ON THE NUMBER OF RESTRICTED PRIME FACTORS OF AN INTEGER. II

BY

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§ 1. Introduction

Let P be the set of all (positive rational) prime numbers, and let E be an arbitrary nonempty subset of P. Throughout this paper, let p denote a general member of P, and for non-negative integers a, write $p^a \| n$ if $p^a | n$ and $p^{a+1} \nmid n$. For each positive integer n, define

$$\omega(n; E) = \sum_{p|n, p\in E} 1, \quad \Omega(n; E) = \sum_{p^a\|n, p\in E} a.$$

We usually write $\omega(n; P) = \omega(n)$, $\Omega(n; P) = \Omega(n)$. In a previous paper [37], we obtained sharp inequalities for the frequencies of large deviations of $\omega(n; E)$ and $\Omega(n; E)$ from their normal order of magnitude. Those inequalities included refinements of a special case of a general theorem due to Elliott [11, Theorem 6] concerning large deviations of $f(g(n))$, where f is a strongly additive arithmetic function and $g(n)$ is a positive-valued polynomial in n with integral coefficients. Elliott's result was in turn a refinement (under stronger hypotheses) of a theorem of Uždavinis [55]. (The result of Uždavinis is stated as Theorem 3.3 in Kubilius [28].)

The methods used in [37] were "almost" elementary. Here we shall use more difficult methods to obtain asymptotic formulas for large deviations of $\omega(n; E)$ and $\Omega(n; E)$. We shall also generalize some of the results of [37] and give some applications. For a partial survey of the literature in this area, see [39].

In order to state our main theorems, it is necessary to introduce further notation which will be used throughout this paper. First, we define

$$Q(t) = t - (1+t)\log(1+t) \quad \text{for real } t > -1, \tag{1.1}$$
$$Q(-1) = -1 = \lim_{t\to -1+} Q(t).$$

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Note that

$$Q(t) = \sum_{n=2}^{\infty} \frac{(-1)^{n-1} t^n}{(n-1)n} \quad \text{for } |t| \leq 1. \tag{1.2}$$

Next, if v, β are real, let

$$R_v(\beta) = \exp\{vQ(\beta v^{-\frac{1}{2}})\} \quad \text{for } v > 0, \quad \beta \geq -v^{\frac{1}{2}}, \tag{1.3}$$

$$G(\beta) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\beta} \exp(-t^2/2)\, dt, \tag{1.4}$$

$$F_v(\beta) = \exp(\beta^2/2)\, G(-|\beta|)\, R_v(\beta) \quad \text{for } v > 0, \quad \beta \geq -v^{\frac{1}{2}}. \tag{1.5}$$

Now define

$$E(x) = \sum_{p \leq x,\, p \in E} p^{-1} \quad (x \text{ real}). \tag{1.6}$$

In [37], it was observed that if $E(x) \to +\infty$ as $x \to +\infty$, then both the average order and the normal order of $\omega(n; E)$ are equal to $E(n)$, and the same statement holds for $\Omega(n; E)$. However, it is often more convenient to discuss the distributions of $\omega(n; E)$ and $\Omega(n; E)$ when $n \leq x$ in terms of some approximation to $E(x)$ which is more elementary or easier to calculate than $E(x)$ itself. For example, if $E = P$, one usually uses $\log_2 x = \log\log x$ as an approximation to $E(x)$ and considers the size of $\omega(n) - \log_2 x$ or $\Omega(n) - \log_2 x$ for values of $n \leq x$. In this paper, we shall compare the sizes of $\omega(n; E)$ and $\Omega(n; E)$ (for $n \leq x$) with a number v which we think of as an approximation to $E(x)$. The degree of approximation will be specified in the theorems. We assume throughout that

(1.7) x, v, β are real with $x \geq 1$, $v > 0$. E is a nonempty set of primes, to be regarded as arbitrary unless further assumptions are stated. (E may depend on x or on various parameters.)

(In many applications, it is convenient to take v to be a functional value $v(x; E)$, the function being defined for all $x \geq c_1(v)$.) Lastly, we define

$$\Lambda = \Lambda(x, v; E) = \max\{2, |E(x) - v|\}. \tag{1.8}$$

We can now state our first main result.

THEOREM 1.9. *Assume* (1.7), *and let*

$$\begin{cases} g(n) = \omega(n; E) & (\textit{for all } n) \quad \textit{or} \\ g(n) = \Omega(n; E) & (\textit{for all } n). \end{cases} \tag{1.10}$$

Define

$$(1.11)\qquad T_v(x, \beta;\, E, g) = x^{-1}\,\mathrm{card}\,\{n\colon n \leqslant x \quad \text{and } g(n) \leqslant v + \beta v^{\frac{1}{2}}\},$$

where card B *means the number of members of the set* B. *Suppose that*

$$(1.12)\qquad v \geqslant \Lambda^4$$

and that

$$(1.13)\qquad |\beta| \leqslant \Lambda^{-2} v^{\frac{1}{2}}.$$

If $\beta \leqslant 0$, *then*

$$(1.14)\qquad T_v(x, \beta;\, E, g) = F_v(\beta) - (2\pi)^{-\frac{1}{2}}\{E(x) - v\} R_v(\beta) v^{-\frac{1}{2}} + O(R_v(\beta) v^{-\frac{1}{2}}),$$

and if $\beta \geqslant 0$, *then*

$$(1.15)\qquad 1 - T_v(x, \beta;\, E, g) = F_v(\beta) + (2\pi)^{-\frac{1}{2}}\{E(x) - v\} R_v(\beta) v^{-\frac{1}{2}} + O(R_v(\beta) v^{-\frac{1}{2}}).$$

In (1.14) *and* (1.15), *the constants implied by* O *are absolute.*

By (4.7), the right-hand sides of both (1.14) and (1.15) can be written in the slightly less precise form

$$F_v(\beta)\{1 + O(\Lambda\{|\beta| + 1\} v^{-\frac{1}{2}})\},$$

so that Theorem 1.9 actually gives asymptotic formulas for $T_v(x, \beta; E, g)$ if v and β are functions of x such that $v \to +\infty$ and $\beta = o(\Lambda^{-2} v^{\frac{1}{2}})$ as $x \to +\infty$. It also gives sharp upper and lower bounds for $T_v(x, \beta; E, g)$ if $|\beta|\Lambda^2 v^{-\frac{1}{2}}$ is less than a sufficiently small absolute constant. For somewhat less precise upper and lower bounds holding over larger β-intervals (roughly $|\beta| < v^{\frac{1}{2}}$), see § 3, Theorem 4.27, and [37]. Upper bounds valid for even larger values of β can be obtained in the same way as [37, (5.15) and (5.16)].

Theorem 1.9 is best possible in a rather strong sense. The error terms in (1.14) and (1.15) cannot be improved. Furthermore, the functions $F_v(\beta)$ (for $\beta \leqslant 0$) and $1 - F_v(\beta)$ (for $\beta \geqslant 0$) are essentially the best possible continuous approximations to $T_v(x, \beta; E, g)$, since the latter (considered as a function of β) has a jump discontinuity of size $\gg R_v(\beta) v^{-\frac{1}{2}}$ when $v + \beta v^{\frac{1}{2}}$ is a positive integer and $|\beta|$ is not too large. (For a more precise formulation, see the end of § 5.)

Theorem 1.9 is of "large deviation" type, so called because it gives precise approximations when β is allowed to range over a rather large interval whose size may vary with x. Asymptotic formulas for large deviations of additive arithmetic functions have been obtained previously by other authors, but Theorem 1.9 and its proof differ significantly from their work. In the special case $E=P$, $g(n)=\omega(n)$, $v=\log_2 x$, the result is due to Kubilius [28, Theorem 9.2]. It was later extended by Kubilius [31] to real-valued additive functions $f(n)$ of a somewhat more general type than $\omega(n)$, and Laurinčikas [32] obtained Kubilius's conclusions for such functions under weaker hypotheses. However, both Kubilius and Laurinčikas assumed that $f(p)$ is very near a fixed number λ for "most" primes p. Consequently, a result like Theorem 1.9 (or Theorem 4.27 below) does not follow from their theorems unless the set E satisfies a condition somewhat stronger than

$$\sum_{p\leqslant x,\, p\in E} p^{-1}\log p \sim \sum_{p\leqslant x} p^{-1}\log p \quad \text{as } x\to+\infty.$$

Furthermore, it seems doubtful that our theorems could be obtained by their methods, due to the possibly irregular distribution of E (see the comments on p. 168 of [28]).

Whereas Kubilius and Laurinčikas used probabilistic methods, our proof of Theorem 1.9 does not require the use of any idea from probability theory. One reason for this is the availability of powerful results of Halász [18], [19] on the local distribution of $\omega(n; E)$ and $\Omega(n; E)$, the proofs of which require only classical real and complex analysis and some prime number theory. We obtain Theorem 1.9 by combining his results with certain estimates for partial sums of the exponential series (Lemma 4.20). As we showed in [38], the latter estimates can be obtained by an elementary *ad hoc* method. However, it was also shown in [38] that they can be derived (in a slightly weaker form) from the difficult Cramér–Petrov theorem on large deviations of sums of independent random variables. Thus there does exist a connection between the present work and probability theory, and Theorem 1.9 (slightly weakened by the requirement of additional assumptions that $v=v(x; E)\to+\infty$ and $\beta=o(v^{\frac{1}{6}})$ as $x\to+\infty$) can be regarded as apparently the first application of the Cramér–Petrov theorem to number theory.

Theorem 1.9 will be derived from Theorem 4.27 below, in which the hypotheses and conclusions are slightly weaker. We shall show that the following result is also a corollary of Theorem 4.27:

THEOREM 1.16. *Assume* (1.7) *and* (1.10). *If* $|\beta|\leqslant\min\{v^{\frac{1}{6}}, \Lambda^{-1}v^{\frac{1}{6}}\}$, *then*

$$T_v(x,\beta; E, g) = G(\beta)+O\,(\exp(-\beta^2/2)\{\beta^2+\Lambda\}v^{-\frac{1}{2}}), \tag{1.17}$$

and hence if β is any real number, we have

$$T_v(x,\beta; E, g) = G(\beta)+O(\Lambda v^{-\frac{1}{2}}). \tag{1.18}$$

In (1.17) *and* (1.18), *the implied constants are absolute.*

Since Lemma 4.4 shows that

$$1-G(|\beta|) = G(-|\beta|) \sim (2\pi)^{-\frac{1}{2}}|\beta|^{-1}\exp(-\beta^2/2) \quad \text{as } |\beta| \to +\infty,$$

(1.17) gives an asymptotic formula for $T_v(x, \beta; E, g)$ whenever v and β are functions of x such that $v \to +\infty$ and $\beta = o(\min\{v^{\frac{1}{6}}, \Lambda^{-1}v^{\frac{1}{2}}\})$ as $x \to +\infty$. The estimate (1.18) is much weaker than (1.17) (if $|\beta|$ is moderately large) but holds without restriction on β. Results like (1.18) have been obtained by many authors beginning with Erdős and Kac [15], who showed that if $v = E(x) \to +\infty$ as $x \to +\infty$, then for each fixed real β, $T_v(x, \beta; E, g) \to G(\beta)$ as $x \to +\infty$. They actually proved a more general result on the distribution of values of additive functions, but they did not estimate the difference $T_v(x, \beta; E, g) - G(\beta)$. The latter quantity was estimated in various ways by LeVeque [33], Kubilius [26], [27], Barban [3], Uždavinis [56] (cf. Kubilius [28, pp. 108, 113]), and Barban and A. I. Vinogradov [4]. Rényi and Turán [47] were the first to obtain an error term like that in (1.18) for the special case $E = P$ (and also in a somewhat more general result on additive functions which does not include (1.18)). Certain generalizations of (1.18) for additive functions, with error terms of similar strength, were obtained by N. M. Timofeev [54] (whose proof was incomplete), Elliott [12] (whose result essentially implies (1.18) for $g(n) = \omega(n; E)$ but not for $g(n) = \Omega(n; E)$), Dubovik [10], and Popov [45]. In his book [13, Chap. 20], Elliott obtains a very general theorem on additive functions which implies (1.18) with $v = E(x)$ and either choice of g. For further discussion and other references, see Norton [39].

We now indicate several applications of our main results.

THEOREM 1.19. *Let x, u, β be real numbers with $3 \leq u \leq x$, write* $\log\log u = \log_2 u$, *and define*

$$A(x, u, \beta) = x^{-1}\,\text{card}\,\{n: n \leq x \text{ and } \sum_{p|n,\, p\leq u} 1 \leq \log_2 u + \beta(\log_2 u)^{\frac{1}{2}}\}.$$

If $(-\frac{1}{2})(\log_2 u)^{\frac{1}{2}} \leq \beta \leq 0$, *then*

$$(1.20) \quad A(x, u, \beta) = G(-|\beta|)\exp\{\beta^2/2 + (\log_2 u)Q(\beta(\log_2 u)^{-\frac{1}{2}})\}\{1 + O(\{|\beta|+1\}(\log_2 u)^{-\frac{1}{2}})\}.$$

When $0 \leq \beta \leq \frac{1}{2}(\log_2 u)^{\frac{1}{2}}$, (1.20) *still holds if* $A(x, u, \beta)$ *is replaced by* $1 - A(x, u, \beta)$. *In both cases, the implied constants are absolute.*

To prove this, let $E = \{p: p \leq u\}$, take $v = \log_2 u$, and note that $0 < E(x) - v < 2$ (see Rosser and Schoenfeld [49, (2.10), (3.19), (3.20)]). Hence $\Lambda = 2$, and the result follows from Theorem 4.27. Theorem 1.16 can also be applied here; it shows, for example, that

$$A(x, u, \beta) = G(\beta) + O((\log_2 u)^{-\frac{1}{2}})$$

if $3 \leqslant u \leqslant x$ and β is real. Such results yield interesting estimates for the sizes of certain prime factors of n. This will be the subject of a later paper.

The special case $u = x$, $\beta = o\,((\log_2 x)^{\frac{1}{2}})$ of Theorem 1.19 is due to Kubilius [28, Theorem 9.2] (as mentioned above, his result was extended by Kubilius [31] and Laurinčikas [32], but their extensions do not include Theorem 1.19). Novoselov [40, p. 266] obtained a weaker version of another special case by showing that for each fixed real β, $A(x, \log x, \beta) \to G(\beta)$ as $x \to +\infty$. His method was quite different from ours (he used topological ideas and Liapounov's central limit theorem), and he did not estimate the rate of convergence. Several other authors have used probabilistic methods to obtain results which are superficially related to Theorem 1.19. For example, if q denotes a prime, then (see Billingsley [6, p. 765])

$$\lim_{x \to +\infty} x^{-1} \operatorname{card} \{n: 3 \leqslant n \leqslant x \quad \text{and} \quad \max_{q \leqslant n} \big(\sum_{p \mid n,\, p \leqslant q} 1 - \log_2 q \big) \leqslant \beta (\log_2 n)^{\frac{1}{2}} \} = 2G(\beta) - 1 \tag{1.21}$$

for each fixed $\beta > 0$. For theorems of the same type as (1.21) (some of them stated in greater generality), see Kubilius [28, Theorem 7.3], Babu [1], [2, p. 331], Billingsley [5, pp. 1113–1114], and Philipp [43, pp. 235–236]. (None of these authors estimated the rate of convergence in results like (1.21).)

Another application of Theorems 1.9, 4.27, and 1.16 concerns prime factors lying in various arithmetic progressions with the same modulus. It is an easy consequence of the following result:

LEMMA 1.22. *Let k be a positive integer, and let L be a nonempty set of integers such that for each $l \in L$, we have $1 \leqslant l \leqslant k$ and $(k, l) = 1$. Write* $\operatorname{card} L = \lambda$, *and let*

$$E = \bigcup_{l \in L} \{p: p \equiv l (\operatorname{mod} k)\}.$$

Then for $x \geqslant 2$,

$$E(x) = \lambda \varphi(k)^{-1} \log_2 x + \sum_{p \leqslant x,\, p \in L} p^{-1} + O(\lambda \varphi(k)^{-1} \log (3k)),$$

where the implied constant is absolute, φ is Euler's function, and $\log_2 x = \log \log x$. *Also,*

$$\sum_{p \leqslant x,\, p \in L} p^{-1} \leqslant \log_2 (3\lambda) + O(1).$$

This lemma is due to the author [37, Lemma 6.3], whose proof depended on the Brun–Titchmarsh and Siegel–Walfisz theorems. We refer to [37, pp. 698–701] for background and remarks on how to improve the result in certain cases. It should be noted that the special case $\lambda = 1$ of this lemma was discovered independently by Pomerance [44]. A somewhat weaker version of this special case was obtained earlier by Rieger [48, Hilfssatz 1].

Lemma 1.22 leads immediately to applications of Theorems 1.9, 4.27, and 1.16 with $v = \lambda\varphi(k)^{-1}\log_2 x$. (Note the importance here of the second-order terms in (1.14) and (1.15).) As a rather special illustration, we mention here the following consequence of (1.18) and Lemma 1.22: if k and l are positive integers with $(k, l) = 1$, then for all real x, β with $x \geq 3$, we have

$$\text{(1.23)} \qquad x^{-1}\,\mathrm{card}\,\Big\{n : n \leq x \text{ and } \sum_{p|n,\, p\equiv l (\mathrm{mod}\, k)} 1 \leq \varphi(k)^{-1}\log_2 x + \beta(\varphi(k)^{-1}\log_2 x)^{\frac{1}{2}}\Big\} = G(\beta) + O(\{\varphi(k)/\log_2 x\}^{\frac{1}{2}}),$$

the implied constant being absolute. The error term here is best possible when β is near 0, and this result improves a theorem of Gyapjas and Kátai [17], who obtained (1.23) with the error term $O(c(k)\,(\log_2 x)^{-\frac{1}{2}})$, where $c(k)$ is an unspecified function of k. (It should be noted that Gyapjas and Kátai obtained some similar results which do not follow from ours. Also, Maĭ-Thuk-Ngoĭ and Tuljaganov [36] apparently announced without proof an estimate for the left-hand side of (1.23), but their work has been unavailable to the present author.)

There are also applications to prime factors in arithmetic progressions with different moduli. For example, suppose that $k_1, \ldots, k_r$ are positive integers which are pairwise relatively prime, and suppose $l_1, \ldots, l_r$ are integers such that $1 \leq l_j \leq k_j$ and $(k_j, l_j) = 1$ for $1 \leq j \leq r$. If

$$E = \bigcup_{j=1}^{r} \{p : p \equiv l_j (\mathrm{mod}\ k_j)\},$$

then by successive application of the inclusion–exclusion principle, the Chinese remainder theorem, and Lemma 1.22, we obtain

$$E(x) = \Big\{1 - \prod_{j=1}^{r}(1 - \varphi(k_j)^{-1})\Big\}\log_2 x + O(2^r) \quad \text{for } x \geq 3,$$

the implied constant being absolute. This can be combined with Theorem 1.9, Theorem 1.16, or Theorem 4.27 in an obvious way.

For a final application, suppose that whenever m, n are positive integers, $d_m(n)$ denotes the number of ordered m-tuples $(t_1, \ldots, t_m)$ of positive integers such that $t_1 \ldots t_m = n$. (Thus

$d_2(n) = d(n)$ is the number of distinct positive divisors of n.) Then $m^{\omega(n)} \leqslant d_m(n) \leqslant m^{\Omega(n)}$ (see [37, pp. 683–684]), and hence our main results give asymptotic formulas for large deviations of $\log d_m(n)$ from its normal order $(\log m) \log_2 n$. This application is not new, since it follows from Kubilius's theorems on large deviations of $\omega(n)$ and $\Omega(n)$ (see [28, Theorem 9.2] and [31]). However, our proof is quite different from his (in particular, it requires no probability theory). The result on $\log d_m(n)$ which can be obtained from (1.18) is due to Rényi and Turán [47]. It improves earlier work of Kac [25], LeVeque [33], and Kubilius [26], [27]. See [37, pp. 683–684] for further information on the distribution of $d_m(n)$.

It seems appropriate to indicate the limitations of our main results. Our methods depend heavily on the properties of the particular functions $\omega(n; E)$ and $\Omega(n; E)$, and we have nothing new to say about other additive functions. Furthermore, we are unable to prove similar results concerning the distribution of $\omega(|f(n)|; E)$ or $\Omega(|f(n)|; E)$ (where f is a polynomial with integral coefficients), nor can we deal with the joint distribution of, say, $\omega(n; E)$ and $\omega(n+1; E)$, nor with the distribution of a sum of such functions. Finally, we have not been able to obtain asymptotic formulas for $T_v(x, \beta; E, g)$ in larger β-intervals than those indicated in our main theorems, nor can we derive asymptotic expansions. See [37, § 7] for a few additional remarks along these lines and some references. Further discussion and references will appear in [39].

This work was begun while I held a visiting research position in the Mathematics Department of the University of Geneva. I am grateful to Professor John Steinig, who arranged my visit and helped to make it a pleasant one. The work was continued at the Mathematics Department of the University of York (England), where my visit was financed by a grant from the Science Research Council of Great Britain. I extend my sincere appreciation to Dr Maurice Dodson for arranging this grant. Finally, I thank Professor P. D. T. A. Elliott for stimulating and informative conversations about various aspects of this research.

§ 2. Notation

The symbols k, l, m, n always denote positive integers, while p always means a prime. $v, x, y, \alpha, \beta, \gamma, \delta, \varepsilon$ denote real numbers. $[x]$ means the largest integer $\leqslant x$, and $\log_2 x$ means $\log \log x$. Empty sums mean 0, empty products 1. The notation $x_1 \dots x_m / y_1 \dots y_n$ is sometimes used instead of $(x_1 \dots x_m)(y_1 \dots y_n)^{-1}$.

In this paper, the notations O (without subscripts), $\ll$, $\gg$ always indicate implied constants which are *absolute*. (Thus $A = O(B)$ is equivalent to $A \ll B$.) The notation $O_{\delta, \varepsilon, \dots}$ indicates an implied constant depending at most on $\delta, \varepsilon, \dots$. For $i = 1, 2, \dots, c_i(\delta, \varepsilon, \dots)$

means a positive number depending at most on $\delta, \varepsilon, \ldots$, while c_i means a positive absolute constant.

Most of the remaining symbols and functions were defined in the first few paragraphs of § 1 (prior to (1.12)). A few further notations will be introduced as needed.

§ 3. Upper and lower bounds for $T_v(x, \beta; E, g)$

To avoid constant repetition, *we assume throughout this section that* (1.7) *and* (1.10) *hold*. Our concern here is to obtain upper and lower bounds for $T_v(x, \beta; E, g)$ which generalize somewhat the main results of [37]. Although the inequalities of this section are not quite as precise as our main asymptotic formulas, they are of interest in themselves because they are valid over larger β-intervals and because their proofs are simpler. Furthermore, some of these inequalities will be used in deriving the asymptotic formulas.

Instead of dealing directly with $T_v(x, \beta; E, g)$, we shall find it more convenient to consider (as in [37]) the related functions

$$L_v(x, \delta; E, g) = \text{card}\,\{n: n \leqslant x \quad \text{and } g(n) \leqslant (1-\delta)v\},$$
$$R_v(x, \delta; E, g) = \text{card}\,\{n: n \leqslant x \quad \text{and } g(n) \geqslant (1+\delta)v\},$$

where δ is real. (There should be no confusion between $R_v(x, \delta; E, g)$ and the function $R_v(\beta)$ defined by (1.3).)

Since we shall often make use of the function $Q(t)$ (defined by (1.1)), it seems appropriate to state here the following simple lemma (cf. [37, Lemma 2.1]):

LEMMA 3.1. *$Q(t)$ is strictly increasing on $[-1, 0]$ and strictly decreasing on $[0, +\infty)$ (thus $Q(t) < 0$ for $t \neq 0$). Also,*

$$-t^2 < Q(t) < -t^2/2 \quad \textit{for } -1 < t < 0,$$
$$-t^2/2 < Q(t) < (1 - 2\log 2)t^2 < (-0.386)t^2 \quad \textit{for } 0 < t < 1.$$

Now define

$$N(m, x; E, g) = \text{card}\,\{n: n \leqslant x \quad \text{and } g(n) = m\} \tag{3.2}$$

for $m = 0, 1, 2, \ldots$ We refer to the problem of estimating $N(m, x; E, g)$ as the *local distribution problem* for g. Note the obvious formulas

$$L_v(x, \delta; E, g) = \sum_{0 \leqslant m \leqslant (1-\delta)v} N(m, x; E, g), \tag{3.3}$$

$$R_v(x, \delta; E, g) = \sum_{m \geqslant (1+\delta)v} N(m, x; E, g). \tag{3.4}$$

In order to use these formulas to estimate $L_v(x, \delta; E, g)$ and $R_v(x, \delta; E, g)$, we need the following remarkable result of Halász [19]:

LEMMA 3.5. *Let* $0<\delta\leqslant 2$. *If* $0\leqslant m\leqslant (2-\delta)E(x)$, *then*

$$N(m, x; E, g)\leqslant c_2(\delta)\, x\frac{E(x)^m}{m!}e^{-E(x)}. \tag{3.6}$$

Furthermore, if $E(x)\geqslant c_3(\delta)$ *and* $0\leqslant m\leqslant (2-\delta)E(x)$, *then*

$$N(m, x; E, g)+N(m+1, x; E, g)\geqslant c_4(\delta)\, x\frac{E(x)^m}{m!}e^{-E(x)}. \tag{3.7}$$

In [19], Halász proved Lemma 3.5 only for $g(n)=\Omega(n; E)$. His proof can be extended to the case $g(n)=\omega(n; E)$; see Norton [37, pp. 687–689] for a few remarks on this extension and on the literature dealing with such results, and see also the remarks below and in § 4. (In [37], the hypothesis $E(x)\geqslant c_3(\delta)$ was replaced by a less general condition.) It should be noted that Sárközy [51] has recently improved (3.7) by showing that the left-hand side can be replaced by $N(m, x; E, g)$ when $\delta E(x)\leqslant m\leqslant(2-\delta)E(x)$. We shall not need Sárközy's result in this paper.

The first uniform upper bounds like (3.6) were proved by Hardy and Ramanujan [23] (reprinted in [46, pp. 262–275]) for the special case $E=P$ (the set of all primes). Since their results have been stated and applied incorrectly several times in the subsequent literature, it seems appropriate to mention here that they proved the sharp estimate

$$N(m, x; P, \omega)\leqslant\frac{c_5\, x(\log_2 x+c_6)^{m-1}}{(m-1)!\,\log x}\quad\text{for } x\geqslant 2, m=1, 2, \ldots, \tag{3.8}$$

but they observed that an inequality of this strength *does not hold for* $N(m, x; P, \Omega)$ without some restriction on the size of m. In [23, Lemma C], they obtained an inequality for $N(m, x; P, \Omega)$ which is more complicated than (3.8); from this, it can be deduced that if $x\geqslant 3$ and $\delta>0$, then

$$N(m, x; P, \Omega)\leqslant\frac{c_7\delta^{-1}x(\log_2 x+c_8)^{m-1}}{(m-1)!\,\log x}\quad\text{for } 1\leqslant m\leqslant(10/9-\delta)\log_2 x. \tag{3.9}$$

An inequality similar to (3.9) (with $c_7\delta^{-1}$ replaced by $c_9(\delta)$) was later shown by Sathe [52, IV, p. 77, (iv)] to hold over the wider range $1\leqslant m\leqslant(2-\delta)\log_2 x$. Sathe actually obtained an asymptotic formula. Selberg [53] gave a different proof of Sathe's result and showed [53,

p. 87] that such an upper estimate for $N(m, x; P, \Omega)$ does not hold if $(2+\delta)\log_2 x \leqslant m \leqslant c_{10}\log_2 x$.

The Hardy–Ramanujan estimates (3.8) and (3.9) have recently been generalized in one sense by Warlimont and Wolke [57], who estimated from above the number of integers n such that $y < n \leqslant y+x$ and $\omega(n)=\Omega(n)=m$. Upper bounds for the local distribution of more general additive functions have been given by Erdős, Ruzsa, and Sárközy [16] and by Halász [20] (see also Ruzsa [50] for related work, and see § 4 below).

The first uniform lower bounds like (3.7) were obtained independently by Erdős and S. S. Pillai for the special case $E=P$. Pillai's work was done about 1940 but apparently was never published; the standard reference for a statement of his results is [22, p. 56]. Erdős [14] actually obtained asymptotic formulas for $N(m, x; P, \omega)$ and $N(m, x; P, \Omega)$ in the range $|m-\log_2 x| \leqslant c_{11}(\log_2 x)^{\frac{1}{2}}$. For further comments on asymptotic formulas for local distribution, see § 4 below.

In order to use (3.3) and (3.4), we first obtain a more convenient form of Lemma 3.5. (Recall that Λ is defined by (1.8).)

LEMMA 3.10. *Let* $0<\beta\leqslant\delta\leqslant 2$. *If*

$$0 \leqslant m \leqslant (2-\delta)v, \tag{3.11}$$

then

$$N(m, x; E, g) \leqslant c_{12}(\beta)\, x \frac{v^m}{m!} e^{-v+\Lambda}. \tag{3.12}$$

Furthermore, if $E(x) \geqslant c_{13}(\beta)$, *if*

$$v \geqslant 3\beta^{-1}\Lambda, \tag{3.13}$$

and if (3.11) *holds, then*

$$N(m, x; E, g) + N(m+1, x; E, g) \geqslant c_{14}(\beta)\, x \frac{v^m}{m!} e^{-v-\Lambda}. \tag{3.14}$$

Proof. Write $E(x)=v+z$, so $|z| \leqslant \Lambda$. First we assert that

$$0 \leqslant m \leqslant (2-\beta/3)\,E(x) \quad \text{if (3.11) and (3.13) hold.} \tag{3.15}$$

For we have $E(x) \geqslant v-\Lambda \geqslant (3\beta^{-1}-1)\Lambda$, so $v \leqslant E(x)+\Lambda \leqslant 3(3-\beta)^{-1}E(x)$, and (3.15) follows easily from the inequality $0 \leqslant m \leqslant (2-\beta)v$.

Assume in this paragraph that (3.11) and (3.13) both hold. By (3.15) and (3.6),

$$N(m, x; E, g) \leqslant c_2(\beta/3)\, x \frac{v^m}{m!} e^{-v}(1+z/v)^m e^{-z}.$$

But

$$e^{z/v} \geqslant 1 + z/v \geqslant 1 - \Lambda/v \geqslant 1/3$$

by (3.13), so

$$(1+z/v)^m e^{-z} \leqslant e^{z(m/v-1)} \leqslant e^{\Lambda}.$$

Hence (3.12) follows in this case. Furthermore, if $E(x) \geqslant c_3(\beta/3)$, then by (3.15) and (3.7),

$$N(m, x; E, g) + N(m+1, x; E, g) \geqslant c_4(\beta/3)\, x \frac{v^m}{m!} e^{-v}(1+z/v)^m e^{-z}.$$

We now apply the inequality $\log(1+y) \geqslant y(1+y)^{-1}$ (valid for $y > -1$). Since $z/v \geqslant -2/3$, we get

$$m \log(1+z/v) - z \geqslant z\left(\frac{m}{v+z} - 1\right) = z\left(\frac{m}{E(x)} - 1\right).$$

By (3.15), $z(m/E(x)-1) \geqslant -\Lambda$, and (3.14) follows.

It remains to be shown that (3.12) holds under the assumptions (3.11) and

$$0 < v < 3\beta^{-1}\Lambda. \tag{3.16}$$

If $m=0$, then (3.12) follows directly from (3.6), so we assume $m \geqslant 1$. It may not be true that $m \leqslant (2-\varepsilon)E(x)$ for some $\varepsilon > 0$, so we can no longer use (3.6). However, we assert that

$$y^m N(m, x; E, g) \leqslant \sum_{n \leqslant x} y^{g(n)} \leqslant c_{15}(\beta)\, x e^{(y-1)E(x)} \quad \text{for } 0 < y \leqslant 2-\beta.$$

The first of these inequalities is trivial, while the second follows for $y \leqslant 1$ from an elementary result of Hall [21] and for $1 \leqslant y \leqslant 2-\beta$ from Norton [37, Lemmas 3.10, 3.11] (or from a much more general and difficult result of Halász [18, Theorem 2]). From this, we immediately obtain

$$N(m, x; E, g) \leqslant c_{15}(\beta) x \exp\{(y-1)v - m\log y + |y-1|\Lambda\}$$

for $0 < y \leqslant 2-\beta$ and any m. The right-hand side is approximately minimized by taking $y = m/v$ (which is permissible by (3.11)). Using this value of y and applying Stirling's formula, we obtain

$$N(m, x; E, g) \leqslant c_{16}(\beta)\, x \frac{v^m}{m!} e^{-v} m^{\frac{1}{2}} \exp\left(\left|\frac{m}{v} - 1\right|\Lambda\right). \tag{3.17}$$

Considering separately the cases $1 \leqslant m \leqslant v$, $v \leqslant m \leqslant (2-\beta)v$, and using (3.16), we find that (3.12) follows from (3.17). Q.E.D.

THEOREM 3.18. *If* $0<\delta<1$, *then*

$$L_v(x,\delta;\, E,g) \ll \delta^{-1}(1-\delta)^{-\frac{1}{2}}xv^{-\frac{1}{2}}e^{Q(-\delta)v+\Lambda}.$$

Proof. Combine (3.3) and (3.12) (with $\beta=1$ and δ replaced by $\delta'=1+\delta$). The result then follows immediately from [37, Lemma 4.5]. Q.E.D.

THEOREM 3.19. *Suppose that* $E(x)\geqslant c_{17}$, $v\geqslant 3\Lambda$, *and* $v^{-\frac{1}{2}}\leqslant\delta\leqslant 1-3v^{-1}$. *Then*

$$L_v(x,\delta;\, E,g) \gg \delta^{-1}(1-\delta)^{\frac{3}{2}}xv^{-\frac{1}{2}}e^{Q(-\delta)v-\Lambda}.$$

Proof. Define $n=[(1-\delta)v]-1$. By (3.3),

$$L_v(x,\delta;E,g) \gg \sum_{m=0}^{n}\{N(m,x;E,g)+N(m+1,x;E,g)\}.$$

We combine this with (3.14), taking $\beta=1$ and replacing δ by $\delta'=1+\delta$. The result then follows from [37, Lemma 4.6]. Q.E.D.

THEOREM 3.20. *If* $0<\delta\leqslant\beta<1$, *then*

$$R_v(x,\delta;\, E,g) \leqslant c_{18}(\beta)\delta^{-1}xv^{-\frac{1}{2}}e^{Q(\delta)v+\Lambda}.$$

Proof. Use (3.4), (3.12), and the method of proof of [37, Theorem 5.12]. Q.E.D.

THEOREM 3.21. *Suppose that* $0<\beta<1$, $E(x)\geqslant c_{19}(\beta)$, $v\geqslant 4(1-\beta)^{-1}\Lambda$, *and* $v^{-\frac{1}{2}}\leqslant\delta\leqslant\beta$. *Then*

$$R_v(x,\delta;\, E,g) \geqslant c_{20}(\beta)\delta^{-1}xv^{-\frac{1}{2}}e^{Q(\delta)v-\Lambda}.$$

Proof. Write $n=[(1+\delta)v]+1$, $\gamma=(2v\delta)^{-1}$. By (3.4),

$$R_v(x,\delta;E,g) \gg \sum_{n\leqslant m\leqslant(1+\gamma)n}\{N(m,x;E,g)+N(m+1,x;E,g)\}. \tag{3.22}$$

Since $v>4\Lambda$, we have $E(x)\leqslant v+\Lambda<\frac{5}{4}v$. Also, $v\delta\geqslant v^{\frac{1}{2}}$. Hence if $c_{19}(\beta)$ is sufficiently large,

$$\begin{aligned}(1+\gamma)n &\leqslant v\{1+\delta+v^{-1}+(2v\delta)^{-1}+(2v)^{-1}+(2v^2\delta)^{-1}\}\\ &\leqslant v\{1+\beta+\tfrac{1}{4}(1-\beta)\}=v\{2-\tfrac{3}{4}(1-\beta)\}.\end{aligned}$$

We apply Lemma 3.10 with β replaced by $\beta'=\frac{3}{4}(1-\beta)$, and we assume $c_{19}(\beta)\geqslant c_{13}(\beta')$. Since $v\geqslant 4(1-\beta)^{-1}\Lambda\geqslant 3(\beta')^{-1}\Lambda$, it follows from (3.14) that

$$N(m,x;E,g)+N(m+1,x;E,g)\geqslant c_{14}(\beta')\,x\frac{v^m}{m!}e^{-v-\Lambda}$$

for each m such that $n \leqslant m \leqslant (1+\gamma)n$. Substituting this estimate in (3.22) and using [37, Lemma 4.8], we get the result. Q.E.D.

In order to obtain the asymptotic formulas of Theorem 4.27 and Theorem 1.9, we need the following slight sharpening of the upper bounds given in Theorems 3.18 and 3.20:

THEOREM 3.23. *If* $0 < \delta \leqslant \frac{1}{2}$ *and*

$$v \geqslant 3\delta^{-1}\Lambda, \tag{3.24}$$

then

$$L_v(x, \delta; E, g) \ll \delta^{-1} x v^{-\frac{1}{2}} e^{Q(-\delta)v + \delta\Lambda}, \tag{3.25}$$

$$R_v(x, \delta; E, g) \ll \delta^{-1} x v^{-\frac{1}{2}} e^{Q(\delta)v + \delta\Lambda}. \tag{3.26}$$

Proof. Write $E(x) = v + z$. By (3.24),

$$v \geqslant 6\Lambda \geqslant 6|z|, \tag{3.27}$$

so

$$5v/6 \leqslant E(x) \leqslant 7v/6. \tag{3.28}$$

Define γ and ε by

$$(1-\delta)v = (1-\gamma)E(x), \quad (1+\delta)v = (1+\varepsilon)E(x), \tag{3.29}$$

so

$$\gamma = \delta + (1-\delta)z(v+z)^{-1}, \quad \varepsilon = \delta - (1+\delta)z(v+z)^{-1}. \tag{3.30}$$

By (3.27) and (3.24),

$$|(1+\delta)z(v+z)^{-1}| \leqslant \tfrac{3}{2}|z|(5v/6)^{-1} \leqslant 3\delta/5. \tag{3.31}$$

Hence

$$2\delta/5 \leqslant \gamma \leqslant 8\delta/5 \leqslant \tfrac{4}{5}, \tag{3.32}$$

$$2\delta/5 \leqslant \varepsilon \leqslant 8\delta/5 \leqslant \tfrac{4}{5}. \tag{3.33}$$

Now by (3.29), (3.3), and (3.4),

$$L_v(x, \delta; E, g) = L_{E(x)}(x, \gamma; E, g), \tag{3.34}$$

$$R_v(x, \delta; E, g) = R_{E(x)}(x, \varepsilon; E, g). \tag{3.35}$$

We apply Theorem 3.18 and use (3.32) and (3.28) to get

$$L_{E(x)}(x, \gamma; E, g) \ll \gamma^{-1} x E(x)^{-\frac{1}{2}} e^{Q(-\gamma)E(x)} \ll \delta^{-1} x v^{-\frac{1}{2}} e^{Q(-\gamma)E(x)}. \tag{3.36}$$

Now if $a > -1$ and $b > -1$, Taylor's theorem yields

$$Q(b) = Q(a) + (a-b)\log(1+a) - (a-b)^2/2(1+\xi), \tag{3.37}$$

where ξ is between a and b. We take $a = -\delta$, $b = -\gamma$, and use (3.30) to get

$$Q(-\gamma) \leqslant Q(-\delta) + (1-\delta) z (v+z)^{-1} \log(1-\delta).$$

By (1.1), it follows that

$$Q(-\gamma)E(x) = Q(-\gamma)(v+z) \leqslant Q(-\delta)v - \delta z \leqslant Q(-\delta)v + \delta\Lambda. \tag{3.38}$$

Combining (3.34), (3.36), and (3.38), we obtain (3.25). (3.26) can be obtained in the same way from (3.35) and Theorem 3.20. Q.E.D.

The next theorem is not needed later but is an interesting complement to Theorem 3.23.

THEOREM 3.39. *Suppose that* $E(x) \geqslant c_{21}$, $v \geqslant \Lambda^2$, *and* $v^{-\frac{1}{2}} \leqslant \delta \leqslant \frac{1}{2}$. *Then*

$$L_v(x, \delta; E, g) \gg \delta^{-1} x v^{-\frac{1}{2}} e^{Q(-\delta)v - 2\delta\Lambda}, \tag{3.40}$$

$$R_v(x, \delta; E, g) \gg \delta^{-1} x v^{-\frac{1}{2}} e^{Q(\delta)v - 2\delta\Lambda}. \tag{3.41}$$

Proof. Write $E(x) = v + z$. We shall show that the theorem holds if we take

$$c_{21} = \max\{c_{17}, c_{19}(\tfrac{4}{5}), 42\}. \tag{3.42}$$

First suppose that $3v^{-\frac{1}{2}} \leqslant \delta \leqslant \frac{1}{2}$. The hypothesis $v \geqslant \Lambda^2$ then shows that (3.24) holds. (3.27) and (3.28) follow, and if we define γ and ε by (3.29), so do all of the remaining steps in the proof of Theorem 3.23. By (3.32) and (3.28),

$$\gamma \geqslant 2\delta/5 \geqslant \tfrac{6}{5} v^{-\frac{1}{2}} > E(x)^{-\frac{1}{2}}.$$

We apply Theorem 3.19 with v replaced by $E(x)$, $\Lambda = 2$, and δ replaced by γ. (3.42) and (3.32) show that the hypotheses of Theorem 3.19 are satisfied, and the result is

$$L_{E(x)}(x, \gamma; E, g) \gg \gamma^{-1} x E(x)^{-\frac{1}{2}} e^{Q(-\gamma)E(x)}. \tag{3.43}$$

We now apply (3.37) with $a = -\delta$, $b = -\gamma$, then use (3.30) to get

$$Q(-\gamma) = Q(-\delta) + (1-\delta)z(v+z)^{-1}\log(1-\delta) - \frac{(1-\delta)^2 z^2}{(v+z)^2 \cdot 2(1+\xi)},$$

where $\xi \geqslant -\frac{4}{5}$. By (3.27) and (3.24),

$$\frac{(1-\delta)^2 z^2}{2(v+z)(1+\xi)} \leqslant 3\Lambda^2 v^{-1} \leqslant \delta\Lambda,$$

so

$$Q(-\gamma)E(x) = Q(-\gamma)(v+z) \geqslant Q(-\delta)v - 2\delta\Lambda.$$

(3.40) follows from this and (3.43), (3.34), (3.28), and (3.32). Still assuming that $3v^{-\frac{1}{2}} \leqslant \delta \leqslant \frac{1}{2}$, we can derive (3.41) in the same way from Theorem 3.21 (with $\beta = \frac{4}{5}$).

Finally, suppose that $v^{-\frac{1}{2}} \leqslant \delta \leqslant 3v^{-\frac{1}{2}} = \delta_0$. (Note that $\delta_0 \leqslant \frac{1}{2}$ by (3.28) and (3.42).) Using what we have just proved and the fact that $Q(t) \ll t^2$ for $|t| \leqslant 1$ (see (1.2) or Lemma 3.1), we find that

$$L_v(x, \delta; E, g) \geqslant L_v(x, \delta_0; E, g) \gg x \gg \delta^{-1} x v^{-\frac{1}{2}} e^{Q(-\delta)v - 2\delta\Lambda},$$

and similarly for $R_v(x, \delta; E, g)$. Q.E.D.

§ 4. Preliminary asymptotic formulas

We assume throughout this section that (1.7) *and* (1.10) *hold*. Up to this point, our work has been based on Halász [19] and Norton [37] and has been essentially elementary. However, the upper and lower bounds given in § 3 are so near to each other as to suggest the existence of asymptotic formulas, and in order to obtain such formulas, we need to use the more difficult results of Halász [18]. These we shall combine with one of the main theorems of Norton [38], which is essentially a special case of the Cramér–Petrov theorem of probability theory but which can also be proved in an elementary way (as was shown in [38]). Before beginning this work, we state here two easy lemmas from [38, § 2]. (Recall that $G(\beta)$ is defined by (1.4).)

LEMMA 4.1. *For any positive real numbers v and z, define*

$$h_v(z) = e^{-v}(ev/z)^z z^{-\frac{1}{2}} = z^{-\frac{1}{2}} \exp\{vQ(zv^{-1}-1)\}. \tag{4.2}$$

Then for any positive integer m, we have

$$(2\pi)^{-\frac{1}{2}} h_v(m)(1-1/12m) < e^{-v}v^m/m! < (2\pi)^{-\frac{1}{2}} h_v(m). \tag{4.3}$$

LEMMA 4.4. *For real $y \neq 0$ and $n = 0, 1, 2, \ldots$, we have*

$$\exp(y^2/2)G(-|y|) = T_n(y) + O(1\cdot 3\cdot 5\cdots(2n-1)|y|^{-2n-1}), \tag{4.5}$$

where

$$T_n(y) = (2\pi)^{-\frac{1}{2}} \sum_{m=1}^{n} (-1)^{m-1} 1\cdot 3\cdot 5\cdots(2m-3)|y|^{-2m+1}. \tag{4.6}$$

Hence

$$1 \ll \exp(y^2/2)G(-|y|)\{|y|+1\} \ll 1 \quad \text{for all real } y. \tag{4.7}$$

We need the following deep and beautiful result of Halász [18], which should be compared with Lemma 3.5 above.

LEMMA 4.8. *Let $0 < \delta \leq 1$. If $E(x) \geq 2$ and $\delta E(x) \leq m \leq (2-\delta)E(x)$, then*

$$N(m, x; E, g) = x\frac{E(x)^m}{m!} e^{-E(x)}\{1 + O_\delta(|mE(x)^{-1} - 1| + E(x)^{-\frac{1}{2}})\}.$$

In [18], Halász proves Lemma 4.8 only for the function $g(n) = \Omega(n; E)$. The proof is based on Theorems 2 and 3 of [18], which we need not state here. These theorems give estimates for $\sum_{n \leq x} f(n)$, where f is a complex-valued completely multiplicative function (i.e., $f(mn) = f(m)f(n)$ for all positive integers m, n). In his application to Lemma 4.8, Halász takes $f(n)$ to be $z^{\Omega(n;E)}$, where z is complex. In order to establish Lemma 4.8 for $g(n) = \omega(n; E)$, one needs to consider $z^{\omega(n;E)}$ instead of $z^{\Omega(n;E)}$, and hence it is necessary to generalize Halász's Theorems 2 and 3 to the case in which f is merely multiplicative. This can be done by considering the completely multiplicative function f^* determined by defining $f^*(p) = f(p)$ for all p. Let h be the multiplicative function determined by taking $h(p^c) = f(p^c) - f(p)f(p^{c-1})$ (for each prime p and $c = 1, 2, \ldots$). It is then easy to verify that $f(n) = \sum_{d|n} h(d)f^*(n/d)$ for all n (each side of this identity is a multiplicative function). If $x \geq y \geq 1$, it follows that

$$\sum_{n \leq x} f(n) = \sum_{lm \leq x} h(l)f^*(m) = \sum_{l \leq y} h(l) \sum_{m \leq x/l} f^*(m) + \sum_{y < l \leq x} h(l) \sum_{m \leq x/l} f^*(m). \tag{4.9}$$

It is convenient to take $y = x^{\frac{1}{2}}$. Then the inner sums on the right-hand side of (4.9) can be estimated by using Theorem 2 or Theorem 3 of [18], and it turns out that if $\sum_{l=1}^{\infty} |h(l)| l^{-\sigma}$ converges for some $\sigma < 1$, each of those theorems has a generalization of the desired type (i.e., for functions which are multiplicative but not completely multiplicative). The details are elementary but a bit lengthy. Finally, the proof of Lemma 4.8 for $g(n) = \omega(n; E)$ is completed as in [18, pp. 230–232].

The first uniform asymptotic formula similar to Lemma 4.8 was obtained by Erdős [14] for the special case $E=P$ (the set of all primes). He showed that

$$N(m, x; P, \omega) = \{1+o(1)\}\frac{x(\log_2 x)^{m-1}}{(m-1)!\log x} \quad \text{as } x\to+\infty$$

for $|m-\log_2 x| \leqslant c_{11}(\log_2 x)^{\frac{1}{2}}$, and he remarked that his methods establish the same result for $N(m, x; P, \Omega)$. Erdős's results were improved by Sathe [52, IV, pp. 77, 79], who obtained the formula

$$N(m, x; P, g) = B_g(m/\log_2 x)\frac{x(\log_2 x)^{m-1}}{(m-1)!\log x}\left\{1+O_\delta\left(\frac{1}{\log_2 x}\right)\right\}$$

for $\delta>0$, $1\leqslant m\leqslant(2-\delta)\log_2 x$, where

$$B_\omega(z) = \frac{1}{\Gamma(z+1)}\prod_p\left\{\left(1-\frac{1}{p}\right)^z\left(1+\frac{z}{p-1}\right)\right\},$$

$$B_\Omega(z) = \frac{1}{\Gamma(z+1)}\prod_p\left\{\left(1-\frac{1}{p}\right)^z\left(1-\frac{z}{p}\right)^{-1}\right\},$$

and Γ is the gamma function. It is easy to see that these results of Sathe imply a slightly more precise form of Lemma 4.8 when $E=P$, the error term $O_\delta(E(x)^{-\frac{1}{2}})$ being replaced by $O_\delta(1/\log_2 x)$. Sathe's proof was essentially elementary but very lengthy and complicated. A simpler but nonelementary proof was given by A. Selberg [53], whose analytic method formed the basis for much of the later work in this area. Delange [8] stated without proof a generalization of the Sathe-Selberg formulas to the case in which P is replaced by the set E_1 of all primes in a union of finitely many arithmetic progressions. He stated also an asymptotic expansion for $N(m, x; E_1, g)$, provided m is *fixed*. (In a later paper [9, § 6.5], Delange obtained general theorems which he asserted were enough to prove all of the results in [8].)

Levin and Faĭnleĭb [35, Theorem 2.2.3] obtained an asymptotic expansion of card $\{n: n\leqslant x \text{ and } h(n)=m\}$ for fixed m, where h is an additive function which takes positive integer values and the numbers $h(p)$ are distributed fairly regularly. (See also Delange [9, § 5.1 and § 6.5] for results on this problem.) The theorem of Levin and Faĭnleĭb applies to $h(n)=\omega(n)$, for example, but not to $h(n)=\omega(n; E)$ unless E has sufficiently regular distribution. (The same comment applies to Delange's results in [9].)

Kubilius [29] derived asymptotic formulas and asymptotic expansions for card $\{n: n\leqslant x \text{ and } f(n)=m\}$ which are uniform in m, where f is an integral-valued additive function such that $f(p)$ is "usually" equal to 1. In particular, his Theorem 3 leads to a slightly more precise version of Lemma 4.8 under the rather restrictive conditions $\sum_{p \notin E} p^{-1} \log p < +\infty$ and $m \sim \log_2 x$ (as $x \to +\infty$).

Many other authors have contributed results on $N(m, x; E, g)$ and similar functions, although none except Sárközy [51] has achieved the generality with respect to E that is evident in Halász's theorems (Lemmas 3.5 and 4.8 above). For references to some of this related work, see Norton [37, p. 688]. Additional work (not mentioned in [37]) concerning asymptotic formulas for local distribution of additive functions has been done by S. Selberg (1940, 1942, 1943, 1947, 1951), Richert (1953), Rényi (1955), Hornfeck (1956), Delange (1957), Rieger (1958), Lu Hong-Wen (1964), Kubilius [28], Kalecki (1965), Kátai (1969), Faĭnleĭb (1970), Kubilius (1970) and [30], and Lucht (1970). Specific references for the papers listed only by date can be found in LeVeque [34, Sections N24, N28, N60].

The form of Lemma 4.8 is a little awkward for our purposes. The following corollary will be more convenient:

Lemma 4.10. *If*

$$v \geqslant 3\Lambda^2 \tag{4.11}$$

and

$$|m - v| \leqslant \Lambda^{-1} v, \tag{4.12}$$

then

$$N(m, x; E, g) = x \frac{v^m}{m!} e^{-v} \{1 + O(\Lambda\{|mv^{-1} - 1| + v^{-\frac{1}{2}}\})\}, \tag{4.13}$$

and hence

$$N(m, x; E, g) = (2\pi v)^{-\frac{1}{2}} x \exp\{vQ(mv^{-1} - 1)\}\{1 + O(\Lambda\{|mv^{-1} - 1| + v^{-\frac{1}{2}}\})\}. \tag{4.14}$$

Proof. Write $E(x) = v + z$. By (4.11),

$$v \geqslant 6\Lambda \geqslant 6|z|, \tag{4.15}$$

so

$$10 \leqslant 5v/6 \leqslant E(x) \leqslant 7v/6. \tag{4.16}$$

By (4.12) and (4.16),

$$3E(x)/7 \leqslant v/2 \leqslant m \leqslant 3v/2 \leqslant 9E(x)/5,$$

so we can apply Lemma 4.8 with $\delta = \frac{1}{5}$. Observe that by (4.11),

$$|m - E(x)| \leqslant |m - v| + \Lambda < |m - v| + v^{\frac{1}{2}},$$

so by Lemma 4.8 and (4.16),

$$N(m, x; E, g) = x \frac{v^m}{m!} e^{-v}(1 + z/v)^m e^{-z}\{1 + O(|mv^{-1} - 1| + v^{-\frac{1}{2}})\}. \tag{4.17}$$

Using (4.15), (4.12), and (4.11), we get

$$\begin{aligned} m \log(1 + z/v) - z &= m\{z/v + O(z^2/v^2)\} - z \\ &= (mv^{-1} - 1)z + O(z^2/v) \ll \Lambda(|mv^{-1} - 1| + v^{-\frac{1}{2}}) \ll 1. \end{aligned}$$

It follows that

$$(1 + z/v)^m e^{-z} = 1 + O(\Lambda\{|mv^{-1} - 1| + v^{-\frac{1}{2}}\}),$$

and thus (4.17) implies (4.13). Finally, (4.14) follows from (4.13), Lemma 4.1, and the fact that

$$m^{-\frac{1}{2}} = \{v(1 + mv^{-1} - 1)\}^{-\frac{1}{2}} = v^{-\frac{1}{2}}\{1 + O(|mv^{-1} - 1|)\}. \qquad \text{Q.E.D.}$$

We need to state one more preliminary result. When v and β are real numbers with $v > 0$, we write

$$v_\beta = v + \beta v^{\frac{1}{2}}, \tag{4.18}$$

$$S_v(\beta) = \sum_{0 \leqslant m \leqslant v_\beta} \frac{e^{-v} v^m}{m!}. \tag{4.19}$$

Recall that the functions $R_v(\beta)$, $F_v(\beta)$ are defined by (1.3) and (1.5), respectively.

LEMMA 4.20. *Let v, ε be real with $v \geqslant 10$, $\frac{2}{3} \leqslant \varepsilon \leqslant 1 - v^{-\frac{1}{2}}$. If $-\varepsilon v^{\frac{1}{2}} \leqslant \beta \leqslant 0$, then*

$$|S_v(\beta) - F_v(\beta)| < 0.8(1 - \varepsilon)^{-\frac{1}{2}} R_v(\beta) v^{-\frac{1}{2}}. \tag{4.21}$$

If $0 \leqslant \beta \leqslant v^{\frac{1}{2}}$, then

$$|1 - S_v(\beta) - F_v(\beta)| < 0.7 R_v(\beta) v^{-\frac{1}{2}}. \tag{4.22}$$

This is Theorem 1.8 of Norton [38], where an elementary proof was given. As was observed there, the constant factors $0.8(1 - \varepsilon)^{-\frac{1}{2}}$ and 0.7 are not far from best possible, but their values are irrelevant for our present purpose (in view of the undetermined constants in Lemmas 3.5 and 4.8). When v is large and $\beta = o(v^{\frac{1}{2}})$, a slightly less specific version of Lemma 4.20 can be derived from the Cramér–Petrov theorem on large deviations of sums of independent random variables. For proofs of the Cramér–Petrov theorem, see Cramér [7], Petrov [41], [42, Chap. 8 and p. 323], and Ibragimov and Linnik [24, Chaps. 6, 7, 8]. For an expository account of the Cramér–Petrov theorem and its connection with Lemma 4.20, see Norton [38, §§ 3, 4].

LEMMA 4.23. *Define* $T_v(x, \beta; E, g)$ *by* (1.11). *If* $-\Lambda^{-1}v^{\frac{1}{2}} \leqslant \alpha \leqslant \beta \leqslant 0$, *then*

$$T_v(x, \beta; E, g) - T_v(x, \alpha; E, g) = F_v(\beta) - F_v(\alpha) + O(\Lambda R_v(\beta) v^{-\frac{1}{2}}). \tag{4.24}$$

If $0 \leqslant \alpha \leqslant \beta \leqslant \Lambda^{-1}v^{\frac{1}{2}}$, *then*

$$T_v(x, \beta; E, g) - T_v(x, \alpha; E, g) = F_v(\alpha) - F_v(\beta) + O(\Lambda R_v(\alpha) v^{-\frac{1}{2}}). \tag{4.25}$$

Proof. First suppose that $v < 3\Lambda^2$. By (1.2) (or Lemma 3.1), $Q(t) \ll t^2$ for $|t| < 1$. Hence if γ is real with $|\gamma| \leqslant \Lambda^{-1}v^{\frac{1}{2}}$, we have $vQ(\gamma v^{-\frac{1}{2}}) \ll 1$, so $1 \ll R_v(\gamma) \ll 1$ and $1 \ll F_v(\gamma) \ll 1$. Thus (4.24) and (4.25) both follow from the trivial inequalities $0 \leqslant T_v(x, \gamma; E, g) \leqslant 1$.

For the remainder of the proof, assume that $v \geqslant 3\Lambda^2$. Our starting point is the obvious formula (cf. (4.18))

$$T_v(x, \beta; E, g) = \sum_{0 \leqslant m \leqslant v_\beta} x^{-1} N(m, x; E, g). \tag{4.26}$$

We shall prove only (4.24), since the proof of (4.25) is almost identical. Let $k = [v_\alpha] + 1$, $l = [v_\beta]$. Suppose that $-\Lambda^{-1}v^{\frac{1}{2}} \leqslant \alpha \leqslant \beta \leqslant 0$, so $v(1 - \Lambda^{-1}) \leqslant v_\alpha \leqslant v_\beta \leqslant v$. It follows from Lemma 4.10 that for $k \leqslant m \leqslant l$,

$$x^{-1} N(m, x; E, g) = \frac{e^{-v} v^m}{m!} \{1 + O(\Lambda\{1 - mv^{-1} + v^{-\frac{1}{2}}\})\}.$$

If $k \leqslant l$, it follows from this, (4.26), and (4.19) that

$$\begin{aligned} T_v(x, \beta; E, g) - T_v(x, \alpha; E, g) &= \sum_{m=k}^{l} x^{-1} N(m, x; E, g) \\ &= S_v(\beta) - S_v(\alpha) + O\left(\Lambda\left\{\frac{e^{-v}v^l}{l!} - \frac{e^{-v}v^{k-1}}{(k-1)!}\right\} + \Lambda v^{-\frac{1}{2}}\{S_v(\beta) - S_v(\alpha)\}\right). \end{aligned}$$

Furthermore, by Lemmas 4.1 and 3.1, $e^{-v}v^l/l! \ll R_v(\beta) v^{-\frac{1}{2}}$. By (4.7) and (4.21), $S_v(\beta) \ll R_v(\beta)$. Thus if $k \leqslant l$, we get

$$T_v(x, \beta; E, g) - T_v(x, \alpha; E, g) = S_v(\beta) - S_v(\alpha) + O(\Lambda R_v(\beta) v^{-\frac{1}{2}}),$$

and this is trivial if $k > l$ (in which case $k = l+1$ and

$$T_v(x, \beta; E, g) - T_v(x, \alpha; E, g) = 0 = S_v(\beta) - S_v(\alpha)).$$

Finally, we apply (4.21) and use the inequality $R_v(\alpha) \leqslant R_v(\beta)$, which follows from Lemma 3.1. We obtain (4.24). Q.E.D.

We now come to the main result of this section. It can be viewed as a preliminary version of Theorem 1.9. Note, however, that in the following theorem, there is no assumption like (1.12) about the relative sizes of v and Λ (we merely assume (1.7), as always). Also, the assumption here about the size of $|\beta|$ is slightly weaker than (1.13).

THEOREM 4.27. *If* $-\Lambda^{-1}v^{\frac{1}{2}} \leqslant \beta \leqslant 0$, *then*

$$T_v(x, \beta; E, g) = F_v(\beta) + O(\Lambda R_v(\beta) v^{-\frac{1}{2}}) = F_v(\beta)\{1 + O(\Lambda\{|\beta| + 1\} v^{-\frac{1}{2}})\}. \tag{4.28}$$

If $0 \leqslant \beta \leqslant \Lambda^{-1} v^{\frac{1}{2}}$, *then*

$$1 - T_v(x, \beta; E, g) = F_v(\beta) + O(\Lambda R_v(\beta) v^{-\frac{1}{2}}) = F_v(\beta)\{1 + O(\Lambda\{\beta + 1\} v^{-\frac{1}{2}})\}. \tag{4.29}$$

Proof. If $v < 3\Lambda^2$, the results are trivial (see the first paragraph of the proof of Lemma 4.23, and note (4.7)). For the rest of this proof, assume $v \geqslant 3\Lambda^2$.

To derive (4.28), we apply (4.24) with $\alpha = -\Lambda^{-1}v^{\frac{1}{2}}$. Note that by (4.7) and Lemma 3.1,

$$F_v(\alpha) \ll |\alpha|^{-1} R_v(\alpha) = \Lambda R_v(\alpha) v^{-\frac{1}{2}} \leqslant \Lambda R_v(\beta) v^{-\frac{1}{2}}.$$

Furthermore, in the notation defined at the beginning of § 3, $T_v(x, \alpha; E, g) = x^{-1} L_v(x, \Lambda^{-1}; E, g)$, and hence Theorem 3.23 and Lemma 3.1 yield $T_v(x, \alpha; E, g) \ll \Lambda R_v(\beta) v^{-\frac{1}{2}}$. Thus the first part of (4.28) follows from (4.24). The second part of (4.28) follows from the first part and (4.7).

We now prove (4.29) with β replaced by α (for convenience). We take $0 \leqslant \alpha \leqslant \Lambda^{-1} v^{\frac{1}{2}} = \beta$ and use the identity

$$1 - T_v(x, \alpha; E, g) = 1 - T_v(x, \beta; E, g) + T_v(x, \beta; E, g) - T_v(x, \alpha; E, g). \tag{4.30}$$

First observe that

$$\begin{aligned} 1 - T_v(x, \beta; E, g) &= 1 - [x]x^{-1} + [x]x^{-1} - T_v(x, \beta; E, g) \\ &< x^{-1} + x^{-1} R_v(x, \Lambda^{-1}; E, g), \end{aligned} \tag{4.31}$$

in the notation of § 3. By Theorem 3.23,

$$x^{-1} R_v(x, \Lambda^{-1}; E, g) \ll \Lambda R_v(\beta) v^{-\frac{1}{2}}. \tag{4.32}$$

Next, we assert that

$$x^{-1} < R_v(\beta)\, v^{-\frac{1}{2}}. \tag{4.33}$$

To prove this, first note that since $v \geqslant 3\Lambda^2 \geqslant 6\Lambda$, we have

$$v \leqslant E(x) + \Lambda \leqslant \sum_{2\leqslant n\leqslant x} n^{-1} + \Lambda \leqslant \log x + \Lambda \leqslant \log x + v/6,$$

so $x^{-1} \leqslant \exp(-5v/6)$. On the other hand, Lemma 3.1 shows that

$$v^{-\frac{1}{2}} R_v(\beta) > v^{-\frac{1}{2}} \exp\{vQ(1)\} > \exp\{-0.4v - \tfrac{1}{2}\log v\} \geqslant \exp\{-0.4v - (1/2e)v\} > \exp\{-0.7v\},$$

and (4.33) follows. (4.31), (4.32), and (4.33) yield

$$1 - T_v(x, \beta; E, g) \ll \Lambda R_v(\beta)\, v^{-\frac{1}{2}}. \tag{4.34}$$

Now, $F_v(\beta) \ll \beta^{-1} R_v(\beta)$ by (4.7), and $R_v(\beta) \leqslant R_v(\alpha)$ by Lemma 3.1, so (4.30), (4.34), and (4.25) combine to give the first part of (4.29) (with β replaced by α). The second part of (4.29) follows from the first and (4.7). Q.E.D.

It is possible to rewrite Theorem 4.27 in various less precise forms. For example, $F_v(\beta)$ can be rewritten by expressing the factor $\exp(\beta^2/2)\, G(-|\beta|)$ in the form (4.5). It is also possible to rewrite $F_v(\beta)$ by using partial sums of the series (1.2) to obtain a representation of the factor $\exp(\beta^2/2)\, R_v(\beta)$. We shall prove here only one relatively simple result of the latter type, namely Theorem 1.16.

Proof of Theorem 1.16. First assume that $|\beta| \leqslant \min\{v^{\frac{1}{6}}, \Lambda^{-1} v^{\frac{1}{2}}\}$. Since $Q(t) = -t^2/2 + O(|t|^3)$ for $|t| \leqslant 1$ (by (1.2)), we have

$$R_v(\beta) = \exp\{-\beta^2/2 + O(|\beta|^3 v^{-\frac{1}{2}})\},$$

so

$$F_v(\beta) = G(-|\beta|)\{1 + O(|\beta|^3 v^{-\frac{1}{2}})\}.$$

Using (4.7), we get

$$F_v(\beta)\{1 + O(\Lambda\{|\beta| + 1\} v^{-\frac{1}{2}})\} = G(-|\beta|) + O\,(\exp(-\beta^2/2)(\beta^2 + \Lambda)\, v^{-\frac{1}{2}}).$$

(1.17) now follows from Theorem 4.27 and the identity $G(\beta) + G(-\beta) = 1$.

We now prove (1.18) for all real β. This is trivial when $v \leqslant \Lambda^2$, since $T_v(x, \beta; E, g) = O(1)$ and $G(\beta) = O(1)$. Suppose $v > \Lambda^2$, and define $\gamma = \Lambda^{-\frac{1}{3}} v^{\frac{1}{6}}$, so $1 < \gamma < \Lambda^{-1} v^{\frac{1}{2}}$. If $|\beta| \leqslant \gamma$, then (1.18) follows from (1.17). Now suppose that $|\beta| \geqslant \gamma$. Then by (4.7),

$$G(-|\beta|) = 1 - G(|\beta|) \leqslant 1 - G(\gamma) = G(-\gamma) \ll \gamma^{-1} \exp(-\gamma^2/2) \ll \gamma^{-3} = \Lambda v^{-\frac{1}{2}}.$$

It follows that if $\beta \leqslant -\gamma$, then

$$0 \leqslant T_v(x, \beta; E, g) \leqslant T_v(x, -\gamma; E, g) = G(-\gamma) + O(\Lambda v^{-\frac{1}{2}}) \ll \Lambda v^{-\frac{1}{2}},$$

while if $\beta \geqslant \gamma$, then

$$0 \leqslant 1 - T_v(x, \beta; E, g) \leqslant 1 - T_v(x, \gamma; E, g) = 1 - G(\gamma) + O(\Lambda v^{-\frac{1}{2}}) \ll \Lambda v^{-\frac{1}{2}}. \qquad \text{Q.E.D.}$$

§ 5. Proof of Theorem 1.9

In order to derive Theorem 1.9 from Theorem 4.27, we need the following lemma:

Lemma 5.1. *Define $f(y) = \exp(y^2/2)G(-|y|)$ for all real y. If β, γ are real with $\beta \neq 0$ and $\beta\gamma \geqslant 0$, then*

$$f(\gamma) = f(\beta) + (\gamma - \beta)\{\beta f(\beta) - (2\pi)^{-\frac{1}{2}} \operatorname{sgn} \beta\} + O(\{\gamma - \beta\}^2\{|\beta| + |\gamma - \beta| + 1\}). \tag{5.2}$$

If $\beta \neq 0$ and $\beta\gamma \leqslant 0$, then

$$f(\gamma) = f(\beta) + (2\pi)^{-\frac{1}{2}}(\beta + \gamma) \operatorname{sgn} \beta + O(\beta^2\{|\beta| + 1\} + \gamma^2\{|\gamma| + 1\}). \tag{5.3}$$

Proof. Consider the functions f_1, f_2 defined by

$$f_j(y) = \exp(y^2/2)G((-1)^j y) \quad (j = 1, 2;\ y \text{ real}).$$

Note that for each j and y,

$$f_j'(y) = (-1)^j(2\pi)^{-\frac{1}{2}} + y f_j(y), \quad f_j''(y) = (-1)^j(2\pi)^{-\frac{1}{2}} y + y^2 f_j(y) + f_j(y).$$

If $\beta < 0$ and $\gamma \leqslant 0$, or if $\beta > 0$ and $\gamma \geqslant 0$, we can use the Taylor expansions of $f_1(y)$ and $f_2(y)$ to obtain

$$f(\gamma) = f(\beta) + (\gamma - \beta)\{\beta f(\beta) - (2\pi)^{-\frac{1}{2}} \operatorname{sgn} \beta\} + \tfrac{1}{2}(\gamma - \beta)^2\{-(2\pi)^{-\frac{1}{2}}|\xi| + \xi^2 f(\xi) + f(\xi)\},$$

where ξ is between β and γ. By (4.7),

$$-(2\pi)^{-\frac{1}{2}}|\xi| + \xi^2 f(\xi) + f(\xi) \ll |\xi| + 1, \tag{5.4}$$

and since $|\xi - \beta| \leqslant |\gamma - \beta|$, we obtain (5.2).

Now suppose that $\beta < 0$ and $\gamma \geqslant 0$, or that $\beta > 0$ and $\gamma \leqslant 0$. We cannot apply Taylor's theorem directly to $f(\gamma) - f(\beta)$ since f is not differentiable at 0. However, we can apply Taylor's theorem to $f_1(y)$ and $f_2(y)$ at $y = 0$, then use (5.4) and the fact that $f(0) = \frac{1}{2}$ to get

$$f(y) = \tfrac{1}{2} - (2\pi)^{-\frac{1}{2}}|y| + O(y^2\{|y| + 1\}) \quad (y \text{ real}). \tag{5.5}$$

Using (5.5) to estimate $f(\beta)$ and $f(\gamma)$, we obtain (5.3). Q.E.D.

Proof of Theorem 1.9. Throughout this proof, we think of x, v, β, E as being arbitrary but fixed, subject only to the assumptions (1.7), (1.12), and (1.13). Write $E(x)=w=v+z$, so $|z|\leqslant\Lambda$ and $w\geqslant\Lambda^4-\Lambda\geqslant 14$. Define γ by the equation

$$v+\beta v^{\frac{1}{2}}=w+\gamma w^{\frac{1}{2}}, \tag{5.6}$$

so

$$T_v(x,\beta;E,g)=T_w(x,\gamma;E,g). \tag{5.7}$$

From (5.6), we obtain

$$\gamma=w^{-\frac{1}{2}}(\beta v^{\frac{1}{2}}-z). \tag{5.8}$$

From (1.12), it follows that $v\geqslant 8\Lambda$, so $v(1-2\Lambda^{-2})\geqslant v/2\geqslant 4\Lambda$, and hence (1.13) implies that $2|\beta|v^{\frac{1}{2}}\leqslant 2\Lambda^{-2}v\leqslant v-4\Lambda$. Thus (5.8) yields

$$2|\gamma|w^{\frac{1}{2}}\leqslant 2|\beta|v^{\frac{1}{2}}+2\Lambda\leqslant v-2\Lambda<w,$$

so $|\gamma|<\frac{1}{2}w^{\frac{1}{2}}$. Hence we can apply Theorem 4.27 with v replaced by $E(x)=w$, Λ replaced by 2, and β replaced by γ. Combining the result with (5.7), we get

$$T_v(x,\beta;E,g)=F_w(\gamma)+O(R_w(\gamma)w^{-\frac{1}{2}})\quad\text{if } \gamma\leqslant 0, \tag{5.9}$$

$$T_v(x,\beta;E,g)=1-F_w(\gamma)+O(R_w(\gamma)w^{-\frac{1}{2}})\quad\text{if } \gamma\geqslant 0. \tag{5.10}$$

The idea of the proof is now very simple: we must estimate $R_w(\gamma)$ and $F_w(\gamma)$ in terms of $R_v(\beta)$ and $F_v(\beta)$. Because of the somewhat complicated nature of these functions, an extended series of calculations is needed to finish the proof. First,

$$w^{-\frac{1}{2}}=(v+z)^{-\frac{1}{2}}=v^{-\frac{1}{2}}\{1+O(\Lambda v^{-1})\}=v^{-\frac{1}{2}}+O(\Lambda v^{-\frac{3}{2}}). \tag{5.11}$$

Hence by (5.8), (1.13), and (1.12),

$$\gamma=\beta-zv^{-\frac{1}{2}}+O(v^{-\frac{1}{2}}). \tag{5.12}$$

Next, we apply (3.37) and recall that $|\gamma|<\frac{1}{2}w^{\frac{1}{2}}$ to get

$$Q(\gamma w^{-\frac{1}{2}})=Q(\beta v^{-\frac{1}{2}})+(\beta v^{-\frac{1}{2}}-\gamma w^{-\frac{1}{2}})\log(1+\beta v^{-\frac{1}{2}})+O(\{\beta v^{-\frac{1}{2}}-\gamma w^{-\frac{1}{2}}\}^2). \tag{5.13}$$

By (5.12), (5.11), (1.12), and (1.13),

$$\gamma w^{-\frac{1}{2}}=\beta v^{-\frac{1}{2}}-zv^{-1}+O(v^{-1}). \tag{5.14}$$

Using (5.14), (1.13), and the estimate $\log(1+y) = y + O(y^2)$ (for $|y| \leqslant \tfrac{1}{2}$), we obtain

$$(\beta v^{-\frac{1}{2}} - \gamma w^{-\frac{1}{2}}) \log(1 + \beta v^{-\frac{1}{2}}) = \beta z v^{-\frac{3}{2}} + O(|\beta| v^{-\frac{3}{2}}). \tag{5.15}$$

Utilizing (5.13), (5.15), (5.14), (1.12), (1.13), and the estimate $Q(t) \ll t^2$ for $|t| \leqslant 1$ (see (1.2)), we get

$$wQ(\gamma w^{-\frac{1}{2}}) = (v+z)Q(\gamma w^{-\frac{1}{2}}) = vQ(\beta v^{-\frac{1}{2}}) + \beta z v^{-\frac{1}{2}} + O(|\beta| v^{-\frac{1}{2}} + v^{-\frac{1}{2}}).$$

Exponentiating this and using (1.13) and the estimate $e^y = 1 + y + O(y^2)$ (for $y \ll 1$), we obtain

$$R_w(\gamma) = R_v(\beta)\{1 + \beta z v^{-\frac{1}{2}} + O(|\beta| v^{-\frac{1}{2}} + v^{-\frac{1}{2}})\}. \tag{5.16}$$

In particular,

$$R_w(\gamma) \ll R_v(\beta). \tag{5.17}$$

Now define $f(y) = \exp(y^2/2)G(-|y|)$ for all real y, so $F_w(\gamma) = f(\gamma)R_w(\gamma)$. We shall apply Lemma 5.1 to estimate $f(\gamma)$. For the remainder of this proof, write $(2\pi)^{-\frac{1}{2}} = c$ for simplicity. First suppose that

$$\beta \neq 0 \quad \text{and } \beta\gamma \geqslant 0. \tag{5.18}$$

We apply (5.2), using (5.12), (1.13), (4.7), and (1.12) to estimate $\gamma - \beta$ and the error term. The result is

$$f(\gamma) = f(\beta) + \{c \operatorname{sgn} \beta - \beta f(\beta)\} z v^{-\frac{1}{2}} + O(v^{-\frac{1}{2}}). \tag{5.19}$$

Multiplying the expressions (5.19) and (5.16), noting the cancellation of the terms $\pm \beta f(\beta) R_v(\beta) z v^{-\frac{1}{2}}$, and simplifying by the use of (5.17), (4.7), (1.13), and (1.12), we get

$$F_w(\gamma) = F_v(\beta) + cR_v(\beta) z v^{-\frac{1}{2}} \operatorname{sgn} \beta + O(R_v(\beta) v^{-\frac{1}{2}}) \tag{5.20}$$

if (5.18) holds.

Now suppose that

$$\beta \neq 0 \quad \text{and } \beta\gamma \leqslant 0. \tag{5.21}$$

If $\beta < 0$ and $\gamma \geqslant 0$, then by (5.8), $|\beta| \leqslant \Lambda v^{-\frac{1}{2}}$, and the same inequality holds if $\beta > 0$ and $\gamma \leqslant 0$. Thus by (5.12),

$$|\beta| \leqslant \Lambda v^{-\frac{1}{2}} \quad \text{and } |\gamma| \ll \Lambda v^{-\frac{1}{2}}. \tag{5.22}$$

We combine (5.3) and (5.12), then estimate the resulting error term by using (5.22) and (1.12). The result is

$$f(\gamma) = f(\beta) + c(2\beta - zv^{-\frac{1}{2}}) \operatorname{sgn} \beta + O(v^{-\frac{1}{2}}). \tag{5.23}$$

Observe that (5.22) and (1.12) imply $|\beta z| \leqslant |\beta| \Lambda \leqslant 1$, so (5.16) becomes

$$R_w(\gamma) = R_v(\beta) + O(R_v(\beta) v^{-\frac{1}{2}}). \tag{5.24}$$

Multiply the expressions (5.23) and (5.24), then use (4.7), (5.22), and (1.12) to estimate the error terms. The result is

$$F_w(\gamma) = F_v(\beta) + c(2\beta - zv^{-\frac{1}{2}}) R_v(\beta) \operatorname{sgn} \beta + O(R_v(\beta) v^{-\frac{1}{2}}) \tag{5.25}$$

if (5.21) holds.

We need to deduce from (5.25) an appropriate expression for $1 - F_w(\gamma)$. Recall that $Q(t) \ll t^2$ for $|t| \leqslant 1$. Hence if $|y| \leqslant 1$, we have

$$R_v(y) = \exp\{vQ(yv^{-\frac{1}{2}})\} = 1 + O(y^2) = 1 + O(y^2 R_v(y)).$$

From this and (5.5), it follows that

$$F_v(y) = \tfrac{1}{2} R_v(y) - c|y| R_v(y) + O(y^2 R_v(y)) = \tfrac{1}{2} - c|y| R_v(y) + O(y^2 R_v(y))$$

if $|y| \leqslant 1$, and in particular,

$$F_v(\beta) = \tfrac{1}{2} - c|\beta| R_v(\beta) + O(R_v(\beta) v^{-\frac{1}{2}}) \tag{5.26}$$

if (5.21) holds, by (5.22) and (1.12). From (5.25) and (5.26), we obtain finally

$$1 - F_w(\gamma) = F_v(\beta) + cR_v(\beta) zv^{-\frac{1}{2}} \operatorname{sgn} \beta + O(R_v(\beta) v^{-\frac{1}{2}}) \tag{5.27}$$

if (5.21) holds.

Now by (5.17) and (5.11), the error terms in (5.9) and (5.10) are both $O(R_v(\beta) v^{-\frac{1}{2}})$. Hence if $\beta \neq 0$, we can derive (1.14) and (1.15) immediately from (5.9), (5.10), (5.20), and (5.27).

Finally, consider the case $\beta = 0$. If $\beta_1 < 0 < \beta_2$, then

$$T_v(x, \beta_1; E, g) \leqslant T_v(x, 0; E, g) \leqslant T_v(x, \beta_2; E, g).$$

If we use (1.14) and (1.15) and let β_1 and β_2 tend to 0, we get

$$T_v(x, 0; E, g) = \tfrac{1}{2} - czv^{-\frac{1}{2}} + O(v^{-\frac{1}{2}}),$$

and hence (1.14) and (1.15) both hold for $\beta = 0$. Q.E.D.

Theorem 1.9 and Theorem 4.27 are best possible in a rather strong sense. To see this, suppose that c_{22} is a sufficiently small (positive absolute) constant, and assume that $v \geqslant \max\{c_{22}^{-2}, 3\Lambda^2\}$, $|\beta| \leqslant c_{22}\Lambda^{-1}v^{\frac{1}{2}}$, $v+\beta v^{\frac{1}{2}}=m$ is a positive integer, and $\beta-\alpha$ is positive and sufficiently small. Then by (4.14),

$$T_v(x, \beta; E, g) - T_v(x, \alpha; E, g) = x^{-1}N(m, x; E, g) \gg R_v(\beta)v^{-\frac{1}{2}}. \tag{5.28}$$

Keeping x fixed, let $H_w(\gamma)$ be any real-valued function which is defined for $w \geqslant \max\{c_{22}^{-2}, 3\Lambda^2\}$ and γ in the interval $(\alpha, \beta]$ and which has the property that for each fixed w, $H_w(\gamma)$ is left-continuous at $\gamma=\beta$. It follows easily from (5.28) that

$$|T_v(x, \gamma; E, g) - H_v(\gamma)| \geqslant c_{23} R_v(\gamma) v^{-\frac{1}{2}}$$

for some $\gamma \in (\alpha, \beta]$. Thus (for any fixed x) Theorems 4.27 and 1.9 show that in this sense, the functions $F_v(\beta)$ (for $\beta \leqslant 0$) and $1-F_v(\beta)$ (for $\beta \geqslant 0$) are essentially the best possible continuous approximations to $T_v(x, \beta; E, g)$. Likewise, (1.18) is best possible if β is near zero.

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