EXPANSIONS FOR SPHERICAL FUNCTIONS ON NONCOMPACT SYMMETRIC SPACES

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Section 0

Let G be a connected noncompact semisimple Lie group with finite center and real rank one; fix a maximal compact subgroup K. Our concern in this paper is Fourier analysis on the Riemannian symmetric space G/K. We shall analyze the local and global behavior of spherical functions, the boundedness of multiplier operators, and the inversion of differential operators. The core of the paper, however, is an analysis of how the size of a function is controlled by the size of its Fourier transform.

There is an extensive literature on such topics, centered about the Paley-Wiener and Plancherel theorems. Our work relies heavily on these earlier ideas and techniques, to which detailed reference will be made in the body of the paper. The problems we wish to solve, however, require estimates more precise and of a different nature than are necessary for the Plancherel or Paley-Wiener theorem. Thus the first two sections of this paper are devoted to the construction of various asymptotic expansions for spherical functions; in later sections we show how these expansions may be applied to the Fourier analysis of G/K.

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Section 1

Let G be a connected noncompact semisimple Lie group with finite center. The Lie algebra of G has a Cartan decomposition $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$; fix a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$.

We shall assume throughout this paper that $\mathfrak{a}$ is one-dimensional.

We fix some order on the non-zero restricted roots; there are at most two roots which are positive with respect to this order, which we denote by α and 2α. Let p and q be the multiplicity of these roots, and define the number ϱ as $\varrho=(p+2q)/2$.

Let K be the maximal compact subgroup of G with Lie algebra $\mathfrak{k}$, and form the Riemannian symmetric space G/K. We may compute that $n\equiv\dim(G/K)=p+q+1$. The elementary spherical functions for G/K are indexed by $\mathfrak{a}'_+$, which we shall identify with $\mathbf{R}^+$, through the map $\lambda\to\lambda\alpha$. Corresponding to each $\lambda\geqslant 0$ is a spherical function denoted by φ_λ.

We fix an element H_0 in $\mathfrak{a}$ with $\alpha(H_0)=1$, and define $A^+=\{\exp tH_0\,|\,t>0\}$; then G has a polar decomposition $G=K\overline{A}{}^+K$, which leads to an integration formula we now describe. Let $D(t)=D(\exp tH_0)=(\sinh t)^p(\sinh 2t)^q$. For a correct normalization of Haar measures and all sufficiently nice f,

$$\int_G f(x)\,dx=\int_K\int_{A^+}\int_K f(k_1\exp tH_0k_2)\,D(t)\,dt\,dk_1\,dk_2. \tag{1.1}$$

A function f is said to be K bi-invariant if f is invariant under left and right translation by K. We define the Fourier transform for such functions by $\hat{f}(\lambda)=\int_G f(g)\bar{\varphi}_\lambda(g)\,dg$. There exists a measure $|c(\lambda)|^{-2}d\lambda$ on $\mathbf{R}^+$ such that $f(g)=\int_0^\infty\varphi_\lambda(g)\hat{f}(\lambda)|c(\lambda)|^{-2}d\lambda$ (see [5], [6b]).

We now define a concept of Fourier multiplier. To a function m in $L^\infty(\mathbf{R}^+)$ we associate a map $T_m: C_c^\infty(G/K)\to L^2(G/K)$ by $T_mf(g)=\int_0^\infty m(\lambda)f*\varphi_\lambda(g)|c(\lambda)|^{-2}d\lambda$. (Alternatively, if we let $\check{m}$ denote the distribution $f\to\int_0^\infty m(\lambda)f*\varphi_\lambda(e)|c(\lambda)|^{-2}d\lambda$, then $T_mf(g)$ is given by convolution with the distribution $\check{m}$.) The function m is said to be a multiplier of $L^p(G/K)$ if the map T_m may be extended to a bounded operator on $L^p(G/K)$.

Finally, we shall follow the standard practice of allowing c to denote a real or complex constant whose nature we do not wish to specify further; its value may vary from line to line. Dependence of such constants upon parameters of interest will be indicated through the use of subscripts.

Section 2

In this section we shall analyze the behavior of $\varphi_\lambda(\exp tH_0)$ for small t. It is an important heuristic principle that locally, spherical functions on G/K behave like spherical functions on the symmetric space $\mathcal{D}$ associated to the Cartan motion group. We shall state and prove a precise form of this principle.

For compact symmetric spaces of rank one, such a principle was established by Szegö [12], who showed that Legendre functions admit a series expansion in terms of Bessel functions. We shall extend this to G/K. Szegö's idea may be illustrated through the following computation for $SL(2, \mathbf{R})$. A change of contour in Harish-Chandra's [6a] integral formula for the spherical function yields

$$\varphi_\lambda(\exp tH_0) = c\int_0^t \cos(\lambda s)(\cosh t - \cosh s)^{-1/2}\,ds. \tag{2.1}$$

For small t,

$$(\cosh t - \cosh s)^{-1/2} = (t^2 - s^2)^{-1/2} + \text{error}; \tag{2.2}$$

φ_λ therefore behaves like

$$\int_0^t \cos(\lambda s)(t^2 - s^2)^{-1/2}\,ds = J_0(\lambda t). \tag{2.3}$$

For $SL(2, \mathbf{R})$, $K = SO(2)$ and $\mathcal{D} = \mathbf{R}^2$; spherical functions for this action are $J_0(\lambda t)/|\lambda t|^0$.

In general, we define

$$\mathfrak{J}_\mu(z) = \frac{J_\mu(z)}{z^\mu}\Gamma(\mu + \tfrac{1}{2})\Gamma(\tfrac{1}{2})2^{\mu-1}$$

and

$$c_0 = c_0(G) = \pi^{1/2}2^{(q/2)-2}\frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}.$$

We shall prove

THEOREM 2.1. *There exist $R_0 > 1$, $R_1 > 1$ such that for any t with $0 \leqslant t \leqslant R_0$ and any $M \geqslant 0$,*

$$\varphi_\lambda(\exp tH_0) = c_0\left[\frac{t^{n-1}}{D(t)}\right]^{1/2}\sum_{m=0}^{\infty} t^{2m}a_m(t)\,\mathfrak{J}_{(n-2)/2+m}(\lambda t) \tag{2.4}$$

$$\varphi_\lambda(\exp tH_0) = c_0\left[\frac{t^{n-1}}{D(t)}\right]^{1/2}\sum_{m=0}^{M} t^{2m}a_m(t)\,\mathfrak{J}_{(n-2)/2+m}(\lambda t) + E_{M+1}(\lambda t) \tag{2.5}$$

where

$$a_0(t) \equiv 1$$
$$|a_m(t)| \leqslant cR_1^{-m} \tag{2.6}$$

$$\begin{aligned} |E_{M+1}(\lambda t)| &\leqslant c_M t^{2(M+1)} && \textit{if } |\lambda t| \leqslant 1 \\ &\leqslant c_M t^{2(M+1)}(\lambda t)^{-((n-1)/2+M+1)} && \textit{if } |\lambda t| > 1. \end{aligned} \tag{2.7}$$

Remarks.

1. The techniques we shall use in establishing this result were developed by Szegö [12] to analyze the behavior of Legendre functions. When the 2α root does not appear for G, q is equal to zero, and the spherical functions may be viewed as Legendre functions of complex index. These were analyzed by Schindler [11], and in this case Theorem 2.1 follows from her work. In the proof of Theorem 2.1 we shall therefore assume that q is non-zero.

2. As the proof of the theorem is somewhat technical, we decompose it into five parts:

I. Derivation of an integral representation for spherical functions, similar to (2.1).

II. Construction of a series expansion, generalizing (2.2).

III. Proof of (2.4) and justification of all formal manipulations in the proof.

IV. Estimation of the size of the $a_k(t)$.

V. Estimation of the error term E_{M+1}.

Proof of Theorem 2.1.

Part I.

LEMMA 2.2. $(c_0 \sinh 2t)^{-1} D(t)\varphi_\lambda(\exp tH_0)$

$$= \int_0^t (\cosh 2t - \cosh 2s)^{(q/2)-1} \int_0^s (\cosh s - \cosh r)^{(p/2)-1} \cos(\lambda r)\, dr \sinh s\, ds \tag{2.8}$$

$$= (\cosh t)^{(q/2)-1} \int_0^t \cos(\lambda s)(\cosh t - \cosh s)^{((p+q)/2)-1} F\left(\frac{q}{2}, 1-\frac{q}{2}; q+p; \frac{\cosh t - \cosh s}{2\cosh t}\right) ds \tag{2.9}$$

Proof. Formula (2.8) was proved by Koornwinder [9]. Formula (2.9) may be derived through an interchange of integrals in (2.8) and an application of Euler's formula

$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\, dt.$$

Part II. We shall expand the hypergeometric function in (2.9) as

$$\sum_{j=0}^{\infty} d_j \left(\frac{\cosh t - \cosh s}{2\cosh t}\right)^j;$$

the appropriate generalization of (2.2) is a series expansion for functions of the form

$$\left(\frac{\cosh t - \cosh s}{t^2 - s^2}\right)^{(n-3)/2+j}.$$

Let $(u, w)=(t, t^2-s^2)$, $B(r)=\{z\in C \mid |z|<r\}$, and let

$$g(u,w)=\begin{cases}\dfrac{2\cosh u-2\cosh(u^2-w)^{1/2}}{w} & \text{when } w\neq 0\\ \dfrac{\sinh u}{u} & \text{when } w=0\end{cases}$$

PROPOSITION 2.3. *$[g(u,w)]^z$ is holomorphic in w for w in $B(3\pi^2)$, for all z in C and all u in $(-\pi,\pi)$. Then*

$$[g(u,w)]^z=\left(\frac{\sinh u}{u}\right)^z\sum_{k=0}^{\infty}a_k(u,z)\,w^k. \tag{2.10}$$

There exists an $R_1>1$ such that for all $x>0$ and u with $|u|^2\leqslant R_1$,

$$\left|\left(\frac{\sinh u}{u}\right)^x a_k(u,x)\right|\leqslant\left(\frac{4\cosh u}{R_1}\right)^x R_1^{-k}. \tag{2.11}$$

Proof. The analyticity of g in the given region was proved in [11]. To prove (2.11), we first prove

LEMMA 2.4. *There is an $R_1>1$ such that when u is real and $|u|^2\leqslant R_1$,*

$$\sup_\theta\left|\frac{\cosh u-\cosh(u^2-R_1e^{i\theta})^{1/2}}{2\cosh u}\right|<1. \tag{2.12}$$

Proof. The maximum modulus principle and the continuity of the function to be estimated allow us to reduce (2.12) to the estimate

$$\sup_{|u|\leqslant 1}\sup_\theta\left|\frac{\cosh u-\cosh(u^2-e^{i\theta})^{1/2}}{2\cosh u}\right|<1.$$

This estimate follows by computation. For later use, we shall require $R_1<\pi/2$.

We can now prove (2.11). The Cauchy formula shows

$$\left(\frac{\sinh u}{u}\right)^x a_k(u,x)=\frac{1}{2\pi i}\int_{|w|=R_1}\frac{[g(u,w)]^x}{w^{k+1}}\,dw.$$

Then

$$\begin{aligned}\left|\left(\frac{\sinh u}{u}\right)^x a_k(u,x)\right|&\leqslant\sup_\theta|g(u,R_1e^{i\theta})|^x R_1^{-k}\\ &\leqslant\left(\frac{4\cosh u}{R_1}\right)^x R_1^{-k}\sup_\theta\left|\frac{\cosh u-\cosh(u^2-R_1e^{i\theta})^{1/2}}{2\cosh u}\right|^x.\end{aligned}$$

From Lemma 2.4, the supremum factor is bounded by 1, proving (2.12).

Part III. We now estimate equation (2.4). We shall proceed formally, but justify all formal manipulations in Lemma 2.5 below.

$$F\left(\frac{q}{2}, 1-\frac{q}{2}; p+q; \frac{\cosh t - \cosh s}{2\cosh t}\right) = \sum_{j=0}^{\infty} d_j \left(\frac{\cosh t - \cosh s}{2 \cosh t}\right)^j \tag{2.13}$$

where

$$d_j = \frac{\Gamma(p+q)}{\Gamma\left(\frac{q}{2}\right)\Gamma\left(1-\frac{q}{2}\right)} \frac{\Gamma\left(\frac{q}{2}+j\right)}{\Gamma(p+q+j)} \frac{\Gamma\left(1-\frac{q}{2}+j\right)}{\Gamma(j+1)}.$$

Substituting (2.13) into the expression (2.9) for $\varphi_\lambda(\exp tH_0)$, we obtain:

$$\begin{aligned}&(c_0 2^{(3-n)/2} D(t)^{-1} \sinh 2t (\cosh t)^{(q/2)-1})^{-1} \varphi_\lambda(\exp tH_0) \\ &= \sum_{j=0}^{\infty} d_j (4 \cosh t)^{-j} \int_0^t \cos \lambda s (2\cosh t - 2 \cosh s)^{(n+3)/2+j} ds.\end{aligned} \tag{2.14}$$

From formula (2.10), this is

$$\sum_{j=0}^{\infty} d_j (4 \cosh t)^{-j} \int_0^t \cos \lambda s (t^2 - s^2)^{(n-3)/2+j} \sum_{k=0}^{\infty} \left(\frac{\sinh t}{t}\right)^{(n-3)/2+j} a_k\left(t, \frac{n-3}{2}+j\right)(t^2-s^2)^k ds. \tag{2.15}$$

This is

$$\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} d_j (4 \cosh t)^{-j} \left(\frac{\sinh t}{t}\right)^{(n-3)/2+j} a_k\left(t, \frac{n-3}{2}+j\right) \int_0^t \cos \lambda s (t^2-s^2)^{(n-3)/2+j+k} ds. \tag{2.16}$$

But

$$\begin{aligned}\int_0^t \cos \lambda s (t^2 - s^2)^{(n-3)/2+j+k} ds &= t^{n-2} t^{2(j+k)} \int_0^1 \cos(\lambda t r)(1-r^2)^{(n-2)/2+j+k-(1/2)} dr \\ &= t^{n-2} t^{2(j+k)} \frac{\Gamma\left(\frac{n-2}{2}+j+k\right)\Gamma(\frac{1}{2})}{2} \frac{J_{(n-2)/2+j+k}(\lambda t)}{\left|\frac{\lambda t}{2}\right|^{(n-2)/2+j+k}} = t^{n-2} t^{2(j+k)} \mathfrak{J}_{(n-2)/2+j+k}(\lambda t)\end{aligned}$$

(See Erdelyi [2], p. 156). Equation (2.16) therefore becomes

$$t^{n-2} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} d_j (4 \cosh t)^{-j} \left(\frac{\sinh t}{t}\right)^{(n-3)/2+j} a_k\left(t, \frac{n-3}{2}+j\right) t^{2(k+j)} \mathfrak{J}_{(n-2)/2+j+k}(\lambda t). \tag{2.17}$$

Rearranging the series, we obtain

$$(\sinh t)^{(n-3)/2} t^{(n-1)/2} \sum_{m=0}^{\infty} a_m(t) t^{2m} \mathfrak{J}_{(n-2)/2+m}(\lambda t) \tag{2.18}$$

where

$$a_m(t) = \sum_{j=0}^{m} d_j (4\cosh t)^{-j} \left(\frac{\sinh t}{t}\right)^j a_{m-j}\left(t, \frac{n-3}{2}+j\right). \tag{2.19}$$

This establishes equation (2.4) of Theorem 2.1.

LEMMA 2.5. *There is a number $R_0>1$ such that for any t with $0 \leqslant t \leqslant R_0$, the above proof of* (2.4) *is valid.*

Proof. Choose any R_0 with $1 < R_0 < R_1^{1/2}$. As $|(\cosh t - \cosh s)/2\cosh t| < \frac{1}{2}$, the hypergeometric series (2.13) converges uniformly; this justifies the interchange of sum and integral between (2.13) and (2.14).

The expression (2.14) is transformed into (2.15) through an application of (2.10); the series in (2.10) will converge uniformly when $|t^2 - s^2| < 3\pi^2$ and $|t| < \pi$. As $s \leqslant t \leqslant R_0 < R_1^{1/2} < \pi^{1/2}$, we may apply (2.10), and use the uniform convergence of the power series to transform (2.15) into (2.16).

To transform (2.16) into (2.17) we must justify the re-arrangement of the double series; it suffices to establish the absolute convergence of the double series. Using estimate (2.11) of Proposition 2.3 and the trivial estimate $|\mathcal{J}_\mu(\lambda t)| \leqslant 1$, we see that a term in the double sum (2.17) is bounded by $|d_j|\,|4\cosh t|^{(n-3)/2}|t^2/R_1|^{j+k}$. As $t^2/R_1 \leqslant R_0^2/R_1 < 1$, the series

$$\sum_j \sum_k |d_j| \left(\frac{t^2}{R_1}\right)^{j+k}$$

clearly converges. This completes the proof of the lemma and of equation (2.4).

Part IV. We wish to show $a_0 \equiv 1$ and $|a_m(t)| \leqslant CR_1^{-m}$. The first is obvious from the definitions. To estimate the a_m, we note that

$$\begin{aligned}
|a_m(t)| &\leqslant \sum_{j=1}^{m} |d_j|\,|4\cosh t|^{-j} \left|\left(\frac{\sinh t}{t}\right)^j a_{m-j}\left(t, \frac{n-3}{2}+j\right)\right| \\
&\leqslant \left(\frac{\sinh t}{t}\right)^{-(n-3)/2} \sum_{j=0}^{m} |d_j|\,|4\cosh t|^{-j} \left|\frac{4\cosh t}{R_1}\right|^{(n-3)/2+j} R_1^{j-m} \\
&= \left(\frac{4t\cosh t}{R_1 \sinh t}\right)^{(n-3)/2} R_1^{-m} \sum_{j=0}^{m} |d_j|.
\end{aligned}$$

But elementary properties of the Γ function show that

$$\sum_{j=0}^{m} |d_j| \leqslant c \sum_{j=1}^{m} \left| \frac{\Gamma\left(\frac{q}{2}+j\right)\Gamma\left(1-\frac{q}{2}+j\right)}{\Gamma(p+q+j)\Gamma(1+j)} \right| \leqslant c \sum j^{-(p+(q/2))} j^{-q/2}.$$

The assumption $q \neq 0$ implies that $p+q \geqslant 2$, so that the sum is bounded independently of m, and the estimate (2.6) is valid.

Part V. To estimate E_{M+1}, we examine the regions $|\lambda t| \leqslant 1$ and $|\lambda t| > 1$ separately. In the former region, we bound $|\mathfrak{J}_\mu(\lambda t)|$ by 1. Then

$$|E_{M+1}| \leqslant c_0 \left|\frac{t^{n-1}}{D(t)}\right|^{1/2} \sum_{j=M+1}^{\infty} t^{2j} |a_j(t)|.$$

From (2.6) we see this is bounded by

$$c \sum_{j=M+1}^{\infty} t^{2j} R_1^{-2j} \leqslant c\left(\frac{t}{R_1}\right)^{2(M+1)} \sum_{j=0}^{\infty} \left(\frac{R_0^2}{R_1}\right)^j \leqslant ct^{2(M+1)}.$$

In the region $|\lambda t| > 1$, we again start with the estimate

$$|E_{M+1}| \leqslant c_0 \left|\frac{t^{n-1}}{D(t)}\right|^{1/2} \sum_{j=M+1}^{\infty} t^{2j} |a_j(t)|\, |\mathfrak{J}_{(n-2)/2+j}(\lambda t)|.$$

For the first term in the series we employ standard estimates on Bessel functions, to obtain

$$|\mathfrak{J}_{(n-2)/2+M+1}(\lambda t)| \leqslant c \frac{\Gamma\left(\dfrac{n-1}{2} + M + 1\right) \Gamma(\tfrac{1}{2})\, 2^{(n-2)/2}\, 2^M}{|\lambda t|^{(n-1)/2+M+1}}.$$

For higher terms, we must employ sharper estimates. Szegö [12] has shown

$$|\mathfrak{J}_\mu(z)| \leqslant \pi^{1/2} \frac{(\mu - \frac{1}{2})^k\, 2^k}{|z|^k} \cdot \frac{\Gamma(\mu + \frac{1}{2} - k)}{\Gamma(\mu + 1 - k)},$$

an estimate which is valid for all real z and integers k with $0 \leqslant k \leqslant \mu$. We set $k = (n-2)/2 + M + 2$, and find

$$|E_{M+1}| \leqslant c_M t^{2(M+1)} |\lambda t|^{-((n-1)/2+M+1)},$$

where

$$c_M \leqslant c\Bigg[2^{(n-2)/2} \Gamma(\tfrac{1}{2})\, R_1^{-(M+1)}\, 2^M \Gamma\left(\frac{n-1}{2} + M + 1\right) + 2^{(n/2)+1} \pi^{1/2}\, 2^M R_0^{-2(M+1)} \sum_{j=M+2} \left(\frac{R_0^2}{R_1}\right)^j \left(\frac{n-3}{2} + j\right)^{(n/2)+M+1} j^{-1/2}\Bigg] < \infty.$$

This establishes estimate (2.7) and completes the proof of Theorem 2.1.

Section 3

In this section we derive estimates on the growth of the spherical functions and their derivatives near infinity. Our approach depends upon a key result of Harish-Chandra:

THEOREM 3.1.

$$\varphi_\lambda(\exp tH_0) = c(\lambda)e^{(i\lambda-\varrho)t}\phi_\lambda(t) + c(-\lambda)e^{(-i\lambda-\varrho)t}\phi_{-\lambda}(t) \tag{3.1}$$

where

$$c(\lambda) = \Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{q}{2}\right)\frac{\Gamma(i\lambda)\,\Gamma\left(\frac{p+i\lambda}{2}\right)}{\Gamma\left(\frac{p}{2}+i\lambda\right)\Gamma\left(\frac{\varrho+i\lambda}{2}\right)} \tag{3.2}$$

$$\phi_\lambda(t) = \sum_{k=0}^{\infty}\Gamma_k(\lambda)\,e^{-2kt} \tag{3.3}$$

$$\Gamma_0(\lambda) \equiv 1$$

$$(k+1)(k+1-i\lambda)\Gamma_{k+1} = \sum_{j=0}^{k}\frac{p}{2}(\varrho+2j-i\lambda)\Gamma_j + \sum_{\substack{j=k+1-2l\\ l>0,\, j\geqslant 0}} q(\varrho+2j-i\lambda)\Gamma_j. \tag{3.4}$$

Remarks.

1. The series (3.3) converges when $|\operatorname{Im}\lambda| < \varrho$, uniformly on compacta not containing $\exp(0\,H_0)$, the group identity. This follows from unpublished work of Harish-Chandra; see Helgason [7], p. 201. Theorem 3.1 was proved in [6a].

2. Our notation differs slightly from that of [6a]; our Γ_k is Harish-Chandra's Γ_{2k}. In Harish-Chandra's notation, $\Gamma_{2k+1} \equiv 0$.

From equation (3.1), we see that

$$\varphi_\lambda(\exp tH_0) = c(\lambda)e^{i\lambda t}e^{-\varrho t} + c(-\lambda)e^{-i\lambda t}e^{-\varrho t} + \text{error terms};$$

estimates on the size of a function f may therefore easily be obtained by Euclidean Fourier transform techniques, if one has some knowledge of $\hat{f}$ and some control of the above error terms. Gangolli [5] showed that there exist positive numbers c and d such that $|\Gamma_k(\lambda)| \leqslant ck^d$. Such estimates are optimal, and suffice to prove Paley-Wiener type theorems (see Helgason [7]). In the Paley-Wiener theorem, one knows that $\hat{f}(\lambda)$ is rapidly decreasing when $\operatorname{Im}\lambda = 0$; estimates on $\Gamma_k(\lambda)$ which are uniform in λ are sufficient to achieve control of f. We are concerned with controlling f under weaker hypotheses on $\hat{f}$; it is essential to estimate precisely the growth of Γ_k in λ.

We shall prove

THEOREM 3.2. *There is a constant $A = A(G)$ such that, for any $M \geqslant 0$ and any λ with* $\operatorname{Im}\lambda \geqslant 0$

$$\Gamma_k(\lambda) = \sum_{m=0}^{M} \gamma_m^k + E_{M+1}^k, \tag{3.5}$$

where γ_m^k is the sum of terms g_l, and $1/g_l$ is an mth degree polynomial in λ. Further,

$$|\gamma_m^k(\lambda)| \leqslant A \frac{\varrho^m e^{2k}}{|\operatorname{Re}\lambda|^m} \tag{3.6}$$

$$|D_{\operatorname{Re}\lambda}^{\alpha}\, \gamma_m^k| \leqslant A 2^{\alpha} \frac{\varrho^m e^{2k}}{|\operatorname{Re}\lambda|^{m+\alpha}} \tag{3.7}$$

$$|E_{M+1}^k| \leqslant A \frac{\varrho^{M+1} e^{2k}}{|\operatorname{Re}\lambda|^{M+1}}. \tag{3.8}$$

Remark. As with Theorem 2.1, the proof of this result is rather technical. We decompose it into four parts:

I. Construction of a recursion simpler than (3.4).
II. Solution of recursion and expansion in the form (3.5).
III. Estimation of the γ_m^k.
IV. Estimation of the error term E_{M+1}^k.

Proof of Theorem 3.2.

Part I.

PROPOSITION 2.3.

$$(k+1)(k+1-i\lambda)\Gamma_{k+1} = (\varrho+k)(\varrho+k-i\lambda)\Gamma_k + q\sum_{j=0}^{k}(-1)^{k+j+1}(\varrho+2j-i\lambda)\Gamma_j. \tag{3.9}$$

Proof. It suffices to prove that the right-hand side of (3.9) equals the right-hand side of (3.4). This latter is

$$\frac{p}{2}(\varrho+2k-i\lambda)\Gamma_k + q(\varrho+2k-i\lambda)\Gamma_k + \sum_{j=0}^{k-1}\frac{p}{2}(\varrho+2j-i\lambda)\Gamma_j + \sum_{\substack{j=k-2l\\ l>0,\, j\geqslant 0}} q(\varrho+2j-i\lambda)\Gamma_j$$
$$+ \sum_{\substack{j=k+1-2l\\ l>0,\, j\geqslant 0}} q(\varrho+2j-i\lambda) - q(\varrho+2k-i\lambda)\Gamma_k - \sum_{\substack{j=k-2l\\ l>0,\, j\geqslant 0}} q(\varrho+2j-i\lambda)\Gamma_j.$$

This is

$$\varrho(\varrho+2k-i\lambda)\Gamma_k + k(k-i\lambda)\Gamma_k + q\sum_{\substack{j=k+1-2l\\ l>0,\, j\geqslant 0}}(\varrho+2j-i\lambda)\Gamma_j$$
$$= (\varrho+k)(\varrho+k-i\lambda)\Gamma_k + q\sum_{j=0}^{k}(-1)^{k+j+1}(\varrho+2j-i\lambda)\Gamma_j.$$

COROLLARY 3.4.

$$\Gamma_{k+1} = \alpha_k \Gamma_k + \sum_{j=0}^{k-1} \beta_j^k \Gamma_j, \tag{3.10}$$

where

$$\alpha_k = 1 + \frac{\frac{p}{2}-1}{k+1} + \frac{\frac{p}{2}-1+\frac{1}{k+1}\left[\left(\frac{p}{2}-1\right)^2+\frac{pq}{2}\right]}{k+1-i\lambda} \tag{3.11}$$

$$\beta_j^k = (-1)^{k+j+1}\frac{q}{k+1}\left(1+\frac{\varrho+2j-1}{k+1-i\lambda}\right). \tag{3.12}$$

Part II. When $q=0$, (3.9) is trivial to solve, and yields

$$\Gamma_k = \frac{\Gamma(1-i\lambda)}{\Gamma(\varrho-i\lambda)}\frac{\Gamma(\varrho+k)}{\Gamma(\varrho)\Gamma(k+1)}\frac{\Gamma(\varrho-i\lambda+k)}{\Gamma(1-i\lambda+k)} = \prod_{j=0}^{k-1}\left(a_j+\frac{b_j}{j+1-i\lambda}\right). \tag{3.13}$$

To facilitate estimates of $D_\lambda^\alpha \Gamma_k$, we expand the product expression into a sum of 2^k monomials, and it is trivial to estimate the derivative of each monomial.

When q is non-zero, (3.9) admits of no simple solution (see, however, Corollary 3.8). Γ_{k+1} may be expressed as a sum of 2^k terms, each of which is a product of α_i's and β_j^l's. These products may in turn be expanded into monomials, through (3.11) and (3.12). A g_m^{k+1} is a term in this expansion for which $(g_m^{k+1})^{-1}$ is a polynomial in λ of degree m. If $q=0$, there are $\binom{k+1}{m}$ such; if $q\neq 0$, there are $\sum_{i=m}^{k+1}\binom{k}{i-1}\binom{i}{m}$ such. Let γ_m^{k+1} be their sum.

Part III.

LEMMA 3.5. $|\gamma_m^k(\lambda)| \leqslant A\dfrac{\varrho^m}{|\mathrm{Re}\,\lambda|^m}e^{2k}$.

Proof. From (3.11) and (3.12) we see $\alpha_k = a_k + b_k/(k+1-i\lambda)$, where

$$|a_k| \leqslant 1+p/(2k+2),\ |b_k| \leqslant p/2 \text{ for } k \geqslant k_0 \text{ and } |b_k| \leqslant A_0 \text{ for } k<k_0;$$
$$\beta_j^k = c_j^k + d_j^k/(k+1-i_\lambda), \text{ where } |c_j^k| \leqslant q/(k+1),\ |d_j^k| \leqslant q(\varrho+2j-1)/(k+1). \tag{3.14}$$

We shall establish the lemma by induction on k, for all j. Assume first that $m=0$, and that the lemma is valid for all γ_0^j with $j \leqslant k$. From (3.9), $\gamma_0^{k+1} = a_k\gamma_0^k + \sum_{j=0}^{k-1} c_j^k \gamma_0^j$. Then (3.14) and induction show that $|\gamma_0^{k+1}| \leqslant A\,e^{2k}(1+(p+q)/2(k+1))$. When $(p+q)/2 < k+1$, $|\gamma_0^{k+1}| < A\,e^{2(k+1)}$. The smaller γ_0^{k+1} may easily be estimated by choosing

$$A > \prod_{j=0}^{(p+q)/2}\left(1+\frac{p+q}{2(j+1)}\right) \geqslant (2\varrho)^\varrho.$$

This proves the lemma for $m=0$ and all k.

When $m>0$, it is clear that $\gamma_m^k=0$ for $k<m$. We shall therefore prove that (3.6) holds for all m with $0<m\leqslant k$ and all k. Assume (3.6) is valid for γ_j^l when $l\leqslant k$ and $j\leqslant l$. If $m=k+1$, $\gamma_{k+1}^{k+1}=[b_k/(k+1-i\lambda)]\gamma_k^k$. Now $\operatorname{Im}\lambda\geqslant 0$, so that

$$|\operatorname{Re}\lambda|\leqslant|k+1-i\lambda| \text{ and}$$

$$|\gamma_{k+1}^{k+1}|\leqslant\frac{p}{2|\operatorname{Re}\lambda|}A\frac{\varrho^k e^{2k}}{|\operatorname{Re}\lambda|^k}\leqslant A\frac{\varrho^{k+1}e^{2(k+1)}}{|\operatorname{Re}\lambda|^{k+1}}.$$

This estimate is valid when $k\geqslant k_0$; to handle the cases $k<k_0$, we must choose $A\geqslant A_0^{k_0}$.

When $m<k+1$,

$$\gamma_m^{k+1}=a_k\gamma_m^k+\frac{b_k}{k+1-i\lambda}\gamma_{m-1}^k+\sum_{j=m}^{k-1}\left(c_j^k\gamma_m^j+\frac{d_j^k\gamma_{m-1}^j}{k+1-i\lambda}\right).$$

Thus

$$|\gamma_m^{k+1}|\leqslant\frac{Ae^{2k}}{|\operatorname{Re}\lambda|^m}\varrho^m\left(1+\frac{p}{2(k+1)}+\frac{p}{2\varrho}+\frac{2q}{(\varrho-1)(k+1)}+\frac{q}{2\varrho}\,\frac{k}{k+1}\right)$$
$$\leqslant\frac{Ae^{2k}\varrho^m}{|\operatorname{Re}\lambda|^m}\left(1+\frac{p+q}{2\varrho}+\frac{p+2q}{2(k+1)}\right)\leqslant Ae^{2(k+1)}\frac{\varrho^m}{|\operatorname{Re}\lambda|^m}.$$

The estimates on b_k are again valid only for large k; for smaller k extra factors of A_0 are required in A.

LEMMA 3.6. $|D_{\operatorname{Re}\lambda}^{\alpha}\gamma_m^k|\leqslant A2^{\alpha}\dfrac{\varrho^m}{|\operatorname{Re}\lambda|^{m+\alpha}}e^{2k}$.

Proof. The proof is the same as that of the previous lemma, but employs obvious estimates such as $|D_\lambda(b_k/(k+1-i\lambda))|=|b_k/(k+1-i\lambda)^2|$.

Part IV.

LEMMA 3.7. $|E_{M+1}^k|\leqslant A\dfrac{\varrho^{M+1}e^{2k}}{|\operatorname{Re}\lambda|^{M+1}}$.

Proof. We prove the lemma by induction on k, for all $M>0$. The $k=1$ case is trivial. Assume the result is valid for all $j\leqslant k$. The terms which contribute to E_{M+1}^{k+1} are:

(i) $$\alpha_k E_{M+1}^k+\sum_{j=M+1}^{k-1}\beta_j^k E_{M+1}^k$$

(ii) $$\frac{b_k}{k+1-i\lambda}\gamma_M^k+\sum_{j=M}^{k-1}\frac{d_j^k}{k+1-i\lambda}\gamma_M^j.$$

Note

$$|\alpha_k E^k_{M+1}| \leqslant \left(1+\frac{p}{2(k+1)}+\frac{|b_k|}{k+1}\right) A \frac{\varrho^{M+1} e^{2k}}{|\mathrm{Re}\ \lambda|^{M+1}}$$
$$\leqslant \left(1+\frac{p}{k+1}\right) A \frac{\varrho^{n+1} e^{2k}}{|\mathrm{Re}\ \lambda|^{M+1}}.$$

$$|\beta^k_j E^j_{M+1}| \leqslant \frac{q}{k+1}\left(1+\frac{\varrho+2j}{k+1}\right) A \frac{\varrho^{M+1} e^{2j}}{|\mathrm{Re}\ \lambda|^{M+1}}$$

$$\left|\frac{b_k}{k+1-i\lambda}\gamma^k_M\right| \leqslant \frac{p}{2|\mathrm{Re}\ \lambda|} A \frac{\varrho^M e^{2k}}{|\mathrm{Re}\ \lambda|^M} \leqslant A \frac{\varrho^{M+1} e^{2k}}{|\mathrm{Re}\ \lambda|^{M+1}}$$

$$\left|\frac{d^k_j}{k+1-i\lambda}\gamma^j_M\right| \leqslant \frac{q}{k+1}\,\frac{\varrho+2j}{|\mathrm{Re}\,\lambda|} A \frac{\varrho^M e^{2j}}{|\mathrm{Re}\ \lambda|^M} \leqslant A \frac{\varrho+2j}{k+1}\,\frac{\varrho^{M+1} e^{2j}}{|\mathrm{Re}\ \lambda|^{M+1}}.$$

We must therefore have

$$3+\frac{p}{k+1}+\frac{q}{k+1}\left(1+\frac{1}{e^2-1}+\frac{\varrho}{(e^2-1)(k+1)}\right)+\frac{\varrho}{(e^2-1)(k+1)} \leqslant e^2,$$

which holds for sufficiently large k.

This completes the proof of the lemma, and completes the proof of Theorem 3.2.

We may use the above results to derive some further information on the behavior of spherical functions. One would like to have, for example, a representation of Γ_k as a quotient of Γ functions, similar to (3.13), but (3.9) clearly shows the $q \neq 0$ case to be more complex than any $q=0$ case. To solve the recursion (3.10), we note

$$\Gamma_1 = \alpha_0 \Gamma_0 = \alpha_0$$
$$\Gamma_2 = \alpha_1 \Gamma_1 + \beta^1_0 \Gamma_0 = \alpha_1 \alpha_0 + \beta^1_0$$
$$\Gamma_3 = \alpha_2 \alpha_1 \alpha_0 + \alpha_3 \beta^1_0 + \beta^2_0 + \beta^2_1 \alpha_0.$$

Γ_{k+1} may be expressed as a sum of 2^{k+1} terms, each of which is a product of α_j's and β^l_m's. It is useful for computational purposes to know which products may occur; we shall give a simple combinatorial characterization, which allows one to write down Γ_{k+1} without having solved (3.10) for Γ_j, $j \leqslant k$.

We would like each term occurring in Γ_{k+1} to have $k+1$ factors; as this is clearly false we develop a substitute notion.

Definition. The type of α_j is one; the type of β^l_m is $l-m+1$. The type of a product is the sum of the types of its factors.

COROLLARY 3.8. Let

$$\alpha_{j_1} \dots \alpha_{j_i} \beta^{l_1}_{m_1} \dots \beta^{l_n}_{m_n} \tag{*}$$

be one of the 2^{k+1} *terms which occur in solving* (3.10) *for* Γ_{k+1}. *The collection of integers* j, l, m *such that* α_j *or* β^l_m *is a factor in* (*) *satisfies the following conditions:*

(a) $0 \leqslant j \leqslant k,\ 1 \leqslant l \leqslant k,\ 0 \leqslant m < l$.

(b) *If* β^l_m *is a factor in* (*), *no* α_j *or* $\beta^{l'}_{m'}$ *is a factor, for any* j, l', m' *in* $[m, l]$.

(c) *The integers* $j_1, \dots, j_i, l_1, \dots, l_n, m_1, \dots, m_n$ *are distinct.*

(d) *The type of* (*) *is* $k+1$.

Conversely, if (*) *is a product the indices of whose factors satisfy* (a)–(d), *then* (*) *is one of the* 2^{k+1} *terms occurring in the solution of* (3.10) *for* Γ_{k+1}.

Proof. That conditions (a)–(d) are satisfied may easily be proven by induction on k. For example, (3.10) shows that the type of a term in Γ_{k+1} may be the type of a term in Γ_k plus the type of α_k, or it may be the type of a term in Γ_j plus the type of β^k_j; either of these is $k+1$.

The converse is of greater interest; we establish it by induction. To analyze Γ_1, we note that (a) requires the terms in (*) to have indices bounded by zero. The candidates for Γ_1 are thus α_0, β^0_0 and $\alpha_0\beta^0_0$; β^0_0 contradicts (a)–(c), while $\alpha_0\beta^0_0$ contrives to contradict all (a)–(d). The assertion of the corollary is the true statement that $\Gamma_1 = \alpha_0$.

Assume the result holds for all Γ_j with $j \leqslant k$. Let (*) be a candidate or Γ_{k+1}; that is, let (*) satisfy (a)–(d). We claim that (*) contains a factor α_k or β^k_j. Let us assume this result for the moment. If α_k occurs, β^k_j does not, by (c). Let $(**) = (*)/\alpha_k$. Then (**) satisfies (a)–(d) with k replaced by $k-1$: conditions (a) and (c) show that (a) holds for (**); the validity of (b) and (c) is not affected by deleting a term, and type $(**) =$ type $(*) -$ type $\alpha_k = k$. Therefore by induction hypothesis (**) is a term occurring in Γ_k, and by (3.10), (*) occurs in Γ_{k+1} through $\alpha_k \Gamma_k$.

If β^k_j occurs in (*), we set $(**) = (*)/\beta^k_j$. If $j = 0$, type $(*) =$ type $\beta^k_0 +$ type $(**) = k+1+$ type (**). Condition (d) requires type $(*) = k+1$: therefore $(*) = \beta^k_0$. But β^k_0 occurs in Γ_{k+1} through $\beta^k_0 \Gamma_0$. If $j > 0$, the proof that (*) occurs is the same as that for α_k.

To complete the proof of the corollary, we must show that either α_k or β^k_j occurs in (*). Assume neither occurs. Let $m_0 = \max \{l \mid \alpha_l \text{ or } \beta^l_j \text{ is a factor in } (*)\}$. Condition (a) implies $m_0 \leqslant k$; condition (c) and our hypothesis imply that $m_0 < k$. We shall show that type $(*) \leqslant m_0 + 1$; this contradicts (d).

To calculate the type of (*), we replace each $\beta^{l_i}_{m_i}$ occurring as a factor in (*) by a formal product $\alpha'_{m_i} \alpha'_{m_i+1} \dots \alpha'_{l_i}$. The type of $\beta^{l_i}_{m_i}$ is the number of factors in this formal product.

When we have replaced each β_m^l in this manner, we form the set S, the set of integers j which appear as indices of an α_j or an α_j'. We wish to show:

(i) to each integer in S corresponds precisely one α_j or α_j'.

(ii) S has at most m_0+1 elements.

Then (i) and (ii) together imply type (*) $\leqslant$ number of integers in $S \leqslant m_0+1$. But (i) follows immediately from conditions (b) and (c); (ii) follows from (a) and the definition of m_0. This completes the proof of the corollary.

Theorem 2.1 gives an asymptotic expansion for φ_λ (exp tH_0) when t is small. For large t we may use Theorem 3.2 to derive a similar expansion.

COROLLARY 3.9. *There exist functions* $\Lambda_m(\lambda, t)$ *and* $\mathcal{E}_{M+1}(\lambda, t)$ *such that, for any* $M \geqslant 0$ *and* $t \geqslant R_0$, λ *with* $\operatorname{Im} \lambda \geqslant 0$

$$\phi_\lambda(t) = \sum_{m=0}^{\infty} \Lambda_m(\lambda, t)\, e^{-2mt}$$

$$\phi_\lambda(t) = \sum_{m=0}^{M} \Lambda_m(\lambda, t)\, e^{-2mt} + \mathcal{E}_{M+1}(\lambda, t),$$

where

$$|D_\lambda^\alpha D_t^\beta \Lambda_m| \leqslant A \frac{\varrho^m e^{2m}}{|\operatorname{Re} \lambda|^{m+\alpha}} 2^{\alpha+\beta} G_\beta(t)$$

$$|D_t^\beta \mathcal{E}_{M+1}| \leqslant A \frac{e^{2(M+1)} \varrho^{M+1}}{|\operatorname{Re} \lambda|^{M+1}} 2^\beta G_\beta(t)$$

and

$$G_\alpha(t) = \sum_{j=0}^{\infty} j^\alpha e^{2j(1-t)}.$$

Proof. If we set $\Lambda_m(\lambda, t) = \sum_{j=0}^{\infty} \gamma_m^{m+j}(\lambda) e^{-2jt}$, the result follows from Theorem 3.2.

Section 4

In the previous sections we obtained series expansions for spherical functions; we note that the expansions which characterize local and global behavior differ radically, both in statement and proof. In this section, we shall apply Theorems 2.1 and 3.2 to the Fourier analysis of K bi-invariant functions; we shall see once again that the local analysis is essentially that of $\mathfrak{p}$, viewed as the symmetric space of the Cartan motion group, while the global analysis has no Euclidean analogue.

We establish notation to be used in the remainder of the paper. N will denote the least integer greater than $n/2$. Let $\psi(g)$ denote a smooth K bi-invariant function with $0 \leqslant \psi \leqslant 1$; ψ (exp $tH_0) = 1$ if $|t| \leqslant R_0^{1/2}$; ψ (exp $tH_0) = 0$ if $|t| \geqslant R_0$.

PROPOSITION 4.1. *Let $p(\lambda)$ be an even $C^N(\mathfrak{a}'_+)$ function satisfying the estimates*

$$D^\alpha p(0) = 0 \quad \textit{when } 0 \leqslant \alpha \leqslant N$$
$$|D_\lambda^\alpha p(\lambda)| \leqslant C_\alpha (1+|\lambda|)^{-\alpha} \quad \textit{when } 0 \leqslant \alpha \leqslant N. \tag{4.1}$$

Then there exists an $L^1(G/K)$ function $e_0(t)$ such that

$$\check{p}(\exp tH_0)\,\psi(\exp tH_0) \equiv \psi(\exp tH_0) \int_0^\infty p(\lambda)\,\varphi_\lambda(\exp tH_0)\,|c(\lambda)|^{-2}\,d\lambda$$
$$= \psi(\exp tH_0)\,c_0 \left(\frac{t^{n-1}}{D(t)}\right)^{1/2} \int_0^\infty p(\lambda)\,\mathfrak{J}_{(n-2)/2}(\lambda t)\,|c(\lambda)|^{-2}\,d\lambda + e_0(t). \tag{4.2}$$

Remarks.

1. It is not clear from the hypotheses that $\check{p}$ exists, other than in a distributional sense. Throughout the remainder of the paper we shall always assume that functions satisfying estimates such as (4.1) are in fact rapidly decreasing in λ, though none of our estimates will depend upon the rate of decrease. This will allow us to define $\check{p}$ pointwise, and to perform various formal manipulations such as integration by parts. To pass from this to arbitrary functions satisfying (4.1), we need a basic theory of approximate identities. Such a theory may easily be developed, in a manner analogous to the Euclidean theory. Pointwise results may be obtained using the work of Clerc and Stein [3] on maximal functions.

2. The proof of Proposition 4.1 requires repeated integrations by parts. It is therefore essential to estimate derivatives of $|c(\lambda)|^{-2}$.

LEMMA 4.2.
$$|D_\lambda^\alpha |c(\lambda)|^{-2}| \leqslant c_\alpha (1+|\lambda|)^{n-1-\alpha}. \tag{4.3}$$

Proof. The lemma may easily be derived from the following formulae, each of which is a consequence of equation (3.2): $|c(\lambda)|^{-2} =$

$$c \prod_{j=0}^{k-1} (j^2+\lambda^2) \qquad \text{when } q=0 \text{ and } p=2k$$

$$c\lambda \tanh\frac{\pi\lambda}{2} \prod_{j=0}^{k-1} [(\tfrac{1}{2}+j)^2+\lambda^2] \qquad \text{when } q=0 \text{ and } p=2k+1$$

$$c\lambda \coth\frac{\pi\lambda}{2} \prod_{j=0}^{k} \left[j^2+\left(\frac{\lambda}{2}\right)^2\right] \prod_{j=1}^{k+l} \left[j^2+\left(\frac{\lambda}{2}\right)^2\right] \qquad \text{when } q=2l+1 \text{ and } p=4k+2$$

$$c\lambda \tanh\frac{\pi\lambda}{2} \prod_{j=0}^{k} \left[(\tfrac{1}{2}+j)^2+\left(\frac{\lambda}{2}\right)^2\right] \prod_{j=0}^{k+l-1} \left[(\tfrac{1}{2}+j)^2+\left(\frac{\lambda}{2}\right)^2\right] \qquad \text{when } q=2l+1 \text{ and } p=4k.$$

Proof of Proposition 4.1. We shall employ Theorem 2.1, with M chosen to be N. Define

$$e_0(t) = \psi(\exp tH_0)\, c_0 \left[\frac{t^{n-1}}{D(t)}\right]^{1/2} \sum_{m=1}^{N} t^{2m} a_m(t) \int_0^\infty \mathfrak{J}_{(n-2)/2+m}(\lambda t)\, p(\lambda)\, |c(\lambda)|^{-2}\, d\lambda$$
$$+ \psi(\exp tH_0) \int_0^\infty E_{N+1}(\lambda t)\, p(\lambda)\, |c(\lambda)|^{-2}\, d\lambda. \tag{4.4}$$

The estimates (2.6) allow us to bound $a_m(t)$ by a constant. Each term in the expression (4.4) is a compactly supported K bi-invariant function; the integration formula (1.1) shows that $e_0(t)$ will be in L^1 if each term

$$\varepsilon_m(t) = t^{2m} \int \mathfrak{J}_{(n-2)/2+m}(\lambda t)\, p(\lambda)\, |c(\lambda)|^{-2}\, d\lambda, \quad 1 \leqslant m \leqslant N$$
$$\varepsilon_{N+1}(t) = \int E_{N+1}(\lambda t)\, p(\lambda)\, |c(\lambda)|^{-2}\, d\lambda$$

can be bounded by $c/|D(t)|$ or c/t^{n-1}. The term ε_{N+1} is easy to estimate; from the estimate (2.7) on E_{N+1}, we see

$$|\varepsilon_{N+1}(t)| \leqslant c_N \|p\|_\infty \left(\int_0^{1/t} t^{2(N+1)}\, d\lambda + \int_{1/t}^\infty t^{2(N+1)} (\lambda t)^{-((n+1)/2+N)} (1+|\lambda|)^{n-1}\, d\lambda \right).$$

The latter integral is convergent, as $N > (n-1)/2$; then

$$|\varepsilon_{N+1}(t)| \leqslant c[t^{2(N+1)} + t^{(n-1)/2-N} t^{N+1-((n-1)/2)}] \leqslant ct.$$

The remaining estimates are more subtle. We shall apply the formula

$$z^{-1} D_z(\mathfrak{J}_\mu(z)) = -c_\mu \mathfrak{J}_{\mu+1}(z)$$

(see Watson [13], p. 18). When $m \geqslant 2$,

$$\varepsilon_m(t) = c_m t^{2m} \int p(\lambda)\, |c(\lambda)|^{-2} \left(-\frac{1}{\lambda t} D_{\lambda t} \right)^N \mathfrak{J}_{m-2+\varepsilon}(\lambda t)\, d\lambda;$$

here ε is zero when n is even and is $\frac{1}{2}$ when n is odd. Integration by parts shows that

$$\varepsilon_m(t) = c_m t^{2(m-N)} \int \left(D_\lambda \cdot \frac{1}{\lambda} \right)^N (p(\lambda)\, |c(\lambda)|^{-2})\, \mathfrak{J}_{m-2+\varepsilon}(\lambda t)\, d\lambda.$$

As $|\mathfrak{J}_\mu(z)| \leqslant 1$, a typical term in the expansion of the integrand is majorized by $c(1+|\lambda|)^{n-1-2N} \leqslant c(1+|\lambda|)^{-2}$; therefore $|\varepsilon_m(t)| \leqslant ct^{2(m-N)} \leqslant ct^{1-n}$.

If $m=1$ and n is odd, we proceed as above, obtaining

$$\varepsilon_1(t)=ct^2\int p(\lambda)\,|c(\lambda)|^{-2}\left(\frac{1}{\lambda t}D_{\lambda t}\right)^N(\cos\lambda t)\,d\lambda,$$

from which follows $|\varepsilon_1(t)|\leqslant c$. When $m=1$ and n is even, we integrate by parts $n/2$ times, to obtain

$$\varepsilon_1(t)=ct^{2-n}\int\left(D_\lambda\cdot\frac{1}{\lambda}\right)^{n/2}(p(\lambda)\,|c(\lambda)|^{-2})\,\mathcal{J}_0(\lambda t)\,d\lambda.$$

The integral splits into two parts, $|\lambda t|\leqslant 1$ and $|\lambda t|\geqslant 1$. The first part contributes $ct\int_0^{1/t}(1+|\lambda|)^{-1}d\lambda\leqslant ct|\log t|$; for the second part we estimate $|\mathcal{J}_0(\lambda t)|\leqslant c|\lambda t|^{-1/2}$, and the second part then contributes a term ct. This completes the proof of the proposition.

COROLLARY 4.3. *Let $p(\lambda)$ be an even $C^N(\mathfrak{a}'_+)$ function satisfying the estimates*

$$D^\alpha_\lambda p(0)=0,\qquad 0\leqslant\alpha\leqslant N$$
$$|D^\alpha_\lambda p(\lambda)|\leqslant c_\alpha(1+|\lambda|)^{-\alpha},\quad 0\leqslant\alpha\leqslant N.$$

Then for all $t\leqslant R_0$,

$$\check{p}(\exp tH_0)=c_0\int_0^\infty p(\lambda)\,\mathcal{J}_{(n-2)/2}(\lambda t)\,|c(\lambda)|^{-2}\,d\lambda+\sum_{m=1}^N e_m(t)+e(t)\tag{4.5}$$

where

$$|e(t)|\leqslant c$$
$$|e_1(t)|\leqslant c/t^{n-1}$$
$$|e_m(t)|\leqslant ct^{2(m-1)-N},\quad m>1.$$

Proof. The corollary follows immediately from the proof of Proposition 4.1.

Proposition 4.1 allows one to replace the inverse spherical transform on G/K by the radial inverse Fourier transform on $\mathbf{R}^n$, at least locally and up to L^1 error terms. The following result (see [3]) shows that globally, the Fourier transform must behave in a manner entirely different than any Euclidean analogue.

THEOREM 4.4. *Let f be a K bi-invariant function in $L^s(G/K)$ for some $1\leqslant s<2$. Then $\hat{f}(\lambda)$ may be extended to a function analytic in the strip $\mathfrak{e}_s=\{\lambda\,|\,|\mathrm{Im}\,\lambda|<((2/s)-1)\varrho\}$. If f is in L^1, $\hat{f}$ is continuous on the closure of $\mathfrak{e}_1$.*

We shall establish a partial converse to Theorem 4.4.

PROPOSITION 4.5. *Let $p(\lambda)$ be an even, Weyl-group invariant function analytic on the strip $\mathfrak{e}_1$. Assume that*

$$|D^\alpha_\sigma p(\sigma+i\tau)|\leqslant c_{\alpha,\tau}(1+|\sigma|)^{-\alpha}\quad\textit{for } 0\leqslant\alpha\leqslant N\tag{4.6}$$

and all $\sigma+i\tau$ in $\mathfrak{e}_1$. Then $\check{p}(g)\,(1-\psi(g))$ is in $L^s(G/K)$ for all s, $1<s<2$.

Remarks.

1. The first result of this kind was proved by Clerc and Stein [3], who established it for symmetric spaces G/K with G complex. Many of the techniques employed below originated in the work of Clerc and Stein.

2. A simple modification of the proof below allows one to show that if p satisfies the estimate (4.6) in the strip $\mathfrak{e}_{s_0}$, then $\check{p}(1-\psi)$ is in L^s for $s_0 < s < 2$.

Proof of Proposition 4.5. We shall show that for every ε with $0<\varepsilon<1$, there exists a constant c_ε and a function $K_\varepsilon(t)$ such that

$$|\check{p}(\exp tH_0)\,(1-\psi(\exp tH_0))| \leqslant c_\varepsilon e^{-(1+\varepsilon)\varrho t}(1+K_\varepsilon(t)) \tag{4.7}$$

where $\int_0^\infty |K_\varepsilon(t)|^2 dt < \infty$. Assuming these results, we choose $s>1$ and compute

$$\|\check{p}(1-\psi)\|_s \leqslant c_\varepsilon\left(\int e^{-(1+\varepsilon)s\varrho t}|D(t)|\,dt\right)^{1/s} + c_\varepsilon\left(\int_0^\infty e^{-(1+\varepsilon)s\varrho t}|K_\varepsilon(t)|^s\,|D(t)|\,dt\right)^{1/s}.$$

We estimate $|D(t)| \leqslant ce^{2\varrho t}$; the first integral may be made finite by choosing $\varepsilon > (2/s)-1$; the second may be estimated using Hölder's inequality. The proposition therefore follows from (4.7).

To establish (4.7), we note that $\check{p}\,(\exp tH_0) = \int_0^\infty p(\lambda)\varphi_\lambda(g)|c(\lambda)|^{-2}d\lambda$; as p is an even function we may use (3.1) to write this as $\int_{-\infty}^\infty p(\lambda)c^{-1}(-\lambda)e^{(i\lambda-\varrho)t}\phi_\lambda(t)\,d\lambda$. When $t \geqslant R_0^{1/2} > 1$, the expansion $\phi_\lambda(t) = \sum \Gamma_k(\lambda)e^{-2kt}$ converges uniformly, and as p is rapidly decreasing (see the first remark to Proposition 4.1) we may interchange sum and integral. Then

$$|(1-\psi)\check{p}| \leqslant (1-\psi)\sum_{k=0}^{\infty}\left|\int_{-\infty}^{\infty} p(\lambda)c^{-1}(-\lambda)\Gamma_k(\lambda)\,e^{(i\lambda-\varrho)t}\,d\lambda\right|e^{-2kt}.$$

The integrand of each term in this sum is holomorphic in the strip $0 \leqslant \operatorname{Im}\lambda < \varrho$; we may change contours of integration, $\lambda + i0 \to \lambda + i\varepsilon\varrho$, for any ε with $0<\varepsilon<1$. Then $|(1-\psi)p|$ is bounded by

$$(1-\psi)\,e^{-(1+\varepsilon)\varrho t}\sum_{k=0}^{\infty}\left|\int_{-\infty}^{\infty} e^{i\lambda t}p(\lambda+i\varepsilon\varrho)\,c^{-1}(-\lambda-i\varepsilon\varrho)\,\Gamma_k(\lambda+i\varepsilon\varrho)\,d\lambda\right|e^{-2kt}.$$

Let $q(\lambda,\varepsilon) = p(\lambda+i\varepsilon\varrho)c^{-1}(-\lambda-i\varepsilon\varrho)$. To establish (4.7) it suffices to prove that

$$\sum_{k=0}^{\infty}\left|\int_{-\infty}^{\infty} e^{i\lambda t}q(\lambda,\varepsilon)\,\Gamma_k(\lambda+i\varepsilon\varrho)\,d\lambda\right|e^{-2kt} \leqslant c_\varepsilon + K_\varepsilon(t). \tag{4.8}$$

Let $\Phi(\lambda)$ be a smooth even function on $\mathbf{R}^1$ with $0 \leqslant \Phi \leqslant 1$; $\Phi(\lambda)=1$ when $|\lambda|>2$; $\Phi(\lambda)=0$ when $|\lambda|<1$. Then

$$\left|\int e^{i\lambda t}(1-\Phi(\lambda))\,q(\lambda,\varepsilon)\,\Gamma_k(\lambda+i\varepsilon\varrho)\,d\lambda\right|\leqslant 2\sup_{|\lambda|<2}|q(\lambda,\varepsilon)\,\Gamma_k(\lambda+i\varepsilon\varrho)|\leqslant c_\varepsilon k^d,$$

from Gangolli [5]. Terms of the above form therefore contribute $c_\varepsilon\sum k^d e^{-2kt}<c_\varepsilon$, and may therefore be ignored.

To estimate

$$\left|\int\Phi(\lambda)\,e^{i\lambda t}q(\lambda,\varepsilon)\,\Gamma_k(\lambda+i\varepsilon\varrho)\,d\lambda\right|, \tag{4.10}$$

we employ Theorem 3.2, with $M=N$. Then (4.10) is bounded by

$$\sum_{m=0}^{N}\left|\int\Phi(\lambda)\,e^{i\lambda t}\gamma_m^k(\lambda+i\varepsilon\varrho)\,q(\lambda,\varepsilon)\,d\lambda\right|$$
$$+\sup_{\lambda}|p(\lambda+i\varepsilon\varrho)|\int\Phi(\lambda)\,|E_{N+1}^k(\lambda)|\,|c(-\lambda-i\varepsilon\varrho)|^{-1}\,d\lambda.$$

As $|\lambda|>1$, we may use (3.2) and standard estimates on quotients of Γ-functions to estimate $|c(-\lambda-i\varepsilon\varrho)|^{-1}\leqslant c_\varepsilon|\lambda|^{(n-1)/2}$. Then (3.8) shows that the final term above may be bounded by

$$c_\varepsilon A\varrho^{N+1}e^{2k}\int_1^\infty|\lambda|^{-(N+1)}|\lambda|^{(n-1)/2}\,d\lambda\leqslant c_\varepsilon e^{2k},\quad\text{as } N>n/2.$$

Such terms therefore contribute at most

$$c_\varepsilon(1-\psi)\sum_{k=0}^{\infty}e^{2k(1-t)}\leqslant c_\varepsilon(1-\psi)\sum_{k=0}^{\infty}\exp\,(2k(1-R_0^{1/2}))\leqslant c_\varepsilon$$

to (4.10), and may therefore be ignored.

We now define

$$K_\varepsilon(t)=(1-\psi)\sum_{m=0}^{N}\sum_{k=0}^{\infty}e^{-2kt}\left|\int\Phi(\lambda)\,e^{i\lambda t}\,q(\lambda,\varepsilon)\,\gamma_m^k(\lambda+i\varepsilon\varrho)\,d\lambda\right|.$$

Let $f_m^k(\lambda)=\Phi(\lambda)\,q(\lambda,\varepsilon)\gamma_m^k(\lambda+i\varepsilon\varrho)$. Then

$$\begin{aligned}\left(\int|K_\varepsilon(t)|^2\,dt\right)^{1/2}&\leqslant\sum_{m=0}^{N}\sum_{k=0}^{\infty}\left(\int_{R_0^{1/2}}^{\infty}e^{-4kt}\left|\int e^{i\lambda t}f_m^k(\lambda)\,d\lambda\right|^2dt\right)^{1/2}\\&\leqslant R_0^{-N/2}\sum_{m=0}^{N}\sum_{k=0}^{\infty}\exp\,(-2kR_0^{1/2})\left(\int_{-\infty}^{\infty}t^{2N}\left|\int e^{i\lambda t}f_m^k(\lambda)\,d\lambda\right|^2dt\right)^{1/2}\\&=R_0^{-N/2}\sum_{m=0}^{N}\sum_{k=0}^{\infty}\exp\,(-2kR_0^{1/2})\left(\int_{-\infty}^{\infty}\left|\int e^{i\lambda t}D_\lambda^N f_m^k(\lambda)\,d\lambda\right|^2dt\right)^{1/2}\\&=c\sum_{m=0}^{N}\sum_{k=0}^{\infty}\exp\,(-2kR_0^{1/2})\left(\int_{-\infty}^{\infty}|D_\lambda^N f_m^k(\lambda)|^2\,d\lambda\right)^{1/2}.\end{aligned}\tag{4.11}$$

This last equality holds by the Plancherel theorem for $\mathbf{R}^1$.

To estimate (4.11), we note that $D^N f_m^k$ may be expressed as a sum of terms $D^\alpha(p)\, D^\beta(\Phi\cdot c^{-1})\, D^\delta(\gamma_m^k)$ where $\alpha+\beta+\delta=N$. We employ the estimates (3.6), (3.7) and the hypotheses (4.6), as well as obvious estimates on Γ functions, to show

$$|D^\alpha(p)\, D^\beta(\Phi\cdot c^{-1})\, D^\delta(\gamma_m^k)| \leqslant c_\varepsilon |\lambda|^{-\alpha} |\lambda|^{((n-1)/2)-\delta} e^{2k} |\lambda|^{-m-\beta}$$

when $|\lambda|>1$; for $\lambda<1$, $f_m^k=0$. Then

$$\left(\int |K_\varepsilon(t)|^2 dt\right)^{1/2} \leqslant c_\varepsilon \sum_{m=0}^{N} \sum_{k=0}^{\infty} \exp\,(-2kR_0^{1/2})\, e^{2k} \left(\int_1^\infty |\lambda|^{n-1-2N} d\lambda\right)^{1/2}$$
$$\leqslant c_\varepsilon \left(\int_1^\infty |\lambda|^{n-1-2N} d\lambda\right)^{1/2}.$$

As N is an integer greater than $n/2$, this integral is finite, and the proof of the proposition complete.

Section 5

This section is devoted to the proof of

THEOREM 5.1. *Let $p(\lambda)$ be an even, Weyl-group invariant function holomorphic in the region $\{\lambda \mid |\operatorname{Im}\lambda| < \varrho\}$, and satisfying in this region the estimates*

$$|D_\sigma^\alpha p(\sigma+i\tau)| \leqslant c_{\alpha,\tau}(1+|\sigma|)^{-\alpha} \quad \text{for } 0\leqslant\alpha\leqslant N.$$

Then p is a multiplier of $L^s(G/K)$ for $1<s<\infty$.

Remark. The first results of this kind were established for the group S^1 by Marcinkiewicz [10]. The first results established for non-compact symmetric spaces G/K were those of Clerc and Stein [3], who considered the case of complex G. Several of the techniques employed below originated in [3].

Proof of Theorem 5.1. Let $k_1(g)=\check{p}(g)\psi(g)$, and $k_2(g)=\check{p}(g)(1-\psi(g))$. We shall show that $\|k_i * f\|_s \leqslant c_s\|f\|_s$ for $1<s<\infty$ and $i=1, 2$. We examine first k_1. In the previous section, we showed that k_1 behaves like the Euclidean inverse Fourier transform of p; we now relate convolution with k_1 to an Euclidean convolution.

LEMMA 5.2. *Let k be a compactly supported K bi-invariant function. If convolution with $D(t)\,k\,(\exp tH_0)$ is a bounded operator on $L^s(\mathbf{R}^1)$, then convolution with k is a bounded operator on $L^s(G/K)$.*

Proof. This result is due to Coifman and Weiss [4].

It therefore suffices to prove

LEMMA 5.3. *There are functions $k_0(t)$, $\varepsilon_0(t)$ such that*

$$D(t)k_1(\exp tH_0) = k_0(t) + \varepsilon_0(t),$$

where $\varepsilon_0(t)$ is in $L^1(\mathbf{R}^1)$ and k_0 satisfies

$$\left| D_y^\alpha \int_{-\infty}^{\infty} e^{-2\pi i x y} k_0(x)\, dx \right| \leqslant c_\alpha (1+|y|)^{-\alpha} \quad \alpha = 0, 1. \tag{5.1}$$

Therefore, k_0 satisfies the conditions of the Marcinkiewicz multiplier theorem, and convolution with $D(t)k_1(\exp tH_0)$ is a bounded operator on $L^s(\mathbf{R}^1)$, $1 < s < \infty$.

Proof. We shall choose Φ as in the proof of Proposition 4.5. Then

$$k_1(\exp tH_0) = \psi(\exp tH_0) \int \Phi(\lambda)\varphi_\lambda(\exp tH_0)\, p(\lambda)\, |c(\lambda)|^{-2}\, d\lambda$$
$$+ \psi(\exp tH_0) \int (1-\Phi(\lambda))\, \varphi_\lambda(\exp tH_0)\, p(\lambda)\, |c(\lambda)|^{-2}\, d\lambda.$$

The second term is bounded by $\psi \cdot \int_0^2 |p(\lambda)|\,|c(\lambda)|^{-2} d\lambda \leqslant c\psi$; ψ is bounded and compactly supported, and therefore in $L^1(G/K)$; we may henceforth ignore the second term. To treat the first term, we note that $\Phi(\lambda)p(\lambda)$ satisfies the hypotheses of Proposition 4.1; we choose

$$k_0(t) = c_0\, \psi(\exp tH_0)\, D(t) \left(\frac{t^{n-1}}{D(t)}\right)^{1/2} \int \Phi(\lambda)\, p(\lambda)\, \mathfrak{J}_{(n-2)/2}(\lambda t)\, |c(\lambda)|^{-2}\, d\lambda$$

and

$$\varepsilon_0(t) = \psi\,(\exp tH_0)\, D(t)\, e_0(t).$$

Then

$$\|\varepsilon_0\|_{1,\mathbf{R}^1} \leqslant \int |e_0(t)|\,|D(t)|\, dt = \|e_0\|_{1,G/K}.$$

To show that k_0 satisfies (5.1), we shall consider separately the cases n odd and n even.

When n is odd, we may write $\mathfrak{J}_{(n-2)/2}(z) = c(z^{-1}D_z)^{(n-1)/2}(\cos z)$. After $(n-1)/2$ integrations by parts, we see that it suffices to prove $(D_\lambda \cdot 1/\lambda)^{(n-1)/2}(\Phi(\lambda)p(\lambda)|c(\lambda)|^{-2})$ satisfies (5.1), which follows immediately from the estimates (4.3) on $|c(\lambda)|^{-2}$ and the hypotheses on p.

When n is even, we may write

$$\mathfrak{J}_{(n-2)/2}(z) = c(z^{-1}D_z)^{(n-2)/2}\, \mathfrak{J}_0(z),$$

and

$$\mathfrak{J}_0(\lambda t) = \frac{2}{\pi} \int_\lambda^\infty (\mu^2 - \lambda^2)^{-1/2} \sin \mu t\, d\mu$$

(see Watson [13], p. 180). Let $q(\lambda)=(D_\lambda\cdot(1/\lambda))^{(n-2)/2}(\Phi(\lambda)p(\lambda)|c(\lambda)|^{-2})$; then

$$\begin{aligned}k_0(\exp tH_0)&=c\psi(\exp tH_0)\left[\frac{D(t)}{t^{n-1}}\right]^{1/2}t\int q(\lambda)\,\mathcal{J}_0(\lambda t)\,d\lambda\\&=c\psi(\exp tH_0)\left[\frac{D(t)}{t^{n-1}}\right]^{1/2}t\int\sin\mu t\int_0^\mu q(\lambda)\,(\mu^2-\lambda^2)^{-1/2}\,d\lambda\,d\mu\\&=c\psi(\exp tH_0)\left[\frac{D(t)}{t^{n-1}}\right]^{1/2}\int\cos\mu t\frac{d}{d\mu}\int_0^\mu q(\lambda)\,(\mu^2-\lambda^2)^{-1/2}\,d\lambda\,d\mu.\end{aligned}$$

To establish (5.1) it then suffices to show that $(d/d\mu)\int_0^\mu q(\lambda)(\mu^2-\lambda^2)^{-1/2}d\lambda$ satisfies (5.1); this is again a straightforward computation.

To complete the proof of Theorem 5.1, we must show that $\|k_2*f\|_s\leqslant c_s\|f\|_s$ for $1<s<\infty$. The appropriate substitute for Euclidean techniques is the following result of Clerc and Stein [3].

LEMMA 5.4. *Let k be a K bi-invariant function in $L^r(G/K$ for all r satisfying $1<r<1+\delta$, where $\delta>0$. Then $\|k*f\|_s\leqslant c_s\|f\|_s$ for $1<s<\infty$.*

To prove Theorem 5.1, we note that Proposition 4.5 shows k_2 to be in all L^r with $1<r<2$; an application of Lemma 5.4 completes the proof of the theorem.

Section 6

Multiplier theorems such as Theorem 5.1 find application in estimating the L^p behavior of differential operators on G/K. Let ω be the radial part of the Laplace-Beltrami operator on G/K; then $\omega\varphi_\lambda=-(\lambda^2+\varrho^2)\varphi_\lambda$. Define $m_\alpha(\lambda)=(\lambda^2+\varrho^2)^{-\alpha/2}$, and define a bi-invariant distribution k_α on G/K by $\hat{k}_\alpha=m_\alpha$. If f is a good bi-invariant function, $k_2*\omega f=\omega(k_2*f)=-f$. On K bi-invariant functions, the k_α behave like fractional integration kernels, $k_\alpha*-=(-\omega)^{-\alpha/2}$. From the results in sections 1–5, we should expect that the local behavior of the k_α is the same as that of fractional integration for the Laplacian on $\mathbf{R}^n$; we should also expect that the global behavior of the k_α has no Euclidean analogue. We shall prove:

THEOREM 6.1. *Fix $\alpha>0$. Then*

$$\|k_\alpha*f\|_q\leqslant c\|f\|_p \tag{6.1}$$

for all f in $L^p(G/K)$ if and only if $p=q$ and $1<p<\infty$, or $p<q$ and one of the following conditions hold:

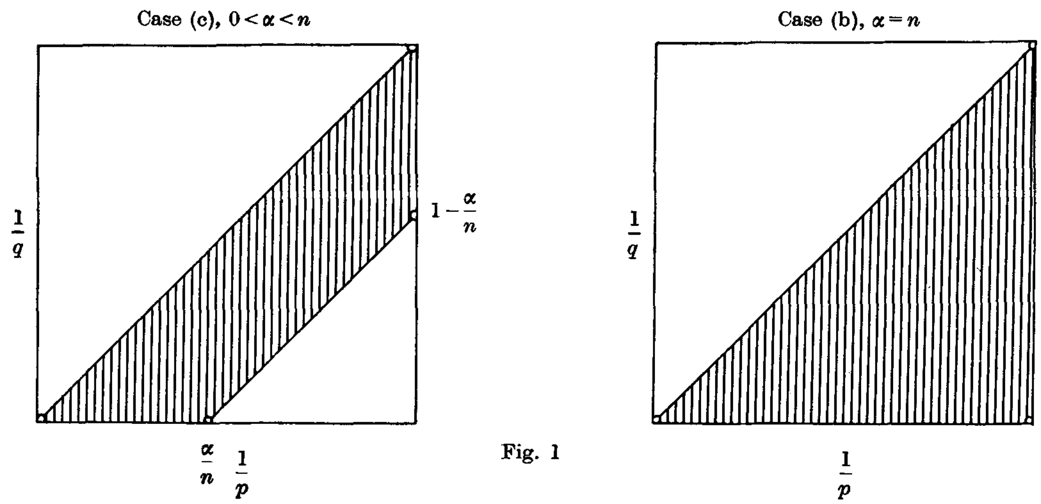

Fig. 1

(a) $\alpha>n$

(b) $\alpha=n$ *and* $q<\infty$

(c) $0<\alpha<n$ *and one of the following holds*

  (i) $p>n/\alpha$

  (ii) $1<p<n/\alpha$ *and* $1/p-\alpha/n\leqslant 1/q$

  (iii) $p=1$ *and* $1-\alpha/n<1/q<1$.

Remarks. 1. Theorem 6.1 may best be understood through reference to Figure 1. Open circles and open areas represent points $(1/p,\ 1/q)$ for which (6.1) does not hold; hatched areas and straight lines represent points for which it does.

2. Set $k_\alpha=f_\alpha+g_\alpha$, where $f_\alpha(g)=k_\alpha(g)\psi(g)$. We shall first prove

LEMMA 6.2.

(I) g_α *is in* L^p *if and only if* $1<p\leqslant\infty$.

(II) *When* $\alpha>n$, f_α *is in* L^p *when* $1\leqslant p\leqslant\infty$.

(III) *When* $\alpha=n$, $c_1\leqslant|f_\alpha(\exp tH_0)/\log t|\leqslant c_2$, *and* f_α *is in* L^p *if and only if* $1\leqslant p<\infty$.

(IV) *When* $0<\alpha<n$, $c_1\leqslant|f_\alpha(\exp tH_0)/t^{\alpha-n}|\leqslant c_2$ *and* f_α *is in* L^p *if and only if* $1\leqslant p<n/(n-\alpha)$.

Proof. To prove (I), we note that $g_\alpha\in L^1$ implies that $\hat{g}_\alpha$ is continuous on the closure of $\mathfrak{e}_1$. But (IV) shows that $f_\alpha\in L^1$; therefore $g_\alpha\in L^1$ implies $\hat{k}_\alpha$ is continuous on $\bar{\mathfrak{e}}_1$, which is manifestly false.

To establish the remainder of part (I), note that m_α satisfies the hypotheses of Proposition 4.5, and therefore g_α is in L^p for $1<p<2$. If suffices to prove, then, that g_α is in L^∞. This follows immediately from Corollary 3.9 and n integrations by parts.

Part II is equally simple. When $\alpha>n$, m_α is in $L^1(\mathfrak{a}'_+, |c(\lambda)|^{-2})$; f_α is therefore a bounded compactly supported function, which is in all L^p classes.

To establish the estimates of parts (III) and (IV), we apply Corollary 4.3 to the function $p(\lambda)=\Phi(\lambda)m_\alpha(\lambda)$. Then $f_\alpha=\check{p}\psi+$bounded terms. As we wish to show that k_α has a singularity near $t=0$, we may ignore any bounded terms. Equation (4.5) then shows that the main singularity of f_α near $t=0$ comes from $\int m_\alpha(\lambda)\Phi(\lambda)\mathcal{J}_{(n-2)/2}(\lambda t)|c(\lambda)|^{-2}d\lambda$. For $|\lambda|>2$, the measure $|c(\lambda)|^{-2}$ behaves like λ^{n-1}, therefore

$$f_\alpha(\exp tH_0)\sim\int\mathcal{J}_{(n-2)/2}(\lambda t)\,(\lambda^2+\varrho^2)^{-\alpha/2}\,\lambda^{n-1}\,d\lambda=ct^{(\alpha-n)/2}\,K_{(\alpha-n)/2}(t),$$

where K_μ is a Bessel function of the third kind; the estimates (III)–(IV) for such functions are classical; see [1].

Proof of Theorem 6.1. The theorem follows from Lemma 6.2 and standard convolution arguments. When $p=q$, the positive results follow from Theorem 5.1. The k_α fail to be bounded on L^1 or L^∞, because the multipliers of L^1 or L^∞ are functions continous on the closure of $\mathfrak{e}_1$.

When $p\neq q$, we must have $p<q$; this is a necessary condition for any translation-invariant operator to be bounded from L^p to L^q when the object G/K is noncompact (see Hörmander [8]).

When $\alpha>n$, we see from parts I and II of Lemma 6.2 that the k_α are in L^p for $1<p\leqslant\infty$. Therefore $\|k_\alpha*f\|_q\leqslant\|k_\alpha\|_q\|f\|_1$ and, dually, $\|k_\alpha*f\|_\infty\leqslant\|f\|_p\|k_\alpha\|_{p'}$. An application of the Riesz-Thorin interpolation theorem to these two results yields part (a) of the theorem.

When $\alpha=n$, k_α is in all L^p classes but L^∞, and all the above arguments are valid but for the estimate $\|k_n*f\|_\infty\leqslant\|f\|_1\|k_n\|_\infty$. It is easy to see this is false, if we choose f to be the δ function (to be precise, we choose a sequence of L^1 functions which approximate the δ function).

When $\alpha<n$, we use the decomposition $k_\alpha=f_\alpha+g_\alpha$. As g_α is in all L^p classes for $1<p\leqslant\infty$, the above arguments show that $\|g_\alpha*f\|_q\leqslant c\|f\|_p$ whenever $p<q$; the boundedness of $k_\alpha*-$ is therefore completely determined by that of $f_\alpha*-$. To analyze this operator, we note that $\|f_\alpha*f\|_p\leqslant\|f_\alpha\|_p\|f\|_1$ and $\|f_\alpha*f\|_\infty\leqslant\|f_\alpha\|_p\|f\|_{p'}$; we may apply the Riesz-Thorin interpolation theorem to these estimates. When $p=n/(n-\alpha)$, f_α is not in L^p, but $f_\alpha*-$ is weakly bounded from L^1 to L^p and $L^{p'}$ to L^∞; to this we may apply the Marcinkiewicz interpolation theorem. This yields the positive results of part (c) of the theorem.

The negative results of part (c) of the theorem are equally simple to prove. The estimates $\|f_\alpha*f\|_p\leqslant c\|f\|_1$ and $\|f_\alpha*f\|_\infty\leqslant c\|f\|_{p'}$ fail for $p\geqslant n/(n-\alpha)$, as may be seen by choosing f to be a δ-function. When $p \neq 1$ or $q \neq \infty$, we may use the relationship $k_\alpha * k_\beta = k_{\alpha+\beta}$. For $\|k_\alpha * k_\beta\|_q \leq c\|k_\beta\|_p$ to hold for some pair p and q, and some $\alpha < n$, all $\beta > n/p'$, part (IV) of Lemma 6.2 shows that α, p and q must be related; a computation exhibits this relationship as part (c) (ii) of the theorem. This completes the proof of Theorem 6.1.

The multipliers $|\lambda^2 + \varrho^2|^{-\alpha+it}$, corresponding to $(-\omega)^{-\alpha+it}$, also satisfy the hypotheses of Theorem 5.1, and presumably an analysis similar to that of Theorem 6.1 may be performed. We may use these oparators to define first order invariant "pseudo-differential" operators, such as $k_\beta \circ \omega$, whereas the only invariant differential operators on G/K are polynomials in ω. It would be of interest to know whether the class of "pseudo-differential" operators defined on $C_c^\infty(G/K)$ through the spherical Fourier transform, co-incides with the class of pseudo-differential operators on the manifold G/K, and, if so, what connection there is between the two different concepts of symbol.

References

[1]. Aronszajn, N. & Smith, K. T., Theory of Bessel potentials I. *Ann. Inst. Fourier,* 11 (1961), 385–475.

[2]. Erdelyi, A. et al., *Higher transcendental functions.* Vol. II, Mc Graw-Hill, New York, 1953.

[3]. Clerc, J. L. & Stein, E. M., L^p-multipliers for noncompact symmetric spaces. *Proc. Nat. Acad. Sci. U.S.A.*, 71 No. 10 (1974), 3911–3912.

[4]. Coifman, R. R. & Weiss, G., Lectures on multipliers. Proc. NSF Regional Conf., Lincoln, Neb. (1976).

[5]. Gangolli, R., On the Plancherel formula and the Paley-Wiener theorem for spherical functions on semisimple Lie groups. *Ann. of Math.*, 93 (1971), 150–165.

[6a]. Harish-Chandra, Spherical functions on a semisimple Lie group I. *Amer. J. Math.*, 80 (1958), 241–310.

[6b]. —— Spherical functions on a semisimple Lie group II. *Amer. J. Math.*, 80 (1958), 553–613.

[7]. Helgason, S., An analogue of the Paley-Wiener theorem for the Fourier transfor mon certain symmetric spaces. *Math. Ann.*, 165 (1966), 297–308.

[8]. Hörmander, L., Estimates for translation invariant operators in L^p spaces. *Acta Math.*, 104 (1960), 93–139

[9]. Koornwinder, T., A new proof of a Paley-Wiener type theorem for the Jacobi transform. *Ark. Mat.*, 13 (1975), 145–159.

[10]. Marcinkiewicz, J., Sur les multiplicateurs des séries de Fourier. *Studia Math.*, 8 (1939), 78–91.

[11]. Schindler, S., Some transplantation theorems for the generalized Mehler transform and related asymptotic expansions. *Trans. Amer. Math. Soc.*, 155 (1971), 257–291.

[12]. Szegö, G., Über einige asymptotische Entwicklungen der Legendreschen Functionen. *Proc. London Math. Soc.* (2), 36 (1932), 427–450.

[13]. Watson, G. N., *A treatise on the theory of Bessel functions.* Cambridge Univ. Press Cambridge, 2nd edition, 1944.

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