{L}$ and let $\{P_\mu : \mu \in E\}$ be the family of solutions to the martingale problem for $\mathfrak{L}$. Then for all $\varphi, \psi \in \mathcal{D}(E)$ and all $t \geq 0$*

$$\int \psi(\mu)\, E^{P_\mu}[\varphi(\mu_t)]\, m(d\mu) = \int \varphi(\mu)\, E^{P_\mu}[\psi(\mu_t)]\, m(d\mu). \tag{5.8}$$

Proof. Set $b_N(y, \mu) = \chi_{(-N,N)}(y) b(y, \mu)$ and $d_N(y, \mu) = \chi_{(-N,N)}(y) d(y, \mu)$ and define $\mathfrak{L}^{(N)}$ accordingly. We first show that if m is time reversible for $\mathfrak{L}$ then it is time reversible for $\mathfrak{L}^{(N)}$. We do this first for $\varphi = \varphi_1 \cdot \varphi_2$ and $\psi = \psi_1 \cdot \psi_2$, where $\varphi_1, \psi_1 \in \mathcal{D}(E; (-N, N))$ and $\varphi_2, \psi_2 \in B(E; (-N, N)^C) \cap \mathcal{D}(E)$. In that case, since $\mathfrak{L}^{(N)}\varphi = \varphi_2 \mathfrak{L}\varphi_1$, $\mathfrak{L}^{(N)}\psi = \psi_2 \mathfrak{L}\psi_1$ and $\mathfrak{L}(\varphi_2\psi_2\psi_1) = \varphi_2\psi_2\mathfrak{L}\psi_1 + \psi_1\mathfrak{L}(\varphi_2\psi_2)$ (see the argument before (4.4)) we have

$$\int \psi \mathfrak{L}^{(N)}\varphi\, dm = \int \varphi_2\psi_2\psi_1 \mathfrak{L}\varphi_1\, dm = \int \varphi_1 \mathfrak{L}(\varphi_2\psi_2\psi_1)\, dm \tag{5.9}$$

$$= \int \varphi_1\varphi_2\psi_2 \mathfrak{L}\psi_1\, dm + \int \varphi_1\psi_1 \mathfrak{L}(\varphi_2\psi_2)\, dm = \int \varphi \mathfrak{L}^{(N)}\psi\, dm.$$

The last equality follows from (4.4). By Lemma 2.9 the set of $\varphi, \psi \in B(E; (-N, N))$ for which (5.9) holds is closed under bounded pointwise convergence and hence by Lemma 1.2

$$\int \psi \mathcal{L}^{(N)} \varphi \, dm = \int \varphi \mathcal{L}^{(N)} \psi \, dm$$

for all φ, $\psi \in B(E)$.

Now if $\{P_\mu^N, \mu \in E\}$ is the family of solutions to the martingale problem for $\mathcal{L}^{(N)}$, then by Theorem 2.10 we have

$$\int \varphi(\mu) \, E^{P_\mu^N}[\psi(\mu_t)] \, m(d\mu) = \int \psi(\mu) \, E^{P_\mu^N}[\varphi(\mu_t)] \, m(d\mu)$$

for φ, $\psi \in \mathcal{D}(E)$. The proof is completed by letting N go to infinity and applying Theorem 3.13. Q.E.D.

5.10. *Remark*. For birth and death processes on the integers the analogue of Theorem 5.7 is also true (see Remark 3.14).

5.11. *Remark*. Let b and d be as in Theorem 5.13 and define $\mathcal{L}$ accordingly. Suppose that m is a probability measure on E such that

$$\int \mathcal{L} f \, dm = 0, \quad f \in \mathcal{G}. \tag{5.12}$$

One can show that (5.12) implies that

$$\int T_t f \, dm = \int f \, dm, \quad f \in B(E), \tag{5.13}$$

where $\{T_t : t \geq 0\}$ is the semi-group determined by $\mathcal{L}$. To see this, define for each interval Λ:

$$b^{m,\Lambda}(y, \cdot) = \chi_\Lambda(y) \, E^m[b(y) \,|\, \mathcal{B}^\Lambda](\cdot),$$
$$d^{m,\Lambda}(y, \cdot) = \chi_\Lambda(y) \, E^m[d(y) \,|\, \mathcal{B}^\Lambda](\cdot),$$

and let $\mathcal{L}^{m,\Lambda}$ be the associated operator. It is easy to check that for $f \in B(E, \Lambda)$

$$\int \mathcal{L}^{m,\Lambda} f \, dm = \int \mathcal{L} f \, dm;$$

and therefore by Theorem 2.11,

$$\int T_t^{m,\Lambda} f \, dm = \int f \, dm, \quad f \in B(E, \Lambda).$$

Thus (5.13) will be proved once we have shown that

$$T_t^{m,\Lambda} f(\mu^0) \to T_t f(\mu^0) \quad \text{as} \quad \Lambda \nearrow R^1 \tag{5.14}$$

for each $f \in \mathcal{D}$, $t > 0$, and $\mu^0 \in E$. But (5.14) is not hard to prove from Theorem 5.13 plus the observation that

$$b^{m,\Lambda}(y,\mu) = b(y,\mu)$$

and

$$d^{m,\Lambda}(y,\mu) = d(y,\mu)$$

so long as $l_\mu(y)$ and $r_\mu(y)$ are in Λ.

The main problem remaining in our study of time reversible equilibrium states is a discussion of whether or not $\mathcal{G}_f$ is empty and an identification of the measures, if any, in $\mathcal{G}_f$. In the next section we do this in the case that f is locally bounded and locally uniformly positive.

6. The set $\mathcal{G}_f$

In this section we follow the route laid out by Spitzer in [11] to find necessary and sufficient conditions for $\mathcal{G}_f$ to be nonempty. We also show that if $\mathcal{G}_f$ is not empty then it contains exactly one point. Except for the technical details, all of the ideas here are due to Spitzer [11].

6.1. Theorem. *Let f be a probability density on $(0, \infty)$ with $\int_0^\infty x f(x)\,dx = \varrho^{-1} < \infty$. Then there is a unique measure m_f on $(E, \mathcal{B}_E)$ such that for $\varphi \in B(E; (a,b))$ satisfying $\varphi(\mu) = 0$ unless $\mu(a,b) = k$:*

$$(6.2)\quad \int \varphi(\mu)\, m_f(d\mu)$$

$$= \begin{cases} \displaystyle\int_{a<x_1<\dots<x_k<b} \int \varphi(\{x_1,\dots,x_k\})\,\varrho(1-F(x_1-a))\prod_{j=2}^{k} f(x_j-x_{j-1})(1-F(b-x_k))\,dx_1\dots dx_k & \text{if } k \geq 1 \\ \displaystyle\varphi(\varnothing)\int_{b-a}^{\infty} \varrho(1-F(x))\,dx & \text{if } k=0. \end{cases}$$

Here $F(t) = \int_0^t f(s)\,ds$ and $\prod_{j=2}^{1} f(x_j - x_{j-1}) \equiv 1$.

Proof. Let $\hat{E} = \{\mu \in \tilde{E}: \mu(\{y\}) \in \{0,1\}, y \in R\}$. From the proof of Lemma 1.1, we see that $\hat{E}$ is a G_δ subset of $\tilde{E}$ and E is a G_δ subset of $\hat{E}$. Therefore, we need only show that a unique $\hat{m}_f$ satisfying (6.2) exists on $(\tilde{E}, \mathcal{B}_{\hat{E}})$ and that $\hat{m}_f(E) = 1$.

Let $\hat{\mathcal{B}}^N$ be the smallest σ-algebra for which all $\varphi \in B(\hat{E}; (-N,N))$ are measurable ($B(\hat{E}; (-N,N))$ is defined by analogy with $B(E; (-N,N))$.) For $\omega \in \hat{E}$, let $[\omega]_N$ denote the atom of $\hat{\mathcal{B}}^N$ which contains ω. Note that if $\bigcap_{k=1}^N [\omega_k]_k \neq \varnothing$ for each N, then $\bigcap_{k=1}^\infty [\omega_k]_k \neq \varnothing$. (This is not true on E and is the reason for introducing $\hat{E}$.) Thus, by a modification of Tulcea's extension theorem (see [15]), the existence of $\hat{m}_f$ on $(\hat{E}, \mathcal{B}_{\hat{E}})$ satisfying (6.2) will follow if we can show that the measure m_f^N defined on $\hat{\mathcal{B}}^N$ by (6.2) (for $\varphi \in B(\hat{E}; (-N, N))$) are consistent. However, this is a straightforward computation if one uses the fact that $F_0(t) + \sum_{k=1}^{\infty} F^{*k} * F_0(t) \equiv \varrho t$ (see Feller [1], Section (11.3)), where $F_0(t) = \int_0^t \varrho(1-F(s))ds$ and $*$ denotes convolution. We leave it to the reader to fill in the details. To show that $\hat{m}_f(E) = 1$, it is enough to show that $\hat{m}_f(\mu(0, N) = k) \to 0$ as $N \to \infty$ for each k (and similarly as $N \to -\infty$). But this is easy from the explicit expression for $\hat{m}_f(\mu(0, N) = k)$ given in (6.2).

Q.E.D.

We say a positive measurable function f on $(0, \infty)$ is locally bounded (l.b.) if $M(t) \equiv \sup_{0<s\leqslant t} f(s) < \infty$ for each $t>0$ and that f is locally positive (l.p.) if $N(t) \equiv \inf_{0<s\leqslant t} f(s) > 0$ for each $t>0$.

6.3. THEOREM. *Let f be a l.b. probability density on $(0, \infty)$ with $\int_0^\infty x f(x)dx = \varrho^{-1} < \infty$. Then $m_f \in \mathcal{G}_f$.*

Proof. If $(a, b) = \Lambda \subset I = (\alpha, \beta)$, let $\mathcal{B}^{\Lambda, I}$ be the smallest σ-algebra such that all $\varphi \in B(E; I \setminus \Lambda)$ are measurable. Let $m^{\mu}_{f, \Lambda, I}$ be the r.c.p.d. of $m_f | \mathcal{B}^{\Lambda, I}$. It is an easy computation, using (6.2), to show that on $\{\mu: \mu(\alpha, a)\mu(b, \beta) \geqslant 1\}$, $m^{\mu}_{f, \Lambda, I} = m^{(\Lambda, \mu)}$, where $m^{(\Lambda, \mu)}$ is given by (5.1). Now, since every $\mu \in E$ satisfies $\mu(\alpha, a)\mu(b, \beta) \geqslant 1$ for all α sufficiently negative and β sufficiently positive, we get the desired conclusion upon letting $I \nearrow R$. Q.E.D.

If f is a l.b. function and $\hat{f}(x) = e^{\lambda x} f(x)$, then it is clear from (5.1) that $\mathcal{G}_f = \mathcal{G}_{\hat{f}}$. Thus, if there is a λ for which $\hat{f}$ becomes a probability density having a finite first moment, then, by Theorem 6.3, $\mathcal{G}_f \neq \emptyset$. Our final goal is to show that if f is a l.b. and l.p. function and if $\mathcal{G}_f \neq \emptyset$, then there is such a λ and $\mathcal{G}_f$ has only one element.

If $\mu \in E$ and $\mu(\alpha, \beta) = k \geqslant 1$, identify $\mu|_{(\alpha, \beta)}$ with an element in $\Sigma_k(\alpha, \beta) \equiv \{(s_1, s_2, ..., s_k): \alpha < s_1 < s\xi < ... < s_k < \beta\}$ in the obvious way.

6.4. LEMMA. *If f is l.b. and l.p. and if $m \in \mathcal{G}_f$, then there is a measurable function $B(x, y)$ on $\{(x, y): x<y\}$ with the property that for $\alpha < \beta$:*

$$(6.5) \quad m(l_\mu(\alpha) \in (\alpha - \varepsilon, \alpha), \mu(\alpha, \beta) = 0, r_\mu(\beta) \in (\beta, \beta + \delta)) = \int_{\alpha-\varepsilon}^{\alpha} dx \int_{\beta}^{\beta+\delta} dy B(x, y) f(y-x)$$

and for measurable $F \subseteq \Sigma_k(\alpha, \beta)$

$$(6.6) \quad m(l_\mu(\alpha) \in (\alpha - \varepsilon, \alpha), \mu|_{(\alpha, \beta)} \in F, r_\mu(\beta) \in (\beta, \beta + \delta))$$

$$= \int_{\alpha-\varepsilon}^{\alpha} dx \int \underset{F}{\cdots} \int ds_1 \ldots ds_k f(s_1 - x) \prod_{j=2}^{k} f(s_j - s_{j-1}) \int_{\beta}^{\beta+\delta} dy B(x, y) f(y - s_k),$$

where $\prod_{j=2}^{1} f(s_j - s_{j-1}) \equiv 1$.

Proof. We only prove (6.6). The proof of (6.5) is similar. If $l<a<x<y<b<r$ define

$$L(l, a, x) = f(x-l) + \int_a^x f(x-s)\,f(s-l)\,ds + \int_{a<s_1<s_2<x}\int f(s_1-l)\,u(s_2-s_1)\,f(x-s_2)\,ds_1\,ds_2$$

and

$$R(y, b, r) = f(r-y) + \int_y^b f(r-s)\,f(s-y)\,ds + \int_{y<s_1<s_2<b}\int f(s_1-y)\,u(s_2-s_1)\,f(r-s_2)\,ds_1\,ds_2.$$

Here $u(t) = u(0, 0, t, t)$ with $u(l, a, b, r)$ as in (5.1).

Now let $a<\alpha-\varepsilon<\beta+\delta<b$ and $F\in\mathcal{B}_{R^k}[\Sigma_k(\alpha,\beta)]$ be given. Then

$$\begin{aligned}(6.7)\quad & m(l_\mu(\alpha)\in(\alpha-\varepsilon,\alpha),\ \mu|_{(\alpha,\beta)}\in F,\ r_\mu(\beta)\in(\beta,\beta+\delta))\\ &= \int_E m(l_\mu(\alpha)\in(\alpha-\varepsilon,\alpha),\ \mu|_{(\alpha,\beta)}\in F,\ r_\mu(\beta)\in(\beta,\beta+\delta)\,|\,\bar{\mathcal{B}}^{(a,b)})(\nu)\,m(d\nu)\\ &= \int_E\left[\int_{\alpha-\varepsilon}^{\alpha}dx\int_\beta^{\beta+\delta}dy\,\frac{L(l_\nu(a),a,x)\,R(y,b,r_\nu(b))}{u(l_\nu(a),a,b,r_\nu(b))}\int\cdots_F\int ds_1\ldots ds_k\,f(s_1-x)\sum_{j=2}^{k}f(s_j-s_{j-1})\right]\\ &\qquad\times m(d\nu)\\ &= \int_{\alpha-\varepsilon}^{\alpha}dx\int\cdots_F\int ds_1\ldots ds_k\,f(s_1-x)\prod_{j=2}^{k}f(s_j-s_{j-1})\int_\beta^{\beta+\delta}dy\,f(y-s_k)\\ &\qquad\times\int_E\frac{L(l_\nu(a),a,x)\,R(y,b,r_\nu(b))}{u(l_\nu(a),a,b,r_\nu(b))}\,m(d\nu).\end{aligned}$$

Set

$$B(x,y) = \int_E\frac{L(l_\nu(a),a,x)\,R(y,b,r_\nu(b))}{u(l_\nu(a),a,b,r_\nu(b))}\,m(d\nu). \tag{6.8}$$

This defines $B(x,y)$ for $a<x<y<b$. But, since the left side of (6.7) is independent of a and b, it follows that $B(x,y)$ is the same, up to sets of measure zero, for any choice of a and b satisfying $a<x<y<b$. Hence $B(x,y)$ is well defined for all $x<y$. Q.E.D.

6.9. Lemma. *If f is l.b. and l.p. and $m\in\mathcal{G}_f$, then $B(x,y)$ satisfies*

$$B(x,y) = \int_y^\infty f(s-y)\,B(x,s)\,ds \quad \text{a.e. } (\lambda_2).$$

Proof. Let $\alpha_1<\alpha_2<\alpha_3<\alpha_4<\beta$ be given, and note that.

$$\begin{aligned}(6.10)\quad & m(l_\mu(\alpha_2)>\alpha_1,\ \mu(\alpha_2,\alpha_3)=0,\ r_\mu(\alpha_3)<\beta)\\ &\quad - m(l_\mu(\alpha_2)>\alpha_1,\ \mu(\alpha_2,\alpha_4)=0,\ r_\mu(\alpha_4)<\beta)\\ &= m(l_\mu(\alpha_2)>\alpha_1,\ \mu(\alpha_2,\alpha_3)=0,\ \mu(\alpha_3,\beta)\geqslant 1,\ r_\mu(\beta)>\beta)\\ &\quad - m(l_\mu(\alpha_2)>\alpha_1,\ \mu(\alpha_2,\alpha_4)=0,\ \mu(\alpha_4,\beta)\geqslant 1,\ r_\mu(\beta)>\beta).\end{aligned}$$

We now use Lemma 6.4 to rewrite (6.10) as:

$$\int_{\alpha_1}^{\alpha_2} dx \int_{\alpha_3}^{\alpha_4} dy B(x,y) f(y-x) = \int_{\alpha_1}^{\alpha_2} dx \int_{\alpha_3}^{\alpha_4} ds_1 f(s_1 - x)$$

$$\times \left[\int_\beta^\infty B(x,y) f(y-s_1)\,dy + \sum_{k=2}^\infty \int_{s_1<\dots<s_k<\beta} ds_2 \dots ds_k \prod_2^k f(s_j - s_{j-1}) \int_\beta^\infty B(x,y) f(y-s_k)\,dy \right].$$

Thus, for each β and a.e. (λ_2) $(x, s_1) \in \Sigma_2(-\infty, \beta)$,

$$B(x, s_1) = \int_\beta^\infty B(x,y) f(y-s_1)\,dy \tag{6.11}$$

$$+ \sum_{k=2}^\infty \int_{s_1<\dots<s_k<\beta} \int ds_2 \dots ds_k \prod_2^k f(s_j - s_{j-1}) \int_\beta^\infty B(x,y) f(y-s_k)\,dy.$$

By Fubini's Theorem, we know that for a.e. (λ_3) $(x, s_1, \beta) \in \Sigma_3(-\infty, \infty)$, equation (6.11) holds. Thus, if A is the set of $(x, s_1) \in \Sigma_2(-\infty, \infty)$ such that (6.11) is true for a.e. (λ_1) $\beta > s_1$, then again by Fubini's Theorem, A has full (λ_2)-measure in $\Sigma_2(-\infty, \infty)$. But for $(x, s) \in A$,

$$B(x,s) \geqslant \int_\beta^\infty B(x,y) f(y-s)\,ds$$

for a.e. (λ_1) $\beta > s$, and therefore for all $\beta > s$. Hence, by monotone convergence,

$$B(x,s) \geqslant \int_s^\infty B(x,y) f(y-s)\,dy, \quad (x,s) \in A. \tag{6.12}$$

Next, define $\tilde{A}$ to be the set of $(x, s_1) \in A$ for which

$$\lim_{\beta \searrow s_1} \sum_{k=2}^\infty \int_{s_1<s_2<\dots<s_k<\beta} \int ds_2 \dots ds_k \prod_2^k f(s_j - s_{j-1}) \int_\beta^\infty B(x,y) f(y-s_k)\,dy = 0. \tag{6.13}$$

It is clear, from the monotone convergence theorem, that equality obtains in (6.12) for $(x, s_1) \in \tilde{A}$. Thus it suffices for us to prove that $\tilde{A}$ has full (λ_2)-measure in $\Sigma_2(-\infty, \infty)$. For each x, set $A_x = \{s: (x, s_1) \in A\}$. By Fubini's Theorem, $B = \{x: \lambda_1((x, \infty) \setminus A_x) = 0\}$ has full (λ_1)-measure in R. Also, if $s \in R$ and $C(s) = \{x: x < s$ and $\int_s^\infty B(x,y) f(y-x)\,dy < \infty\}$, then (6.5) implies that $\lambda_1((-\infty, s) \setminus C(s)) = 0$. Thus, another application of Fubini's Theorem proves that $D \equiv \{(x,s) \in A: x \in B \cap C(s)\}$ has full (λ_2)-measure in $\Sigma_2(-\infty, \infty)$. It is therefore sufficient for us to show that $D \subseteq \tilde{A}$. That is, we must prove that (6.13) holds for $(x, s_1) \in D$. To this end, let $s_1 \in R$ and $x \in B \cap C(s_1)$ be given. Since $x \in B$,

$$\int_\beta^\infty B(x,y) f(y-s_k)\,dy \leqslant \int_{s_k}^\infty B(x,y) f(y-s_k)\,dy \leqslant B(x, s_k)$$

for a.e. (λ_1) $s_k \in (x, \beta)$. Thus if M and N are as in the paragraph preceding Theorem 6.3, then we have:

$$\sum_{k=2}^{\infty} \int_{s_1 < s_2 < \ldots < s_k < \beta} \int ds_2 \ldots ds_k \prod_{j=2}^{\infty} f(s_j - s_{j-1}) \int_{\beta}^{\infty} B(x, y) f(y - s_k)\, dy \tag{6.14}$$

$$\leqslant \frac{M(\beta - s)}{N(\beta - x)} e^{M(\beta - s)(\beta - s)} \int_{s_1}^{\beta} ds_k\, f(s_k - x)\, B(x, s_k).$$

The right side of (6.14) goes to zero as β decreases to s_1 since $x \in C(s_1)$. Q.E.D.

Now define $c_x(t)$ for $x \in R$ and $t > 0$ by

$$c_x(t) = B(x, x+t).$$

According to Lemma 6.9

$$c_x(t) = \int_0^{\infty} c_x(t+s) f(s)\, ds \quad \text{a.e. } (\lambda_2), \tag{6.15}$$

and by changing $c_x(\cdot)$ on a set of measure zero we may assume that for a.e. (λ_1) x, $c_x(\cdot)$ satisfies (6.15) for every $t > 0$. ($c_x(\cdot)$ so modified may be infinite on a set of measure zero.)

6.16. Lemma. *Let f be l.b. and l.p. on $(0, \infty)$ and let $\varphi(t)$ be a non-negative (possibly infinite on a set of measure zero) function such that*

$$\varphi(t) = \int_0^{\infty} \varphi(t+s) f(s)\, ds \quad \textit{for every} \quad t > 0. \tag{6.17}$$

Then either $\varphi \equiv 0$ or there is a λ such that

$$\int_0^{\infty} e^{\lambda t} f(t)\, dt = 1$$

and an $\alpha > 0$ such that $\varphi(t) = \alpha e^{\lambda t}$.

We postpone the proof of this lemma until the end of the section.

6.18. Lemma. *With $B(x, y)$ and f as above, there is a λ such that $\int_0^{\infty} e^{\lambda t} f(t)\, dt = 1$ and a constant $\bar{c}$ such that*

$$B(x, y) = \bar{c} e^{\lambda(y - x)} \quad \textit{a.e. } (\lambda_2).$$

Proof. Since $B(x, y)$ is not zero a.e. (λ_2) the first statement follows from (6.15) and Lemma 6.16. Also from Lemma 6.16 it follows that there is a function $\bar{c}(x)$ such that $c_x(t) = \bar{c}(x) e^{\lambda t}$ a.e. (λ_2).

Now if we had defined $d_y(t) = B(y - t, y)$, then an argument identical to the above would show that $d_y(t) = \bar{d}(y) e^{\lambda t}$ a.e. (λ_2) for some measurable function $d(y)$. Thus

$$B(x, y) = c_x(y - x) = \bar{c}(x) e^{\lambda(y - x)} = \bar{d}(y) e^{\lambda(y - x)} \quad \text{a.e. } (\lambda_2),$$

It follows that there is a constant, $\bar{c}$, such that $\bar{c}(x) = \bar{c}$ a.e. (λ_1). Q.E.D.

6.19. LEMMA. *If f is l.b. and l.p. and $\mathcal{G}_f \neq \emptyset$, then there is a λ such that $e^{\lambda x} f(x)$ is a probability density and*

$$(\bar{c})^{-1} = \int_0^\infty t e^{\lambda t} f(t)\, dt. \tag{6.20}$$

Proof. We already know that there is a λ such that $e^{\lambda x} f(x)$ is a probability density and $B(x, y) = \bar{c} e^{\lambda(y-x)}$. Since m is a measure on E we have

$$1 = m(l_\mu(0) \in (-\infty, 0),\, r_\mu(0) \in (0, \infty)) = \int_{-\infty}^0 dx \int_0^\infty dy B(x, y) f(y-x)$$

$$= \int_{-\infty}^0 dx \int_0^\infty dy\, \bar{c} e^{\lambda(y-x)} f(y-x) = \bar{c} \int_0^\lambda t e^{\lambda t} f(t)\, dt. \qquad \text{Q.E.D.}$$

As we pointed out before, if $\hat{f}(x) = e^{\lambda x} f(x)$ for some λ, then $\mathcal{G}_f = \mathcal{G}_{\hat{f}}$. Lemma 6.19 says that if $\mathcal{G}_f$ is not empty then there is a probability density $\hat{f}$ with a finite first moment such that $\mathcal{G}_f = \mathcal{G}_{\hat{f}}$ and moreover $\hat{f}(x) = e^{\lambda x} f(x)$ for some λ.

6.21. THEOREM. *If f is l.b. and l.p. and $\mathcal{G}_f \neq \emptyset$ then there is a λ such that $\hat{f}(x) = e^{\lambda x} f(x)$ is a probability density with a finite first moment, and $\mathcal{G}$ consists of exactly one element whose marginals are given by (6.2) with $\hat{f}$ in place of f.*

Proof. Everything has been proved except the uniqueness. But this follows from Lemma 6.4 and the equation $B(x, y) = \bar{c} e^{\lambda(y-\tau)}$. Q.E.D.

All that remains is to prove Lemma 6.16. We proceed in a series of lemmas. Notice first that since f is l.p. it follows from (6.17) that $\int_K \varphi(t)\, dt < \infty$ for all compact $K \subset (0, \infty)$.

6.22. LEMMA. *Under the hypotheses of Lemma 6.16 either $\varphi(t) \equiv 0$ or $\varphi(t) > 0$ for all $t > 0$.*

Proof. Since $f(s) > 0$ for all s it follows from (6.17) that if $\varphi(t_0) = 0$ then $\varphi(t) = 0$ for all $t > t_0$. Suppose $\varphi(t_0) = 0$ for some t_0. Let $\psi(t) = \varphi(t_0 - t)$. Then for $0 < t < t_0$

$$\psi(t) = \int_0^\infty \varphi(t_0 - t + s) f(s)\, ds = \int_0^t \varphi(t_0 - t + s) f(s)\, ds \leqslant M(t_0) \int_0^t \psi(t-s)\, ds = M(t_0) \int_0^t \psi(s)\, ds.$$

Since ψ is locally integrable and non-negative it follows that $\psi(t) = 0$ for $0 < t < t_0$. Q.E.D.

6.23. LEMMA. *If f is a strictly positive, continuous function on $(0, \infty)$ and φ is a strictly positive measurable (possibly infinite on a set of measure zero) function on $(0, \infty)$ which satisfies (6.17) and*

$$\int_0^\infty \varphi(s) f(s)\, ds = 1, \tag{6.24}$$

then there is a unique λ such that $\int_0^\infty e^{\lambda t} f(t)\, dt = 1$, and moreover $\varphi(t) = e^{\lambda t}$.

Proof. Let $\mathfrak{V}$ be the vector space of locally finite signed measures on $(0, \infty)$ with the topology of weak convergence on compact subsets of $(0, \infty)$. $\mathfrak{V}$ is locally convex with this topology. Set $\mathfrak{J} = \{\mu \in \mathfrak{V}$: μ is a positive measure on $(0, \infty)$, $\int_0^\infty f(s)\,\mu(ds) \leqslant 1$, and $\mu(K) \geqslant \int_0^\infty \mu(K+s) f(s)\, ds$ for all compact sets $K \subset (0, \infty)\}$.

$$\int_0^\infty \mu(K+s) f(s)\, ds = \int_0^\infty \mu(dt) \int_0^\infty \chi_K(s+t) f(s)\, ds, \tag{6.25}$$

and $\int_0^\infty \chi_K(s+t) f(s)\, ds = \int_0^\infty \chi_K(u) f(u-t)\, du$ is continuous in t. Therefore, since f is also continuous, it follows that $\mathfrak{J}$ is compact. $\mathfrak{J}$ is clearly convex, and $\mathfrak{J}$ is metrizable; therefore, every element of $\mathfrak{J}$ is an average of extreme points of $\mathfrak{J}$. The measure $\mu_0 \in \mathfrak{J}$ with density φ satisfies

$$\int_0^\infty f(s)\,\mu_0(ds) = 1 \quad \text{and} \tag{6.26}$$

$$\mu_0(K) = \int_0^\infty \mu_0(K+s) f(s)\, ds \quad \text{for all compact } K \subset (0, \infty). \tag{6.27}$$

Thus the extreme points of $\mathfrak{J}$ of which μ_0 is an average must also satisfy (6.26) and (6.27). In particular there is an extreme point $\bar{\mu}$ of $\mathfrak{J}$ which satisfies (6.26) and (6.27). Because $\bar{\mu}$ satisfies (6.27) and the right side of (6.27) is given by (6.25) it follows that $\bar{\mu}$ has a density $\psi(t)$ which satisfies (6.17) and (6.24) and thus, by Lemma 6.22, is strictly positive. Obviously,

$$\psi(t) = \int_0^\infty \frac{\psi(t+s)}{\psi(s)} \psi(s) f(s)\, ds.$$

Moreover, for every s for which $\psi(s)$ is finite (which is almost every s) $\psi(t+s)/\psi(s)$ again satisfies (6.17) and (6.24) as a function of t. Thus since $\psi(s) f(s) > 0$ for every s, $\int_0^\infty \psi(s) f(s) ds = 1$, and $\bar{\mu}$ is extreme, it follows that

$$\psi(t)\,\psi(s) = \psi(t+s) \quad \text{for a.e. } (\lambda_2)\ t,\ s > 0.$$

Let $a(s) \in C_0^\infty(0, 1)$ be such that $\int_0^\infty \psi(s) a(s) ds = 1$ and define $\bar{\psi}(t) = \int_0^\infty \psi(t+s)\, a(s)\, ds = \int_0^\infty \psi(u) a(u-t)\, du$. $\bar{\psi}(t) \in C^\infty(0, \infty)$ and for a.e. (λ_1) t we have

$$\bar{\psi}(t) = \int_0^\infty \psi(t)\,\psi(s)\, a(s)\, ds = \psi(t).$$

Thus $\bar{\psi}(t+s) = \bar{\psi}(t)\,\bar{\psi}(s)$ for all t, $s > 0$ and hence $\bar{\psi}(t) = e^{\lambda t}$ for some λ and $\psi(t) = e^{\lambda t}$ a.e. (λ_1). But since ψ satisfies (6.17), $\psi(t) = e^{\lambda t}$ for all $t > 0$.

Since there is at most one λ with $\int_0^\infty e^{\lambda t} f(t)\, dt = 1$, this shows that the measure $\bar{\mu}$ is uniquely determined and hence $\mu_0 = \bar{\mu}$. This shows that $\varphi(t) = e^{\lambda t}$ a.e. (λ_1), and since φ satisfies (6.17), $\varphi(t) = e^{\lambda t}$ for all $t > 0$, where λ is the unique number such that (6.24) holds.

Q.E.D.

6.28. LEMMA. *If f is a l.b. and l.p. function on $(0, \infty)$ and φ is a positive (possibly infinite on a set of measure zero) function which satisfies (6.17) and (6.24) then there is a unique λ such that $\int_0^\infty e^{\lambda t} f(t)\,dt = 1$ and moreover $\varphi(t) = e^{\lambda t}$ for all $t > 0$.*

Proof. Let $g(t) = \int_0^t f(t-s) f(s)\,ds$. Since f is l.b., g is continuous, and since f is l.p., g is strictly positive for $t > 0$. Also

$$\int_0^\infty \varphi(t+h)\, g(t)\,dt = \int_0^\infty \varphi(t+h) \int_0^t f(t-s) f(s)\,ds\,dt$$

$$= \int_0^\infty f(s)\,ds \int_s^\infty \varphi(t+h) f(t-s)\,dt = \int_0^\infty \varphi(s+h) f(s)\,ds = \varphi(h).$$

Therefore by Lemma 6.23, there is a unique λ such that $\int_0^\infty e^{\lambda t} g(t)\,dt = 1$ and $\varphi(t) = e^{\lambda t}$. But

$$\int_0^\infty e^{\lambda t} g(t)\,dt = \left(\int_0^\infty e^{\lambda t} f(t)\,dt\right)^2. \qquad \text{Q.E.D.}$$

Proof of Lemma 6.16. From Lemma 6.22 we see that we may assume that $\varphi(t) > 0$ or all $t > 0$. Also $\varphi(t) < \infty$ for a.e. (λ_1) t. Thus there is a sequence $\varepsilon_n \searrow 0$ and a $\gamma > \varepsilon_1$ such that $\varphi(\gamma) < \infty$ and $\varphi(\varepsilon_n) < \infty$ for all n. Let $\psi_n(t) = \varphi(\varepsilon_n + t)/\varphi(\varepsilon_n)$. One easily checks that ψ_n and f satisfy the hypotheses of Lemma 6.28. Thus there is a λ such that $\int_0^\infty e^{\lambda t} f(t)\,dt = 1$ and $\psi_n(t) = e^{\lambda t}$. Therefore $\varphi(\varepsilon_n + t) = \varphi(\varepsilon_n) e^{\lambda t}$. Since $\varepsilon_n \searrow 0$, this shows that $\varphi(t)$ is continuous. Also $\varphi(\gamma) = \varphi(\varepsilon_n) e^{\lambda(\gamma - \varepsilon_n)}$. Thus

$$\varphi(\varepsilon_n + t) = \varphi(\gamma) e^{-\lambda(\gamma - \varepsilon_n)} e^{\lambda t}.$$

Letting $n \to \infty$ we get the desired result. Q.E.D.

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