ON IVERSEN'S THEOREM FOR MEROMORPHIC FUNCTIONS WITH FEW POLES

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1. Introduction

Suppose that f is a nonconstant entire function. Then a classical theorem of Iversen [10] asserts that $f(z)$ has ∞ as an asymptotic value. In other words there exists a path Γ going from a finite point z_0 to ∞ in the complex plane such that

$$f(z) \to \infty, \quad \text{as } z \to \infty \quad \text{along } \Gamma. \tag{1.1}$$

It is natural to ask whether this result still holds if f has few poles in a suitable sense. Suppose first that f is meromorphic, transcendental and has only a finite number of poles in the open plane. Then

$$f(z) = F(z) + R(z),$$

where $R(z)$ is the sum of the principal parts of $f(z)$ at the poles and $F(z)$ is an entire function which is also transcendental. Thus (1.1) holds for $F(z)$ and so for $f(z)$.

If we ask for stronger results than this, positive theorems become scanty without extra hypotheses. The following theorem is due to Anderson & Clunie [1]

THEOREM A. *Suppose that $f(z)$ is meromorphic and such that*

$$T(r, f) = O\,(\log r)^2 \quad \textit{as } r \to \infty, \tag{1.2}$$

and further that ∞ is deficient in the sense of Nevanlinna, i.e.(1)

$$1 - \delta(\infty) = \overline{\lim_{r\to\infty}} \frac{N(r, \infty)}{T(r, f)} < 1, \tag{1.3}$$

Then ∞ is an asymptotic value of $f(z)$.

(1) We use the standard notation of Nevanlinna Theory. See e.g. [7, Chapter 1].

More strongly Anderson & Clunie proved that under the hypotheses of Theorem A (1.1) holds for almost all rays Γ through the origin.

It is natural to ask whether the condition (1.2) can be weakened. Ter-Israelyan [14] has given examples to show that the conclusion of Theorem A is false in general if we assume merely that $f(z)$ has order zero, instead of (1.2). Gol'dberg & Ostrovskii [5, p. 245] give examples of functions of $f(z)$, such that

$$N(r, \infty) = O(r^k) \tag{1.4}$$

and

$$\varliminf_{r\to\infty} \frac{T(r,f)}{r^\lambda} > 0, \tag{1.5}$$

where $\frac{1}{2} < k < \lambda$, and such that ∞ is not asymptotic.

2. Statement of results

In this paper we prove the following two theorems.

THEOREM 1. *Given any function* $\psi(r)$, *such that*

$$\psi(r) \to \infty, \quad as\ r \to \infty, \tag{2.1}$$

there exists $f(z)$ *meromorphic and not constant in the plane, such that*

$$T(r, f) < \psi(r)\,(\log r)^2, \tag{2.2}$$

for all sufficiently large r *and*

$$\delta(\infty, f) = 1, \tag{2.3}$$

but such that ∞ *is not an asymptotic value of* $f(z)$.

Thus ∞ is deficient, even with deficiency one and $f(z)$ only just exceeds the growth prescribed by (1.2), but the conclusion of Theorem A fails. Theorem 1 sharpens the examples of Ter-Israelyan [14] and shows that Theorem A is essentially best possible.

It turns out that the behaviour of the functions of Theorem 1 is essentially associated with irregular growth. We can show that functions satisfying (1.4) and (1.5) where $k < \inf(\lambda, \frac{1}{2})$ do indeed have ∞ as an asymptotic value. More precisely we prove

THEOREM 2. *Suppose that $f(z)$ is meromorphic and not constant in the plane and that for some a in the closed plane*

$$\lim_{r\to\infty}\left\{T(r,f)-\tfrac{1}{2}r^{1/2}\int_r^\infty \frac{N(t,a)\,dt}{t^{3/2}}\right\}=+\infty. \tag{2.4}$$

Then a is an asymptotic value of $f(z)$.

COROLLARY 1. *Suppose that for some $K<\infty$, we have*

$$\overline{\lim_{r\to\infty}}\,\frac{r^{1/2}}{2T(r,f)}\int_r^\infty \frac{T(t,f)\,dt}{t^{3/2}}=K<\infty. \tag{2.5}$$

Then if $\delta(a,f)>1-K^{-1}$, a is an asymptotic value of f.

COROLLARY 2. *If*

$$\lim_{r\to\infty}\frac{T(2r,f)}{T(r,f)}=1,$$

then any deficient value of f is asymptotic. In particular the conclusion holds under the hypothesis (1.2).

We shall see that under the hypothesis (1.2) a significantly weaker condition than deficiency suffices to make a asymptotic.

COROLLARY 3. *If f has very regular*(1) *growth of order λ, where $0<\lambda<\frac{1}{2}$, and $\delta(a,f)=1$, then a is asymptotic. If f has perfectly regular*(1) *growth the conclusion holds for $\delta(a,f)>2\lambda$.*

COROLLARY 4. *If for some λ, such that $0<\lambda<\frac{1}{2}$, we have*

$$0\leqslant\overline{\lim_{r\to\infty}}\,\frac{N(r,a)}{r^\lambda}<(1-2\lambda)\,\underline{\lim_{r\to\infty}}\,\frac{T(r,f)}{r^\lambda}\leqslant\infty,$$

then a is asymptotic.

The corollaries are all almost immediate deductions of the main theorem. Collingwood [4] and Nevanlinna [12, p. 259] conjectured that deficient values might be asymptotic. The first counterexample was given by H. Laurent-Schwarz [11]. However, Theorem 1, Corollary 2, shows that the result is true for functions of order zero and smooth growth. This result also contains Theorem A as a special case, except that the asymptotic path Γ need no longer be a ray in this case. An example of this will be given in Theorem 7. Corollary 4 gives a positive answer to problem (2.8) of [8]. The question was asked whether (1.4) and (1.5) imply that ∞ is asymptotic if $k<\frac{1}{2}$, and Corollary 4 shows that this is so.

(1) These concepts are due to Valiron [15].

3. Some examples

In order to prove Theorem 1, we shall construct meromorphic functions as products. In order to discuss the factors we need the following simple

LEMMA 1. *If k is an integer, $k > 20$ and*

$$P_k(z) = \frac{1+\{2(z+1)/3\}^k}{\{1+(2/3)^k\}(z+1)},$$

then

(i) $P_k(0) = 1$,

(ii) $N(r, \infty, P_k) \leqslant \dfrac{20}{k} N(r, 0, P_k), \quad 0 < r < \infty$

(iii) $|P_k(z)| < \frac{9}{10}, \quad$ for $|z+1| = \frac{5}{4}$.

The first result is obvious. Next we note that $P_k(z)$ has simple zeros where

$$\frac{2(z+1)}{3} = e^{(2\nu+1)\pi i/k}, \quad \nu = 0, \mp 1, \mp 2, \ldots$$

We write $\theta_\nu = (2\nu+1)\pi/k$, so that the zeros occur at

$$z_\nu = \tfrac{3}{2} e^{i\theta_\nu} - 1.$$

We note that

$$|z_\nu| = |\tfrac{1}{2} + \tfrac{3}{2}(e^{i\theta_\nu} - 1)| \leqslant \tfrac{1}{2} + \tfrac{3}{2}|\theta_\nu| < 1$$

if $|\theta_\nu| < \frac{1}{3}$, i.e. if $|2\nu+1| < k/(3\pi)$. The interval $|x| < k/(3\pi)$ contains at least $k/(3\pi) - 1$ distinct odd integers $2\nu+1$, i.e. at least $k/20$, for $k \geqslant 20$. Thus $n(r, 0) \geqslant k/20$ for $r \geqslant 1$. Since $n(r, \infty) = 0$ for $r < 1$, $n(r, \infty) = 1$ for $r \geqslant 1$, we deduce that for $P_k(z)$

$$n(t, \infty) \leqslant \frac{20}{k} n(t, 0), \quad 0 < t < \infty.$$

On dividing by t and integrating from $t = 0$ to r, we deduce (ii).

It remains to prove (iii). We note that for $|z+1| = \frac{5}{4}$, we have, since $k \leqslant 20$,

$$|P_k(z)| \leqslant \frac{1+(\frac{2}{3}\cdot\frac{5}{4})^k}{\{1+(\frac{2}{3})^k\}(\frac{5}{4})} < \tfrac{4}{5}\{1+(\tfrac{5}{6})^{20}\} < \tfrac{4}{5}(1+\tfrac{1}{8}) = \tfrac{9}{10}.$$

This proves (iii) and completes the proof of Lemma 1.

We now choose sequences of positive numbers r_ν and of positive integers k_ν, and q_ν, and set

$$P(z)=\prod_{\nu=1}^{\infty} P_{k_\nu}\left(\frac{z}{r_\nu}\right)^{q_\nu}. \tag{3.1}$$

We shall see that if the above sequences are suitably defined the product $P(z)$ converges to a meromorphic function for which ∞ is a deficient but not an asymptotic value. We divide the proof into a number of steps.

3.1 Subsidiary results

Let r_ν be positive numbers and q_ν, k_ν positive integers, for $\nu \geqslant 1$, which satisfy the following conditions

$$q_1=r_1=1, \tag{3.2}$$

and for $\nu \geqslant 1$

$$\frac{r_{\nu+1}}{r_\nu} \geqslant 10^7, \tag{3.3}$$

$$20<k_\nu<\frac{1}{20\,000}\frac{r_{\nu+1}}{r_\nu}\Big/\left(\log\frac{r_{\nu+1}}{r_\nu}\right). \tag{3.4}$$

$$100k_\nu \log\left(\frac{r_{\nu+1}}{r_\nu}\right)<\frac{q_{\nu+1}}{q_\nu}<200k_\nu \log\left(\frac{r_{\nu+1}}{r_\nu}\right). \tag{3.5}$$

In view of (3.3) the last term in (3.4) is greater than 30. We assume that q_ν, r_ν and $k_{\nu-1}$ have already been chosen. Then if $r_{\nu+1}$ is chosen to satisfy (3.3) a choice of k_ν is possible to satisfy (3.4), and then a choice of $q_{\nu+1}$ is clearly possible to satisfy (3.5). Thus q_ν, r_ν and k_ν can be chosen inductively to satisfy the above conditions. We shall show that in this case the product $P(z)$ has the required properties.

Before proceeding we need some inequalities.

LEMMA 2. *If k_ν, q_ν, r_ν satisfy the above conditions then we have*

$$\sum_{\nu>\mu}\frac{q_\nu r_\mu}{r_\nu}<\frac{100}{99}\frac{q_{\mu+1}r_\mu}{r_{\mu+1}}<\frac{q_\mu}{99}, \tag{3.6}$$

and

$$\sum_{\nu<\mu} k_\nu q_\nu \log\left(\frac{r_\mu}{r_\nu}\right)<\frac{q_\mu}{99}. \tag{3.7}$$

In fact we deduce from (3.5) and (3.4) that

$$\frac{q_{\nu+1}}{q_\nu}<\frac{1}{100}\frac{r_{\nu+1}}{r_\nu},\quad \text{i.e.}\quad \frac{q_{\nu+1}}{r_{\nu+1}}<\frac{1}{100}\frac{q_\nu}{r_\nu}.$$

Thus

$$\sum_{\nu=\mu+1}^{\infty} \frac{q_\nu}{r_\nu} < \frac{q_{\mu+1}}{r_{\mu+1}} \sum_{\nu=\mu+1}^{\infty} (100)^{\mu-\nu+1} = \frac{100}{99} \frac{q_{\mu+1}}{r_{\mu+1}} < \frac{1}{99} \frac{q_\mu}{r_\mu},$$

and this proves (3.6).

To prove (3.7) we write

$$\delta_\nu = \log\left(\frac{r_{\nu+1}}{r_\nu}\right).$$

Then (3.5) shows that

$$k_\nu q_\nu \delta_\nu < \frac{1}{100} q_{\nu+1}.$$

Thus for $\nu < \mu$ we have

$$\frac{q_\nu k_\nu}{q_\mu} < (100)^{\nu-\mu} \frac{1}{(\delta_\nu \delta_{\nu+1} \dots \delta_{\mu-1})(k_{\nu+1} \dots k_{\mu-1})}.$$

Also $\delta_\nu > 2$, in view of (3.3). Thus if

$$\delta = \max_{\nu \leqslant s \leqslant \mu-1} \delta_s,$$

we have

$$\delta_\nu \delta_{\nu+1} \dots \delta_{\mu-1} \geqslant 2^{\mu-\nu-1} \delta \geqslant (\mu-\nu)\delta \geqslant \sum_{s=\nu}^{\mu-1} \delta_s = \log(r_\mu / r_\nu).$$

Since the $k_{\nu+1}$ are all greater than one we deduce that

$$\frac{q_\nu k_\nu}{q_\mu} < \frac{(100)^{\nu-\mu}}{\log(r_\mu/r_\nu)}.$$

Thus

$$\sum_{\nu<\mu} q_\nu k_\nu \log(r_\mu/r_\nu) < \sum_{\nu=1}^{\mu-1} (100)^{\nu-\mu} q_\mu < \frac{q_\mu}{99}.$$

This proves (3.7) and completes the proof of Lemma 2.

We next need some more inequalities for $P_k(z)$

Lemma 3. *We have*

$$|\log P_k(z)| < 2|z|, \quad \textit{if } |z| \leqslant \tfrac{1}{6} \tag{3.8}$$

and

$$\left(\frac{|z|}{2}\right)^{k-1} < |P_k(z)| < |z|^{k-1}, \quad \textit{if } |z| \geqslant 6. \tag{3.9}$$

To prove the first inequality we write

$$P_k(z) = \frac{1+(\frac{2}{3})^k + (\frac{2}{3})^k\{(z+1)^k - 1\}}{\{1+(\frac{2}{3})^k\}(z+1)},$$

so that

$$P_k(z) - 1 = \frac{(\frac{2}{3})^k\{(z+1)^k - 1\} - (1+(\frac{2}{3})^k)z}{(1+(\frac{2}{3})^k)(z+1)} = \frac{1}{(z+1)}\left\{\frac{2^k}{3^k+2^k}\{(z+1)^k-1\} - z\right\}.$$

Suppose now that $|z| = \frac{1}{6}$. Then

$$|P_k(z) - 1| \leqslant \tfrac{6}{5}\{(\tfrac{2}{3})^k((\tfrac{7}{6})^k + 1) + \tfrac{1}{6}\} < \frac{6}{5}\frac{2^{k+1}}{3^k}\frac{7^k}{6^k} + \tfrac{1}{5} = \tfrac{12}{5}\cdot(\tfrac{7}{9})^k + \tfrac{1}{5} < \tfrac{1}{4},$$

since $k \geqslant 20$. Thus

$$|\log P_k(z)| = |\log\{1 + (P_k(z) - 1)\}| < \sum_{\nu=1}^{\infty} |P_k(z) - 1|^{\nu} < \sum_{\nu=1}^{\infty} 4^{-\nu} = \tfrac{1}{3}.$$

Hence in view of Schwarz's Lemma we deduce (3.8).

Next if $|z| \geqslant 6$, we have

$$\frac{5|z|}{6} \leqslant |z+1| \leqslant \frac{7|z|}{6}.$$

Thus

$$|P_k(z)| < \frac{2}{|z+1|}\left(\frac{2|z+1|}{3}\right)^k = \frac{2^{k+1}}{3^k}|z+1|^{k-1} \leqslant \frac{2^{k+1}7^{k-1}}{3^k 6^{k-1}}|z|^{k-1} = \tfrac{4}{3}\cdot(\tfrac{7}{9})^{k-1}|z|^{k-1} < |z|^{k-1}.$$

This proves the right hand inequality in (3.9). Similarly

$$\begin{aligned} |P_k(z)| &> \frac{1}{2(1+(\frac{2}{3})^k)}(\tfrac{2}{3})^k|z+1|^{k-1} \\ &> \frac{1}{4}\left(\frac{2}{3}\right)^{k-1}\left(\frac{5|z|}{6}\right)^{k-1} = \frac{1}{4}\left(\frac{5|z|}{9}\right)^{k-1} \\ &= \frac{1}{4}\cdot\left(\frac{10}{9}\right)^{k-1}\left(\frac{|z|}{2}\right)^{k-1} > \left(\frac{|z|}{2}\right)^{k-1}, \end{aligned}$$

since $k > 20$. This completes the proof of (3.9) and of Lemma 3.

3.2 Properties of $P(z)$

We are now able to prove that $P(z)$ satisfies the desired conditions. Because of its generality we state our result as

THEOREM 3. *Suppose that k_ν, q_ν, r_ν satisfy* (3.2) *to* (3.5) *for $\nu \geqslant 1$. Then the product $P(z)$, given by* (3.1), *converges locally uniformly in the plane to a meromorphic function with the following properties for $\mu \geqslant 1$.*

$$\log |P(z)| < -\frac{q_\mu}{30}, \quad \textit{if } |z + r_\mu| = \frac{5r_\mu}{4}. \tag{3.10}$$

$$\log |P(z)| > 20q_\mu, \quad \textit{if } |z| = 6r_\mu. \tag{3.11}$$

$$\tfrac{1}{3} q_\mu k_\mu \log \frac{r}{r_\mu} < T(r, P) < 100 k_\mu q_\mu \log \frac{r}{r_\mu} \quad \textit{if } 6r_\mu \leqslant r \leqslant \tfrac{1}{6} r_{\mu+1}. \tag{3.12}$$

Further

$$\delta(\infty, P) \geqslant 1 - 20 \varlimsup_{\nu \to \infty} (k_\nu^{-1}). \tag{3.13}$$

In particular $P(z)$ has no finite or infinite asymptotic values, but ∞ is a deficient value for $P(z)$.

Suppose first that

$$|z| = r \leqslant \tfrac{1}{6} r_{\mu+1}. \tag{3.14}$$

Then for $\nu > \mu$ we deduce from Lemma 3, (3.8) that

$$\left| q_\nu \log P_{k_\nu}\left(\frac{z}{r_\nu}\right) \right| < 2q_\nu \frac{r}{r_\nu}.$$

Thus in view of (3.6) we deduce that if

$$Q(z) = \prod_{\nu=\mu+1}^{\infty} \left\{ P_{k_\nu}\left(\frac{z}{r_\nu}\right) \right\}^{q_\nu},$$

then the product $Q(z)$ converges uniformly from $|z| \leqslant \frac{1}{6} r_{\mu+1}$ to a regular function without zeros which satisfies

$$|\log Q(z)| < 2r \sum_{\nu=\mu+1}^{\infty} \frac{q_\nu}{r_\nu} < \frac{200}{99} \frac{q_{\mu+1} r}{r_{\mu+1}} < \frac{2}{99} \frac{q_\mu r}{r_\mu}. \tag{3.15}$$

Thus $P(z)$ is meromorphic in the open plane.

Next suppose that

$$k_\nu \geqslant k, \quad \nu \geqslant \nu_0,$$

where k and ν_0 are taken as fixed. Then in view of Lemma 1, (ii) we have

$$N(r, \infty, P_{k_\nu}) \leqslant \frac{20}{k} N(r, 0, P_{k_\nu}), \quad \nu \geqslant \nu_0.$$

Thus we have as $r \to \infty$,

$$\begin{aligned} N(r,\infty,P(z)) &= \sum_{\nu=1}^{\mu} N(r,\infty,P_{k_\nu}) \\ &\leqslant \sum_{\nu=1}^{\nu_0} N(r,\infty,P_{k_\nu}) + \frac{20}{k}\sum_{\nu=1}^{\mu} N(r,0,P_{k_\nu}) \\ &\leqslant \frac{20}{k} N(r,0,P(z)) + O(\log r) \\ &\leqslant \frac{20}{k} T(r,P) + O(\log r). \end{aligned}$$

Evidently $P(z)$ has infinitely many poles and zeros so that

$$\frac{T(r,P)}{\log r} \to \infty, \quad \text{as } r \to \infty.$$

We deduce that

$$\overline{\lim_{r\to\infty}} \frac{N(r,\infty,P)}{T(r,P)} \leqslant \frac{20}{k},$$

i.e.

$$\delta(\infty,P) \geqslant 1 - \frac{20}{k}.$$

If $k_\nu \to \infty$, we may take k as large as we please and obtain $\delta(\infty,P)=1$. Otherwise we may take

$$k = \underline{\lim_{\nu\to\infty}}\, k_\nu, \quad \frac{1}{k} = \overline{\lim_{\nu\to\infty}}\, \frac{1}{k_\nu},$$

so that k is an integer and $k \geqslant 21$ in view of (3.4). Thus we again deduce (3.13). Thus (3.13) holds in every case and since

$$\delta(\infty,P) \geqslant \frac{1}{21},$$

$P(z)$ has ∞ as a deficient value.

Suppose next that

$$|z + r_\mu| = \frac{5r_\mu}{4},$$

so that

$$\frac{r_\mu}{4} \leqslant r \leqslant \frac{9r_\mu}{4}. \tag{3.16}$$

Thus (3.14) and so (3.15) holds. Using Lemma 1 (iii) we deduce that

$$\log |P(z)| < q_\mu \log \frac{9}{10} + \frac{2}{99} \frac{q_\mu r}{r_\mu} + \sum_{\nu=1}^{\mu-1} q_\nu \log \left| P_{k_\nu}\left(\frac{z}{r_\nu}\right)\right|.$$

Using (3.9) and (3.16) we deduce

$$\log |P(z)| < q_\mu\{\log \tfrac{9}{10} + \tfrac{1}{22}\} + \sum_{\nu=1}^{\mu-1} q_\nu(k_\nu - 1) \log \frac{r}{r_\nu}.$$

Also in view of (3.3) and (3.16)

$$\log \frac{r}{r_\nu} < \log \frac{r_\mu}{r_\nu} + 1 < \frac{11}{10} \log \frac{r_\mu}{r_\nu}, \quad \nu < \mu.$$

Thus (3.7) yields

$$\sum_{\nu=1}^{\mu-1} q_\nu(k_\nu - 1) \log \frac{r}{r_\nu} < \frac{11}{10} \cdot \frac{q_\mu}{99} = \frac{q_\mu}{90}.$$

We deduce that

$$\log |P(z)| < q_\mu\{\log \tfrac{9}{10} + \tfrac{1}{22} + \tfrac{1}{90}\} < -q_\mu\{\tfrac{1}{10} - \tfrac{1}{90} - \tfrac{1}{22}\} < -\frac{q_\mu}{30}.$$

This proves (3.10).

Next suppose that

$$6r_\mu \leqslant r \leqslant \tfrac{1}{6} r_{\mu+1}. \tag{3.17}$$

Then (3.15) and (3.9) yield

$$\log |P(z)| \geqslant \sum_{\nu=1}^{\mu} q_\nu(k_\nu - 1) \log \left(\frac{r}{2r_\nu}\right) - \frac{2}{99} \frac{q_\mu r}{r_\mu}$$

$$\geqslant q_\mu \left\{(k_\mu - 1) \log \left(\frac{r}{2r_\mu}\right) - \frac{2}{99} \frac{r}{r_\mu}\right\}.$$

Setting $r = 6r_\mu$, we deduce

$$\log |P(z)| \geqslant q_\mu \left\{20 \log 3 - \frac{12}{99}\right\} > 20 q_\mu,$$

which proves (3.11).

Since every path Γ going to ∞ meets both the circles $|z| = 6r_\mu$ and $|z + r_\mu| = \frac{5}{4} r_\mu$ for large μ, it follows from (3.10) and (3.11) that $\log |P(z)|$ is unbounded above and below on Γ, so that $P(z)$ cannot tend to any finite limit nor to ∞ as $z \to \infty$ in Γ. Thus $P(z)$ has no asymptotic values.

It remains to prove (3.12). We obtain first a lower bound for $T(r, P)$. We note that $P(z)$ has $q_\mu k_\mu$ zeros on the circle

$$|z + r_\mu| = \tfrac{3}{2} r_\mu.$$

Thus

$$n(r,0,P) \geqslant q_\mu k_\mu, \quad r \geqslant \frac{5r_\mu}{2},$$

$$N(r,0,P) \geqslant q_\mu k_\mu \int_{(5r_\mu)/2}^{r} \frac{dt}{t} = q_\mu k_\mu \log \frac{2r}{5r_\mu} > \tfrac{1}{3} q_\mu k_\mu \log \frac{r}{r_\mu},$$

since $(r/r_\mu) \geqslant 6 > (\frac{5}{2})^{3/2}$. Further

$$T(r,P) \geqslant N(r,0,P) > \tfrac{1}{3} q_\mu k_\mu \log \frac{r}{r_\mu},$$

and this proves the left hand inequality in (3.12).

To obtain the right hand inequality, suppose that (3.17) holds. Then for $\nu \leqslant \mu$, $P_{k_\nu}(z/r_\nu)$ has no zeros in $|z| < \frac{1}{2} r_\nu$ and at most k_ν zeros altogether. Thus

$$N\left(r, 0, P_{k_\nu}\left(\frac{z}{r_\nu}\right)\right) \leqslant \int_{\frac{1}{2}r_\nu}^{r} k_\nu \frac{dt}{t} = k_\nu \log \frac{2r}{r_\nu}.$$

Thus

$$N(r,0,P(z)) \leqslant \sum_{\nu=1}^{\mu} k_\nu q_\nu \log \frac{2r}{r_\nu} = \sum_{\nu=1}^{\mu} k_\nu q_\nu \left\{\log \frac{r_\mu}{r_\nu} + \log \frac{2r}{r_\mu}\right\}.$$

In view of (3.7) we have

$$\sum_{\nu<\mu} k_\nu q_\nu \log \frac{r_\mu}{r_\nu} < \frac{q_\mu}{99}.$$

Using also (3.5) we deduce that

$$\sum_{\nu<\mu} k_\nu q_\nu < \frac{q_\mu}{1000} < \frac{k_\mu q_\mu}{10\,000}.$$

Thus in the range (3.17) we have

$$N(r,0,P(z)) < \frac{q_\mu}{99} + k_\mu q_\mu (1 + 10^{-4}) \log \frac{2r}{r_\mu}$$
$$< 2k_\mu q_\mu \log \frac{r}{r_\mu}.$$

Next in view of (3.9) we have in the range (3.17)

$$\left| P_{k_\nu}\left(\frac{z}{r_\nu}\right) \right| > 1, \quad 1 \leqslant \nu \leqslant \mu.$$

Thus

$$\log\left|\frac{1}{P(z)}\right| \leqslant \sum_{\nu=\mu+1}^{\infty} -q_\nu \log\left|P_{k_\nu}\left(\frac{z}{r_\nu}\right)\right| < 2\sum_{\nu=\mu+1}^{\infty}\frac{q_\nu r}{r_\nu} < \frac{200r}{99}\frac{q_{\mu+1}}{r_{\mu+1}},$$

in view of (3.8) and (3.6). Thus

$$T(r,P) = N(r,0,P) + m(r,0,P) < 2k_\mu q_\mu \log\frac{r}{r_\mu} + \frac{200}{99} q_{\mu+1}\frac{r}{r_{\mu+1}}.$$

Also in view of (3.17), (3.3) and (3.5) we have

$$r/\log\left(\frac{r}{r_\mu}\right) \leqslant r_{\mu+1}\Big/\left\{6\log\left(\frac{r_{\mu+1}}{6r_\mu}\right)\right\} < r_{\mu+1}/\{5\log(r_{\mu+1}/r_\mu)\} < \frac{40k_\mu q_\mu r_{\mu+1}}{q_{\mu+1}}.$$

Thus

$$T(r,P) < 2k_\mu q_\mu\left(\log\frac{r}{r_\mu}\right)\left(1+\frac{4000}{99}\right) < 100k_\mu q_\mu \log\left(\frac{r}{r_\mu}\right).$$

This proves the right hand inequality of (3.12) if r lies in the range (3.17) and completes the proof of Theorem 3.

3.3. Proof of Theorem 1

To prove Theorem 1, we show that we can choose the quantities k_μ, q_μ and r_μ in Theorem 3 so that $k_\mu \to \infty$, with μ and hence $\delta(\infty, P) = 1$, while at the same time

$$T(r,P) < \psi(r)(\log r)^2, \quad r \geqslant \tfrac{1}{6}r_3, \tag{3.18}$$

where $\psi(r)$ is any function satisfying (2.1). To do this we choose

$$k_1 = 21, \quad k_\mu = 20\mu, \quad \mu \geqslant 2,$$

and suppose that $q_{\mu-1}$, $r_{\mu-1}$ have already been chosen for $\mu \geqslant 2$. We then choose r_μ so large that (3.4) is satisfied, i.e.

$$\frac{r_\mu}{r_{\mu-1}}\Big/\log\left(\frac{r_\mu}{r_{\mu-1}}\right) > 20\,000k_\mu,$$

and further such that

$$\psi(r) > 5\cdot 10^6\mu^2 q_{\mu-1}, \quad r \geqslant \tfrac{1}{6}r_\mu. \tag{3.19}$$

This choice is possible in view of (2.1). We then define

$$q_\mu = \left[100k_\mu q_{\mu-1}\log\frac{r_\mu}{r_{\mu-1}}\right], \tag{3.20}$$

where $[x]$ denotes the integral part of x. Then

$$100k_{\mu-1}q_{\mu-1}\log\left(\frac{r_\mu}{r_{\mu-1}}\right) < q_\mu < 200k_{\mu-1}q_{\mu-1}\log\left(\frac{r_\mu}{r_{\mu-1}}\right),$$

so that (3.5) is satisfied. Thus (3.2) to (3.5) are all satisfied and so $P(z)$ satisfies the conclusions of Theorem 3 with $\delta(\infty, P)=1$. Further we have from (3.12)

$$T(r,P) < 100k_\mu q_\mu \log r, \quad \tfrac{1}{6}r_\mu \leqslant r < \tfrac{1}{6}r_{\mu+1}. \tag{3.21}$$

If $r \geqslant 6r_\mu$ this follows immediately from (3.12). If

$$\frac{r_\mu}{6} \leqslant r \leqslant 6r_\mu$$

we deduce from (3.12) that

$$T(r,P) \leqslant T(6r_\mu, P) < 100k_\mu q_\mu \log 6 < 100k_\mu q_\mu \log r,$$

since $r \geqslant \frac{1}{6}r_3 > 6$. Thus (3.21) holds in this case also and so generally. Further we deduce from (3.20), for $\frac{1}{6}r_\mu \leqslant r \leqslant \frac{1}{6}r_{\mu+1}$, $\mu \geqslant 3$,

$$\begin{aligned} 100k_\mu q_\mu &< 10^4 k_\mu^2 q_{\mu-1}\log\left(\frac{r_\mu}{6}\right) \\ &= 4\cdot 10^6 \mu^2 q_{\mu-1}\log\left(\frac{r_\mu}{6}\right) \\ &< \psi(r)\log r \end{aligned}$$

in view of (3.19). Now (3.18) follows from (3.21) and the proof of Theorem 1 is complete.

3.4. Some further examples

We can use Theorem 3 to construct some other examples which will serve to illustrate Theorem 2.

THEOREM 4. *Given $\varepsilon > 0$ and $0 < \lambda < 1$, there exists a meromorphic function $P(z)$ having very regular growth of order λ and no asymptotic values, while $\delta(\infty, P) > 1-\varepsilon$.*

Theorem 2, Corollary 3, shows that for $\lambda < \frac{1}{2}$ the conclusion is not possible with $\delta(\infty, P)=1$, nor if very regular growth is replaced by perfectly regular growth (see [15]).

We assume $\varepsilon < 1$, and choose a positive integer k, such that

$$k_\nu = k > \frac{20}{\varepsilon}, \quad \nu = 1, \ldots, \infty. \tag{3.22}$$

Thus if $P(z)$ can be chosen in Theorem 3, so as to satisfy the other conditions of Theorems 3 and 4, we shall have

$$\delta(\infty, P) \geqslant 1 - \frac{20}{k} > 1 - \varepsilon.$$

Having chosen k, we define q to be a large positive integer and set

$$q_\nu = q^{\nu-1}, \quad r_\nu = a^{\nu-1} \tag{3.23}$$

where a is given by

$$q = a^\lambda. \tag{3.24}$$

We check that (3.2) to (3.5) are satisfied. This is obvious for (3.2). Also (3.3) is equivalent to

$$q^{1/\lambda} \geqslant 10^7, \tag{3.25}$$

and (3.4) to

$$\frac{\lambda q^{1/\lambda}}{\log q} > 20\,000k. \tag{3.26}$$

Finally (3.5) is equivalent to

$$100k < \frac{\lambda q}{\log q} < 200k. \tag{3.27}$$

All these conditions are satisfied if q is large enough. For we can then choose k to satisfy (3.22) and (3.27) and since $\lambda < 1$, (3.26) is a consequence of (3.27) for large q. Also (3.25) holds for large q.

We now deduce from Theorem 3, (3.12), that

$$\frac{k}{3} q^{\mu-1} \log 6 < T(r, P) < 100kq^{\mu-1} \log \frac{a}{6}, \quad 6r_\mu \leqslant r \leqslant \tfrac{1}{6} r_{\mu+1}.$$

Since $T(r, P)$ increases with r, the right hand inequality is valid also for

$$\tfrac{1}{6} r_\mu \leqslant r < \tfrac{1}{6} r_{\mu+1}$$

and the left hand inequality for $6r_\mu \leqslant r \leqslant 6r_{\mu+1}$. We deduce that

$$\frac{k}{3} q^{\mu-2} \log 6 < T(r, P) < 100kq^\mu \log \left(\frac{a}{6}\right), \quad r_\mu \leqslant r \leqslant r_{\mu+1},$$

i.e.

$$c_1 r_{\mu+1}^\lambda < T(r, P) < c_2 r_\mu^\lambda$$

where c_1, c_2 are constants depending on k and q. Thus

$$c_1 \leqslant \varliminf_{r\to\infty} \frac{T(r,P)}{r^\lambda} \leqslant \varlimsup_{r\to\infty} \frac{T(r,P)}{r^\lambda} \leqslant c_2,$$

so that $P(z)$ has very regular growth of order λ ([15]). This proves Theorem 4.

3.4.1. We have next

THEOREM 5. *Given* $0<\lambda<1$, *there exists* $P(z)$ *having regular growth of order* λ *and no asymptotic values, while at the same time* $\delta(\infty, P)=1$.

We choose

$$q_\nu = q^{(\nu-1)^2}, \quad r_\nu = q^{(\nu-1)^2/\lambda},$$

where q is a sufficiently large positive integer. Then (3.2) and (3.3) are satisfied if q is large enough and also

$$r_{\nu+1}/r_\nu \to \infty.$$

The conditions (3.5), (3.4) become

$$100k_\nu \frac{(2\nu-1)\log q}{\lambda} < q^{2\nu-1} < 200k_\nu \frac{(2\nu-1)\log q}{\lambda},$$

and

$$20\,000k_\nu \frac{(2\nu-1)\log q}{\lambda} < q^{(2\nu-1)/\lambda}$$

and these are again compatible if q is large enough. Further now $k_\nu \to \infty$ with ν and so $\delta(\infty, P)=1$ in Theorem 3. It remains to prove only that $P(z)$ has regular growth of order λ. In fact we have from (3.12)

$$\begin{aligned}\log T(r,P) &= (\mu-1)^2 \log q + O(\mu), \quad r_\mu \leqslant r \leqslant r_{\mu+1}\\ &= \lambda \log r + O\,(\log r)^{1/2}.\end{aligned}$$

Thus

$$\frac{\log T(r,P)}{\log r} \to \lambda, \quad \text{as } r\to\infty,$$

so that $P(z)$ has regular growth of order λ ([15]). This proves Theorem 5.

4. Functions of slowly increasing growth

In this section we provide an example to show that under the hypotheses of Theorem 2, Corollary 2, unlike those of Theorem A, there need not be radial asymptotic values.

We consider entire functions $f(z)$ such that

$$\frac{T(2r,f)}{T(r,f)} \to 1, \quad \text{as } r \to \infty. \tag{4.1}$$

We provide first the following characterization.

THEOREM 6. *If $f(z)$ is an entire function then $f(z)$ satisfies (4.1) if and only if $f(z)$ has genus zero and further for some finite a and hence for every a, we have*

$$n(r, a) = o\{N(r, a)\} \quad \text{as } r \to \infty. \tag{4.2}$$

Suppose first that (4.1) holds. Then it follows from a classical result of Nevanlinna [12, p. 264] that for all a outside a set of capacity zero we have

$$N(r, a) \sim T(r, f) \quad \text{as } r \to \infty. \tag{4.3}$$

We fix a value of a satisfying (4.3) and such that $f(0) \neq a$. Then we deduce from (4.1) and (4.3) that

$$\frac{N(2r, a)}{N(r, a)} \to 1. \tag{4.4}$$

Thus

$$n(r, a) \log 2 \leqslant \int_r^{2r} \frac{n(t, a)\, dt}{t} = N(2r, a) - N(r, a) = o\{N(r, a)\},$$

which yields (4.2). We deduce that for $r_2 > r_1 > r_0(\varepsilon, a)$ we have

$$\log \frac{N(r_2, a)}{N(r_1, a)} = \int_{r_1}^{r_2} \frac{n(r, a)}{N(r, a)} \frac{dr}{r} < \varepsilon \log \frac{r_2}{r_1},$$

i.e.

$$N(r_2, a) \leqslant N(r_1, a) \left(\frac{r_2}{r_1}\right)^{\varepsilon}. \tag{4.5}$$

Thus in particular we deduce, combining (4.3) and (4.5) that

$$T(r_2, f) = O(r_2^{\varepsilon}) \quad \text{as } r_2 \to \infty,$$

so that f has order and so genus zero. Thus the conditions of Theorem 6 are necessary.

We next prove that the conditions are sufficient. Suppose then that f satisfies (4.2) for some value a. We may without loss of generality suppose that $a = 0$, since otherwise we consider $f(z)-a$ instead of $f(z)$. This will not affect (4.1) nor the genus of f. We also write $n(r)$ instead of $n(r, 0)$. Then since $f(z)$ has genus zero

$$f(z)=az^p \prod_{\nu=1}^{\infty}\left(1+\frac{z}{z_\nu}\right), \tag{4.6}$$

where z_ν are the zeros of $f(z)$. If there are only a finite number of zeros then $f(z)$ is a polynomial and so (4.1) holds trivially. Otherwise

$$\frac{T(r,f)}{\log r}\to\infty,$$

and so if $f_0(z)=f(z)/(az^p)$, we deduce that

$$T(r,f)=T(r,f_0)+O(\log r)\sim T(r,f_0).$$

Thus we may suppose that $a=1$ and $p=0$ in (4.6). We write

$$A(r)=\inf_{|z|=r}\log|f(z)|,\quad B(r)=\sup_{|z|=r}|\log f(z)|.$$

and use a technique due to Barry [2]. We have

$$B(r)-A(r)\leqslant\sum_{\nu=1}^{\infty}\log\left|\frac{\left(1+\dfrac{r}{r_\nu}\right)}{\left(1-\dfrac{r}{r_\nu}\right)}\right|.$$

We proceed to estimate

$$\int_r^{2r}\{B(r)-A(r)\}\frac{dr}{r}\leqslant\sum_{\nu=1}^{\infty}\int_r^{2r}\log\left|\frac{1+t/r_\nu}{1-t/r_\nu}\right|\frac{dt}{t}.$$

We consider separately the ranges $r_\nu<4r$, and $r_\nu\geqslant 4r$ and denote the corresponding sums by $\sum_1$, $\sum_2$ respectively. Then in $\sum_1$ we have(1)

$$\int_r^{2r}\log\left|\frac{1+t/r_\nu}{1-t/r_\nu}\right|\frac{dr}{r}\leqslant\int_0^{\infty}\log\left|\frac{1+t/r_\nu}{1-t/r_\nu}\right|\frac{dr}{r}=\int_0^{\infty}\log\left|\frac{1+x}{1-x}\right|\frac{dx}{x}=\frac{\pi^2}{2}.$$

Thus

$$\textstyle\sum_1\leqslant\dfrac{\pi^2}{2}n(2r)=o\{N(r)\},$$

(1) See Barry [2].

since (4.2) and hence (4.4) holds with $a=0$. Also in $\sum_2$ we have

$$\log\left(\frac{1+t/r_\nu}{1-t/r_\nu}\right) \leqslant 2\frac{t}{r_\nu}(1+(\tfrac{1}{2})^2+(\tfrac{1}{4})^2+\dots)=\frac{8}{3}\frac{t}{r_\nu}.$$

Thus

$$\int_r^{2r}\log\left(\frac{1+t/r_\nu}{1-t/r_\nu}\right)\frac{dt}{t} \leqslant \frac{8}{3r_\nu}\int_r^{2r} dt=\frac{8r}{3r_\nu}<3\frac{r}{r_\nu}.$$

Thus

$$\begin{aligned}\textstyle\sum_2 &\leqslant 3r \sum_{r_\nu>4r}\frac{1}{r_\nu}=3r\int_{4r}^{\infty}\frac{1}{t}\,dn(t)\\ &= -\tfrac{3}{4}n(4r)+3r\int_{4r}^{\infty} n(r)\frac{dr}{r^2}\\ &= o\left\{r\int_r^{\infty}\frac{N(t)\,dt}{t^2}\right\}\end{aligned}$$

in view of (4.2). Using also (4.5) with $\varepsilon=\frac{1}{2}$, we deduce that

$$\textstyle\sum_2 = o\left\{rN(r)\int_r^{\infty}\left(\frac{t}{r}\right)^{1/2}\frac{dt}{t^2}\right\}=o\{N(r)\}.$$

On combining this with (4.7) we deduce that

$$\int_r^{2r}\{B(r)-A(r)\}\frac{dr}{r}=o\{N(r)\},\quad \text{as } r\to\infty. \tag{4.8}$$

Evidently

$$A(r)\leqslant N(r)\leqslant T(r)\leqslant B(r).$$

Hence (4.8) shows that for all sufficiently large μ, there exists r_μ, such that $2^\mu \leqslant r_\mu \leqslant 2^{\mu+1}$, and

$$A(r_\mu)\geqslant B(r_\mu)-\varepsilon N(r_\mu)>(1-\varepsilon)N(r_\mu). \tag{4.9}$$

Thus for any finite a, we have for $\mu>\mu(a)$, $A(r_\mu)>1+|a|$ and so

$$m(r_\mu,a)=0,\quad N(r_\mu,a)=T(r_\mu)+O(1),\quad N(r_\mu)=T(r_\mu)+O(1). \tag{4.10}$$

Also for $r_\mu\leqslant r<r_{\mu+1}$, we deduce that

$$N(r_\mu)\leqslant N(r)\leqslant N(r_{\mu+1})\leqslant N(4r_\mu)\sim N(r_\mu).$$

$$N(r)\leqslant T(r)\leqslant N(r_{\mu+1})+O(1)\sim N(r).$$

Thus

$$T(r) \sim N(r)$$

as $r \to \infty$, so that (4.4) with $a = 0$ leads to (4.1). Again (4.10) yields for any fixed a, $r_\mu \leqslant r \leqslant r_{\mu+1}$

$$T(r_\mu) + O(1) \leqslant N(r_\mu, a) \leqslant N(r, a) \leqslant T(r) + O(1) \leqslant T(4r_\mu) + O(1),$$

and in view of (4.1), we deduce that

$$N(r, a) \sim T(r) \quad \text{as } r \to \infty.$$

Thus (4.4) holds for every a and so does (4.2). This completes the proof of Theorem 6. For future reference we also note that (4.9) leads to

$$(1-\varepsilon)\, B(r) < (1 + o(1))\, N(r), \quad \text{as } r \to \infty$$

i.e.

$$B(r) \sim N(r) \sim T(r) \quad \text{as } r \to \infty. \tag{4.11}$$

4.1.

We can now construct our desired example.

THEOREM 7. *There exists an entire function $f(z)$ satisfying the hypotheses of Theorem 6, but such that*

$$\varliminf_{r\to\infty} |f(z_0 + re^{i\theta})| = 0 \tag{4.12}$$

for every fixed complex z_0 and real θ. Thus $f(z)$ cannot have any radial asymptotic values.

We consider the sequence of rationals

$$1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{2}{3}, \tfrac{1}{4}, \tfrac{1}{2}, \tfrac{3}{4} \ldots$$

and denote the νth member of this sequence by θ_ν. We set

$$z_\nu = r_\nu e^{2\pi i \theta_\nu}, \quad \nu = 1, 2, \ldots \tag{4.13}$$

where the positive numbers r_ν and positive integers p_ν are to be determined as follows. We assume

$$r_1 = 4, \quad r_{\nu+1} \geqslant r_\nu^2, \tag{4.14}$$

and then define p_ν inductively by $p_1 = 1$,

$$p_\nu = \left[\frac{p_{\nu-1} \log r_\nu}{(\log \nu)^{1/2}}\right], \quad \nu \geqslant 2 \tag{4.15}$$

where $[x]$ denotes the integral part of x.

We now define $f(z)$ to be the entire function of genus zero with zeros of multiplicity p_ν at $z=z_\nu$, and such that $f(0)=1$. Then

$$r_\nu \geqslant 2^{2^\nu}, \ \log r_\nu \geqslant 2^\nu \log 2,$$

so that

$$\frac{p_\nu}{p_{\nu-1}} \to \infty \quad \text{as } \nu \to \infty. \tag{4.16}$$

Thus

$$n(r_\nu, 0) = \sum_{\mu=1}^{\nu} p_\mu \sim p_\nu, \quad \text{as } \nu \to \infty$$

and so we have for $r_\nu \leqslant r < r_{\nu+1}$

$$n(r, 0) \sim p_\nu,$$

and

$$N(r,0) = \sum_{\mu=1}^{\nu} p_\mu \log \frac{r}{r_\mu} \geqslant p_{\nu-1} \log \frac{r}{r_{\nu-1}} \geqslant \tfrac{1}{2} p_{\nu-1} \log r$$

in view of (4.14). Thus (4.14) shows that

$$n(r,0) = O\left\{\frac{N(r,0)}{(\log \nu)^{1/2}}\right\}$$

so that

$$f(z) = \prod_{\nu=1}^{\infty} \left(1 - \frac{z}{z_\nu}\right)^{p_\nu}$$

satisfy the hypotheses of Theorem 6, with $a=0$ in (4.2).

We suppose that $0 \leqslant \theta < 2\pi$ in (4.12) and note that for any positive integer q, we can find $\theta_\nu = p/q$, such that

$$|2\pi\theta_\nu - \theta| \leqslant \frac{\pi}{q}.$$

Also $\nu \leqslant 1+1+2+\ldots+(q-1) < q^2$, so that $q > \nu^{1/2}$. Thus

$$|2\pi\theta_\nu - \theta| < \frac{\pi}{\nu^{1/2}},$$

for infinitely many ν. We set $z'_\nu = z_0 + r_\nu e^{i\theta}$, and deduce that

$$\begin{aligned} |z'_\nu - z_\nu| &\leqslant |z_0| + r_\nu \left|e^{i\theta} - e^{2\pi i \theta_\nu}\right| \leqslant \frac{\pi r_\nu}{\nu^{1/2}} + |z_0| \\ &\leqslant \frac{r_\nu}{\nu^{1/4}}, \end{aligned}$$

for some arbitrarily large ν.

We now note that in $|z-z_\nu| < r_\nu$, we have, in view of (4.11),

$$\log |f(z)| < B(2r_\nu) < (1+o(1))\,N(2r_\nu) < \{1+o(1)\}\,N(r_\nu)$$
$$= (1+o(1)) \sum_{\mu=1}^{\nu-1} p_\mu \log \frac{r_\nu}{r_\mu} < (2+o(1)) \sum_{\mu=1}^{\nu-1} p_\mu \log r_\nu < 3p_{\nu-1} \log r_\nu,$$

when ν is large. Thus by Schwarz's lemma we deduce

$$\log |f(z'_\nu)| < 3p_{\nu-1} \log r_\nu + p_\nu \log \left\{\frac{|z'_\nu - z_\nu|}{r_\nu}\right\}$$
$$< 3p_{\nu-1} \log r_\nu - \frac{p_\nu}{4} \log \nu$$
$$< -\{\tfrac{1}{4} + o(1)\}\, p_{\nu-1} \log r_\nu \,(\log \nu)^{1/2}$$

in view of (4.15). This inequality holds for a sequence of points $z'_\nu = z_0 + r_\nu e^{i\theta}$ which tend to ∞, and this completes the proof of (4.12). In particular $f(z)$ cannot tend to ∞ along any ray Γ. In view of (4.9) $f(z)$ cannot be bounded on Γ and so $f(z)$ has no radial asymptotic values. This completes the proof of Theorem 7. By allowing r_ν to tend to ∞ sufficiently rapidly, we can in addition satisfy (2.2).

5. Proof of Theorem 2; a topological lemma

We shall deduce Theorem 2 from the following result which is essentially topological.

LEMMA 4. *Suppose that $f(z)$ is a meromorphic function not having ∞ as an asymptotic value. Then $f(z)$ is bounded either on a path Γ going to ∞, or on the union of a sequence Γ_N of analytic Jordan curves which surround the origin and whose distance from the origin tends to ∞ with N.*

Let z_n be the branchpoints of $f(z)$, i.e. the points where $f'(z)=0$. We assume that $|f(z_n)|$ is never equal to a positive integer. If this condition is not satisfied we consider $af(z)$ instead of $f(z)$ where a is a positive number unequal to the numbers $m/|f(z_n)|$ where m, n are positive integers. Let n be a positive integer. It follows from our assumption that the set $|f(z)| = n$ consists either of disjoint closed analytic Jordan curves or of Jordan arcs going from ∞ to ∞. If there are any such arcs Lemma 4 is proved. Thus we may assume that the set $|f(z)| = n$ consists entirely of closed analytic Jordan curves $g_\nu(n)$.

Consider next the open set $|f(z)| > n$. We distinguish two cases. Suppose first that this set contains an unbounded component G_n for every n. Then G_n is unique, since G_n clearly lies exterior to all the closed curves $g_\nu(n)$. If G'_n were another unbounded component of

$|f| > n$ and z_n, z_n' were points of G_n, G_n' respectively we could join z_n, z_n' by a path not meeting any of the $g_\nu(n)$ and on such a path $|f(z)| > n$, which contradicts the assumption that z_n, z_n' lie in separate components of $|f(z)| > n$. Thus G_n is unique. Evidently $G_{n+1} \subset G_n$. Let z_n be a point of G_n such that $|z_n| > n$, $|f(z_n)| > n$. Then $z_{n+1} \in G_{n+1} \subset G_n$, so that we can join z_n to z_{n+1} by a path γ_n in G_n. Let R be a fixed number such that $f(z)$ has no poles on $|z| = R$ and choose n_0 so large that

$$n_0 > R \quad \text{and} \quad n_0 > \max_{|z|=R} |f(z)|.$$

Then, for $n > n_0$, γ_n contains z_n where $|z_n| > R$ and γ_n cannot meet $|z| = R$. Thus γ_n lies outside $|z| = R$ for $n > n_0$ so that γ_n goes to ∞ with n. Thus

$$\Gamma = \bigcup_{n=1}^{\infty} \gamma_n$$

is a path from z_1 to ∞, and $|f(z)| > n$ on γ_n, so that $f(z) \to \infty$ as $z \to \infty$ along Γ.

This contradicts our assumption that ∞ is not an asymptotic value of $f(z)$ and so this case cannot occur. Thus, for some fixed n, every component G_ν of the set $|f(z)| > n$ is bounded.

We may assume that $f(z)$ has infinitely many poles, since otherwise the hypotheses of Lemma 4 imply that $f(z)$ is rational, and finite at ∞, in which case the conclusion is trivial. Since each component G_ν is bounded and each pole lies in one of these G_ν there must be infinitely many components G_ν. The outer boundary g_ν of G_ν will go to ∞ as $\nu \to \infty$ for fixed n. If g_ν surrounds the origin for infinitely many ν, we have established the conclusion of Lemma 4. Thus we may assume that for $\nu > \nu_0$, g_ν does not surround the origin. Choose R_0 so large that the disk $|z| < R_0$ contains g_ν for $\nu \leqslant \nu_0$. Then for $R > R_0$ the circle $|z| = R$ cannot lie in any G_ν since otherwise the outer boundary g_ν of G_ν would be a curve surrounding the origin. We also assume that $|z| = R$ does not touch any of the g_ν.

Let G_ν, $\nu = \nu_1$ to ν_2 be those components which meet $|z| = R$. By our construction $\nu_1 > \nu_0$, so that the origin lies outside each g_ν for $\nu \geqslant \nu_1$ and so outside the corresponding components G_ν. If E is the union of $|z| = R$ and the closures of the G_ν for $\nu_1 \leqslant \nu \leqslant \nu_2$, then E is a compact connected set and so the domains complementary to E are simply connected. Let D_0 be that component of the complement of E which contains the origin. By construction the boundary γ_R of D_0 is a sectionally analytic Jordan curve on which $|f(z)| \leqslant n$. For γ_R consists of arcs of g_ν and of arcs of $|z| = R$ on which $|f(z)| < n$.

Clearly γ_R surrounds the origin, since any path Γ going from 0 to ∞ must meet $|z| = R$ and so goes outside D_0. Thus Γ meets γ_R. Also, for any positive integer M we may choose

R so large that G_ν lies in $|z| < R$ for $\nu = 1$ to M. Then $\nu_1 > M$ and so γ_R is far away from the origin if R is large. Taking $\Gamma_N = \gamma_{R_N}$, where $R_N \to \infty$, we obtain a sequence of Jordan curves with the properties required by Lemma 4 and the proof of that Lemma is complete.

5.1. An example for Lemma 4

It is clear that if $f(z)$ is bounded on a sequence of curves Γ_N as in Lemma 4, then $f(z)$ cannot have ∞ as an asymptotic value since any path going to ∞ must meet the Γ_N for all large N. On the other hand f may very well be bounded on one path and go to ∞ on another, so that the first condition of Lemma 4 does not by itself preclude ∞ from being an asymptotic value. This makes it natural to ask whether the first condition can be omitted from Lemma 4. The following example shows that this is not possible in general.

Example. Let

$$f(z) = \sum_{n=1}^{\infty} \{\tfrac{1}{2} \cos (z - 4in)\}^{-n}.$$

Then $f(z)$ does not have ∞ as an asymptotic value, but $f(z)$ is not bounded on the union of any sequence Γ_N of curves satisfying the conditions of Lemma 4.

We note that if $|y| \geq 2$, then

$$\tfrac{1}{2} |\cos (x + iy)| \geq \tfrac{1}{4}(e^{|y|} - e^{-|y|}) > \tfrac{3}{2}.$$

Thus the series for $f(z)$ converges locally uniformly, and $f(z)$ is meromorphic in the plane. Also if $z = x + iy$, where $|y - 4n| \geq 2$ for every n, we have

$$|f(z)| \leq \sum_{1}^{\infty} (\tfrac{2}{3})^n = 2.$$

In particular this inequality holds for $y \leq 2$, and for $y = 4\nu - 2$, $\nu = 1$ to ∞. Thus if $|f(z)| > 2$ on a path Γ going to ∞, we must have $4\nu - 2 < y < 4\nu + 2$ on Γ for some fixed $\nu \geq 1$. Thus Γ must meet the lines $x = m\pi$ for some arbitrarily large positive or negative integers m. If $z = m\pi + iy$, where $4\nu - 2 < y < 4\nu + 2$, we have

$$|\cos (z - 4i\nu)| = \cosh (y - 4\nu) \geq 1, \quad |\tfrac{1}{2} \cos (z - 4in)| \geq \tfrac{3}{2}, \quad n \neq \nu.$$

Thus

$$|f(z)| \leq 2^\nu + \sum_{n=1}^{\infty} (\tfrac{2}{3})^n = 2^\nu + 2.$$

Hence $f(z)$ cannot tend to ∞ as $z \to \infty$ on Γ, and ∞ is not an asymptotic value of $f(z)$.

Suppose now that Γ_N is a Jordan curve surrounding the origin, whose distance from the origin is at least $4N$. Then Γ_N must meet the line $y = 4N$. On this line we have

$$\left|\tfrac{1}{2}\cos(z-4iN)\right| \leqslant \tfrac{1}{2}, \quad \left|\tfrac{1}{2}(\cos z - 4in)\right| \geqslant \tfrac{3}{2}, \quad n \neq N.$$

Thus

$$|f(z)| \geqslant 2^N - \sum_{n \neq N} \left|\tfrac{1}{2}\cos(z-4in)\right|^{-n} > 2^N - 2.$$

Thus

$$\sup_{z \in \Gamma_N} |f(z)| \geqslant 2^N - 2,$$

and so $f(z)$ is unbounded on $\cup\, \Gamma_N$.

Our example shows that there exist functions $f(z)$ satisfying only the first condition, but not the second condition of Lemma 4.

5.2. Quantitative consequences of Lemma 4

In order to prove Theorem 2 we shall show that the conclusion of Lemma 4 is not compatible with (2.4) when $a = \infty$. We first need an inequality for the Green's function of a simply connected domain.

LEMMA 5. *Suppose that D is a simply connected domain containing the origin and let d be the distance from the origin to the complement of D. Then if $g(0, w)$ is the Green's function of D at the origin, we have for w in D*

$$\log^+ \frac{d}{|w|} \leqslant g(0, w) \leqslant \lambda(|w|/d),$$

where

$$\lambda(t) = \log \frac{1}{t} + 2, \quad t < 1$$

$$\lambda(t) = 2t^{-1/2}, \quad t \geqslant 1.$$

The first inequality is obvious, since D contains the disk $|w| < d$ and $g(0, w) \geqslant 0$ for $|w| > d$. To prove the second inequality suppose that $\xi = \varphi(w)$ maps D onto $|\xi| < 1$, so that $\varphi(0) = 0$. Then

$$g(0, w) = \log \left|\frac{1}{\varphi(w)}\right|.$$

Let

$$w = \psi(\xi) = a_1 \xi + a_2 \xi^2 + \ldots$$

be the inverse function to $\xi = \varphi(w)$, so that $\psi(\xi)$ is univalent in $|\xi| < 1$. It then follows from classical results, [6, pp. 3 and 4] that $|a_1| \leqslant 4d$ and

$$|w| \leqslant \frac{|a_1||\xi|}{(1-|\xi|)^2} = \frac{|a_1|}{(|\xi|^{-1/2} - |\xi|^{1/2})^2} < \frac{4d}{(\log 1/|\xi|)^2}.$$

Thus

$$\log \left|\frac{1}{\xi}\right| = g(0, w) < 2\left(\frac{d}{|w|}\right)^{1/2} = \lambda(|w|/d), \quad |w| \geqslant d.$$

If $|w| \leqslant d$, we may assume $|\xi| \leqslant e^{-2}$, since otherwise Lemma 5 is trivial. Then

$$|w| \leqslant \frac{a_1|\xi|}{(1-e^{-2})^2} \leqslant \frac{4d|\xi|}{(1-e^{-2})^2} < e^2 d|\xi|.$$

Thus

$$\log \left|\frac{1}{\xi}\right| = g(0, w) < 2 + \log \frac{d}{|w|}.$$

This proves Lemma 5.

We shall deduce Theorem 2 from the following more precise result.

THEOREM 8. *Suppose that $f(z)$ is meromorphic in the closure $\bar{D}$ of a simply connected domain D containing the origin, and that $|f(z)| \leqslant M < \infty$, on the finite boundary Γ_0 of D. Let d be the distance from the origin to Γ_0, which is assumed not to be empty. Then either*

$$m(d, f) \leqslant \log (M+1) + d^{1/2} \int_d^\infty \frac{n(t, \infty)\, dt}{t^{3/2}}, \tag{5.1}$$

where $n(r, \infty)$ is the number of poles of $f(z)$ in $\{|z| \leqslant r\} \cap D$ or D is unbounded and (1.1) *holds for some path tending to ∞ in D.*

We assume without loss of generality that the right hand side of (5.1) is finite, since otherwise there is nothing to prove. Thus if b_ν are the poles of $f(z)$ in D we have

$$\int_d^\infty \frac{1}{t^{1/2}} dn(t, \infty) = \sum_{|b_\nu| > d} |b_\nu|^{-1/2} < \infty$$

Using Lemma 5 we deduce that

$$g(z) = \sum_\nu g(z, b_\nu)$$

converges uniformly in any bounded subset of D to a function which is harmonic in D, except at the points b_ν and vanishes continuously on the finite boundary of D. Let $a_\mu(\varphi)$ be the zeros of $f(z)-e^{i\varphi}$ in $|z|\leqslant d$ and set

$$g_\varphi(z)=\sum_\mu g(z,a_\mu(\varphi)).$$

We define

$$u_\varphi(z)=\log|f(z)-e^{i\varphi}|+g_\varphi(z)-g(z)-\log(M+1). \tag{5.2}$$

Then $u_\varphi(z)$ is subharmonic in $\bar{D}$ and $u_\varphi(z)\leqslant 0$ on Γ_0. We now distinguish two cases. Suppose first that $f(0)\neq\infty$, and

$$u_\varphi(z)\leqslant 0,\quad 0\leqslant\varphi\leqslant 2\pi,\quad z\in D. \tag{5.3}$$

In view of Lemma 5 this leads to

$$\log|f(0)-e^{i\varphi}|+\sum_\mu \log^+\frac{d}{|a_\mu(\varphi)|}-g(0)\leqslant\log(M+1),$$

i.e.

$$\log|f(0)-e^{i\varphi}|+N(d,e^{i\varphi})\leqslant g(0)+\log(M+1).$$

We integrate the left hand side with respect to φ and use an identity of Cartan (see e.g. [7], Theorem 1.3, p. 8). This yields

$$T(d,f)\leqslant g(0)+\log(M+1).$$

Next it follows from Lemma 5 that

$$\begin{aligned} g(0)&\leqslant\sum_{|b_\nu|\leqslant d}\left\{\log\left|\frac{d}{b_\nu}\right|+2\right\}+2\sum_{|b_\nu|>d}\left|\frac{d}{b_\nu}\right|^{1/2}\\ &=\int_0^d\left(\log\frac{d}{r}+2\right)dn(r,\infty)+\int_d^\infty 2\left(\frac{d}{r}\right)^{1/2}dn(r,\infty)\\ &=\int_0^d\frac{n(r,\infty)\,dr}{r}+d^{1/2}\int_d^\infty\frac{n(r,\infty)\,dr}{r^{3/2}}.\end{aligned}$$

Thus

$$T(d,f)=m(d,f)+N(d,f\leqslant N(d,f)+d^{1/2}\int_d^\infty\frac{n(r,\infty)\,dr}{r^{3/2}}+\log(M+1).$$

which is (5.1).

If we write

$$N(r,\infty)=\int_0^r\frac{n(t,\infty)\,dt}{t},$$

we may write (5.1) in the equivalent form

$$T(d,f)\leqslant\tfrac{1}{2}d^{1/2}\int_d^\infty\frac{N(r,\infty)\,dr}{r^{3/2}}+\log(M+1), \tag{5.4}$$

after an integration by parts.

We suppose next that (5.3) is false. Since $u_\varphi(z)$ is subharmonic in $\bar{D}$, D must be unbounded in this case. We choose φ, such that (5.2) is false for some z in D and set

$$v(z) = \max\,(u_\varphi(z), 0), \quad z \in D$$
$$v(z) = 0, \quad z \notin D.$$

Then $v(z)$ is subharmonic in the whole plane and not constant, and now we deduce from a theorem of Talpur [13], that there exists a path Γ, going to ∞, such that

$$v(z) \to +\infty, \quad \text{as } z \to \infty \quad \text{along } \Gamma.$$

Since $v(z)=0$ outside D, Γ must lie in D from a certain point onward, so that we may assume that Γ lies entirely in D. Thus

$$u_\varphi(z) \to +\infty, \quad \text{as } z \to \infty \quad \text{on } \Gamma.$$

We recall the definition (5.2) and note that $g(z) > 0$ in 0 and

$$g_\varphi(z) \to 0 \quad \text{as } z \to \infty \quad \text{in } D.$$

Thus we deduce that

$$\log |f(z) - e^{i\varphi}| \to +\infty, \quad \text{as } z \to \infty \quad \text{along } \Gamma,$$

and this yields (1.1).

It is worth noting that in this situation we can actually prove rather more. Since $v(z)$ is bounded on a connected unbounded set it follows from [3, Theorem 3], that

$$\frac{v(|z|)}{\psi(|z|)} \to \infty$$

as $z \to \infty$ along some path Γ, where $\psi(t)$ is any positive increasing function of t, for $t \geq t_0$, which is such that

$$\int_{t_0}^{\infty} \frac{\psi(t)\,dt}{t^{3/2}} < \infty. \tag{5.5}$$

Also since $v(z)$ has bounded minimum on circles $|z| = r$, it follows from a result of Heins [9], that the limit

$$\lim_{r\to\infty} \frac{T(r, v)}{r^{1/2}}$$

exists and is positive, where

$$T(r, v) = \frac{1}{2\pi} \int_0^{2\pi} v(re^{i\theta})\,d\theta.$$

From this we deduce that if $f(z)$ is meromorphic in the plane, f must have at least order $\frac{1}{2}$, mean type. We can sum up by saying that if (5.1) is false then $\log^+ f$ must grow at least like $|z|^{1/2}$ in an average sense and at least like $\psi(|z|)$ along a path in D, where $\psi(t)$ is any function satisfying (5.5).

We have proved Theorem 8 with the hypothesis that $f(0) \neq \infty$. If $f(0) = \infty$, we apply Theorem 8 to $f(z_0 + z)$, where z_0 is small. If for some z_0 (5.1) fails to hold we again deduce (1.1). Otherwise we allow z_0 to tend to zero and then we deduce (5.1) for $f(z)$.

5.3. Completion of proof of Theorem 2

Suppose now that $f(z)$ is meromorphic in the plane and does not have ∞ as an asymptotic value. Then it follows from Lemma 4, that $|f(z)| \leq M$ on a path Γ going to ∞ or on the union of a sequence Γ_n of Jordan curves surrounding the origin. Suppose e.g. the former holds and that the path goes from z_0 to ∞. Then for $d > |z_0|$ the path meets $|z| = d$. Hence there exists an arc of this path joining a point $z_1 = de^{i\theta}$ to ∞ and lying otherwise in $|z| > d$. Thus we may apply Theorem 8 and in particular (5.1) or equivalently (5.4) with any $d > |z_0|$, which contradicts (2.4).

Similarly if $|f(z)| \leq M$ on the sequence Γ_n of Jordan curves surrounding the origin, we obtain (5.4) with $d = d_n$, where d_n is the distance from Γ_n to the origin and this again contradicts (2.4). Thus $f(z)$ must have ∞ as an asymptotic value and Theorem 2 is proved when $a = \infty$. If a is finite we apply the above argument with $(f(z) - a)^{-1}$ instead of $f(z)$.

5.4. Proof of the corollaries of Theorem 2

We proceed to prove the corollaries of Theorem 2. Suppose then that (2.5) holds and that $\delta = \delta(a, f) > 1 - K^{-1}$. Then we have as $t \to \infty$

$$N(t, a) < (1 - \delta + o(1))\, T(t, f),$$

$$\frac{r^{1/2}}{2} \int_r^\infty \frac{N(t, a)\, dt}{t^{3/2}} < \{1 - \delta + o(1)\} \frac{r^{1/2}}{2} \int_r^\infty \frac{T(t, f)\, dt}{t^{3/2}} < \{K(1 - \delta) + o(1)\}\, T(r, f)$$

in view of (2.5). Since $K(1 - \delta) < 1$ and $T(r, f) \to \infty$ with r, we deduce (2.4). This proves Corollary 1.

To prove Corollary 2, it is enough to show that under the hypothesis of Corollary 2, we have (2.5) with $K = 1$. We recall from section 4 that (4.1) implies (4.3) and (4.5) for some finite a and every positive ε. Thus, writing $T(r) = T(r, f)$, we have

$$T(t) \leq (1 + \varepsilon)\, T(r) \left(\frac{r}{t}\right)^\varepsilon, \quad t \geq r \geq r_0(\varepsilon).$$

This yields for $r > r_0(\varepsilon)$

$$\int_r^\infty \frac{T(t)\,dt}{t^{3/2}} \leqslant \frac{(1+\varepsilon)\,T(r)}{r^\varepsilon}\int_r^\infty \frac{dt}{t^{3/2-\varepsilon}} = \frac{(1+\varepsilon)\,T(r)}{(\frac{1}{2}-\varepsilon)\,r^{1/2}}.$$

Thus

$$\overline{\lim_{r\to\infty}}\,\frac{r^{1/2}}{2T(r)}\int_r^\infty \frac{T(t)\,dt}{t^{3/2}} \leqslant \frac{1+\varepsilon}{1-2\varepsilon}.$$

Since ε is any positive number we obtain (2.5) with $K \leqslant 1$, and we deduce from Corollary 1 that any deficient value is asymptotic.

If $f(z)$ satisfies (1.2) i.e.

$$\overline{\lim_{r\to\infty}}\,\frac{T(r,f)}{(\log r)^2} = A < \infty$$

we deduce that

$$n(r,a) \leqslant \frac{1}{\log r}\int_r^{r^2} \frac{n(t,a)\,dt}{t} \leqslant (4A + o(1))\log r.$$

Thus if $f(z)$ is transcendental, we deduce that for every a with at most one exception

$$\frac{N(r,a)}{\log r} \to \infty,$$

and so

$$\frac{n(r,a)}{N(r,a)} \to 0, \quad \text{as } r \to \infty.$$

Thus in view of Theorem 6, $f(z)$ satisfies (4.1) and so every deficient value is asymptotic. However the condition (2.4) yields more than this. We have

$$\begin{aligned} T(r,f) - \tfrac{1}{2}r^{1/2}\int_r^\infty \frac{N(t,a)\,dt}{t^{3/2}} &= m(r,a) - \tfrac{1}{2}r^{1/2}\int_r^\infty \frac{\{N(t,a)-N(r,a)\}\,dt}{t^{3/2}} + O(1) \\ &= m(r,a) - r^{1/2}\int_r^\infty \frac{n(t,a)\,dt}{t^{3/2}} + O(1) \\ &\geqslant m(r,a) - (4A + o(1))\,r^{1/2}\int_r^\infty \frac{\log t\,dt}{t^{3/2}} \\ &\geqslant m(r,a) - (8A + o(1))\log r. \end{aligned}$$

Thus the condition (2.4) is satisfied in this case as soon as

$$\underline{\lim}_{r\to\infty}\,\frac{m(r,a)}{\log r} > 8A.$$

If a is deficient we deduce that

$$\varliminf_{r\to\infty} \frac{m(r,a)}{T(r,f)} > 0,$$

so that

$$\frac{m(r,a)}{\log r} = \frac{m(r,a)}{T(r,f)} \cdot \frac{T(r,f)}{\log r} \to \infty, \quad \text{as } r \to \infty.$$

We have thus proved Corollary 2 in a somewhat stronger form.

We next prove Corollary 3. We recall (c.f. [15]) that $f(z)$ has very regular growth of order λ if there exist positive constants c_1, c_2 such that

$$c_1 r^\lambda < T(r) < c_2 r^\lambda$$

for all sufficiently large r, where $T(r) = T(r, f)$. This implies

$$\int_r^\infty \frac{T(t)\,dt}{t^{3/2}} < c_2 \int_r^\infty \frac{dt}{t^{3/2-\lambda}} = \frac{2c_2}{(1-2\lambda)} r^{\lambda - 1/2}.$$

Thus

$$\frac{r^{1/2}}{2T(r)} \int_r^\infty \frac{T(t)\,dt}{t^{3/2}} < \frac{c_2}{c_1(1-2\lambda)} = K.$$

Hence in view of Corollary 1, a is asymptotic if

$$\delta(a,f) > 1 - K^{-1} = 1 - \frac{c_1(1-2\lambda)}{c_2}.$$

In particular the conclusion holds if $\delta(a, f) = 1$.

The example of Theorem 4 shows that we cannot in this corollary replace $\delta(a, f) = 1$ by $\delta(a, f) > 1 - \varepsilon$, for ε independent of c_1, c_2. If f has perfectly regular growth [15], we may choose the ratio c_2/c_1 as near one as we please. In this case the conclusion holds as soon as

$$\delta(a, f) > 1 - (1 - 2\lambda) = 2\lambda.$$

I do not know whether this conclusion is sharp, or whether all deficient values are necessarily asymptotic for this class of functions.

It remains to prove Corollary 4. We suppose that

$$(1 - 2\lambda) \varliminf_{r\to\infty} \frac{T(r,f)}{r^\lambda} = c_1, \ \varlimsup_{r\to\infty} \frac{N(r,a)}{r^\lambda} = c_2,$$

where $0 \leqslant c_2 < c_1 \leqslant \infty$. Then

$$\tfrac{1}{2} r^{1/2} \int_r^\infty \frac{N(t,a)\,dt}{t^{3/2}} \leqslant r^{1/2}\left(\frac{c_2}{2} + o(1)\right) \int_r^\infty t^{\lambda - 3/2}\,dt = \left\{\frac{c_2}{1-2\lambda} + o(1)\right\} r^\lambda$$

$$< \left\{\frac{c_2}{c_1} + o(1)\right\} T(r,f), \quad \text{as } r \to \infty.$$

Since $c_2/c_1 < 1$, we deduce that (2.4) holds and so Corollary 4 follows from Theorem 1.

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