RIEMANN-ROCH AND TOPOLOGICAL K-THEORY FOR SINGULAR VARIETIES[1]

BY

PAUL BAUM, WILLIAM FULTON and ROBERT MACPHERSON

Brown University, Providence, Rhode Island, U.S.A.

Contents

§ 0. Introduction

0.1. Summary

The basic Riemann–Roch problem is to give, for any sheaf S of O_X modules on an algebraic variety X, a formula for $\chi(X, S)$, the alternating sum of the ranks of the sheaf cohomology groups $H^i(X, S)$. Perhaps the most striking fact about $\chi(X, S)$ is that it is constant in a flat family: while the individual ranks of the $H^i(X, S)$ may vary, their alternating sum does not. This invariance under deformation leads one to suspect that $\chi(X, S)$ may be a topological invariant. In this paper we will present the Riemann–Roch Theorem as a transition from algebra to topology; one consequence will be a topological formula for $\chi(X, S)$.

---

(1) Research partially supported by NSF grants MCS 76-09817 and MCS 76-09753.

One novelty of our approach is that we consistently emphasize the fundamental role played by this invariance under deformations. Deformation of a given imbedding to the imbedding in the normal bundle is the kernel of the proof. Explicit deformations are also used to verify the Riemann–Roch formula on projective space, rather than power series calculations with Chern classes as in former proofs. An extension of these ideas can be used [9] to characterize the arithmetic genus as the only deformation invariant: Suppose an integer $I(X)$ is assigned to each non-singular projective variety X, which satisfies

$$I(X) = I(X_1) + I(X_2) - I(Z)$$

whenever X is a divisor on a non-singular variety which can be deformed in a linear equivalence to a sum of two divisors X_1 and X_2 which meet transversally in Z. Then $I(X)$ is a constant multiple of $\chi(X, O_X)$.

Grothendieck enriched the Riemann–Roch circle of ideas to the study of two algebraic K functors. $K^0_{\mathrm{alg}}(X)$ is the Grothendieck group of algebraic vector bundles on X; it is a ring-valued contravariant functor. $K_0^{\mathrm{alg}}(X)$ is the Grothendieck group of coherent sheaves of O_X modules; it becomes covariant for proper maps f by sending a sheaf S to the alternating sum of $R^i f_* S$. If f maps X to a point, S is sent to $\chi(X, S) \in \mathbf{Z} = K_0^{\mathrm{alg}}(\mathrm{pt.})$. (The map $K^0_{\mathrm{alg}}(X) \to K_0^{\mathrm{alg}}(X)$ which takes a vector bundle to its sheaf of sections is an isomorphism for non-singular varieties, so the original treatment could identify K^0_{alg} and K_0^{alg}. Since we are including singular varieties for which this Poincaré duality isomorphism fails, we must keep them separate.)

The main point of this paper is to use topological K-theory as the topological receiver for the Riemann–Roch information. For a topological space X, let $K^0_{\mathrm{top}}(X)$ be the Grothendieck group of topological vector bundles on X; K^0_{top} is a ringvalued contravariant functor. $K_0^{\mathrm{top}}(X)$ is defined to be the Grothendieck group of complexes of vector bundles on $\mathbf{C}^n$ exact off X for some closed embedding of X in $\mathbf{C}^n$; it becomes a covariant functor for proper maps by using Bott periodicity. (See [2] for a description of homology K-theory better suited to Riemann–Roch questions.)

One of our aims is to emphasize the analogy between the algebraic and topological versions of K-theory.

In each case there are: cap products $K^0X \otimes K_0X \to K_0X$, with the usual projection formula; exterior products $K_0X_1 \otimes K_0X_2 \to K_0(X_1 \times X_2)$; and restriction homomorphisms $K_0X \to K_0U$ for U open in X. (See § 3 for precise statements.)

For any complex algebraic variety X there is a ring homomorphism

$$\alpha^{\cdot}\colon K^0_{\mathrm{alg}}X \to K^0_{\mathrm{top}}X$$

defined by taking an algebraic vector bundle to its underlying topological vector bundle; $\alpha^{\cdot}$ is a natural transformation of contravariant functors.

The Riemann-Roch theorem we prove in this paper constructs, for each quasi-projective variety X, a homomorphism

$$\alpha_{.}: K_0^{\mathrm{alg}} X \to K_0^{\mathrm{top}} X$$

of abelian groups, which is covariant for proper morphisms, and is compatible with cap products, exterior products, and restrictions. In case X is non-singular, $\alpha.$ takes the structure sheaf O_X of X to the K-theory orientation $\{X\}$ of X determined by its complex structure. (See § 4 for the precise statement of the theorem.) This Riemann-Roch theorem generalizes the theorem of Aityah and Hirzebruch [1] to possibly singular (and possibly non-compact) varieties. The class $\{X\} \subset \alpha.[O_X]$ determines a K-theory orientation even for a singular variety (see § 6).

Riemann-Roch in terms of topological K-theory is certainly more natural than the previous versions using ordinary homology (or Chow groups or graded K-groups). No characteristic classes or formal power series calculations are needed for either the statement or the proof of the theorem; they are replaced by elementary geometric constructions. All the formulas are simpler in the K-theory version since no Todd class correction terms are necessary. The present theorem also includes torsion, which is lost in the homology version.

For certain applications, however, it is preferable to use homology as the topological receiver of the Riemann-Roch information because of its computational facility. The homology version follows directly from the K-theory version using standard purely topological techniques, as follows.

The Chern character

$$ch^{\cdot}: K^0_{\mathrm{top}} \to H^{\cdot}(\ ;\mathbf{Q})$$

determines a corresponding transformation of homology theories

$$ch_{.}: K_0^{\mathrm{top}} \to H_{.}(\ ;\mathbf{Q})$$

with values in homology with rational coefficients (Borel-Moore homology in the non-compact case). The composite $\tau. = ch.\circ\alpha.$ is the Riemann-Roch mapping from K_0^{alg} to $H.(\ ;\mathbf{Q})$ which was constructed in [3]; this generalized the Grothendieck Riemann-Roch theorem to singular varieties. When we map a projective variety X to a point, we recover a Hirzebruch-Riemann-Roch formula for singular varieties:

$$\sum (-1)^i \dim H^i(X, E) = \varepsilon(ch(E) \cap \tau(X))$$

for an algebraic vector-bundle E on X. Here ε takes the degree of the zero-dimensional component of a homology class, and $\tau(X) = ch.\{X\}$ is the homology Todd class of X; when X is non-singular, $\tau(X)$ is Poincaré dual to the Todd class of the tangent bundle to X, giving Hirzebruch's original formula.

0.2. The Riemann–Roch map

Here is a sketch of the construction of $\alpha.[\mathcal{S}]$ in $K_0^{\mathrm{top}}X$ for a coherent sheaf $\mathcal{S}$ on a projective variety X. Choose a closed imbedding $i: X \to Y$ of X into a non-singular projective variety Y. Choose a resolution of $i_*\mathcal{S}$ by a complex of algebraic vector bundles (locally free sheaves) $E.$ on Y:

$$0 \to E_m \to \dots \to E_0 \to i_*\mathcal{S} \to 0.$$

Choose a C^∞ imbedding of Y in a sphere S^{2r}, and give the normal bundle N to Y in S^{2r} the complex structure induced from the tangent bundle to Y. Let $\pi: N \to Y$ be the projection, and let $\Lambda^{\cdot *}N^\vee$ be the Koszul–Thom complex on N. Then $\pi^*E.\otimes\Lambda^{\cdot}\pi^*N^\vee$ is a complex of vector bundles on N which is exact off X, so by the difference bundle construction (or § 1) it determines an element in the relative (topological) K-group $K^0(N, N-X)$, which by choice of tubular neighborhood and excision is identified with $K^0(\mathrm{S}^{2r}, \mathrm{S}^{2r}-X)$.

When we identify $K^0(\mathrm{S}^{2r}, \mathrm{S}^{2r}-X)$ with $K_0^{\mathrm{top}}X$, we have the desired element $\alpha.[\mathcal{S}]$ in $K_0^{\mathrm{top}}X$; much of the task of proving Riemann-Roch amounts to showing that $\alpha.[\mathcal{S}]$ is independent of all the choices. The element constructed in $K^0(\mathrm{S}^{2r}, \mathrm{S}^{2r}-X)$ also determines an element in $\tilde{K}^0(\mathrm{S}^{2r}) = \mathbf{Z}$ by Bott periodicity, and this integer is the Euler characteristic $\Sigma(-1)^i \dim H^i(X, \mathcal{S})$ of the coherent sheaf $\mathcal{S}$.

0.3. Plan of the paper

In the above description of the Riemann–Roch map there is a mixture of ingredients from algebraic geometry and topology. We have made an effort to separate the argument into its essential pieces. In algebraic geometry and topology there are relative groups $K_X^{\mathrm{alg}}Y$ and $K_X^{\mathrm{top}}Y$ for X a closed subvariety (resp. sub-space) of Y which are constructed out of complexes of algebraic (resp. topological) vector-bundles on Y that are exact off X. Both those relative groups have natural pull-back maps and products, and most important, "Thom-Gysin maps" from K_XY to K_XZ for suitable imbeddings of Y in Z (§ 1). The essential step in Riemann–Roch (§ 2) is to show that the natural map α from $K_X^{\mathrm{alg}}Y$ to $K_X^{\mathrm{top}}Y$—which takes a complex of algebraic vector bundles to its underlying complex of topological vector bundles—is compatible with these Thom–Gysin maps when

Y and Z are non-singular varieties. This being obvious when a neighborhood of Y in Z is algebraically (or complex analytically) isomorphic to the imbedding of Y in the normal bundle, we deform the given imbedding to the normal imbedding, as in [3]. In § 2 we include an alternate description of this deformation, following [15]. Another innovation is the use of a concept of "transversality" suggested by the work in [11], which allows us to study how the relative K-groups vary in a family.

In both algebraic geometry and topology there are duality or "homology maps" h from $K_X Y$ to $K_0 X$, at least when Y is non-singular, and the Riemann–Roch map $\alpha.$ is then the composite

$$K_0^{\mathrm{alg}} X \underset{\simeq}{\xleftarrow{\ h\ }} K_X^{\mathrm{alg}} Y \xrightarrow{\ \alpha\ } K_X^{\mathrm{top}} Y \underset{\simeq}{\xrightarrow{\ h\ }} K_0^{\mathrm{top}} X.$$

We have tried to make the paper quite self-contained. The results of [3] are not assumed. Two appendices are included to provide algebraic geometers and topologists with an elementary discussion of some standard, but rather inaccessible, results from the other field; the first relates the definition of $K_X^{\mathrm{top}} Y$ using complexes of vector bundles on Y, exact off X, to a more common definition of $K^0_{\mathrm{top}}(Y, Y-X)$, while the second discusses some homological algebra of complexes of sheaves. A third appendix describes an algebraic deformation of the diagonal in $\mathbf{P}^n \times \mathbf{P}^n$ to its Künneth decomposition, which is used to show directly that algebraic and topological K-theory assign the same genus to projective space.

We point out several respects in which we have done more than is required to prove the Riemann–Roch theorem. (1) We have formalized much of the arguments so that it may be used in other instances where there are relative groups with properties analogous to those discussed here. In a following paper [4] this will be carried out to give a simple treatment of a Lefschetz–Riemann–Roch theorem for singular varieties. It is hoped that the higher K-groups of Quillen, as well as the Chow groups, will have relative groups with similar properties, so that the proof given here will apply without essential change to these situations. (2) Although we do not include complete proofs of the Riemann–Roch theorems for local complete intersections (§ 4.2) the preliminaries are carried out in sufficient generality so that such a proof can be completed along the same lines; primarily this means that we allow the ambient spaces to have singularities, whereas for the main theorem they could all be taken to be non-singular. (3) The reader interested only in compact varieties could simplify some of the argument. An occasional vector bundle would have to be compactified, and the discussion of orientations in § 6 is a little more complicated if one cannot restrict to open subvarieties, however.

0.4. Acknowledgements

We thank M. F. Atiyah for many enlightening comments, and for suggesting that K_*^{top} is the most natural homology theory to use for Riemann–Roch. We also benefitted from discussions with A. Landman, E. Laitinen, and G. Quart. Some of this work appeared in preprint form (Aarhus Universitet Preprint Series 1976/77 No. 42) while the authors were visiting at IAS, Aarhus, and IHES respectively.

0.5. Convention

We will use the word "variety" throughout to mean an arbitrary complex quasi-projective scheme, as in [12]. The reader may also take the word to mean reduced, irreducible complex quasi-projective variety, following Serre [13], with slight loss in generality, but with no change needed in statements or proofs.

§ 1. Relative K-groups

In this chapter we define and study the groups $K_X^{\text{alg}}\, Y$ and $K_X^{\text{top}}\, Y$ which play central roles in the proof of the Riemann–Roch theorem.

1.1. Definitions

Let X be a closed algebraic subset of a variety Y. Consider complexes E.

$$0 \longrightarrow E_n \xrightarrow{d_n} E_{n-1} \longrightarrow \cdots \xrightarrow{d_1} E_0 \longrightarrow 0$$

of algebraic vector bundles (locally free sheaves) on Y which are exact off X; thus the boundary maps are morphisms of algebraic vector bundles on Y, with $d_{i-1} \circ d_i = 0$, and for all $y \notin X$, the induced complex on the fibres at y is an exact sequence of vector spaces. Define $K_X^{\text{alg}}\, Y$ to be the free abelian group on the isomorphism classes of such complexes modulo the relations:

(i) For each exact sequence

$$0 \to E.' \to E. \to E." \to 0$$

of such complexes, set the class of $E.$ equal to the sum of the class of $E.'$ and the class of $E."$.

(ii) For any complex $E.$ that is exact on all of Y, set the class of $E.$ equal to zero.

Denote by $[E.]$ the element in $K_X^{\text{alg}}\, Y$ represented by a complex $E.$.

Let X be a closed subspace of a locally compact space Y. For simplicity we will always assume that the pair (Y^c, X^c) of one-point compactifications is homeomorphic to pair of finite simplicial complexes—an assumption that holds when X is a subvariety of a complex variety Y. Consider complexes

$$0 \to E_n \to E_{n-1} \to \dots \to E_0 \to 0$$

on Y which are exact on $Y - X$. Define $K_X^{\mathrm{top}} Y$ to be the free abelian group on the isomorphism classes of such complexes $E.$, modulo the relations (i) and (ii) as in the algebraic case, and

(iii) If $E.$ is a complex on $Y \times [0, 1]$ which is exact off $X \times [0, 1]$, and $E.(t)$ denotes the induced complex on $Y = Y \times \{t\}$, set the class of $E.(0)$ equal to the class of $E.(1)$.

Since an exact sequence as in (i) is homotopic to a split exact sequence, it is only necessary to include direct sums in relation (i).

For any closed algebraic subset X of a complex variety Y we have a canonical homomorphism

$$\alpha: K_X^{\mathrm{alg}} Y \to K_X^{\mathrm{top}} Y$$

which is defined by taking a complex of algebraic vector bundles to its underlying complex of topological vector bundles.

1.2. Pull-backs

If $X \subset Y$ and $X' \subset Y'$ are in § 1.1, and $f: Y' \to Y$ is a morphism such that $f^{-1}(X) \subset X'$, there are pull-back maps

$$f^*: K_X Y \to K_{X'} Y'.$$

In the algebraic geometry setting f will be a morphism of varieties (schemes). If $E.$ is a complex on Y exact off X, then $f^*E.$ is a complex on Y' exact off X', and

$$f^*: K_X^{\mathrm{alg}} Y \to K_{X'}^{\mathrm{alg}} Y'$$

is determined by setting $f^*[E.] = [f^*E.]$.

In topology f should be a continuous mapping. Then

$$f^*: K_X^{\mathrm{top}} Y \to K_{X'}^{\mathrm{top}} Y'$$

is defined by the analogous formula $f^*[E.] = [f^*E.]$.

In both cases the pull-back maps are *functorial*, i.e., if we also have $g\colon Y'' \to Y'$ with $g^{-1}(X') \subset X''$, then $(fg)^* = g^* f^*$ as homomorphisms from $K_X Y$ to $K_{X''} Y''$.

Here and in the rest of the paper we use one statement with the notation K instead of two entirely similar statements for K^{alg} and K^{top}. This is to emphasize the properties they have in common, as well as to avoid unnecessary duplication.

1.3. Products

If $X_1 \subset Y_1$ and $X_2 \subset Y_2$ are as in § 1.1, we have *exterior products*

$$K_{X_1} Y_1 \otimes K_{X_2} Y_2 \to K_{X_1 \times X_2}(Y_1 \times Y_2).$$

In algebraic geometry if E^1 and E^2 are coherent sheaves or complexes on Y_1 and Y_2 respectively, then $E^1 \boxtimes E^2$ denotes the tensor product $pr_1^* E^1 \otimes_{\mathcal{O}_{Y_1 \times Y_2}} pr_2^* E^2$. Note that this tensor product is exact in each variable, and that a tensor product of complexes is exact where either complex is exact. The exterior product is determined by setting $[E^1_\cdot] \times [E^2_\cdot] = [E^1_\cdot \boxtimes E^2_\cdot]$, where $E^i_\cdot$ is a complex on Y_i exact off X_i.

The same formula determines the product in topology.

The exterior products are *associative*: if $a_i \in K_{X_i} Y_i$, then $(a_1 \times a_2) \times a_3 = a_1 \times (a_2 \times a_3)$.

The products are *compatible with pull-backs*: if $f_i\colon Y_i' \to Y_i$, $f_i^{-1}(X_i) \subset X_i'$ as in § 1.2, and $a_i \in K_{X_i} Y_i$, then $(f_1 \times f_2)^*(a_1 \times a_2) = f_1^* a_1 \times f_2^* a_2$.

If X_1 and X_2 are closed in Y as in § 1.1, we define as usual internal cup products

$$K_{X_1} Y \otimes K_{X_2} Y \xrightarrow{\ \cup\ } K_{X_1 \cap X_2} Y$$

to be the composite of the external product

$$K_{X_1} Y \otimes K_{X_2} Y \xrightarrow{\ \times\ } K_{X_1 \times X_2}(Y \times Y)$$

followed by the pull-back

$$K_{X_1 \times X_2}(Y \times Y) \xrightarrow{\ \delta^*\ } K_{X_1 \cap X_2} Y$$

determined by the diagonal mapping $\delta\colon Y \to Y \times Y$. These internal products are also associative and compatible with pull-backs.

1.4. Koszul–Thom classes

Let E be a complex vector bundle (algebraic or topological) on Y (a variety or a topological space. Regard Y as a sub-space of E by the zero section. The *Koszul-Thom* *class* λ_E in $K_Y E$ can be described as follows. Let d be the rank of E, $\pi: E \to Y$ the projection. The bundle $\pi^* E$ has a canonical section, which determines a homomorphism from $\pi^* E^\vee$ to the trivial line bundle, and this determines a complex

$$0 \to \Lambda^d \pi^* E^\vee \to \Lambda^{d-1} \pi^* E^\vee \to \ldots \to \Lambda^1 \pi^* E^\vee \to \Lambda^0 \pi^* E^\vee \to 0.$$

This is a complex on E which is exact off Y, and the element of $K_Y E$ it determines is called the *Koszul-Thom class* and denoted λ_E.

1.5. Thom–Gysin maps

For $X \subset Y$ as in § 1, and suitably nice closed imbeddings $i: Y \to Z$ we define *Thom–Gysin* maps

$$i_*: K_X Y \to K_X Z.$$

In algebraic geometry we define these Thom-Gysin homomorphisms when i is a closed imbedding of quasi-projective schemes which is of *finite Tor dimension*; this means that the structure sheaf O_Y, regarded as an O_Z-module, has a finite resolution by locally free sheaves of O_Z-modules. Two important cases are when Z is nonsingular and Y is arbitrary, or when Y is a local complete intersection in Z. If $E.$ is a complex of vector bundles on Y exact off X, extension by zero gives a complex of sheaves $i_* E.$ on Z. The finite Tor dimension assumption implies that there is a complex of vector bundles $F.$ on Z and a surjective map of complexes $F. \to i_* E.$ whose kernel is an acyclic complex of sheaves on Z; equivalently the induced map $H_i(F.) \to H_i(i_* E.)$ on the homology sheaves is an isomorphism. We call $F. \to i_* E.$ a *resolution* of $E.$ on Z. Any two such resolutions $F.$ are dominated by a third, so $[F.]$ is a well-defined element in $K_X^{\mathrm{alg}} Z$. And an exact sequence of complexes on Y can be resolved by an exact sequence of complexes on Z, so the Thom–Gysin map

$$i_*: K_X^{\mathrm{alg}} Y \to K_X^{\mathrm{alg}} Z$$

is determined by setting $i_*[E.] = [F.]$ for $E.$ and $F.$ as above. See Appendix 2 for homological details.

In topology we define Thom–Gysin homomorphisms

$$i_*: K_X^{\mathrm{top}} Y \to K_X^{\mathrm{top}} Z$$

when $i: Y \to Z$ is a closed imbedding at C^∞ manifolds, and the normal bundle N to Y in Z has a given complex structure. Let $\pi: N \to Y$ be the projection, and let $\bar{\imath}: Y \to N$ be the imbedding of Y as the zero section in N. We have the pull-back map $\pi^*: K_X^{\text{top}} Y \to K_{\pi^{-1}(X)}^{\text{top}} N$ (§ 1.2), the Koszul–Thom class $\lambda_N \in K_Y^{\text{top}} N$ (§ 1.4), and the cup product

$$K_{\pi^{-1}(X)}^{\text{top}} N \otimes K_Y^{\text{top}} N \xrightarrow{\smile} K_X^{\text{top}} N$$

(§ 1.3), so that formula $a \to \pi^* a \smile \lambda_N$ gives a homomorphism

$$\bar{i}_*: K_X^{\text{top}} Y \to K_X^{\text{top}} N.$$

If $\theta: N \to Z$ maps N onto a tubular neighborhood of Y in Z, then the pull-back mapping

$$\theta^*: K_X^{\text{top}} Z \to K_X^{\text{top}} N$$

is an isomorphism by excision (Appendix 1), and independent of θ: a homotopy $H: N \times [0,1] \to Z$ from one tubular neighborhood to another determines $H^*: K_X^{\text{top}} Z \to K_{X \times [0,1]}^{\text{top}}(N \times [0,1])$, and the result of specializing to 0 and 1 are equal by the relation (iii) defining K^{top}. Then the Thom–Gysin map $i_*: K_X^{\text{top}} Y \to K_X^{\text{top}} Z$ is defined to be the composite

$$K_X^{\text{top}} Y \xrightarrow{\bar{i}_*} K_X^{\text{top}} N \xrightarrow{(\theta^*)^{-1}} K_X^{\text{top}} Z.$$

Note that in topology the Thom–Gysin maps are always isomorphisms (Appendix 1).

The Thom–Gysin maps are *functorial*: if X is closed in Y, and $i: Y \to Z$ and $j: Z \to W$ are suitably nice imbeddings as above, then $j \circ i$ is also suitably nice, and $(j \circ i)_*$ and $j_* i_*$ define the same homomorphism from $K_X Y$ to $K_X W$. Let us explain this.

In algebraic geometry the assumption that i and j have finite Tor dimension implies that $j \circ i$ also has finite Tor dimension, and the proof that $(j \circ i)_* = j_* i_*$ follows from the fact that if $E.$ is a complex on X, $F. \to i_* E.$ is a resolution on Y, and $G. \to j_* F.$ is a resolution on Z, then $G. \to j_* F. \to j_* i_* E.$ is a resolution on Z.

In topology we assume that the normal bundle N_{ji} to Y in W is given a complex structure so that the canonical exact sequence

$$0 \to N_i \to N_{ji} \to N_j \,|\, Y \to 0$$

is an exact sequence of complex vector bundles. The equation $(ji)_* = j_* i_*$ can then be proved by choosing compatible tubular neighborhoods for the two imbeddings, and choosing a splitting of the exact sequence of normal bundles. Note that if $N = N_1 \oplus N_2$ is a direct sum of vector bundles, then the Koszul–Thom complexes satisfy $\Lambda^{\cdot} \pi^* N^{\vee} = \Lambda^{\cdot} \pi_1^* N_1^{\vee} \otimes \Lambda^{\cdot} \pi_2^* N_2^{\vee}$, so $\lambda_N = \lambda_{N_1} \smile \lambda_{N_2}$.

The Thom–Gysin maps are also *compatible with products*. If X_j is closed in Y_j, and i_j: $Y_j \to Z_j$ are suitably nice imbeddings as above, $j=1, 2$, then $i_1 \times i_2$ is also suitably nice, and the diagram

$$\begin{array}{ccc} K_{X_1} Y_1 \otimes K_{X_2} Y_2 & \xrightarrow{\times} & K_{X_1 \times X_2}(Y_1 \times Y_2) \\ \downarrow{\scriptstyle i_{1*} \otimes i_{2*}} & & \downarrow{\scriptstyle (i_1 \times i_2)_*} \\ K_{X_1} Z_1 \otimes K_{X_2} Z_2 & \xrightarrow{\times} & K_{X_1 \times X_2}(Z_1 \times Z_2) \end{array}$$

commutes. In both cases this follows easily from the definitions.

1.6. Transversality

Let X be closed in Y, and let i: $Y \to Z$ be a suitably nice imbedding (in algebraic geometry or topology) as in § 1.5. Let f: $Z' \to Z$ be a morphism, and form the fibre square

$$\begin{array}{ccc} Y' & \xrightarrow{i'} & Z' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[i]{} & Z \end{array}$$

If f is suitably transversal to i, then i' will also be a suitably nice imbedding, and for $X'=g^{-1}(X)$, the diagram

$$\begin{array}{ccc} K_X Y & \xrightarrow{i_*} & K_X Z \\ {\scriptstyle g^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ K_{X'} Y' & \xrightarrow[i'_*]{} & K_{X'} Z' \end{array}$$

commutes.

In algebraic geometry the morphisms f and i are suitably transversal for this purpose if f and i are *Tor-independent*, i.e., $\mathrm{Tor}_k^{O_Z}(O_Y, O_{Z'})=0$ for all $k>0$. If i has finite Tor-dimension, it follows that i' also has finite Tor-dimension, and if $E.$ is a complex of locally free sheaves on Y, and $F. \to i_* E.$ is a resolution on Z, then $f^*F. \to i'_*(g^*E.)$ is a resolution on Z' (Appendix 2), from which the commutativity of the diagram follows.

In topology we assume f is transversal to i in the sense of C^∞ maps of C^∞ manifolds, so Y' is a submanifold of Z', and its normal bundle N' is the pull-back of the normal bundle N to Y in Z; we give N' the complex structure induced from that on N. The Koszul–Thom complex for N therefore pulls back to the Koszul–Thom complex for N', and the commutativity of the diagram follows easily.

§ 2. Deformation to the normal bundle

It follows immediately from the definitions that the canonical homomorphism

$$\alpha\colon K_X^{\mathrm{alg}}\,Y \to K_X^{\mathrm{top}}\,Y$$

of § 1.1 is compatible with the pull-backs of § 1.2 and the products of § 1.3. The corresponding assertion for Thom–Gysin maps is proved in the following theorem, which is the essential step in the proof of the Riemann–Roch theorem.

THEOREM. *Let $i\colon Y \to Z$ be a closed imbedding of non-singular complex varieties. Then for any closed algebraic subset X in Y, the diagram*

$$\begin{array}{ccc} K_X^{\mathrm{alg}}\,Y & \xrightarrow{i_*} & K_X^{\mathrm{alg}}Z \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha} \\ K_X^{\mathrm{top}}\,Y & \xrightarrow{i_*} & K_X^{\mathrm{top}}Z \end{array}$$

commutes.

We first note that the theorem is true in case i is the imbedding of Y as the zero section of a vector bundle N on Y. If $\pi\colon N \to Y$ is the projection, the Koszul–Thom complex $\Lambda^{\cdot}\pi^*N^{\vee}$ is a resolution of the structure sheaf of Y on N; it follows that if $E.$ is a complex of algebraic vector bundles on Y, then $\pi^*E.\otimes\Lambda^{\cdot}\pi^*N^{\vee}$ is a resolution of $i_*E.$ on N, so $i_*[E.] = [\pi^*E.\otimes\Lambda^{\cdot}\pi^*N^{\vee}]$, and this agrees with the definition of the Thom–Gysin map in topology.

For the general case we use the method of deformation to the normal bundle, as developed in [3]. Here we give a shorter construction of the same deformation space, following Verdier [15].

LEMMA. *Let $i\colon Y \to Z$ be a closed imbedding of non-singular complex varieties with normal bundle N. Let $\bar{\imath}\colon Y \to N$ be the zero section imbedding of Y in N. Let $j_t\colon Y \to Y \times \mathbf{C}$ be the imbedding given by $j_t(y) = (y, t)$. Then there is a non-singular variety W, with* $\dim W = \dim Z + 1$, *and a commutative diagram of closed imbeddings:*

$$\begin{array}{ccccc} Y & \xrightarrow{j_1} & Y \times \mathbf{C} & \xleftarrow{j_0} & Y \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\bar{\imath}} \\ Z & \xrightarrow[k_1]{} & W & \xleftarrow[k_0]{} & N \end{array}$$

where the squares are transversal—as squares of complex manifolds, and hence in both the algebraic and topological senses of § 1.6.

Proof. Define W to be the flow-up of $Z\times \mathbf{C}$ along the subvariety $Y\times\{0\}$. Since the normal bundle to $Y\times\{0\}$ in $Z\times\mathbf{C}$ is $N\oplus 1$, the exceptional divisor of the blow-up is the projective completion $P(N\oplus 1)$ of N; the map k_0 of the above diagram is the inclusion of N in $P(N\oplus 1)$ followed by the inclusion of the exceptional divisor in W.

In general if $A\subset B\subset C$ are closed imbeddings of algebraic varieties, the blow-up $\hat{B}$ of B along A is imbedded in the blow-up $\hat{C}$ of C along A. If E_B, E_C are the exceptional divisors, there is a fibre square

$$\begin{array}{ccc} E_B & \longrightarrow & \hat{B} \\ \downarrow & & \downarrow \\ E_C & \longrightarrow & \hat{C} \end{array}$$

Since a local equation for E_C on $\hat{C}$ pulls back to a local equation for E_B on $\hat{B}$, this square is always Tor independent. If A, B, and C are non-singular, all the spaces constructed are complex manifolds, and the square is transversal in the C^∞ sense as well. Note also that if A is a divisor on B, the blow-up of B along A is trivial and one can identify B with $\hat{B}$ and A with E_B.

If these facts are applied to the inclusions

$$Y\times\{0\}\subset Y\times\mathbf{C}\subset Z\times\mathbf{C}$$

we get an imbedding $\psi\colon Y\times\mathbf{C}\to W$, and the right square of the desired diagram is the above blow-up diagram—except that we have thrown away the complement of N in $P(N\oplus 1)$.

The blow-down map from W to $Z\times\mathbf{C}$ is an isomorphism off $Y\times\{0\}$. In particular the restriction to $Z\times\mathbf{C}^*$, $\mathbf{C}^*=\mathbf{C}-\{0\}$, can be identified with an open set in W. The left square of the required diagram then comes from

$$\begin{array}{ccccc} Y & \xrightarrow{j_1} & Y\times\mathbf{C}^* & \subset & Y\times\mathbf{C} \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i\times id} & & \downarrow{\scriptstyle\psi} \\ Z & \longrightarrow & Z\times\mathbf{C}^* & \subset & W \end{array}$$

where Z is imbedded in $Z\times\mathbf{C}$ by $z\to(z,1)$.

Remark. The blow-down map $W\to Z\times\mathbf{C}$ followed by the projection to C gives a flat family $\varrho\colon W\to\mathbf{C}$, with a family of imbeddings

$$\begin{tikzcd} Y \times \mathbf{C} \arrow[rr, "\psi"] \arrow[dr, "pr"'] & & W \arrow[dl, "\varrho"] \\ & \mathbf{C} & \end{tikzcd}$$

The imbeddings ψ_t of Y in $W_t = \varrho^{-1}(t)$ are all isomorphic to the given imbedding for $t \neq 0$. For $t=0$, W_0 is the union of $P(N \oplus 1)$ and the blow-up $\hat{Z}$ of Z along Y, meeting transversally in $P(N)$. For details of this description see [10].

We now prove that α is compatible with the Thom–Gysin maps. From the diagram constructed in the lemma we obtain diagrams

$$\begin{tikzcd} K_X Y \arrow[d, "i_*"'] & K_{X\times\mathbf{C}}(Y \times \mathbf{C}) \arrow[l, "j_1^*"'] \arrow[r, "j_0^*"] \arrow[d, "\psi_*"] & K_X Y \arrow[d, "\bar{\imath}_*"] \\ K_X Z & K_{X\times\mathbf{C}} W \arrow[l, "k_1^*"] \arrow[r, "k_0^*"'] & K_X N \end{tikzcd}$$

in both algebraic geometry and topology; the horizontal maps are pull-backs, and the vertical maps are Thom–Gysin maps. The squares commute by the transversality properties proved in § 1.6. We need two additional facts:

(i) j_1^* is surjective in the algebraic geometry diagram;

(ii) $\operatorname{Ker}(k_0^*) \subset \operatorname{Ker}(k_1^*)$ in the topology diagram.

The first assertion follows from the fact that $j_1^* p^* = (p \circ j_1)^*$ is the identity, where p is the projection of $Y \times \mathbf{C}$ to Y. (In fact j_1^* is an isomorphism.) The second follows most simply from the obvious fact that $\operatorname{Ker}(j_0^*) = \operatorname{Ker}(j_1^*)$, and the fact that the vertical Thom–Gysin maps are all isomorphisms in topology (Appendix 1). (One could also use the fact that the family of imbeddings is topologically trivial near $Y \times \mathbf{C}$, as in [3] § 1.2.)

Now consider the map that α induces from the algebraic geometry diagram to the topological diagram, regarding the two diagrams as vertical with the topological diagram behind the algebraic one. The top and bottom squares of the resulting double cube commute since α commutes with pullbacks. The right side face commutes by the vector bundle case considered earlier.

The proof can now be completed by a diagram chase. For

$$k_0^*(\alpha\psi_* - \psi_*\alpha) = \alpha k_0^*\psi_* - k_0^*\psi_*\alpha = \alpha\bar{\imath}_* j_0^* - \bar{\imath}_* j_0^*\alpha = (\alpha\bar{\imath}_* - \bar{\imath}_*\alpha) j_0^* = 0.$$

From (ii), this implies that $k_1^*(\alpha\psi_* - \psi_*\alpha) = 0$, and the same sequence of steps on the left side of the diagram shows that $(\alpha i_* - i_*\alpha) j_1^* = 0$. The desired equation $\alpha i_* - i_*\alpha = 0$ follows from (i).

Remark. The same deformation can be used to simplify the proof of Riemann–Roch for imbeddings of complex analytic manifolds in [1].

§ 3. K-cohomology and homology

In § 3.1 we describe the K-cohomology groups K^0X and K-homology groups K_0X in algebraic geometry and topology. The relations with the relative groups of § 1 are discussed in § 3.2 and § 3.3. Poincaré duality and a preliminary discussion of orientations are in § 3.4.

3.1. Definitions

If X is an algebraic variety (resp. a topological space), $K^0_{\text{alg}}X$ (resp. $K^0_{\text{top}}X$) denotes the Grothendieck group of algebraic (resp. topological) vector bundles on X. That is, form the free abelian group on the isomorphism classes of vector bundles, modulo the relations given by exact sequences of vector bundles. In each case K^0X is a commutative ring with unit, the product given by tensor product of vector bundles, and K^0 is a contravariant functor on the respective category. In fact, K^0X may be identified with the ring K_XX of § 1.

In each case there is a corresponding functor K_0, which is covariant for proper morphisms, with a *cap product*

$$K^0X \otimes K_0X \xrightarrow{\frown} K_0X$$

making K_0X into a K^0X-module, and satisfying a *projection formula*

$$f_*(f^*b \frown a) = b \frown f_*a$$

if $f: X \to Y$ is a proper morphism, $b \in K^0Y$, $a \in K_0X$. We also have *exterior products*

$$K_0X_1 \otimes K_0X_2 \xrightarrow{\times} K_0(X_1 \times X_2)$$

which are compatible with proper maps, i.e.,

$$(f_1 \times f_2)_*(a_1 \times a_2) = f_{1*}(a_1) \times f_{2*}(a_2)$$

if $f_i: X_i \to Y_i$ is proper, $a_i \in K_0X_i$. And each theory has *restriction* homomorphisms

$$K_0X \xrightarrow{j^*} K_0U$$

if $j\colon U \to X$ is the inclusion of an open subscheme or subspace U in X. Restriction maps are also compatible with pushing forward, cap products, and exterior products, but we will not need these properties.

For a variety X, $K_0^{\mathrm{alg}} X$ is the Grothendieck group of coherent algebraic sheaves on X; write $[\mathcal{F}]$ for the element in $K_0^{\mathrm{alg}} X$ determined by a sheaf $\mathcal{F}$. The cap product is given by $[\mathrm{E}] \frown [\mathcal{F}] = [\mathrm{E} \otimes \mathcal{F}]$ for a vector bundle E (regarded as a locally free sheaf) and coherent sheaf $\mathbf{F}$ on X. The exterior product is given by the formula $[\mathcal{F}_1] \times [\mathcal{F}_2] = [\mathcal{F}_1 \boxtimes \mathcal{F}_2]$ for sheaves $\mathcal{F}_i$ on X_i, where the tensor product is the one discussed in § 1.3. The restriction homomorphism $j^*\colon K_0^{\mathrm{alg}} X \to K_0^{\mathrm{alg}} U$ takes $[\mathcal{F}]$ to $[\mathcal{F} | U]$, where $\mathcal{F} | U$ is the restriction of the sheaf $\mathcal{F}$ to U. If $f\colon X \to Y$ is a proper morphism, then the homomorphism

$$f_*\colon K_0^{\mathrm{alg}} X \to K_0^{\mathrm{alg}} Y$$

is defined by setting $f_*[\mathcal{F}] = \sum (-1)^i [R^i f_* \mathcal{F}]$, where the $R^i f_* \mathcal{F}$ are Grothendieck's higher direct image sheaves: for an affine open subset U of Y, $\Gamma(U, R^i f_* \mathcal{F}) = H^i(f^{-1}(U), \mathcal{F})$.

In topology we continue to work with locally compact spaces whose one-point compactification is homeomorphic to a finite simplicial complex. Such a space X may be imbedded as a closed subspace of some $\mathbf{C}^n$ so that $(\mathbf{C}^n, X)$ is a simplicial pair as in § 1.1; then $K_0^{\mathrm{top}} X$ may be defined by making Alexander duality a definition:

$$K_0^{\mathrm{top}} X = K_X^{\mathrm{top}} \mathbf{C}^n$$

where the right side is defined in § 1, and discussed in Appendix 1. That this is independent of the imbedding follows from the Thom–Gysin isomorphism $K_X^{\mathrm{top}} \mathbf{C}^n \cong K_X^{\mathrm{top}} \mathbf{C}^{n+1}$ for the standard imbedding of $\mathbf{C}^n$ in $\mathbf{C}^{n+1}$, together with the fact that two imbeddings are isotopic if n is large.

To describe the cap product in K^{top}, let $X \subset \mathbf{C}^n$ as above. For any neighborhood U of X in $\mathbf{C}^n$, we have a product

$$K_{\mathrm{top}}^0 U \otimes K_X^{\mathrm{top}} U \to K_X^{\mathrm{top}} U$$

from § 1.3. Since $K_X^{\mathrm{top}} U = K_X^{\mathrm{top}} \mathbf{C}^n = K_0^{\mathrm{top}} X$ by excision (Appendix 1), and any element in $K_{\mathrm{top}}^0 X$ extends to an element in $K_{\mathrm{top}}^0 U$ for some neighborhood U of X, this determines the desired product $K_{\mathrm{top}}^0 X \otimes K_0^{\mathrm{top}} X \to K_0^{\mathrm{top}} X$.

For the exterior product, let $X_i \subset \mathbf{C}^{n_i}$; then the product

$$K_{X_1}^{\mathrm{top}} \mathbf{C}^{n_1} \otimes K_{X_2}^{\mathrm{top}} \mathbf{C}^{n_2} \to K_{X_1 \times X_2}^{\mathrm{top}} (\mathbf{C}^{n_1} \times \mathbf{C}^{n_2})$$

of § 1.3 translates to give the required product.

To describe f_* for a proper mapping $f: X \to Y$, note first that when f is a closed imbedding with (Y, X) triangulable, and we imbed Y in some $\mathbf{C}^n$ as above, we have a natural map $K_X^{\text{top}}\mathbf{C}^n \to K_Y^{\text{top}}\mathbf{C}^n$ by the pull-back of § 1.2, and this is $f_*: K_0^{\text{top}} X \to K_0^{\text{top}} Y$ in this case. In the general case f may be factored as a closed imbedding $i: X \to Y \times D$, where D is a closed disk in some $\mathbf{C}^m$, followed by a projection $p: Y \times D \to Y$. Then $f_* = p_* \circ i_*$, and p_* may be identified with the isomorphism

$$j_t^*: K_{Y \times D}(\mathbf{C}^n \times \mathbf{C}^m) \to K_Y \mathbf{C}^n$$

where $Y \subset \mathbf{C}^n$, $t \in D$, and j_t is the imbedding $v \to (v, t)$ of $\mathbf{C}^n$ in $\mathbf{C}^n \times \mathbf{C}^m$.

For the restriction mapping from $K_0^{\text{top}} X$ to $K_0^{\text{top}} U$, for Y open in X, imbed X in $\mathbf{C}^n$ as above, and let V be open in $\mathbf{C}^n$ with $V \cap X = U$. Then $K_0^{\text{top}} U$ may be identified with $K_U^{\text{top}} V$ (see the more general duality in § 3.2), so the desired map is the pull-back $K_X^{\text{top}} \mathbf{C}^n \to K_U^{\text{top}} V$ induced by the inclusion of V in $\mathbf{C}^n$.

An alternate approach to K_0^{top} is to use the correspondence between cohomology and homology theories, as given by Whitehead [16]. For finite CW complexes K_{top}^0 is the even part of the cohomology theory determined by the unitary spectrum, and K_0^{top} can be taken to be the even part of the corresponding homology theory. For non-compact X, let $X^c = X \cup \{\infty\}$ be the one-point compactification, and define $K_0^{\text{top}} X$ to be $K_0^{\text{top}}(X^c, \{\infty\})$. The desired properties can then be reduced to standard properties for finite CW complexes given in [16]. See [2] for a geometric description more suitable to Riemann–Roch.

3.2. Relative and absolute K-groups

The algebraic and topological theories each have *homology maps*

$$h: K_X Y \to K_0 X$$

when X is closed in a suitable Y; these are compatible with the constructions of § 1 and § 3.1.

In algebraic geometry there is no restriction on Y, and $h: K_X^{\text{alg}} Y \to K_0^{\text{alg}} X$ is defined by setting

$$h[E.] = \sum (-1)^i [H_i(E.)]$$

Here the $H_i(E.)$ are the homology sheaves of the complex $E.$ of locally free sheaves on Y. They are coherent sheaves on Y which are supported on X and thus determine elements $[H_i(E.)]$ in $K_0^{\text{alg}} X$, (cf. Appendix 2).

In topology we will define

$$h: K_X^{\text{top}} Y \to K_0^{\text{top}} X$$

under the conditions that Y is a C^∞ manifold whose tangent bundle has a given complex structure, and X is a closed subspace of Y with (Y, X) triangulable as in § 1.1. Imbed Y as a closed subcomplex of some $\mathbf{C}^n$. We have the Thom–Gysin isomorphism

$$K_X^{\mathrm{top}}\, Y \xrightarrow{\sim} K_X^{\mathrm{top}}\, \mathbf{C}^n,$$

which is the desired map h when we identify the right side with $K_0^{\mathrm{top}} X$.

3.3. Properties of the homology maps

We need some basic properties of these homology maps $h: K_X Y \to K_0 X$ in algebraic geometry and topology.

Property 1. Let X be closed in Y, with Y, Z as above, and assume $i: Y \to Z$ is a suitably nice closed imbedding as in § 1.5. In topology assume the complex structure on the normal bundle to Y in Z is compatible with the complex structures on the tangent bundles. Then the diagram

$$\begin{tikzcd} K_X Y \arrow[d, "i_*"'] \arrow[dr, "h"] & \\ K_X Z \arrow[r, "h"'] & K_0 X \end{tikzcd}$$

commutes.

This is obvious from the definition in algebraic geometry, and in topology it follows from the functoriality of the Thom–Gysin maps. The next five properties are likewise simple consequences of the definitions; for the last, in algebraic geometry, see Appendix 2.

Property 2. With $X_i \subset Y_i$ as above, the diagram

$$\begin{tikzcd} K_{X_1} Y_1 \otimes K_{X_2} Y_2 \arrow[r, "\times"] \arrow[d, "h \otimes h"] & K_{X_1 \times X_2}(Y_1 \times Y_2) \arrow[d, "h"] \\ K_0 X_1 \otimes K_0 X_2 \arrow[r, "\times"] & K_0(X_1 \times X_2) \end{tikzcd}$$

commutes.

Property 3. Let $X \subset Y$ as above, and let j be the inclusion of X in Y. Then the diagram

$$\begin{tikzcd} K^0 Y \otimes K_X Y \arrow[r, "\cup"] \arrow[d, "j^* \otimes h"] & K_X Y \arrow[d, "h"] \\ K^0 X \otimes K_0 X \arrow[r, "\cap"] & K_0 X \end{tikzcd}$$

commutes.

Property 4. Let $X \subset Y$ as above, let Y_0 be open in Y, and let $X_0 = Y_0 \cap X$. Then the diagram

$$\begin{array}{ccc} K_X Y & \xrightarrow{h} & K_0 X \\ \downarrow & & \downarrow \\ K_{X_0} Y_0 & \xrightarrow{h} & K_0 X_0 \end{array}$$

commutes, where the left vertical map is the pull-back (§ 1.2), and the right vertical map is restriction (§ 3.1).

Property 5. If $X' \subset X \subset Y$ is a sequence of closed imbeddings, with Y as above, then the diagram

$$\begin{array}{ccc} K_{X'} Y & \xrightarrow{h} & K_0 X' \\ \downarrow & & \downarrow \\ K_X Y & \xrightarrow{h} & K_0 X \end{array}$$

commutes, where the left vertical map is the pull-back of § 1.1, and the right vertical map is induced by the inclusion of X' in X.

Property 6. The homology mapping

$$K_X Y \xrightarrow{h} K_0 X$$

is an isomorphism, provided—in the algebraic case—that Y is non-singular.

3.4. Poincaré duality

For any variety X, or any C^∞ manifold X with complex tangent bundle we may apply the above homology mapping to the imbedding of X in itself. Since $K^0X = K_X X$ in either case, there results the *Poincaré duality map*

$$h: K^0X \to K_0 X.$$

This is an isomorphism in algebraic geometry if X is non-singular (§ 3.3 Property 6), and also in topology by the description of h in § 3.2. The image of 1 by the Poincaré duality map will be called the *fundamental class* of X. In $K_0^{\text{alg}} X$ it is $[O_X]$, where O_X is the structure sheaf of X. In topology it is the K-theory orientation class, which we denote by $\{X\}$. See § 6 for a general discussion of orientations.

The Poincaré duality map takes an element in K^0X to its cap product with the fundamental class, as follows, e.g., from Property 3 in § 3.2.

§ 4. The Riemann–Roch theorem

4.1. The theorem

Let $\alpha^{\cdot}: K^0_{\mathrm{alg}} X \to K^0_{\mathrm{top}} X$ be the homomorphism which takes an algebraic vector bundle to its underlying topological vector bundle. This is a natural transformation of contravariant functors from the category of complex quasi-projective schemes to rings.

Define the corresponding homomorphism

$$\alpha_{\cdot}: K_0^{\mathrm{alg}} X \to K_0^{\mathrm{top}} X$$

as follows. Choose an imbedding of X in a non-singular variety Y; then $\alpha_{\cdot}$ is the composite

$$K_0^{\mathrm{alg}} X \xleftarrow[\cong]{h} K_X^{\mathrm{alg}} Y \xrightarrow{\alpha} K_X^{\mathrm{top}} Y \xrightarrow[\cong]{h} K_0^{\mathrm{top}} X$$

Here h and α are the maps defined in § 3.2 and § 1.1 respectively.

RIEMANN-ROCH THEOREM. *The homomorphism $\alpha_{\cdot}$ is independent of the imbedding, and satisfies*

1. (covariance). *For every proper morphism $f: X \to X'$, the diagram*

$$\begin{array}{ccc} K_0^{\mathrm{alg}} X & \xrightarrow{\alpha_{\cdot}} & K_0^{\mathrm{top}} X \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\ K_0^{\mathrm{alg}} X' & \xrightarrow{\alpha_{\cdot}} & K_0^{\mathrm{top}} X' \end{array}$$

commutes.

2. (module). *For every X the diagram*

$$\begin{array}{ccc} K^0_{\mathrm{alg}} X \otimes K_0^{\mathrm{alg}} X & \xrightarrow{\alpha^{\cdot} \otimes \alpha_{\cdot}} & K^0_{\mathrm{top}} X \otimes K_0^{\mathrm{top}} X \\ \downarrow{\scriptstyle \frown} & & \downarrow{\scriptstyle \frown} \\ K_0^{\mathrm{alg}} X & \xrightarrow{\alpha_{\cdot}} & K_0^{\mathrm{top}} X \end{array}$$

commutes.

3. (product). *For any X_1, X_2, the diagram*

$$\begin{array}{ccc} K_0^{\mathrm{alg}} X_1 \otimes K_0^{\mathrm{alg}} X_2 & \xrightarrow{\alpha_{\cdot} \otimes \alpha_{\cdot}} & K_0^{\mathrm{top}} X_1 \otimes K_0^{\mathrm{top}} X_2 \\ \downarrow{\scriptstyle \times} & & \downarrow{\scriptstyle \times} \\ K_0^{\mathrm{alg}}(X_1 \times X_2) & \xrightarrow{\alpha_{\cdot}} & K_0^{\mathrm{top}}(X_1 \times X_2) \end{array}$$

commutes.

4. (restriction). *If U is an open subscheme of X, then the diagram*

$$\begin{tikzcd} K_0^{\mathrm{alg}} X \arrow[r, "\alpha."] \arrow[d] & K_0^{\mathrm{top}} X \arrow[d] \\ K_0^{\mathrm{alg}} U \arrow[r, "\alpha."] & K_0^{\mathrm{top}} U \end{tikzcd}$$

commutes; the vertical maps are the restriction homomorphisms.

5. (orientation). *If X is non-singular, then $\alpha.([O_X]) = \{X\}$ is the K-theory orientation class.*

The vertical maps in the above diagrams are defined in § 3.1, the K-theory orientation in § 3.4.

Proof. Until we have proved that $\alpha.$ is independent of the imbedding, we denote by $\alpha_.^Y$ the homomorphism from $K_0^{\mathrm{alg}} X$ to $K_0^{\mathrm{top}} X$ defined before the statement of the theorem by imbedding X in a non-singular variety Y. Before giving the proof itself we prove several preliminary results. We emphasize the formal nature of the proof, assuming the results of § 1–3.

(1) If $X \subset Y$, and $i\colon Y \to Z$ is a closed imbedding of non-singular varieties, then $\alpha_.^Y = \alpha_.^Z$. This follows from the commutative diagram

$$\begin{tikzcd} & K_X^{\mathrm{alg}} Y \arrow[r, "\alpha"] \arrow[dl, "h"'] \arrow[dd, "i_*"] & K_X^{\mathrm{top}} Y \arrow[dr, "h"] \arrow[dd, "i_*"] & \\ K_0^{\mathrm{alg}} X & & & K_0^{\mathrm{top}} X \\ & K_X^{\mathrm{alg}} Z \arrow[r, "\alpha"] \arrow[ul, "h"] & K_X^{\mathrm{top}} Z \arrow[ur, "h"'] & \end{tikzcd}$$

The square commutes by § 2, the triangles by Property 1 of § 3.3.

(2) For any closed imbedding $j\colon X \to Y$, with Y non-singular, and any $b \in K^0_{\mathrm{alg}} Y$, $a \in K_0^{\mathrm{alg}} X$, $\alpha_.^Y(j^* b \frown a) = j^* \alpha^{\cdot}(b) \frown \alpha_.^Y(a)$. This follows from Property 3 of § 3.3 and the fact that α is compatible with products (cf. § 2).

(3) If $X_i \subset Y_i$, Y_i non-singular, then

$$\begin{tikzcd} K_0^{\mathrm{alg}} X_1 \otimes K_0^{\mathrm{alg}} X_2 \arrow[r, "\alpha_.^{Y_1} \otimes \alpha_.^{Y_2}"] \arrow[d, "\times"] & K_0^{\mathrm{top}} X_1 \otimes K_0^{\mathrm{top}} X_2 \arrow[d, "\times"] \\ K_0^{\mathrm{alg}}(X_1 \times X_2) \arrow[r, "\alpha_.^{Y_1 \times Y_2}"] & K_0^{\mathrm{top}}(X_1 \times X_2) \end{tikzcd}$$

commutes.

This follows from Property 2 of § 3.3 and the fact that α commutes with products (§ 2).

(4) If $X \subset Y$ as above, and $f: X' \to X$ is a closed imbedding, then the diagram

$$\begin{array}{ccc} K_0^{\mathrm{alg}} X' & \xrightarrow{\alpha_\cdot^Y} & K_0^{\mathrm{top}} X' \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ K_0^{\mathrm{alg}} X & \xrightarrow{\alpha_\cdot^Y} & K_0^{\mathrm{top}} X \end{array}$$

commutes. This diagram is the outside of a diagram

$$\begin{array}{ccccccc} K_0^{\mathrm{alg}} X' & \xrightarrow{h} & K_{X'}^{\mathrm{alg}} Y & \xrightarrow{\alpha} & K_{X'}^{\mathrm{top}} Y & \xrightarrow{h} & K_0^{\mathrm{top}} X' \\ {\scriptstyle f_*}\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle f_*} \\ K_0^{\mathrm{alg}} X & \xrightarrow{h} & K_X^{\mathrm{alg}} Y & \xrightarrow{\alpha} & K_X^{\mathrm{top}} Y & \xrightarrow{h} & K_0^{\mathrm{top}} X \end{array}$$

where the unmarked vertical maps are the pull-backs of § 1.1 The middle square commutes by § 2, the outside squares by § 3.3 Property 5.

(5) Let $q: \mathbf{P}^n \to \mathrm{pt.}$ be the map from projective n-space to a point. Then

$$\begin{array}{ccc} K_0^{\mathrm{alg}} \mathbf{P}^n & \xrightarrow{\alpha_\cdot^{\mathbf{P}^n}} & K_0^{\mathrm{top}} \mathbf{P}^n \\ {\scriptstyle q_*}\downarrow & & \downarrow{\scriptstyle q_*} \\ K_0^{\mathrm{alg}}(\mathrm{pt.}) & \xrightarrow{\alpha_\cdot^{\mathrm{pt.}}} & K_0^{\mathrm{top}}(\mathrm{pt.}) \end{array}$$

commutes.

Proof. (For a proof using Chern classes and formal calculations, see [1]). Since $K_0^{\mathrm{alg}} \mathbf{P}^n$ is generated by the classes $[\mathcal{O}_{\mathbf{P}^n}(d)]$ (cf. [7]), it suffices to prove that the two routes around the diagram agree for them. Let $\mathbf{P}^{n-1}$ be the hyperplane in $\mathbf{P}^n$ defined by setting the coordinate $X_0 = 0$, and consider the exact sequences

$$0 \to \mathcal{O}_{\mathbf{P}^n}(d-1) \to \mathcal{O}_{\mathbf{P}^n}(d) \to \mathcal{O}_{\mathbf{P}^{n-1}}(d) \to 0.$$

Assuming the result inductively for $\mathbf{P}^{n-1}$, and using step (4), it suffices to prove it for the one case $d = 0$. This is done in Appendix 3.

(6) Let $X \subset Y$ as above, and let Y_0 be an open neighborhood of X in Y. Then $\alpha_\cdot^Y = \alpha_\cdot^{Y_0}$. This follows from Property 4 of § 3.3 and the fact that α commutes with pullbacks (§ 2).

(7) Let $X \subset Y$ as above, and let P be a complex projective n-space, and let p: $X \times P \to X$ be the projection. Then the diagram

$$\begin{array}{ccc} K_0^{\mathrm{alg}}(X \times P) & \xrightarrow{\alpha_.^{Y\times P}} & K_0^{\mathrm{top}}(X \times P) \\ \downarrow{p_*} & & \downarrow{p_*} \\ K_0^{\mathrm{alg}} X & \xrightarrow{\alpha_.^{Y}} & K_0^{\mathrm{top}} X \end{array}$$

commutes. Consider the commutative diagram (from (5) above):

$$\begin{array}{ccc} K_0^{\mathrm{alg}} X \otimes K_0^{\mathrm{alg}} P & \xrightarrow{\alpha_.^{Y} \otimes \alpha_.^{P}} & K_0^{\mathrm{top}} X \otimes K_0^{\mathrm{top}} P \\ \downarrow{1 \otimes q_*} & & \downarrow{1 \otimes q_*} \\ K_0^{\mathrm{alg}} X \otimes K_0^{\mathrm{alg}}(\mathrm{pt.}) & \xrightarrow{\alpha_.^{Y} \otimes \alpha_.^{\mathrm{pt.}}} & K_0^{\mathrm{top}} X \otimes K_0^{\mathrm{top}}(\mathrm{pt.}) \end{array}$$

The exterior product maps this square to the above square. The sides of the resulting cube commute by § 3.1, the top and bottom by (3) above, and desired commuting of the back square follows from the fact that the product map $K_0^{\mathrm{alg}} X \otimes K_0^{\mathrm{alg}} P \to K_0^{\mathrm{alg}}(X \times P)$ is surjective ([7], Prop. 9).

We now turn to the proof of the Riemann–Roch theorem. To show that $\alpha.$ is independent of the imbedding, suppose $X \subset Y_1$ and $X \subset Y_2$ were two imbeddings in non-singular quasi-projective varieties. By (1), we may assume Y_1 and Y_2 are open subschemes in complex projective spaces P_1 and P_2 respectively. Consider the diagonal imbedding i: $X \to X \times P_2$ given by $x \to (x, x)$, the projection p: $X \times P_2 \to X$, and the diagram

$$\begin{array}{ccc} K_0^{\mathrm{alg}} X & \xrightarrow{\alpha_.^{Y_1\times Y_2}} & K_0^{\mathrm{top}} X \\ \downarrow{i_*} & & \downarrow{i_*} \\ K_0^{\mathrm{alg}}(X \times P_2) & \xrightarrow{\alpha_.^{Y_1\times P_2}} & K_0^{\mathrm{top}}(X \times P_2) \\ \downarrow{p_*} & & \downarrow{p_*} \\ K_0^{\mathrm{alg}} X & \xrightarrow{\alpha_.^{Y_1}} & K_0^{\mathrm{top}} X\ . \end{array}$$

The top map $\alpha_.^{Y_1\times Y_2}$ is equal to the map $\alpha_.^{Y_1\times P_2}$ obtained by imbedding X in $Y_1 \times P_2$ by (6), and then the upper square commutes by (4). The lower square commutes by (7). But since $p_* i_*$ is the identity on $K_0 X$ this shows that $\alpha_.^{Y_1\times Y_2} = \alpha_.^{Y_1}$, and hence that $\alpha_.^{Y_1} = \alpha_.^{Y_2}$ by symmetry.

Since any morphism f: $X \to Y$ factors into an imbedding $X \to Y \times \mathbf{P}^n$ followed by a projection $Y \times \mathbf{P}^n \to Y$, the covariance property 1 follows from (4) and (7).

The module property 2 follows from (2), and the fact that for any algebraic vector bundle E on a projective variety X, there is an imbedding of X in a non-singular projective variety Y so that E is the restriction of a vector bundle on Y ([3], Appendix § 3.2).

The product property 3 follows from (3), and the restriction property 4 from property 4 of § 3.3 and the fact that α commutes with pullbacks. Finally, the orientation property 5 is obvious from the definition if we use the imbedding of X in itself.

4.2. Complements

The results sketched in this section are not essential to the rest of the paper.

(1) *Uniqueness.* The functor α is uniquely determined by the covariance property 1 and the orientation property 5. This follows from the fact that $K_0^{\mathrm{alg}} X$ is generated by elements of the form $\pi_X[O_V]$, where V is a non-singular variety and π: $V \to X$ is proper. Note that resolution of singularities is not used anywhere else in this paper. (See [3] for other uniqueness results.)

(2) *Local complete intersections.* If f: $X \to Y$ is a morphism of algebraic varieties which is a local complete intersection, there are Gysin morphisms

$$f_*: K^0_{\mathrm{alg}} X \to K^0_{\mathrm{alg}} Y \quad (\text{for } f \text{ proper})$$

and

$$f^*: K_0^{\mathrm{alg}} Y \to K_0^{\mathrm{alg}} X.$$

For the case of a closed imbedding f_* is defined as in § 1.5; $f^*[\mathfrak{F}] = \sum (-1)^i [\mathrm{Tor}_i^{O_Y}(\mathfrak{F}, O_X)]$. (See [5] for general properties of these Gysin maps.) The analogous maps in topology were constructed in [3], § 4.

Riemann–Roch theorem for local complete intersections. Let f: $X \to Y$ be a local complete intersection morphism of quasi-projective complex schemes. Then the diagrams

(1)
$$\begin{array}{ccc} K^0_{\mathrm{alg}} X & \xrightarrow{\alpha^{\cdot}} & K^0_{\mathrm{top}} X \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\ K^0_{\mathrm{alg}} Y & \xrightarrow{\alpha^{\cdot}} & K^0_{\mathrm{top}} Y \end{array}$$

and

(2)
$$\begin{array}{ccc} K_0^{\mathrm{alg}} Y & \xrightarrow{\alpha_{\cdot}} & K_0^{\mathrm{top}} Y \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\ K_0^{\mathrm{alg}} X & \xrightarrow{\alpha_{\cdot}} & K_0^{\mathrm{top}} X \end{array}$$

commute.

These can be proved using the properties of K^{alg} and K^{top} developed in § 1 and § 3, and the deformation construction of § 2. The proof of (1) follows closely the proof of the theorem in § 2, and in fact they have a common generalization; (2) can be proved by an argument that is nearly dual to the proof of (1). We omit the details.

§ 5. The Chern character

The discussion in § 1 and § 3 concerning topological K-theory has a completely analogous, and better known, form for ordinary homology. Let $H^{\cdot}X$ be ordinary singular cohomology with rational coefficients; only the groups of even degree need be included. If X is closed in Y, let $H_X^{\cdot}Y = H^{\cdot}(Y, Y - X)$ be the local cohomology of Y with support in X. This theory has pull-backs, products, and Thom–Gysin maps just as in § 1, satisfying the same properties. The homology groups $H.X$ can be defined to be Borel–Moore homology [6], but it is more in keeping with our treatment here to define $H_i X$ to be $H_X^{2n-i}\mathbf{C}^n$ for a simplicial imbedding of X in some $\mathbf{C}^n$ (our standing assumptions on our spaces making such an imbedding possible.) All the properties of § 3 are then valid, with the same hypothesis as for K^{top}.

Let $ch^{\cdot}\colon K^0_{\mathrm{top}} \to H^{\cdot}$ be the usual Chern character. This is a natural transformation of contravariant functors from spaces to rings, characterized by the fact that $ch^{\cdot}(L) = \exp(c) = \sum (1/n!)(c)^n$ when L is a line bundle on X with first Chern class c in $H^2(X)$. There is a canonical extension of the Chern character to homomorphisms

$$ch\colon K_X^{\mathrm{top}} Y \to H_X^{\cdot} Y$$

for X closed in Y, which is compatible with pull-backs and products; B. Iversen [13] has given a nice construction for this. Although this local Chern character is compatible with pull-backs and products, it does not commute with the Thom–Gysin maps. If $i\colon Y \to Z$ is a closed imbedding of C^∞ manifolds with complex normal bundle N, and X is closed in Y, then the diagram

$$\begin{array}{ccc} K_X^{\mathrm{top}} Y & \xrightarrow{ch} & H_X^{\cdot} Y \\ {\scriptstyle i_*}\downarrow & & \downarrow{\scriptstyle \bar{i}_*} \\ K_X^{\mathrm{top}} Z & \xrightarrow{ch} & H_X^{\cdot} Z \end{array}$$

commutes only if the usual Thom–Gysin map $i_*\colon H_X^{\cdot} Y \to H_X^{\cdot} Z$ is modified by defining

$$\bar{i}_*(a) = i_*(\mathrm{td}\,(N)^{-1} \cup a),$$

where td (N) is the Todd class of N. (The Todd class of a vector bundle is a characteristic class—i.e. it is contravariant—which takes sums of vector bundles to products, and if c is the first chern class of a line bundle, its Todd class is $c(1-\exp(-c))^{-1}$. It is also determined as the characteristic class which makes the above diagrams commute.) Note that the multiplicative property of the Todd class insures that these modified Thom–Gysin maps are also functorial.

Corresponding to the cohomology Chern character $ch^{\cdot}$ there is a homomorphism

$$ch.\colon K_0^{\mathrm{top}} X \to H.X$$

such that such that $ch^{\cdot}$ and $ch.$ satisfy the obvious analogues of Properties 1–4 of the Riemann–Roch theorem in § 4. The analogue to Property 5 is that

$$ch.\{Y\} = \mathrm{td}\,(T_Y) \cap [Y]$$

when Y is a C^∞ $2n$-manifold with complex tangent bundle T_Y, and $[Y]$ is the fundamental, or homology orientation class, in $H_{2n}Y$. We may define the homology Chern character by imbedding X in some $\mathbf{C}^n$, and then $ch.$ is the composite

$$K_0^{\mathrm{top}} X \cong K_X^{\mathrm{top}} \mathbf{C}^n \xrightarrow{ch.} H_X^{\cdot} \mathbf{C}^n \cong H.X$$

where the isomorphisms are the defining (Alexander) isomorphisms.

Define $\tau^{\cdot}\colon K^0_{\mathrm{alg}} \to H^{\cdot}$ to be the composite of the homomorphism $\alpha^{\cdot}\colon K^0_{\mathrm{alg}} \to K^0_{\mathrm{top}}$ and the Chern character $ch^{\cdot}\colon K^0_{\mathrm{top}} \to H^{\cdot}$. Similarly $\tau.\colon K_0^{\mathrm{alg}} \to H.$ is defined by $\tau. = ch.\circ\alpha$.

The following theorem of [3] is then an immediate consequence of the theorem in § 4.

THEOREM. *The mapping*

$$\tau.\colon K_0^{\mathrm{alg}} X \to H.X$$

is covariant for proper morphisms, $\tau^{\cdot}$ and $\tau.$ are compatible with cap products, and $\tau.$ is compatible with cartesian products and restriction to open subvarieties. If X is non-singular,

$$\tau.[O_X] = \mathrm{td}\,(T_X) \cap [X].$$

Remarks. (1) If one wants only this result, one may ignore topological K-theory and work directly with the composite $K^{\mathrm{alg}} \to H$. This gives a simplification of the proof given in [3].

(2) If $f\colon X \to Y$ is a local complete intersection morphism, it follows from the construction of the Gysin maps ([3] § 4.4) that the diagrams

(1)
$$\begin{tikzcd} K^0_{\mathrm{top}}X \arrow[r, "ch^{\cdot}"] \arrow[d, "f_*"'] & H^{\cdot}X \arrow[d, "f_*(\mathrm{td}\,(T_f)\cup\ )"] \\ K^0_{\mathrm{top}} \arrow[r, "ch^{\cdot}"] & H^{\cdot}Y \end{tikzcd}$$

and

(2)
$$\begin{tikzcd} K_0^{\mathrm{top}}Y \arrow[r, "ch."] \arrow[d, "f_*"'] & H.Y \arrow[d, "\mathrm{td}\,(T_f)\cap f^*"] \\ K_0^{\mathrm{top}}X \arrow[r, "ch."] & H.X \end{tikzcd}$$

commute, where T_f is the virtual tangent bundle of f. If these diagrams are combined with the corresponding diagrams of § 4.2 we recover theorems of [3] § 4.3 and [15] § 10.

Riemann–Roch without denominators, for values in cohomology with integer coefficients, (cf. [3] § 4.5) is similarly an easy corollary of diagram (1) in § 4.2.

§ 6. Orientations

We consider locally compact spaces X whose one-point compactification is homeomorphic to a finite simplicial complex, so that the homology K-groups $K_0^{\mathrm{top}}X$ are defined as in § 3.1. For a point x in X, let

$$K_0^{\mathrm{top}}(X)_x = \varinjlim K_0^{\mathrm{top}}U$$

where the limit is over the open neighborhoods U of x, and the maps are given by restrictions. If X is triangulated near x, this is isomorphic to the homology K-group of the pair (St (x), Lk (x)) consisting of the star and link of x in X.

At any point x at which X is a topological $2n$-manifold, $K_0^{\mathrm{top}}(X)_x$ is infinite cyclic.

Definition. Assume X has a dense open set which is an even dimensional manifold. An element η in $K_0^{\mathrm{top}}X$ is a K^{top}*-orientation* for X if the image of η in $K_0^{\mathrm{top}}(X)_x$ is a generator for each manifold point x in X. (It evidently suffices if this condition holds for a dense open set of manifold points in X.).

If X is a C^∞ manifold with a complex tangent bundle, it is clear that the orientation $\{X\}$ defined in § 3.4 is a K^{top}-orientation for X in the above sense.

For a quasi-projective variety X, the homomorphism $\alpha.$ of § 4 takes the element $[O_X]$ in $K_0^{\mathrm{alg}}X$ to an element in $K_0^{\mathrm{top}}X$ that we denote by $\{X\}$:

$$\{X\} = \alpha.[O_X].$$

We deduce from the Riemann–Roch theorem the following facts about the orientation class:

(1) $\{X_1 \times X_2\} = \{X_1\} \times \{X_2\}$ for any varieties X_1, X_2.

(2) If U is an open subscheme of X, then $\{X\}$ maps to $\{U\}$ by the restriction homomorphism from $K_0^{\mathrm{top}} X$ to $K_0^{\mathrm{top}} U$.

(3) If X is non-singular, $\{X\}$ is the orientation class determined by its complex tangent bundle.

The following proposition follows immediately from (2) and (3).

PROPOSITION. *For any reduced quasi-projective variety X, $\{X\}$ is a K^{top}-orientation for X.*

We remark that if $\pi\colon \tilde{X} \to X$ is a resolution of singularities for X, then $\pi_*[O_{\tilde{X}}]$ differs from $[O_X]$ in $K_0^{\mathrm{alg}} X$ by terms supported on the singular locus of X. Therefore $\pi_*\{\tilde{X}\}$ differs from $\{X\}$ by a term—usually *not* zero—supported on the singular locus. (Thus resolution of singularities shows directly that any complex analytic space has K^{top}-orientations. The Riemann–Roch theorem produces a *canonical* K^{top}-orientation at least for quasi-projective varieties. See [9] for an application of this idea.)

It follows from the assertions of § 4.2 that $f^*\{Y\} = \{X\}$ for any local completion morphism $f\colon X \to Y$, where f^* is the Gysin homomorphism.

If one defines $H.(X)_x$ analogously to the definition of $K_0^{\mathrm{top}}(X)_x$, the homology Chern character induces a homomorphism from $K_0^{\mathrm{top}}(X)_x$ to $H.(X)_x$. If X is a $2n$-manifold at x, then $K_0^{\mathrm{top}}(X)_x \cong \mathbf{Z}$ is mapped isomorphically to the integer homology in $H_{2n} X \cong \mathbf{Q}$. It follows that if X is a $2n$-circuit, an element $\eta \in K_0^{\mathrm{top}} X$ is a K-top-orientation for X if and only if $ch_n(\eta) \in H_{2n} X$ is an orientation for X as a $2n$-circuit.

For an algebraic variety X, the image $ch.\{X\} = \tau.[O_X]$ in $H.X$ is called the *homology Todd class* of X, and denoted $\tau(X)$. See ([3] § 4) for properties of the homology Todd class. When the Chern character is applied to the formula $f^*\{Y\} = \{X\}$ for a local complete intersection morphism, there results the formula

$$\mathrm{td}\,(T_f) \cap f^*\tau(Y) = \tau(X)$$

which was conjectured in [3] and proved by Verdier [15].

There are many even-dimens'onal C^∞ manifolds which are orientable in the usual sense but which have no K^{top}-orientation. (This happens when the second Stiefel-Whitney class is not the mod 2 reduction of any integral class; e.g., the Grassmannians of oriented

k-planes in $\mathbf{R}^n$, k, n odd, $k \geqslant 3$, $n-k \geqslant 3$.) In contrast we have seen that any complex quasi-projective algebraic variety, whether singular or not, has a canonical K^{top}-orientation.

Appendix 1. Complexes of vector bundles

Let $K_X^{\mathrm{top}} Y$ be the group defined in § 1.1 from complexes of topological vector bundles on Y, exact off X; for simplicity, we continue to assume the pair of one-point compactifications (Y^c, X^c) is homeomorphic to a pair of finite simplicial complexes. Let $K_X^{\mathrm{res}} Y$ be the free abelian group on restricted complexes, i.e., complexes of length 2,

$$0 \to E_1 \to E_0 \to 0$$

exact off X, such that E_1 is a trivial vector bundle, modulo the same three relations as in § 1.1.

LEMMA. *The map $K_X^{\mathrm{res}} Y \to K_X^{\mathrm{top}} Y$ induced by inclusion on the object level is an isomorphism.*

Proof. The main step is to show that a complex

$$0 \longrightarrow E_n \xrightarrow{d_n} E_{n-1} \longrightarrow \dots \xrightarrow{d_1} E_0 \longrightarrow 0$$

of length $n > 1$ is equivalent, modulo the three relations, to one of length $n-1$. Put metrics on the bundles E_n and E_{n-1}. On $Y-X$, choose a mapping d_n^{-1}: $E_{n-1} \to E_n$ so that $d_n^{-1} \circ d_n$ is the identity on E_n. For example, since d_n imbeds E_n in E_{n-1} on $Y-X$, d_n^{-1} could be orthogonal projection. Let the norm of d_n^{-1}, denoted $|d_n^{-1}|$, be the continuous, positive real-valued function on $Y-X$ whose value at $y \in Y-X$ is given by

$$|d_n^{-1}|(y) = \max_{\substack{|v|=1 \\ v \in E_{n-1}(y)}} |d_n^{-1}(v)|.$$

Let ε: $Y \to \mathbf{R}$ be a continuous function so that $\varepsilon(y) > 0$ if $y \in Y-X$, $\varepsilon(X) = 0$, and $|d_n^{-1}| \cdot \varepsilon$ extends to a continuous function on Y which vanishes on X. For example let

$$\varepsilon(y) = \min\left(d(y, X), \frac{d(y, X)}{|d^{-1}|(y)}\right)$$

where $d(y, X)$ is the distance from y to X in some metric on Y.

Now the given complex is equivalent to the complex

$$0 \longrightarrow E_n \xrightarrow{\begin{bmatrix} d_n \\ 0 \end{bmatrix}} \begin{matrix} E_{n-1} \\ \oplus \\ E_n \end{matrix} \xrightarrow{\begin{bmatrix} d_{n-1} & 0 \\ 0 & \varepsilon \end{bmatrix}} \begin{matrix} E_{n-2} \\ \oplus \\ E_n \end{matrix} \xrightarrow{[d_{n-2} \ 0]} E_{n-3} \longrightarrow \dots$$

where ε means ε times the identity. This is by using the first relation, plus the fact that

$$0 \longrightarrow E_n \xrightarrow{\varepsilon} E_n \longrightarrow 0$$

is homotopic to the exact complex

$$0 \longrightarrow E_n \xrightarrow{\mathrm{id}} E_n \longrightarrow 0$$

by the homotopy $(1-t(1-\varepsilon))\cdot \mathrm{id}$. Now consider the following homotopy:

$$E_n \xrightarrow{\begin{bmatrix} d_n \\ t \end{bmatrix}} \begin{matrix} E_{n-1} \\ \oplus \\ E_n \end{matrix} \xrightarrow{\begin{bmatrix} d_{n-1} & 0 \\ -t\varepsilon d_n & \varepsilon \end{bmatrix}} \begin{matrix} E_{n-2} \\ \oplus \\ E_n \end{matrix} \longrightarrow E_{n-3}$$

Note that εd_n^{-1} is a continuous vector bundle map on all of Y, vanishing on X. At $t=0$ we have the previous complex, while at $t=1$ we have a complex which contains

$$0 \longrightarrow E_n \xrightarrow{\mathrm{id}} E_n \longrightarrow 0$$

(with the first E_n in the nth position) as a subcomplex. Then the quotient complex has length $n-1$, as desired.

To complete the proof of the lemma it suffices to show that a complex

$$0 \longrightarrow E_1 \xrightarrow{d_1} E_0 \longrightarrow 0$$

is equivalent to a complex with the first bundle trivial. For this choose any vector bundle E so that $E_1 \oplus E$ is trivial, and take the direct sum of the given complex and

$$0 \longrightarrow E \xrightarrow{\mathrm{id}} E \longrightarrow 0.$$

We list several consequences of this lemma.

(1) *Excision.* If U is an open neighborhood of X in Y, then the restriction homomorphism

$$K_X^{\mathrm{top}} Y \to K_X^{\mathrm{top}} U$$

is an isomorphism. For any restricted complex on U has a canonical extension to one on Y.

(2) Let C be a closed subspace of Y contained in $Y-X$ such that the inclusion of C in $Y-X$ is a deformation retract. Then $K_X^{\mathrm{top}} Y$ is canonically isomorphic to $\tilde{K}^0(Y/C)$. Elements of $\tilde{K}^0(Y/C)$ are determined by vector bundles on Y which are trivialized near C. Such a trivialization can be extended to all of $Y-X$, and by making the map from

the trivial bundle to the given bundle vanish as we approach X (as in the lemma), we obtain a restricted complex on Y.

(3) *Thom isomorphism.* If E is a complex vector bundle on Y, there is a Thom isomorphism

$$K_X^{\mathrm{top}} Y \to K_X^{\mathrm{top}} E.$$

If $\pi: R \to Y$ is the projection, the map is given by $a \to \pi^*(a) \cup \lambda_E$ where λ_E is the Koszul-Thom complex (§ 1.4). Choose a metric on E, and let D and S be the unit disk and sphere bundles in E. For any pair (A, B) of finite CW complexes in $(Y, Y-X)$, the induced map

$$\tilde{K}^0(A/B) \to \tilde{K}^0(D \cap \pi^{-1}A/(S \cap \pi^{-1}A) \cup (D \cap \pi^{-1}B))$$

is an isomorphism by the Thom isomorphism for finite CW complexes (cf. [8], p. 45). Now choose (A, B) so that $B \subset C \subset Y-X$, with C as in (2), and so that the map $A/B \to Y/C$ is a homotopy equivalence. Then we have isomorphisms

$$K_X^{\mathrm{top}} Y \xrightarrow[\cong]{} \tilde{K}^0(Y/C) \xrightarrow[\cong]{} \tilde{K}^0(A/B),$$

the first by (2), the second since $\tilde{K}^0$ is a homotopy invariant. Similarly

$$\tilde{K}_X^{\mathrm{top}} E \xrightarrow[\cong]{} \tilde{K}^0(E/\pi^{-1}C \cup (E - \mathrm{int}\,(D))) \xrightarrow[\cong]{} \tilde{K}^0(D \cap \pi^{-1}A/(S \cap \pi^{-1}A) \cup (D \cap \pi^{-1}B))$$

so the result follows from the isomorphism for finite CW complexes.

Appendix 2. Complexes of sheaves

One may find generalizations of the results sketched here, e.g., in [5].

1. Resolutions

Consider a complex of coherent sheaves $E.$ on a quasi-projective scheme X:

$$\ldots \to E_n \to \ldots \to E_0 \to 0.$$

By a *resolution* of $E.$ we mean a complex $F.$ of locally free sheaves on X and a homomorphism of complexes $\varphi.: F. \to E.$ which is a *quasi-isomorphism*, i.e. $\varphi.$ induces an isomorphism $H_i F. \to H_i E.$ in homology sheaves for all i; for convenience we also require $\varphi.$ to be surjective.

That such resolutions exist follows from the fact that any coherent sheaf is the image of a locally free sheaf. In fact, choose any surjection $F_0 \to E_0$ to start, and if $F.$ has been constructed to the nth stage, let Z_n be the kernel of the map from F_n to F_{n-1}, and let

$$K_{n+1} = \mathrm{Ker}\,(Z_n \oplus E_{n+1} \to E_n),$$

where the map takes (z_n, e_{n+1}) to $\varphi_n(z_n) - d_{n+1}(e_{n+1})$. Choose any surjection of a locally free F_{n+1} to K_{n+1} to continue the complex one step further.

To resolve an exact sequence of complexes $0 \to E'_. \to E. \to E''_. \to 0$, first choose a resolution $F''_. \to E''_. \to 0$, and then choose a resolution $F.$ of the complex $\mathrm{Ker}\,(E. \oplus F''_. \to E''_.)$. Then $F.$ resolves $E.$ and maps onto $F''_.$, and the kernel of $F. \to F''_.$ gives a resolution of $F'_.$ of $E'_.$. The same reasoning shows that any two resolutions of a complex can be dominated by a third.

If the complex $E.$ is bounded, i.e., $E_i = 0$ for $i \gg 0$, and the sheaves E_i have finite Tor dimension, i.e., each E_i has a finite resolution by locally free sheaves, then the resolving complex $F.$ may be chosen to be bounded. One way to see this is by induction on the length of the complex $E.$, applying the result of the previous paragraphs to the exact sequence

$$0 \to E'_. \to E. \to E''_. \to 0$$

where $E'_.$ is the truncation of $E.$ at term k, and $E''_.$ is the remainder.

2. Tor independence

Let

$$\begin{tikzcd} Y' \arrow[r, "j"] \arrow[d, "g"'] & Z' \arrow[d, "f"] \\ Y \arrow[r, "i"'] & Z \end{tikzcd}$$

be a Tor-independent fibre square, i.e. $\mathrm{Tor}_k^{O_Z}(O_Y, O_{Z'}) = 0$ for all $k > 0$. Assume also that i is a closed imbedding of finite Tor dimension, i.e. $i_* O_Y$ has a finite resolution on Z. The condition of Tor independence means that if $\mathcal{L}.$ is a resolution of $i_* O_Y$ on Z, then $f^* \mathcal{L}.$ is a resolution of $j_* O_{Y'}$ on Z'. It follows that j is also a closed imbedding of finite Tor dimension. If $E.$ is a bounded complex of locally free sheaves on Y, then $i_* E.$ has a bounded resolution $F.$ on Z. We claim that the canonical homomorphism from $f^* F.$ to $j_* g^* E.$ is a resolution. For an induction on the length of $E.$ as before reduces the assertion to the case where $E.$ is a locally free module, and then by restricting to open subsets to the case where $E.$ is a free module, thence to $E. = O_Y$, which finishes the proof.

3. Homology of complexes

We first sketch the argument that if X is a closed subscheme of Y, then $K_0^{\text{alg}} X$ can be identified with the Grothendieck group $'K_0^{\text{alg}} X$ of coherent sheaves on Y which are supported on X. Write $O_X = O_Y/I$, where I is the idealsheaf of X. For a coherent sheaf F supported on Y, choose a filtration $F = F_0 \supset F_1 \supset \ldots \supset F_n = 0$ so that $I \cdot (F_i/F_{i+1}) = 0$ for all i; $F_i = I^i F$ is such a sequence. Then F_i/F_{i+1} is a coherent sheaf on X, and the map $[F] \to \sum [F_i/F_{i+1}]$ gives an inverse to the canonical map $K_0^{\text{alg}} X \to {'K_0^{\text{alg}}} X$. The essential point is that, by the usual Jordan–Hölder argument, this is independent of the filtration. Then if $0 \to F' \to F \to F'' \to 0$ is an exact sequence of sheaves supported on X, we can give F and F'' the I-adic filtrations as above, and F' the filtration induced from that on F; the fact that the map $'K_0^{\text{alg}} X \to K_0^{\text{alg}} X$ is well-defined follows easily. In particular if X is a non-reduced scheme, $K_0^{\text{alg}} X \cong K_0^{\text{alg}}(X_{\text{red}})$.

Next, let $'K_X^{\text{alg}} Y$ be the group constructed from all complexes $F.$:

$$0 \to F_n \to \ldots \to F_0 \to 0$$

of coherent sheaves on Y which are exact off X, dividing by relations for short exact sequences of such complexes, and for complexes that are exact everywhere on Y, just as we did to construct $K_X^{\text{alg}} Y$ from complexes of locally free sheaves in § 1.1. There is a homomorphism $'h\colon {'K_X^{\text{alg}}} Y \to {'K_0^{\text{alg}}} X$ given by $'h[F.] = \sum (-1)^i [H_i(F.)]$, where $H_i(F.)$ are the homology sheaves of the complex. In fact $'h$ is always an isomorphism; the inverse map takes a sheaf F to the complex $F.$ with $F_0 = F$, $F_i = 0$ for $i \neq 0$. To prove this one uses an exact sequence (with the complexes written vertically)

$$\begin{array}{ccccccccc}
0 & \to & Z_n & \to & F_n & \to & B_{n-1} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & 0 & \to & F_{n-1} & \to & F_{n-1} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & &
\end{array}$$

to show that a complex of length n is equal to one of length zero plus one with $H_n = 0$; for a complex $F.$ of length $n > 0$ with $H_n = 0$, consider

$$\begin{array}{ccccccccc}
0 & \to & F_n & \to & F_n & \to & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & F_n & \to & F_{n-1} & \to & B_{n-2} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & 0 & \to & F_{n-2} & \to & F_{n-2} & \to & 0
\end{array}$$

to see that it is equivalent to a shorter one.

13 – 792908 *Acta mathematica* 143. Imprimé le 28 Décembre 1979.

We now have a commutative diagram

$$\begin{array}{ccc} K_X^{\mathrm{alg}}\,Y & \longrightarrow & 'K_X^{\mathrm{alg}}\,Y \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle 'h} \\ K_0^{\mathrm{alg}}\,X & \xrightarrow[\approx]{} & 'K_0^{\mathrm{alg}}\,X \end{array}$$

We claim that the map $K_X^{\mathrm{alg}}\,Y \to 'K_X^{\mathrm{alg}}\,Y$ is an isomorphism when Y is non-singular; then h is also an isomorphism when Y is non-singular. The inverse map $'K_X^{\mathrm{alg}}\,Y \to K_X^{\mathrm{alg}}\,Y$ is constructed just as in § 9.1, mapping $[F.]$ to $[E.]$, where $E. \to F.$ is a resolution by locally free sheaves on Y; note that all the sheaves now have finite resolutions since Y is non-singular.

4. Intersections and unions

Let $X_0, \ldots, X_n$ be closed subschemes of a scheme X, defined by ideal sheaves $I_0, \ldots, I_n$. The *union* $X_0 \cup \ldots \cup X_n$ is the subscheme of X defined by the ideal sheaf $I_0 \cap \ldots \cap I_n$, while the *intersections* $X_{i_1} \cap \ldots \cap X_{i_r}$ are defined by ideal sheaves $I_{i_1} + \ldots + I_{i_r}$.

LEMMA. *Assume that for any two disjoint subsets S, T of $\{0, \ldots, n\}$, if $J_S = \sum_{i \in S} I_i$, the ideal sheaves satisfy*

$$(*) \qquad J_S + \bigcap_{j \in T} I_j = \bigcap_{j \in T} (J_S + I_j).$$

Then

$$[O_{X_0 \cup \ldots \cup X_n}] + \sum_{k=1}^{n+1} (-1)^k \sum_{i_1 < i_2 < \ldots < i_k} [O_{X_{i_1} \cap \ldots \cap X_{i_k}}] = 0$$

in $K_0^{\mathrm{alg}}\,X$.

Proof. The case $n=1$ follows from the exact sequence

$$0 \longrightarrow O_{X_0 \cup X_1} \xrightarrow{\alpha} O_{X_0} \oplus O_{X_1} \xrightarrow{\beta} O_{X_0 \cap X_1} \longrightarrow 0$$

where α (resp. β) is a sum (resp. difference) of two restriction maps. Apply the case $n=1$ to the subschemes $Y_0 = X_0$, $Y_1 = X_1 \cup \ldots \cup X_n$. The assumption implies that

$$Y_0 \cap Y_1 = (X_0 \cap X_1) \cup \ldots \cup (X_0 \cap X_n),$$

and the result follows by applying the inductive case of n subschemes to $X_1, \ldots, X_n$ and to $X_0 \cap X_1, \ldots, X_0 \cap X_n$.

Remark. The equation (*) is satisfied whenever $I_0, \ldots, I_n$ are ideals in a polynomial ring $\mathbf{C}[Z_1, \ldots, Z_m]$ with each I_i generated by some subset of the coordinate functions $Z_1, \ldots, Z_n$. For then any polynomial can be written uniquely as a sum $P+Q$ for $P \in J_S$, and Q involving none of the variables that generate J_S; then $P+Q$ belongs to $\bigcap (J_S+I_i)$ only if $Q \in \bigcap I_i$, which gives (*).

The ideals in (*) always have the same radical, since the corresponding equation for algebraic subsets is a set-theoretic identity. The scheme-theoretic assumption of (*) fails, for example, for three lines in the plane which pass through a point.

Appendix 3. The genus of projective space

Denote by $\varkappa(X)$ the integer that topological K-theory assigns to a non-singular projective variety by the process described in § 0.2. More generally, for any projective subscheme X of a non-singular Y, let $\varkappa^Y(X)$ be the image of $[O_X]$ under the composite

$$K_0^{\text{alg}} X \xrightarrow{\alpha_\cdot^Y} K_0^{\text{top}} X \xrightarrow{p_*} K_0^{\text{top}}(\text{pt.}) = \mathbf{Z}$$

where $\alpha_\cdot^Y$ is the map defined in § 4.1, and p maps X to a point. Set $\varkappa(X) = \varkappa^X(X)$, if X is non-singular. It follows from the Riemann–Roch Theorem (§ 4) that $\alpha_\cdot^Y$ and hence $\varkappa^Y$ is independent of Y, and that in fact $\varkappa^Y(X)$ is the arithmetic genus of X, but as the results of this appendix are used in step (5) of § 4.1, we will use only results proved earlier. In particular,

(i) $\varkappa^Y(X) = \varkappa(X)$ if X is non-singular.

(ii) $\varkappa(X_1 \times X_2) = \varkappa(X_1)\varkappa(X_2)$ if X_1, X_2 are non-singular; these follow from steps (1) and (3) of § 4.1 respectively.

LEMMA 1. *Let $\mathfrak{X} \subset Y \times T$ be a family of closed subschemes of a non-singular projective variety Y, flat over a connected variety T. Then all the $\varkappa^Y(\mathfrak{X}_t)$ are equal for $t \in T$.*

Proof. The structure sheaf $O_{\mathfrak{X}}$ has a resolution by a complex of locally free sheaves $\mathcal{E}.$ on $Y \times T$. The restriction $\mathcal{E}._t$ to the fibre over t resolves $O_{\mathfrak{X}_t}$, and therefore the associated complex $E._t$ of vector bundles represents the image of $[O_{\mathfrak{X}_t}]$ under the composite

$$K_0^{\text{alg}} \mathfrak{X}_t \xrightarrow[\cong]{h} K_{\mathfrak{X}_t}^{\text{alg}} Y \xrightarrow{\alpha} K_{\mathfrak{X}_t}^{\text{top}} Y.$$

The complexes $E._t$ all define the same element in $K_Y^{\text{top}} Y = K^0_{\text{top}} Y$ (§ 1.1, 3.1), and the desired result follows from the commutativity of

$$\begin{array}{ccccc} K^{\mathrm{top}}_{\mathfrak{X}_t} Y & \xrightarrow{h} & K^{\mathrm{top}}_0 \mathfrak{X}_t & \longrightarrow & K^{\mathrm{top}}_0(\mathrm{pt.}) \\ \downarrow & & \downarrow & & \| \\ K^{\mathrm{top}}_Y Y & \xrightarrow{h} & K^{\mathrm{top}}_0 Y & \longrightarrow & K^{\mathrm{top}}_0(\mathrm{pt.}) \end{array}$$

(§ 3.3 Property 5).

We will use this lemma for a family in $\mathbf{P}^n \times \mathbf{P}^n$ which deforms the diagonal to its Künneth decomposition, to show that $\varkappa(\mathbf{P}^n)=1$. The verification that this is a flat family requires an algebraic lemma.

LEMMA 2. *Let I be the ideal in the polynomial ring $\mathbf{C}[X_0, \ldots, X_n, Y_0, \ldots, Y_n, U]$ generated by the polynomials*

$$X_i Y_j - U^{j-i} X_j Y_i, \quad 0 \leqslant i < j \leqslant n.$$

Then the ring $\mathbf{C}[X, Y, U]/I$ is a torsion-free $\mathbf{C}[U]$-module.

Proof. It suffices to show that every element of $\mathbf{C}[X, Y, U]/I$ is uniquely represented by a linear combination of monomials of the form

$$(*) \qquad Y_0^{\alpha_0} Y_1^{\alpha_1} \cdot \ldots \cdot Y_k^{\alpha_k} X_k^{\beta_k} \cdot \ldots \cdot X_n^{\beta_n} U^{\gamma}.$$

To see this, define the $\mathbf{C}$-linear mapping L from $\mathbf{C}[X, Y, U]$ to itself which sends a monomial

$$Y_0^{a_0} \cdot \ldots \cdot Y_n^{a_n} X_0^{b_0} \cdot \ldots \cdot X_n^{b_n} U^c$$

to the monomial of the form (*) determined by the following procedure: k is the largest integer with $0 \leqslant k \leqslant n$ and $a_k + \ldots + a_n > b_0 + \ldots + b_{k-1}$, and

$$\begin{aligned} \alpha_i &= a_i + b_i \quad \text{for } i<k \\ \beta_i &= a_i + b_i \quad \text{for } i>k \\ \alpha_k &= (a_k + \ldots + a_n) - (b_0 + \ldots + b_{k-1}) \\ \beta_k &= (b_0 + \ldots + b_k) - (a_{k+1} \ldots + a_n) \\ \gamma &= c + \sum_{j>k} (j-k) a_k + \sum_{j>k} (k-j) b_j. \end{aligned}$$

A simple computation shows that L is a projection ($L^2 = L$), and that the value of L on a monomial is unchanged if any $X_i Y_j$ which appears with $i<j$ is replaced by $X_j Y_i U^{j-i}$. It follows that L vanishes on I. For any $P \in \mathbf{C}[X, Y, U]$, $L(P)$ is a representative of P modulo I of the required form, and if Q were another such representative, $L(P) - Q = L(P) - L(Q) = L(P-Q) = 0$, showing uniqueness.

PROPOSITION. *For all* $n \geqslant 0$, $\varkappa(\mathbf{P}^n)=1$.

Proof. This is obvious for $n=0$. Assume that $\varkappa(\mathbf{P}^m)=1$ for all $m<n$. Consider the subscheme $\mathfrak{X}$ of $\mathbf{P}^n \times \mathbf{P}^n \times \mathbf{A}^1$ defined by the ideal I of Lemma 2, where $\mathbf{A}^1=\mathbf{C}$ is the affine line.

This family of subschemes of $Y=\mathbf{P}^n \times \mathbf{P}^n$ is flat over $\mathbf{A}^1$ by Lemma 2, and so $\varkappa^Y(\mathfrak{X}_1)=\varkappa^Y(\mathfrak{X}_0)$ by Lemma 1. Now $\mathfrak{X}_1 \cong \mathbf{P}^n$ is the diagonal subvariety of $\mathbf{P}^n \times \mathbf{P}^n$, and $\mathfrak{X}_0$ is the subscheme defined by the ideal generated by all monomials $X_i Y_j$ for $0 \leqslant i<j \leqslant n$. This ideal may also be written as the intersection $I_0 \cap \ldots \cap I_n$, where

$$I_k=(X_0, \ldots, X_{k-1}, Y_{k+1}, \ldots, Y_n).$$

Thus $\mathfrak{X}_0$ is the union of the corresponding subschemes $X_0, \ldots, X_n$, with $X_k \cong \mathbf{P}^{n-k} \times \mathbf{P}^k$. Note that any intersection $X_{i_1} \cap \ldots \cap X_{i_k}$ with $k>1$ is a product $\mathbf{P}^a \times \mathbf{P}^b$ with $0 \leqslant a<n$, $0 \leqslant b<n$. The lemma in Appendix 2.4 implies the equation

$$\varkappa^Y(\mathfrak{X}_0)+\sum_{k=1}^{n+1}(-1)^k \sum_{i_1<\ldots<i_k} \varkappa^Y(X_{i_1} \cap \ldots \cap X_{i_k})=0.$$

The terms $\varkappa^Y(X_{i_1} \cap \ldots \cap X_{i_k})$ are all known to be one by induction and equations (i), (ii), except for the terms $\varkappa^Y(X_0)$ and $\varkappa^Y(X_n)$, which are $\varkappa(\mathbf{P}^n)$.

The left side of the equation simplifies to

$$\varkappa^Y(\mathfrak{X}_1)-2\varkappa(\mathbf{P}^n)-(n-1)+\binom{n+1}{2}-\binom{n+1}{3}+\ldots+(-1)^{n+1}\binom{n+1}{n+2}=-\varkappa(\mathbf{P}^n)-(n-1)+n$$

which gives the required equation $\varkappa(\mathbf{P}^n)=1$.

Remark. The same proof applies to the arithmetic genus, since it too is invariant in flat families.

References

[1]. ATIYAH, M. F. & HIRZEBRUCH, F., The Riemann-Roch theorem for analytic embeddings. *Topology*, 1, (1961), 151–166.

[2]. BAUM, P., K-homology. To appear.

[3]. BAUM, P., FULTON, W. & MACPHERSON, R., Riemann-Roch for singular varieties. *Publ. Math. IHES*, 45 (1975), 101–167.

[4]. BAUM, P., FULTON, W. & QUART, G., *Lefschetz-Riemann-Roch for singular varieties*. Following article.

[5]. BERTHELOT, P., GROTHENDIECK, A., ILLUSIE, L., et al., *Théorie des intersections et théorème de Riemann-Roch*. Springer Lecture Notes in Mathematics, 225 (1971).

[6]. BOREL, A. & MOORE, J. C., Homology theory for locally compact spaces. *Michigan Math. J.*, 7 (1960), 137–159.

[7]. Borel, A. & Serre, J.-P., Le théorème de Riemann-Roch, d'après A. Grothendieck. *Bull. Soc. Math. France*, 86 (1958), 97–136.

[8]. Dyer, E., *Cohomology theories*. W. A. Benjamin, New York (1969).

[9]. Fulton, W., A Hirzebruch-Riemann-Roch formula for analytic spaces and non-projective algebraic varieties. *Compositio Math.*, 34 (1977), 279–283.

[10]. —— *A note on the arithmetic genus*. To appear in *Amer. J. Math.*

[11]. Fulton, W. & MacPherson, R., Intersecting cycles on an algebraic variety. *Real and complex singularities*, Oslo, 1976, Sijthoff & Noordhoff (1978), 179–197.

[12]. Grothendieck, A. & Dieudonne, J., Eléments de géométrie algébrique. *Publ. Math. IHES Nos.* 4 (1960), 8 (1961), 11 (1961).

[13]. Iversen, B., Local Chern classes. *Ann. Sci. École Norm. Sup.*, 4e série, 9 (1976), 155–169.

[14]. Serre, J.-P., Faisceaux algébriques cohérents. *Ann. of Math.*, 61 (1955), 197–278.

[15]. Verdier, J.-L., Le théorème de Riemann-Roch pour les intersections complètes. *Seminaire de géometrie analytique de l'École Normale Supérieure* 1974–75. Exposé IX, *Astérique* 36–37 (1976), 189–228.

[16]. Whitehead, G. W., Generalized homology theories. *Trans. Amer. Math. Soc.*, 102 (1962), 227–283.

Received August 13, 1978