MINIMAL SURFACES WITH FREE BOUNDARIES

BY

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1. Problem and main results

The classical work of R. Courant and H. Lewy has initiated the study of minimal surfaces with free or at least partially free boundaries on prescribed, not necessarily planar surfaces. During the last decade, several authors including S. Hildebrandt, W. Jäger, J. C. C. Nitsche, K. H. Goldhorn, F. P. Harth and J. E. Taylor have investigated the boundary behavior of a minimal surface on its free boundary; see in particular [6], [11], [12], [16], [17], [19]. A survey of the results up to 1975, with an appended bibliography can be found in chapter VI.2, pp. 447–474, and on p. 707 of [18].

Let us consider a typical problem. Given a configuration in Euclidean 3-space $\mathbf{R}^3$ consisting of a smooth 2-dimensional surface $\mathcal{S}$ and of a smooth Jordan arc Γ having its end points P_1 and P_2 on $\mathcal{S}$, but no other points in common with $\mathcal{S}$. We introduce the class $\mathfrak{C} = \mathfrak{C}(\Gamma, \mathcal{S})$ of all surfaces $x = x(w) = (x^1(u, v), x^2(u, v), x^3(u, v))$ in $C^0 \cap H_2^1(B, \mathbf{R}^3)$, $w = u + iv$, which are parametrized over the semi-disc $B = \{w;\ |w| < 1,\ v > 0\}$ and are bounded by the configuration $\langle \Gamma, \mathcal{S} \rangle$ in the following sense:

Denote by C the closed circular arc $\{w;\ |w| = 1,\ v \geq 0\}$ and by I the open interval $\{w;\ |u| < 1,\ v = 0\}$. Moreover, fix a third point P_3 on Γ, different from P_1 and P_2. Let x_C and x_I be the L_2-traces ("boundary values") of $x \in H_2^1(B, \mathbf{R}^3)$ on C and I, respectively. Then, for any surface x in $\mathfrak{C}$ we assume that x_C maps C continuously and in weakly monotonic manner onto Γ such that $x_C(-1) = P_1$, $x_C(1) = P_2$ and $x_C(i) = P_3$, while $x_I(w) \in \mathcal{S}$ almost everywhere on I.

We look for a surface $x(w)$ which minimizes the Dirichlet integral

$$D(x) = \iint_B |\nabla x|^2 \, du \, dv \tag{1.1}$$

in the class $\mathfrak{C}(\Gamma, \mathcal{S})$. It is well known that this variational problem, to be denoted by $\mathcal{D}(\Gamma, \mathcal{S})$, always has at least one solution $x \in \mathfrak{C}$. The position vector x is real analytic in

B and satisfies there the conditions

$$\Delta x = 0 \tag{1.2}$$

and

$$|x_u|^2 = |x_v|^2, \quad x_u \cdot x_v = 0. \tag{1.3}$$

In other words, every solution of $\mathcal{D}(\Gamma, S)$ is a minimal surface which is bounded in a weak sense by the configuration $\langle \Gamma, S \rangle$. It is a matter of record that $x(w)$ also minimizes the area functional

$$A(x) = \iint_B |x_u \wedge x_v| \, du \, dv$$

in the class $\mathfrak{C}(\Gamma, S)$; see [6], [16].

There exists satisfactory information concerning the boundary behavior of a solution x of $\mathcal{D}(\Gamma, S)$ near the "fixed" boundary; see [18], chapter V.2.1, pp. 281–325. In particular, if Γ is a regular arc of class $C^{s+\alpha}$, $s = 1, 2, \ldots, 0 < \alpha < 1$, then $x \in C^{s+\alpha}(B \cup C_0, \mathbf{R}^3)$ where $C_0 = \operatorname{int} C = \{w; |w| = 1, v > 0\}$. However, the investigations regarding the behavior of a solution surface near the free part of its boundary and the nature of its trace have not yet reached a final stage. For the present, the best result is the following ([17], [19]):

Every solution of $\mathcal{D}(\Gamma, S)$ belongs to the class $C^{s+\alpha}(B \cup I, \mathbf{R}^3)$, provided that S is a regular surface of class $C^{s+\alpha}$, $s = 1, 2, \ldots, 0 < \alpha < 1$, without boundary and satisfies a local chord-arc condition.

That is, S can be for instance a sphere, a torus or a plane, but the theorem generally does not apply to surfaces S with boundary. A typical example of such a surface is the finite portion of a plane. On the other hand, these are just the examples with which the experimenter is often confronted; cf. figures 1–3(1). It is the aim of the present paper to supply a regularity theorem for the solutions of $\mathcal{D}(\Gamma, S)$ yielding regularity up to the free boundary even in cases when the boundary of the supporting surface S is non-void. For this we shall assume that S is a part of a larger complete surface $\mathcal{T}$ without boundary which is obtained from $\mathcal{T}$ by finitely many cuts along closed and mutually non-intersecting Jordan curves $\Gamma_1, \Gamma_2, \ldots, \Gamma_N$. As examples we can consider the finite simply-connected portion of a plane, or a hemi-sphere S as part of a sphere cut out by an equator Γ_1, or a triply-connected plane domain S which is cut from a plane $\mathcal{T}$ by three closed curves Γ_1, Γ_2 and Γ_3 (figures 1, 2). Our approach consists in treating $\mathcal{D}(\Gamma, S)$ as a Signorini problem, that is, as a variational problem with a "thin obstacle" on the supporting sur-

(1) In experiments with the configuration of figure 2 one may observe occasionally also a three-sheeted surface system having two free traces which follow either side of "hole", as well as a branch line along which the three sheets meet at an angle of 120°. Such an aggregate of minimal surfaces is not a solution of problem $\mathcal{D}(\Gamma, S)$, but can be transformed into one if one of the surfaces is broken.

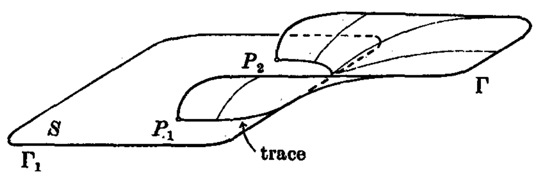

Fig. 1.

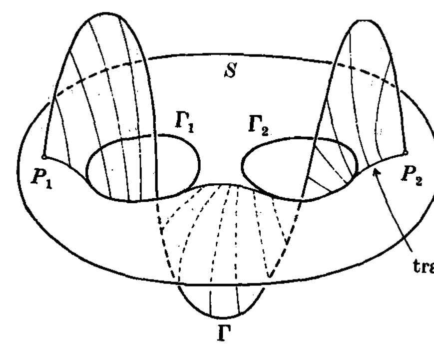

Fig. 2.

face $\mathcal{T}$, the obstacle being formed by the curves $\Gamma_1, \Gamma_2, \ldots, \Gamma_N$. For scalar equations the Signorini problem has been studied by many authors; cf. [3] and [4] for a bibliography. We know that under natural conditions on the variational problem there exist Lipschitz continuous solutions. The results include the case of harmonic functions and that of non-parametric minimal surfaces. In two recent papers [3], [4], J. Frehse found an important improvement. By a remarkable combination of Widman's hole filling technique and Moser's iteration procedure he proved that a Lipschitz continuous solution of a scalar Signorini problem is in fact of class C^1 up to the thin obstacle. C. Gerhardt [5] has considered the problem for non-parametric minimal surfaces. The results mentioned do not apply to the problems considered in the present paper, however, since we deal with parametric minimal surfaces, that is, with systems of differential equations. We shall prove an analogue to Frehse's result proceeding in four steps:

First, we prove that every solution $x(w)$ of $\mathfrak{D}(\Gamma, S)$ satisfies a Morrey condition on $B \cup I$. From this it follows that x is Hölder continuous in $B \cup I$. This fact is already contained in [11], [16], but for the sake of readability we include a self contained proof. We then apply the technique of [10] to derive L_2-bounds for the second derivatives of x on every compact subset of $B \cup I$. The crucial step is the third one. Introducing new

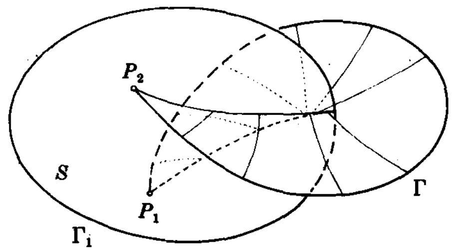

Fig. 3.

coordinates in $\mathbf{R}^3$ near S, which are chosen in an appropriate way, we are able to split off from the variational inequality for $\mathcal{D}(\Gamma, S)$ a boundary value problem for only two functions which can be treated by potential theoretic means. The derivatives of the third coordinate function are connected with the derivatives of the two other coordinate functions via certain conformality relations. In this way it will be seen that x belongs to the regularity class C^1. In the final step we discuss the differential geometric behavior of the trace curve $x(u, 0)$, $u \in I$, along which the minimal surface x intersects the supporting surface S. Adapting an idea from [8], [9], [15] and using the well known asymptotic formula due to Hartman and Wintner [7], we obtain an asymptotic representation for x_w on I which in turn yields the desired properties of the trace. We find that the trace is a regular C^1-curve except in the (isolated) branch points of odd order where the non-oriented tangent is still continuous, but the tangent direction jumps by 180 degrees. Both phenomena occur in the experiments, as can be seen from figures 1 and 3.

Nevertheless, it is interesting to describe conditions on Γ and S which exclude the appearance of cusps in the trace. This matter will be discussed in a forthcoming paper of the authors.

An explicit example of a minimal surface whose trace has a cusp on the boundary of S can be obtained from Henneberg's surface ([18], § 154):

$$\begin{aligned} x^1 &= 2 \sinh u \cos v - \tfrac{2}{3} \sinh 3u \cos 3v \\ x^2 &= 2 \cosh 2u \cos 2v - 2 \\ x^3 &= 2 \sinh u \sin v + \tfrac{2}{3} \sinh 3u \sin 3v. \end{aligned}$$

This surface intersects the plane $x^3 = 0$ in Neill's parabola $9(x^1)^2 = (x^2)^3$. Figures 4, 5 depict two views of parts of Henneberg's surface.(1) The part in figure 4 corresponds to the domain $\{w;\ |w| \leqslant 0.64,\ v \geqslant 0\}$ with the square $|x^1| \leqslant 1$, $|x^2 - 1| \leqslant 1$ in the plane $x^3 = 0$ as supporting surface S. The x^3-axis is a line of symmetry for the surface, and the arc Γ—image of $\{w;\ |w| = 0.64,\ v \geqslant 0\}$—has a closed convex curve as its projection onto the (x^2, x^3)-plane. In view of a new uniqueness theorem, to be published elsewhere, the surface is, in fact, a solution of problem $\mathcal{D}(\Gamma, S)$. For small values of the variable w we have expansions

$$x^1 = \operatorname{Re}\{-\tfrac{8}{3}w^3 + \ldots\}, \quad x^2 = \operatorname{Re}\{4w^2 + \ldots\}, \quad x^3 = \operatorname{Re}\{-4iw^2 + \ldots\} \tag{1.4}$$

from which it is seen that $w = 0$ is a branch point of order one on the minimal surface.

We wish to point out that our technique works as well in the case of minimal surfaces

(1) Figure 5 was kindly prepared by Dr. I. Haubitz at the Rechenzentrum of the University Würzburg.

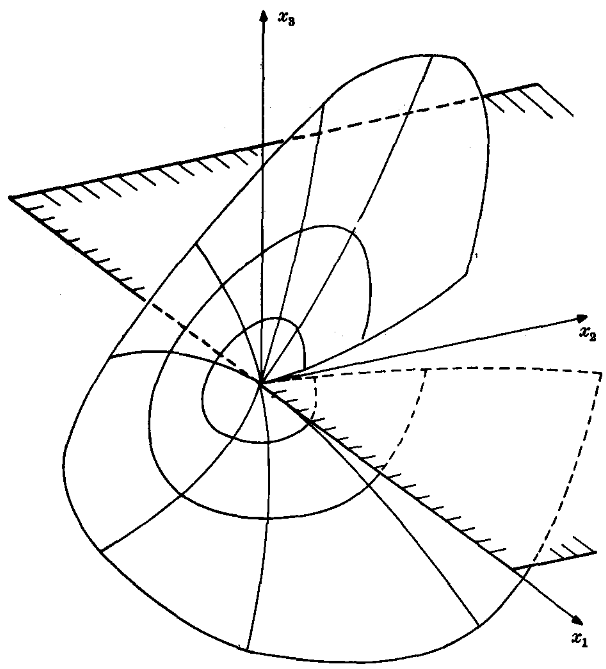

Fig. 4.

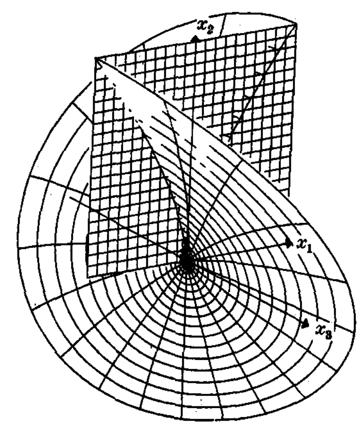

Fig. 5.

which may assume a completely free boundary on a surface S. A typical example is shown in figure 6 where S has the shape of a distorted napkin-ring.

Finally, we mention that the actual minimum property of a solution of $\mathcal{D}(\Gamma, S)$ is only used in the first step, while the other steps of the regularity proof rest merely on the assumption that x is a stationary minimal surface bounded by the configuration $\langle\Gamma, S\rangle$.

2. Growth of the Dirichlet integral of a solution of $\mathcal{D}(\Gamma, S)$

Definition. A set S in $\mathbf{R}^3$ is said to satisfy a (local) chord-arc condition if there exist constants $M \geq 1$ and $\delta > 0$ such that any two points x_1 and x_2 on S of distance $|x_2 - x_1| \leq \delta$ can be connected in S by an arc whose length is bounded by $M|x_2 - x_1|$.

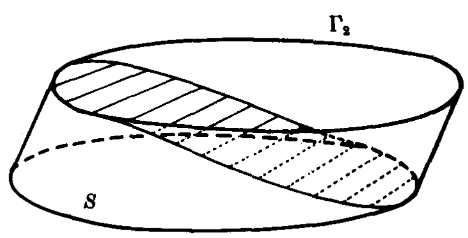

Fig. 6.

It is not difficult to see that every compact regular C^1-surface, with or without boundary, satisfies a chord-arc condition. On the other hand, an unbounded surface will in general not satisfy such a condition, even if it is of the regularity class C^∞.

The aim of the present section is the proof of the following result:

THEOREM 1. *Suppose that $x = x(u, v)$ is a solution of the problem $\mathcal{P}(\Gamma, S)$. Assume that S satisfies a chord-arc condition with constants M and δ, and set $e = \inf\{D(z); z \in \mathfrak{C}(\Gamma, S)\} > 0$. Let d be a number in the interval $0<d<1$, and $Z_d = \{w \in B; |w| < 1-d\}$, $B_r(w_0) = \{w; |w-w_0| < r\}$. For every $w_0 \in \bar{Z}_d$ and r in $0<r<\infty$ we have*

$$\iint_{B\cap B_r(w_0)} |\nabla x|^2\, du\, dv \leqslant (2r/d)^{2\mu} D(x) \tag{2.1}$$

where

$$\mu = \min\{1/(1+M^2), \delta^2/(\pi e)\}. \tag{2.2}$$

It follows that $x(u, v) \in C^{0+\mu}(\bar{Z}_d, \mathbf{R}^3)$.

Proof. A solution x of $\mathcal{P}(\Gamma, S)$ is harmonic in B and satisfies there the conformality relations (1.3), as well as the condition

$$D(x) = e. \tag{2.3}$$

For any point $w_0 \in \bar{B}$ we define

$$\Phi(r, w_0) = \iint_{B\cap B_r(w_0)} |\nabla x|^2\, du\, dv. \tag{2.4}$$

We shall prove first that, given any $d \in (0, 1)$, the inequality

$$\Phi(r, w_0) \leqslant (r/d)^{2\mu}\Phi(d, w_0) \tag{2.5}$$

holds for all $r \in [0, d]$ and for every $w_0 \in I$ satisfying $|w_0| \leqslant 1-d$.

For this purpose we fix an arbitrary $w_0 \in I$ with $|w_0| \leqslant 1-d$ and introduce the abbreviations $B_r = B_r(w_0)$, $\hat{S}_r = B \cap B_r(w_0)$, $\Phi(r) = \Phi(r, w_0)$. Introducing polar coordinates r, θ around w_0, we write $x(w) = x(w_0 + re^{i\theta}) = \xi(r, \theta)$. Then

$$\Phi(r) = \int_0^r \left\{\int_0^\pi \left[|\xi_\varrho(\varrho, \theta)|^2 + \frac{1}{\varrho^2}|\xi_\theta(\varrho, \theta)|^2\right] \varrho\, d\theta\right\} d\varrho. \tag{2.6}$$

There is a 1-dimensional null set $\mathcal{N}$ such that

$$\int_0^\pi |\xi_r(r, \theta)|^2 d\theta < \infty \quad \text{for } r \in (0, d) - \mathcal{N}$$

and that the absolutely continuous function $\Phi(r)$ satisfies

$$\Phi'(r) = \int_0^\pi \left\{r|\xi_r(r, \theta)|^2 + \frac{1}{r}|\xi_\theta(r, \theta)|^2\right\} d\theta$$

for $r \in (0, d) - \mathcal{N}$. In particular, the limits

$$x_1(r) = \lim_{\theta \to \pi - 0} \xi(r, \theta), \quad x_2(r) = \lim_{\theta \to \pi + 0} \xi(r, \theta)$$

exist for $r \in (0, d) - \mathcal{N}$. On account of (1.3), we have $r|\xi_r| = |\xi_\theta|$, $\xi_r \cdot \xi_\theta = 0$, so that

$$\Phi'(r) = (2/r) \int_0^\pi |\xi_\theta(r, \theta)|^2 d\theta \quad \text{for} \quad r \in (0, d) - \mathcal{N}. \tag{2.7}$$

Consider an arbitrary $r \in [0, d] - \mathcal{N}$ for which

$$\int_0^\pi |\xi_\theta(r, \theta)|^2 d\theta \leqslant \delta^2/\pi. \tag{2.8}$$

Since

$$|x_1(r) - x_2(r)| \leqslant \int_0^\pi |\xi_\theta(r, \theta)| d\theta \leqslant \sqrt{\pi} \left[\int_0^\pi |\{\xi_\theta(r, \theta)|^2 d\theta \right]^{1/2}, \tag{2.9}$$

we see that

$$|x_1(r) - x_2(r)| \leqslant \delta.$$

Since the supporting surface S satisfies a chord-arc condition, there exists a rectifiable arc $\gamma = \{\eta(s);\ 0 \leqslant s \leqslant l(\gamma)\}$ on S which connects the points $x_1(r)$, $x_2(r)$ and whose length is subject to the inequality

$$l(\gamma) \leqslant M |x_1(r) - x_2(r)|. \tag{2.10}$$

If s is chosen as parameter of the arc length on γ, then $|\eta'(s)| = 1$ almost everywhere on $[0, l(\gamma)]$. Introducing the reparametrization of γ,

$$\zeta(\theta) = \eta\left(\frac{\theta - \pi}{\pi} l(\gamma)\right), \quad \pi \leqslant \theta \leqslant 2\pi,$$

we obtain

$$|\zeta_\theta(\theta)| = \text{const} = l(\gamma)/\pi \quad \text{a.e. on } [\pi, 2\pi],$$

and also

$$l(\gamma) = \int_\pi^{2\pi} |\zeta_\theta| d\theta.$$

This implies that

$$\pi \int_\pi^{2\pi} |\zeta_\theta|^2 d\theta = \left(\int_\pi^{2\pi} |\zeta_\theta| d\theta \right)^2 = l(\gamma)^2. \tag{2.11}$$

From (2.9)–(2.11) it can be concluded that

$$\int_\pi^{2\pi} |\zeta_\theta|^2 d\theta \leqslant M^2 \int_0^\pi |\xi_\theta(r, \theta)|^2 d\theta. \tag{2.12}$$

Consider the harmonic function $h(w) = h(u, v)$ on B_r with the boundary value function $H(\theta) = h(w_0 + re^{i\theta})$ which is defined by

$$H(\theta) = \begin{cases} \xi(r, \theta) & \text{for} \quad 0 \leqslant \theta \leqslant \pi, \\ \zeta(\theta) & \phantom{\text{for}} \quad \pi \leqslant \theta \leqslant 2\pi. \end{cases}$$

The function $H(\theta)$ is absolutely continuous on $[0, 2\pi]$ and periodic: $H(0)=H(2\pi)$, and, by (2.12),

$$\int_0^{2\pi} |H_\theta|^2 d\theta \leqslant (1+M^2)\int_0^{\pi} |\xi_\theta(r,\theta)|^2 d\theta. \tag{2.13}$$

Furthermore,

$$\iint_{B_r} |\nabla h|^2 du\,dv \leqslant \int_0^{2\pi} |H_\theta(\theta)|^2 d\theta \tag{2.14}$$

(cf. [14], Lemma 9.4.2, p. 375). Combining this inequality with (2.7) and (2.13), we find

$$\iint_{B_r} |\nabla h|^2 du\,dv \leqslant (1+M^2)\,(r/2)\,\Phi'(r). \tag{2.15}$$

We now consider the function $y=y(u, v)=y(w)$ on $B \cup B_r$ which is defined as $x(w)$ for $w\in B-B_r$ and as $h(w)$ for $w \in B_r$. Clearly, $y(w)$ is of the regularity class $C^0 \cap H_2^1$ on $B \cup B_r$. Let τ be the homeomorphism of $\bar{B}$ onto $\overline{B \cup B_r}$ which maps B conformally onto $B \cup B_r$ leaving the points $1, -1, i$ fixed. Then, the composition $z=y\circ\tau$ is in $\mathfrak{C}(\Gamma, S)$, that is, z is a comparison surface for the minimum problem $\mathcal{P}(\Gamma, S)$ so that

$$\iint_B |\nabla x|^2 du\,dv \leqslant \iint_B |\nabla z|^2 du\,dv. \tag{2.16}$$

By virtue of the conformal invariance of the Dirichlet integral,

$$\iint_B |\nabla z|^2 du\,dv = \iint_{B\cup B_r} |\nabla y|^2 du\,dv. \tag{2.17}$$

Because of the definition of y, (2.16) and (2.17) imply that

$$\Phi(r) = \iint_{S_r} |\nabla x|^2 du\,dv \leqslant \iint_{B_r} |\nabla h|^2 du\,dv. \tag{2.18}$$

From (2.15) and (2.18) we derive the relation

$$\Phi(r) \leqslant \tfrac{1}{2}(1+M^2) r \Phi'(r) \tag{2.19}$$

for every $r\in(0,d)-\mathcal{N}$ for which (2.8) is satisfied.

On the other hand, if for some $r \in (0, d)-\mathcal{N}$

$$\int_0^{\pi} |\xi_\theta(r,\theta)|^2 d\theta > \delta^2/\pi, \tag{2.20}$$

then, trivially,

$$\Phi(r) \leqslant D(x) = e < e\pi\delta^{-2}\int_0^{\pi} |\xi_\theta(r,\theta)|^2 d\theta.$$

Hence, (2.7) yields the estimate

$$\Phi(r) \leqslant \tfrac{1}{2}\pi e \delta^{-2} r \Phi'(r). \tag{2.21}$$

Defining μ as in (2.2), we finally obtain the estimate

$$2\mu\Phi(r) \leqslant r\Phi'(r) \quad \text{for } r\in(0,d)-\mathcal{N}. \tag{2.22}$$

It follows by integration that

$$\Phi(r) \leqslant (r/d)^{2\mu}\Phi(d) \quad \text{or } r\in[0,d], \tag{2.23}$$

thus (2.5) is proved.

Next, we choose $w_0=u_0+iv_0$ with $v_0\geqslant R$ and $|w_0|\leqslant 1-R$ for some $R\in(0,1)$. Then we have (cf. [14], Lemma 4.9.2, p. 375)

$$\Phi(r,w_0)=\iint_{B_r(w_0)}|\nabla x|^2\,du\,dv\leqslant \int_0^{2\pi}|\xi_\theta(r,\theta)|^2\,d\theta=\tfrac{1}{2}r\Phi'(r) \tag{2.24}$$

for almost all $r\in(0,R)$, so that

$$\Phi(r,w_0)\leqslant (r/R)^2\,\Phi(R,w_0) \quad \text{for } r\in[0,R]. \tag{2.25}$$

Finally, let w_0 be an arbitrary point in $\bar{Z}_d$, for some $d\in(0,1)$, i.e., $w_0\in\bar{B}$ and, $|w_0|\leqslant 1-d$. We distinguish two cases.

Case 1. $d/2\leqslant v_0$.

With the choice $R=d/2$, (2.25) implies that

$$\Phi(r,w_0)\leqslant (2r/d)^2 D(x) \quad \text{for } 0\leqslant r\leqslant d/2. \tag{2.26}$$

Case 2. $0\leqslant v_0<d/2$.

(a) If $v_0\leqslant r\leqslant d/2$, then $B_r(w_0)\subset B_{2r}(u_0)$. Then

$$\Phi(r,w_0)\leqslant \Phi(2r,u_0).$$

On the other hand, (2.5) implies that

$$\Phi(2r,u_0)\leqslant (2r/d)^{2\mu}\Phi(d,u_0),$$

and therefore,

$$\Phi(r,w_0)\leqslant (2r/d)^{2\mu}D(x). \tag{2.27}$$

In particular, we note that

$$\Phi(v_0,w_0)\leqslant (2v_0/d)^{2\mu}D(x). \tag{2.28}$$

(b) If $0\leqslant r<v_0$, we may apply (2.25) with $R=v_0$ to obtain

$$\Phi(r,w_0)\leqslant (r/v_0)^2\,\Phi(v_0,w_0)$$

and, combining this with (2.28)

$$\begin{aligned}\Phi(r,w_0) &\leqslant (r/v_0)^2(2v_0/d)^{2\mu}D(x)\\ &\leqslant (2r/d)^{2\mu}(v/v_0)^{2(1-\mu)}D(x)\\ &\leqslant (2r/d)^{2\mu}D(x).\end{aligned}$$

Thus inequality (2.27) is established also for the subcase *2(b)*.

The above discussion of the two cases 1 and 2 yields that

$$\Phi(r, w_0) \leqslant (2r/d)^{2\mu} D(x) \quad \text{for } 0 \leqslant r \leqslant d/2 \tag{2.29}$$

and for all $w_0 \in \bar{Z}_d$. This in turn implies the estimate (2.1).

It is a consequence of the Morrey condition (2.1) that $x(u, v)$ satisfies a uniform Hölder condition in $\bar{Z}_d$ with exponent μ (the proof is similar to the proof of Theorem 3.5.2 on p. 79 of [14]).

Theorem 1 is proved.

3. L_2-estimates of the second derivatives up to the free boundary

We repeat some notations: $w=u+iv=(u, v)$, $B=\{w;\ |w|<1,\ v>0\}$, $I=\{w;\ |u|<1,\ v=0\}$, $Z_d=\{w \in B;\ |w|<1-d\}$, $B_r(w_0)=\{w;\ |w-w_0|<r\}$, $S_r(w_0)=B \cap B_r(w_0)$.

The first aim of this section is the proof of the fact that every solution $x=x(u, v)$ of $\mathcal{D}(\Gamma, \mathcal{S})$ belongs to the regularity class H_2^2 on Z_d for every $d<1$. For this purpose it is clearly sufficient to prove that for every $w_0 \in I$, there is a number $r_0>0$ such that the second derivatives $\nabla^2 x$ of x are square integrable on $S_{r_0}(w_0)$.

Suppose now that $\mathcal{S}$ satisfies a chord-arc condition, and is a part of a regular C^3-surface $\mathcal{J}$ in $\mathbf{R}^3$ without boundary which is cut out of $\mathcal{J}$ by finitely many closed regular non-intersecting Jordan curves $\Gamma_1, \Gamma_2, \ldots, \Gamma_N$ of class C^3. In the previous section 2 it has been shown that $x(u, v)$ satisfies a Hölder condition on Z_d for every $d<1$. Therefore, the following discussion can be carried out locally, that is, around small pieces of $\mathcal{S}$ which can be flattened.

Consider an arbitrary point $w_0 \in I$, and set $x_0=x(w_0)$. Then, $x_0 \in \mathcal{S}$. If $x_0 \notin \bigcup_{j=1}^N \Gamma_j$, that is, $x_0 \in \operatorname{int} \mathcal{S}$, we may find a number $\varrho>0$ such that the image of $\{w \in I:\ |w-w_0| \leqslant \varrho\}$ under the mapping by the vector $x(u, v)$ is contained in int $\mathcal{S}$. Hence, we can infer that x belongs to the regularity class C^2 on $\overline{S_{r_0}(w_0)}$ for every $r_0 \in (0, \varrho)$; see [17].

In view of the above, it suffices to consider the case that $x_0 \in \partial \mathcal{S}=\Gamma_1+\Gamma_2+\ldots+\Gamma_N$. Assume that x_0 is a point on Γ_k. We shall linearize the boundary conditions by "flattening" the supporting surface $\mathcal{S}$, and by "straightening" the obstacle curve Γ_k. This is done in the following way:

There is a 3-dimensional neighborhood $U(x_0)$ of x_0, and a C^3-diffeomorphism $g: x \to y$ of $\mathbf{R}^3$ onto itself mapping x_0 onto y_0, and $U(x_0)$ onto the open ball $\mathcal{K}_R(y_0)=\{y \in \mathbf{R}^3;\ |y-y_0|<R\}$, such that $\mathcal{J} \cap U(x_0)$ is mapped onto $\{y;\ y^3=0,\ |y-y_0|<R\}$, $\mathcal{S} \cap U(x_0)$ is mapped onto the convex set $\mathcal{C}_R=\{y \in \mathbf{R}^3;\ y^1 \geqslant 0,\ y^3=0,\ |y-y_0|<R\}$ and $\Gamma_k \cap U(x_0)$ onto the interval

$\{y; y^1=y^3=0,\ |y-y_0|<R\}$ on he y^2-axis. Since $x(u, v)$ is continuous on $B\cup I$, there is a $\varrho>0$ such that $\overline{S_\varrho(w_0)}$ is mapped by $x(u, v)$ into $U(x_0)$.

Let h be the inverse of g. The transformed surface with the position vector $y(u, v)=g(x(u, v))$ is connected with the minimal surface by virtue of the relation

$$x(u, v)=h((y(u, v))$$

from which it follows that

$$\nabla x=H(y)\nabla y,\quad \nabla=\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right),\quad H(y)=\frac{\partial h}{\partial y}.$$

Therefore

$$|\nabla x|^2=|x_u|^2+|x_v|^2=y_u\cdot \mathcal{G}(y)y_u+y_v\cdot\mathcal{G}(y)y_v,$$

where we have set

$$\mathcal{G}(y)=H^T(y)\cdot H(y)=(g_{jk}(y)),\quad 1\leqslant j, k\leqslant 3.$$

Let us introduce the new functional

$$D^*(z)=\iint_B\{z_u\cdot\mathcal{G}(z)z_u+z_v\cdot\mathcal{G}(z)z_v\}\,du\,dv.$$

Then,

$$D(x)=D^*(y).$$

Let ε_0 be a positive number, and $\varphi=\varphi(u, v)=\{\varphi^1(u, v),\ \varphi^2(u, v),\ \varphi^3(u, v)\}$ a $\mathbf{R}^3$-valued function on $\bar B$ with the property that

$$z_\varepsilon=h(y-\varepsilon\varphi)\in\mathfrak{C}(\Gamma, S)\quad\text{for all } \varepsilon \text{ in the interval } (0, \varepsilon_0). \tag{3.1}$$

Then,

$$D(z_\varepsilon)=D^*(y-\varepsilon\varphi),$$

and the minimum property of x implies that

$$D(x)\leqslant D(z_\varepsilon)\quad\text{for } 0\leqslant\varepsilon<\varepsilon_0.$$

Consequently,

$$D^*(y)\leqslant D^*(y-\varepsilon\varphi)\quad\text{for } 0\leqslant\varepsilon<\varepsilon_0, \tag{3.2}$$

and, therefore,

$$\lim_{\varepsilon\to+0}\frac{1}{\varepsilon}[D^*(y-\varepsilon\varphi)-D^*(y)]\geqslant 0.$$

If $\varphi(u, v)$ is essentially bounded and of the class H_2^1 on B, this limit exists, and is $-\delta D^*(y, \varphi)$ where $\delta D^*(y, \varphi)$ is the first variation of the functional D^* in y, in direction of the vector field φ. Thus we infer that

$$\iint_B g_{jk}(y)\cdot[y_u^j\varphi_u^k+y_v^j\varphi_v^k]\,du\,dv\leqslant-\frac{1}{2}\iint_B\frac{\partial g_{jk}}{\partial y^l}(y)\cdot\{y_u^jy_u^k+y_v^jy_v^k\}\varphi^l\,du\,dv \tag{3.3}$$

for all admissible test functions $\varphi(u, v)$, that is, for vector functions $\varphi\in H_2^1\cap L_\infty(B, \mathbf{R}^3)$ which satisfy (3.1) for some $\varepsilon_0=\varepsilon_0(\varphi)>0$.

Since $x(u, v)$ is harmonic in B, we know that $y(u, v)$ is of class $C^3(B, \mathbf{R}^3)$. By virtue of the fundamental lemma of the calculus of variations, we infer from (3.2) and (3.3) that $y(u, v)$ satisfies in B the Euler equations

$$\Delta y^l + \Gamma^l_{jk}(y)\{y^j_u y^k_u + y^j_v y^k_v\} = 0 \tag{3.4}$$

where

$$\Gamma_{jkl} = \frac{1}{2}\left(\frac{\partial g_{kl}}{\partial y^j} - \frac{\partial g_{jl}}{\partial y^k} + \frac{\partial g_{jk}}{\partial y^l}\right), \quad \Gamma^l_{jk} = g^{lm}\Gamma_{jmk}$$

The conformality relations (1.3) are transformed into

$$y_u \cdot \mathcal{G}(y) y_u = y_v \cdot \mathcal{G}(y) y_v, \quad y_u \cdot \mathcal{G}(y) y_v = 0. \tag{3.5}$$

Now we choose some $r \in (0, \varrho/2)$, and some "friend" $\eta = \eta(w)$ in $C^\infty_c(B_{2r}(w_0), \mathbf{R}^3)$ satisfying $\eta(w) \equiv 1$ for $w \in B_r(w_0)$, and $|\nabla \eta| \leqslant k/r$, $0 \leqslant \eta \leqslant 1$ on $\mathbf{C}$. Moreover, let us denote by Δ_h the tangential difference quotient which, for a function $\psi(w) = \psi(u, v)$, is defined by

$$(\Delta_h \psi)(w) = \frac{1}{h}[\psi(u+h, v) - \psi(u, v)]$$
$$= \frac{1}{h}[\psi(w + h\zeta) - \psi(w)]$$

where $h \neq 0$, and $\zeta = (1, 0)$.

We claim that

$$\varphi = -\Delta_{-h}\{\eta^2 \Delta_h y\} \tag{3.6}$$

is an admissible test function for (3.3) provided that $|h|$ is sufficiently small. In fact, φ is clearly in $L_\infty \cap H^1_2(B, \mathbf{R}^3)$ for small $|h|$, and

$$y(w) - \varepsilon\varphi(w) = y(w) + \varepsilon\Delta_{-h}\{\eta^2(w)(\Delta_h y)(w)\} = \lambda_1 y(w + h\zeta) + \lambda_2 y(w - h\zeta) + [1 - \lambda_1 - \lambda_2] y(w)$$

where

$$\lambda_1 = (\varepsilon/h^2)\eta^2(w), \quad \lambda_2 = (\varepsilon/h^2)\eta^2(w - h\zeta).$$

Hence, for $0 < \varepsilon < h^2/2$, we have that

$$0 \leqslant \lambda_1, \lambda_2 \leqslant \tfrac{1}{2}.$$

Thus we infer that, for every $w \in B \cup I$, the difference $y(w) - \varepsilon\varphi(w)$ is a convex combination of the three points $y(w + h\zeta)$, $y(w)$, $y(w - h\zeta)$.

Since $\eta(w) = 0$ for $|w - w_0| \geqslant 2r$, we get $\lambda_1(w) = 0$ and $\lambda_2(w) = 0$ for $|w - w_0| \geqslant 2r + |h|$. Therefore,

$$y(w) - \varepsilon\varphi(w) = y(w) \quad \text{for } |w - w_0| \geqslant 2r + |h|.$$

If $|w - w_0| < 2r + |h|$, we obtain $|w \pm h\zeta - w_0| \leqslant 2r + 2|h|$. Hence, for $|h| < \varrho/2 - r$ and $|w - w_0| < 2r + |h|$, it is seen that w and $w \pm h\zeta$ are in $\overline{S_\varrho(w_0)}$, and, therefore, $x(w)$ and $x(w \pm h\zeta)$ are in $U(x_0)$. Thus, the points $y(w)$ and $y(w \pm h\zeta)$ are contained in $\mathbf{C}_R$ provided that $w \in I$ and $|w-w_0|<2r+|h|$, so that also $y(w)-\varepsilon\varphi(w)\in \mathcal{C}_R$, since $\mathcal{C}_R$ is a convex set. We note that $\mathcal{S}\cap U(x_0)=h(\mathcal{C}_R)$. Thus we have proved that

$$h[y(w)-\varepsilon\varphi(w)]\in \mathcal{S} \quad \text{for } w\in I \quad \text{and } 0<\varepsilon<h^2/2$$

provided that $|h|<\varrho/2-r$.

Now it follows immediately that the test function φ defined in (3.6) will satisfy (3.1) for $\varepsilon_0=h^2/2$. It follows that φ is admissible in the variational inequality (3.3). From this point on, we may proceed as in [10], section 7. Inserting (3.6) into (3.3), and choosing $r>0$ sufficiently small, we obtain an estimate of the form

$$\iint_B \eta^2|\nabla\Delta_h y|^2\,du\,dv\leqslant c \tag{3.7}$$

with a bound c independent of h as $h\to 0$. The main difficulty in deriving (3.7) consists in proving an estimate of the type

$$\iint_B \eta^2|\nabla y|^2|\Delta_h y|^2\,du\,dv\leqslant \varepsilon(r)\iint_B \eta^2\{|\nabla\Delta_h y|^2\,du\,dv+c^*(r) \tag{3.8}$$

where $\varepsilon(r)$ and $c^*(r)$ are numbers independent of r such that $\varepsilon(r)\to 0$, $c^*(r)\to\infty$ as $r\to 0$ (cf. [10], inequality (7.9), p. 66). The proof of (3.8) can be based on a reproducing property for the Morrey norm due to Morrey [14] (cf. Lemma 5.4.1, p. 144). The essential ingredient is the growth property (2.1) of the Dirichlet integral of x proved in section 2 which implies a similar property for the Dirichlet integral of y, taking into account that

$$|\nabla x|^2=y_u\cdot \mathcal{G}(y)y_u+y_v\cdot\mathcal{G}(y)y_v$$

and that the matrix $\mathcal{G}$ is positive definite. For $h\to 0$, the estimate (3.7) implies that

$$\iint_B \eta^2|\nabla y_u|^2\,du\,dv\leqslant c. \tag{3.9}$$

Hence y_{uu} and y_{uv} are in L_2 on $S_r(w_0)$ for sufficiently small $r>0$. Furthermore, (3.5), (3.8), and (3.9) yield the estimate

$$\iint_B \eta^2|\nabla y|^4\,du\,dv\leqslant c'. \tag{3.10}$$

Solving equation (3.4) for y_{vv} and using (3.9), (3.10), we finally obtain

$$\iint_B \eta^2|y_{vv}|^2\,du\,dv\leqslant c''. \tag{3.11}$$

Thus we infer that $\nabla^2 y$ is in L_2 on $S_r(w_0)$, for sufficiently small $r>0$. Since $x(w)=h(y(w))$, the chain rule and $\nabla y\in L_4$, $\nabla^2 y\in L_2$ on $S_r(w_0)$ imply that also $\nabla^2 x$ is in L_2 on $S_r(w_0)$.

For the details of the proof, the reader is referred to [10], sections 2, 6, and 7.

Hence we have proved that $x \in H_2^2(Z_d, \mathbf{R}^3)$ for every $d<1$. By virtue of Sobolev's lemma, also $x \in H_p^1(Z_d, \mathbf{R}^3)$ for every $p<\infty$ and each $d<1$. Let $I_d=\{w \in I;\ |w|<d\}$. A well known imbedding theorem (cf. [14], Theorem 3.4.5, p. 76) implies that x_u and x_v have an L_2-trace on I_d. Analogously, y_u, $y_v \in L_2(I_d, \mathbf{R}^3)$ for $d<1$. If we perform in (3.3) a partial integration, and take into account (3.4) as well as the regularity results, which were stated before, we arrive at the inequality

$$\int_{I_\varrho(w_0)} g_{jk}(y)\, y_v^j \varphi^k \, du \geq 0, \tag{3.12}$$

where $I_\varrho(w_0)=I \cap B_\varrho(w_0)=\{w \in I;\ |w-w_0|<\varrho\}$ and $w_0 \in I$, and $\varphi(w)=(\varphi^1(w), \varphi^2(w), \varphi^3(w))$ is an arbitrary admissible test vector of the class $C_c^\infty(B_\varrho(w_0), \mathbf{R}^3)$.

By our construction, φ^2 is free on $I_\varrho(w_0)$, while φ^1 is free only on the open part $\{w \in I_\varrho(w_0);\ y^1(w)>0\}$, and φ^3 has to be zero on $I_\varrho(w_0)$. Thus we conclude from (3.12) that

$$\begin{aligned} g_{j1}(y)\, y_v^j &= 0 \quad \text{a.e. on } I_\varrho(w_0) \cap \{y^1(w)>0\}, \\ g_{j2}(y)\, y_v^j &= 0 \quad \text{a.e. on } I_\varrho(w_0). \end{aligned} \tag{3.13}$$

Now, that we have proved that $x(u, v)$ is in $H_2^2(Z_d, \mathbf{R}^3)$, we observe that it suffices to assume that g is a C^2-diffeomorphism to obtain (3.4), (3.5), and (3.13). Then, we have the following:

THEOREM 2. *Suppose that $x=x(u, v)$ is a solution of $\mathcal{P}(\Gamma, S)$ where S satisfies a chord-arc condition and is a part of a regular C^3-surface $\mathcal{T}$ in $\mathbf{R}^3$ without boundary which is cut out of $\mathcal{T}$ by finitely many, closed, regular, non-intersecting Jordan curves $\Gamma_1, \ldots, \Gamma_N$ of the class C^3. Then, $x(u, v)$ is in $H_2^2 \cap H_p^1(Z_d, \mathbf{R}^3)$ for each $d<1$ and every $p<\infty$, and x_u and x_v have an L_2-trace on every compact subinterval of I.*

Let $w_0 \in I$, and $x_0=x(w_0) \in S$, and suppose that $U(x_0)$ is mapped into a neighborhood of the origin by a C^3-diffeomorphism $g\colon U(x_0) \to V(0)$ which maps $\mathcal{T} \cap U(x_0)$ into the plane $\{y \in \mathbf{R}^3\colon y^3=0\}$ such that $g(x_0)=0$. If $x_0 \in \Gamma_k$ for some k, we assume in addition that g maps $\Gamma_k \cap U(x_0)$ into the y^2-axis, and $S \cap U(x_0)$ into the half plane $\{y \in \mathbf{R}^3;\ y^1 \geq 0,\ y^3=0\}$. Finally, let $\varrho>0$ be so small that $x(u, v)$ maps $\overline{S_\varrho(w_0)}$ into $U(x_0)$. Then, $y(u, v)=g(x(u, v))$ is in $C^2(S_\varrho(w_0), \mathbf{R}^3)$ as well as in $H_2^2 \cap H_p^1(S_\varrho(w_0), \mathbf{R}^3)$ for each $p \in (1, \infty)$, and satisfies (3.4) and (3.5) on $S_\varrho(w_0)$. Moreover the normal derivative y_v has an L_2-trace on $I_\varrho(w_0)=\{w \in I;\ |w-w_0|<\varrho\}$, and satisfies the following boundary conditions:

$$\left.\begin{aligned} g_{j1}(y)\, y_v^j &= 0 \quad \text{a.e. on } I_\varrho(w_0) \cap \{y^1(w)>0\} \\ g_{j2}(y)\, y_v^j &= 0 \quad \text{a.e. on } I_\varrho(w_0) \\ y^3 &= 0 \qquad \text{on } I_\varrho(w_0). \end{aligned}\right\} \tag{3.14}$$

4. Continuity of the derivatives at the free boundary

Let w_0, x_0, ϱ, and g be chosen as in Theorem 2, and let h be the inverse of g. To simplify the boundary conditions (3.14), we now choose the diffeomorphism $x=h(y)$ in a special way. Consider a Gauss representation

$$t(\xi,\eta)=(t^1(\xi,\eta),\,t^2(\xi,\eta),\,t^3(\xi,\eta)),$$

$|\xi|^2+|\eta|^2<R^2$, of a piece $\mathcal{T}\cap U(x_0)$ of $\mathcal{T}$ around x_0, by a C^3-function $t(\xi,\eta)$. Set

$$E=|t_\xi|^2,\quad F=t_\xi\cdot t_\eta,\quad G=|t_\eta|^2,$$

and let

$$n=\frac{t_\xi\wedge t_\eta}{|t_\xi\wedge t_\eta|}$$

be the associated unit normal vector on $\mathcal{T}$. Moreover, set

$$W=|t_\xi\wedge t_\eta|=\sqrt{EG-F^2},$$

and

$$\begin{aligned}L&=-t_\xi\cdot n_\xi &&=n\cdot t_{\xi\xi}\\ M&=-\tfrac{1}{2}(t_\xi\cdot n_\eta+t_\eta\cdot n_\xi)&&=n\cdot t_{\xi\eta}\\ N&=-t_\eta\cdot n_\eta &&=n\cdot t_{\eta\eta}.\end{aligned}$$

The Weingarten equations are

$$n_\xi=\frac{1}{W^2}\{(FM-GL)t_\xi+(FL-EM)t_\eta\}$$

$$n_\eta=\frac{1}{W^2}\{(FN-GM)t_\xi+(FM-EN)t_\eta\}.$$

We may choose ξ, η as orthogonal coordinates on $\mathcal{T}$, that is, $F=0$ and in such a way that the part $\Gamma_k\cap U(x_0)$ of the "obstacle curve" Γ_k on $\mathcal{T}$ is represented by $\{\xi=0\}$, while the "admissible domain" $\mathcal{S}\cap U(x_0)$ on $\mathcal{T}\cap U(x_0)$ is described by $\{\xi\geqslant 0\}$.

Set

$$y^1=\xi,\quad y^2=\eta,\quad y^3=\zeta,\quad y=(y^1,y^2,y^3),$$

and define $x=h(y)$ by

$$x=t(\xi,\eta)+\zeta n(\xi,\eta).$$

Then, the matrix elements

$$g_{jk}(y)=\frac{\partial h}{\partial y^j}\frac{\partial h}{\partial y^k}$$

of $\mathcal{G}(y) = H^T(y)H(y)$, $H(y) = \partial h/\partial y$, are computed as

$$\begin{aligned}
g_{11} &= |h_\xi|^2 = E - 2\zeta L + \zeta^2 |n_\xi|^2 \\
g_{22} &= |h_\eta|^2 = G - 2\zeta N + \zeta^2 |n_\eta|^2 \\
g_{33} &= |h_\zeta|^2 = |n|^2 = 1 \\
g_{12} &= g_{21} = h_\xi \cdot h_\eta = F - 2\zeta M + \zeta^2 n_\xi \cdot n_\eta \\
g_{13} &= g_{31} = h_\xi \cdot h_\zeta = t_\xi \cdot n + \zeta n_\xi \cdot n = 0 \\
g_{23} &= g_{32} = h_\eta \cdot h_\zeta = t_\eta \cdot n + \zeta n_\eta \cdot n = 0.
\end{aligned}$$

In view of the relation $F = 0$ and of the Weingarten equations, we find

$$\begin{aligned}
g_{11} &= E - 2\zeta L + \zeta^2 \left\{ \frac{L^2}{E} + \frac{M^2}{G} \right\} \\
g_{22} &= G - 2\zeta N + \zeta^2 \left\{ \frac{M^2}{E} + \frac{N^2}{G} \right\} \\
g_{12} &= -2\zeta M + \zeta^2 \left\{ \frac{LM}{E} + \frac{MN}{G} \right\} = g_{21} \\
g_{33} &= 1, \quad g_{13} = g_{31} = 0, \quad g_{23} = g_{32} = 0.
\end{aligned}$$

We see that $\zeta = 0$ lies on $\mathcal{J} \cap U(x_0)$, and that

$$G(\xi, \eta, 0) = \begin{pmatrix} E(\xi,\eta) & 0 & 0 \\ 0 & G(\xi,\eta) & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $E(\xi, \eta) \neq 0$, and $G(\xi, \eta) \neq 0$, since $W^2 = EG \neq 0$. Hence, the boundary conditions (3.14) reduce to

$$\left.\begin{aligned}
\xi_v &= 0 \quad \text{a.e. on } I_\varrho(w_0) \cap \{\xi(w) > 0\} \\
\eta_v &= 0 \quad \text{a.e. on } I_\varrho(w_0) \\
\zeta &= 0 \qquad \text{on } I_\varrho(w_0)
\end{aligned}\right\} \tag{4.1}$$

for the transformed minimal surface

$$y(u, v) = h(x(u, v)) = (\xi(u, v), \eta(u, v), \zeta(u, v)).$$

Furthermore,

$$\Delta y^l = -\Gamma^l_{jk}(y) \{ y^j_u y^k_u + y^j_v y^k_v \} \quad \text{in } S_\varrho(w_0)$$

for $1 \leq l \leq 3$, and $|\nabla y|^2 \in L_p$ on $S_\varrho(w_0)$, for each $p \in (0, 1)$. Thus, we obtain

$$\Delta \eta \in L_p(S_\varrho(w_0), \mathbf{R}), \quad \eta_v = 0 \text{ on } I_\varrho(w_0) \quad \text{for each } p \in (1, \infty) \tag{4.2}$$

and

$$\Delta \zeta \in L_p(S_\varrho(w_0), \mathbf{R}), \quad \zeta = 0 \text{ on } I_\varrho(w_0) \quad \text{for each } p \in (1, \infty). \tag{4.3}$$

Well known potential-theoretic results yield that $\eta(u, v)$ and $\zeta(u, v)$ are in H_p^2 on $S_r(w_0)$ for each $r \in (0, \varrho)$, and every $p \in (1, \infty)$. Then we infer from Sobolev's embedding theorem that $\eta(u, v)$ and $\zeta(u, v)$ are of the regularity class $C^{1+\alpha}(\overline{S_r(w_0)}, \mathbf{R})$ for each $\alpha \in (0, 1)$, and $0 < r < \varrho$.

We still need an information about the first derivatives of $\xi(u, v)$, however. We know that

$$\Delta\xi \in L_p(S_\varrho(w_0), \mathbf{R}) \quad \text{for all } p \in (1, \infty)$$

and that

$$\xi = 0 \text{ on } I'_\varrho(w_0), \quad \xi_v = 0 \text{ on } I''_\varrho(w_0),$$

where

$$I'_\varrho(w_0) = \{w \in I_\varrho(w_0), \xi(w) = 0\}, \quad I''_\varrho(w_0) = I_\varrho(w_0) - I'_\varrho(w_0).$$

While it seems to be impossible to draw any conclusions from these relations, we can fortunately still exploit the conformality relations (3.5) which can be written in the form

$$y_w \cdot G(y) y_w = 0, \quad \frac{\partial}{\partial w} = \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right). \tag{4.4}$$

Since

$$g_{13} = g_{31} = g_{23} = g_{32} = 0, \quad g_{33} = 1,$$

(4.4) becomes

$$g_{11}(y)\xi_w^2 + 2g_{12}(y)\xi_w \cdot \eta_w + g_{22}(y)\eta_w^2 + \zeta_w^2 = 0. \tag{4.5}$$

If $\varrho > 0$ is sufficiently small, then $y(w)$ is sufficiently close to $(0, 0, 0)$; consequently, $g_{11}(y(w))$ is close to $E(\xi(w), \eta(w))$, and therefore, $g_{11}(y(w)) \geq c > 0$ for some number c, provided that $w \in \overline{S_\varrho(w_0)}$.

Set

$$f(w) = \xi_w + \frac{g_{12}(y)}{g_{11}(y)}\eta_w, \quad y = y(w). \tag{4.6}$$

We see from (4.5) that

$$f^2(w) = \left\{\frac{g_{12}(y)}{g_{11}(y)}\eta_w\right\}^2 - \frac{g_{22}(y)}{g_{11}(y)}\eta_w^2 - \frac{1}{g_{11}(y)}\zeta_w^2. \tag{4.7}$$

Since η_w and ζ_w are continuous on $\overline{S_r(w_0)}$ for $0 < r < \varrho$, (4.6) and (4.7) imply that $f(w)$ is a complex valued, continuous function on the open, connected set $\Omega = S_r(w_0)$ and that the square of $f(w)$ is continuous on $\overline{\Omega}$. With the help of the following lemma of E. Heinz (cf. [8], [9]) it will be seen that $f(w)$ is continuous in the closure $\overline{S_r(w_0)}$.

LEMMA. *Let $f(w)$ be a complex valued continuous function on an open connected set Ω in $\mathbf{C}$, such that $f^2(w)$ has a continuous extension to $\overline{\Omega}$. Then also $f(w)$ can be extended continuously to $\overline{\Omega}$ provided that $\partial\Omega$ is non-degenerate, that is, for every $w_0 \in \partial\Omega$ there exists a $\delta > 0$ such that $\Omega_\delta(w_0) = \Omega \cap B_\delta(w_0)$ is connected.*

Proof. Let w_0 be an arbitrary point on $\partial\Omega$. Then, there exists an $\alpha\in\mathbf{C}$ such that $f^2(w)\to\alpha$ as $w\to w_0$. If $\alpha=0$ then $|f(w)|^2\to 0$, and also $f(w)\to 0$ as $w\to w_0$. If $\alpha\neq 0$, set $\alpha=\beta^2$ for some $\beta\in\mathbf{C}$, $\beta\neq 0$. Pick an $\varepsilon>0$ such that $0<\varepsilon<|\beta|$. There is a $\delta>0$ such that $\Omega_\delta(w_0)$ is connected, and that $f(w)$ maps the set $\Omega_\delta(w_0)$ into the disconnected set $B_\varepsilon(\beta)+B_\varepsilon(-\beta)$. Since $f(w)$ is continuous on Ω, the image set $f[\Omega_\delta(w_0)]$ is connected, and therefore already contained in one of the discs $B_\varepsilon(\beta)$, $B_\varepsilon(-\beta)$. Thus, $\lim_{w\to w_0} f(w)$ exists and is equal to β or $-\beta$. We now define a function $F(w)$ in the following way:

$$F(w)=\begin{cases} f(w), & w\in\Omega \\ \lim\limits_{\substack{\tilde{w}\to w\\ \tilde{w}\in\Omega}} f(\tilde{w}), & w\in\partial\Omega \end{cases}$$

This function, an extension of $f(w)$, is continuous in $\overline{\Omega}$. The lemma is proved.

From the continuity of $f(w)$ it now follows from (4.6) that ξ_w is continuous in $\overline{S_r(w_0)}$. Thus $y(w)$ belongs to class C^1 in $\overline{S_r(w_0)}$. Since w_0 is an arbitrary point on I, we finally see that $x(w)\in C^1(B\cup I,\mathbf{R}^3)$.

We summarize the results of section 4:

THEOREM 3. *Suppose that the assumptions of Theorem 2 are satisfied. Then every solution of problem* $\mathfrak{P}(\Gamma, S)$ *belongs to the regularity class* $C^1(B\cup I,\mathbf{R}^3)$.

5. Asymptotic expansions around branch points on the free boundary

We continue to employ the notations and assumptions of section 4.

It follows from (4.6) and (4.7) that, for sufficiently small $\varrho>0$, there is a number $c_0>0$ such that

$$|\nabla\xi|^2\leqslant c_0\{|\nabla\eta|^2+|\nabla\zeta|^2\} \quad \text{on } \overline{S_\varrho(w_0)}. \tag{5.1}$$

By virtue of

$$\Delta y^l=-\Gamma^l_{jk}(y)\{y^j_u y^k_u+y^j_v y^k_v\},\quad 1\leqslant l\leqslant 3,$$

therefore also

$$|\Delta\eta|+|\Delta\zeta|\leqslant c\{|\nabla\eta|^2+|\nabla\zeta|^2\} \quad \text{on } \overline{S_\varrho(w_0)}, \tag{5.2}$$

where $c>0$ denotes another suitable constant.

Moreover, from (4.1),

$$\eta_v=0 \quad\text{and}\quad \zeta=0 \quad \text{on } I_\varrho(w_0). \tag{5.3}$$

Denote by $\bar{w}=u-iv$ the image point of the point $w=u+iv$ under a reflection on the real axis. We know that η and ζ lie in $C^1(\overline{S_\varrho(w_0)},\mathbf{R}^3)$ provided that $\varrho>0$ is sufficiently small. Thus it follows from (5.3) that $\eta(w)$ and $\zeta(w)$ can be extended as C^1-functions $\mathrm{H}(w)$, $\mathrm{Z}(w)$ across the real axis onto $\overline{B_\varrho(w_0)}$ by the definitions

$$\mathrm{H}(w)=\begin{cases}\eta(w) & \text{if } w\in\overline{S_\varrho(w_0)},\\ \eta(\bar w) & \text{if } \bar w\in\overline{S_\varrho(w_0)},\end{cases}$$

and

$$\mathrm{Z}(w)=\begin{cases}\zeta(w) & \text{if } w\in\overline{S_\varrho(w_0)},\\ -\zeta(\bar w) & \text{if } \bar w\in\overline{S_\varrho(w_0)}.\end{cases}$$

The vector valued function

$$Y(w)=(\mathrm{H}(w),\mathrm{Z}(w))$$

has continuous first derivatives on $\overline{B_\varrho(w_0)}$, and continuous second derivatives on $B_\varrho(w_0)-I_\varrho(w_0)$. The second derivatives are in L_p on $B_\varrho(w_0)$ for each $p\in(1,\infty)$. (5.2) implies that $Y_{w\bar w}=\frac{1}{4}\Delta Y$ is essentially bounded on $B_\varrho(w_0)$, and that

$$|Y_{w\bar w}|\leqslant k|Y_w| \quad \text{on } B_\varrho(w_0)-I_\varrho(w_0) \tag{5.4}$$

for some constant $k>0$.

On the other hand, a well known application of the Gauss integration formula yields the identity

$$\frac{1}{2i}\oint_{\partial D} Y_w\cdot\varphi\, dw=\iint_D (Y_w\cdot\varphi_{\bar w}+Y_{w\bar w}\cdot\varphi)\,du\,dv \tag{5.5}$$

for each domain D in $B_\varrho(w_0)$ with piecewise smooth boundary, and for all $\varphi\in C^1(\overline{D},\mathbf{C}^2)$. On account of (5.4), we derive from (5.5) the inequality

$$\left|\oint_{\partial D} Y_w\cdot\varphi\,dw\right|\leqslant 2\iint_D |Y_w|\{|\varphi_{\bar w}|+k|\varphi|\}\,du\,dv \quad \text{for all } \varphi\in C^1(\overline{D},\mathbf{C}^2). \tag{5.6}$$

As in [8], p. 103 or [15], p. 331 the technique of P. Hartman and A. Wintner [7] can be applied to (5.6). Hence, if $Y_w(w_0)=0$, and if $Y_w\not\equiv 0$, then there exists a vector $A=(a^2,a^3)\in\mathbf{C}^2$ with $A\neq 0$, and an integer $\nu\geqslant 1$ such that

$$Y_w(w)=A(w-w_0)^\nu+o(|w-w_0|^\nu) \quad \text{as } w\to w_0. \tag{5.7}$$

Let us consider now a boundary branch point $w_0\in I$ of $x(u,v)$, that is,

$$x_u(w_0)=0 \quad \text{and} \quad x_v(w_0)=0.$$

Then $y_w(w_0)=0$, and in particular, $Y_w(w_0)=0$.

Since the branch points $w\in B$ of $x(u,v)$ are isolated, we have $x_w\not\equiv 0$ on each open subset of B. It follows from (4.5), that also $Y_w\not\equiv 0$ on each open subset of $B_\varrho(w_0)$. Thus the Hartman–Wintner device can be applied, to obtain (5.7).

We now consider the function

$$h(w)=(w-w_0)^{-\nu}f(w);$$

$h(w)$ is continuous on $\{w; 0<|w-w_0|<\varrho\}$, $f(w)$ being defined by (4.6). Note that $y(w_0)=0$, and that

$$\mathcal{G}(0)=\begin{pmatrix} E & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & 1 \end{pmatrix}_{\xi=0,\,\eta=0.}$$

In view of (4.7) and (5.7) the limit $\lim_{w\to w_0} h^2(w)$ exists. Applying the lemma of section 4, we see that also $\lim_{w\to w_0} h(w)$ exists. Set

$$a^1=\lim_{w\to w_0} h(w)=\lim_{w\to w_0}(w-w_0)^{-\nu}\,\xi_w(w). \tag{5.8}$$

Then it follows from (4.7) and (5.7) that

$$y_w(w)=a(w-w_0)^\nu+o(|w-w_0|^\nu) \quad \text{as } w\to w_0 \tag{5.9}$$

where $a=(a^1, a^2, a^3)\neq 0$ is a vector in $\mathbf{C}^3$ which satisfies

$$g_{jk}(0)\,a^j a^k=0. \tag{5.10}$$

Since

$$x_w=H(y)\,y_w, \quad H=\frac{\partial h}{\partial y}, \tag{5.11}$$

we obtain the following result:

THEOREM 4. *Suppose that the assumptions of Theorem 2 are satisfied. Let $w_0\in I$ be a branch point of a solution $x(u, v)=x(w)$ of $\mathcal{D}(\Gamma, S)$ on the free boundary. Then there exist a vector $b=(b^1, b^2, b^3)\in\mathbf{C}^3$ with $|b|\to 0$ and*

$$b\cdot b=0, \tag{5.12}$$

and an integer $\nu\geqslant 1$, such that

$$x_w(w)=b(w-w_0)^\nu+o(|w-w_0|^\nu) \quad \text{as } w\to w_0. \tag{5.13}$$

A geometric consequence of (5.13) is the following. Let

$$2b=\alpha-i\beta, \quad \text{where } \alpha, \beta\in\mathbf{R}^3. \tag{5.14}$$

By (5.12),

$$|\alpha|=|\beta|\neq 0, \quad \alpha\cdot\beta=0. \tag{5.15}$$

For $w_0=u_0\in I$ and $w\to w_0$ we have

$$\begin{aligned} x_u(w) &= \alpha\,\mathrm{Re}\,(w-w_0)^\nu+\beta\,\mathrm{Im}\,(w-w_0)^\nu+o(|w-w_0|^\nu) \\ x_v(w) &= -\alpha\,\mathrm{Im}\,(w-w_0)^\nu+\beta\,\mathrm{Re}\,(w-w_0)^\nu+o(|w-w_0|^\nu) \end{aligned} \tag{5.16}$$

and therefore

$$x_u(w)\wedge x_v(w)=(\alpha\wedge\beta)\,|w-w_0|^{2\nu}+o(|w-w_0|^{2\nu}). \tag{5.17}$$

Denote by

$$\mathfrak{N}(w) = \frac{x_u(w) \wedge x_v(w)}{|x_u(w) \wedge x_v(w)|}, \quad w \neq w_0, \tag{5.18}$$

the unit normal vector of the minimal surface $x(w)$. Then $\mathfrak{N}(w)$ converges to a limit vector as $w \to w_0$. In fact,

$$\lim_{w \to w_0} \mathfrak{N}(w) = \frac{\alpha \wedge \beta}{|\alpha \wedge \beta|}. \tag{5.19}$$

Therefore, the tangent plane of the minimal surface $x(w)$ tends to a limiting position as w tends to the branch point w_0 on the free boundary w_0.

We further consider the trace $x(u, 0)$ on the supporting surface $\mathcal{S}$. By (5.16),

$$x_u(u, 0) = \alpha(u - u_0)^\nu + o(|u - u_0|^\nu) \quad \text{as } u \to u_0. \tag{5.20}$$

Then we obtain for the tangent vector

$$\mathfrak{T}(u) = \frac{x_u(u, 0)}{|x_u(u, 0)|} \tag{5.21}$$

of the trace curve $x(u, 0)$ the asymptotic representation

$$\mathfrak{T}(u) = \frac{\alpha}{|\alpha|} \left(\frac{u - u_0}{|u - u_0|} \right)^\nu + o(1) \quad \text{as } u \to u_0. \tag{5.22}$$

Therefore, the non-oriented tangent moves continuously through a boundary branch point while the oriented tangent is continuous for branch points of even order ν, but, for branch points of odd order, the tangent direction jumps by 180 degrees.

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