BOUNDARY BEHAVIOUR OF MEROMORPHIC FUNCTIONS OF SEVERAL VARIABLES

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0. Introduction

The aim of this paper is to prove an n-dimensional generalization of a theorem of R. Nevanlinna. This theorem says that a meromorphic function of bounded characteristic in the unit disc $\{z: |z|<1\}$ has (finite) nontangential limits in almost every point of the circumference $\{z: |z|=1\}$. A function f is of bounded characteristic if

$$\int_0^{2\pi} \log^+ |f(re^{i\varphi})|\, d\varphi = O(1) \quad (0 \leqslant r < 1) \tag{1}$$

and

$$\sum_j (1-|b_j|) < \infty \tag{2}$$

where the b_j's are the poles of f (counted with multiplicity). $\log^+$ stands for the maximum of log and zero.

As a matter of fact, Nevanlinna showed that functions of bounded characteristic can be represented as the quotient of two bounded holomorphic functions: $f=g/h$. Now, Fatou's theorem tells us that g and h have nontangential limits a.e. on the circumference; hence the theorem of Nevanlinna. To be quite rigorous, it should be added that, by a theorem of F. and M. Riesz, the boundary values of h are a.e. different from zero.

In several variables Fatou's theorem generalizes straightforwardly, not only to functions in the unit ball, but to functions defined in domains with smooth boundary. The point is that this theorem holds for bounded harmonic functions too, and there is much less difference between harmonic functions in $\mathbf{C}$ and in $\mathbf{C}^n$ $(n>1)$, then between analytic functions in $\mathbf{C}$ and in $\mathbf{C}^n$.

The situation turns out to be more complicated if meromorphic functions of bounded characteristic are considered. These are defined by the conditions

$$\int_{\partial D_\varepsilon} \log^+ |f(z)|\, d\Omega(z) = O(1) \quad (0 < \varepsilon < \varepsilon_0) \tag{3}$$

and

$$\int_P \delta(z, \partial D)\, d\mu(z) < \infty, \tag{4}$$

where:

1. D is a bounded domain in $\mathbf{C}^n$ with C^2 boundary, f is a meromorphic function defined on it;
2. $\delta(z, \partial D)$ denotes the distance of a point $z \in \mathbf{C}^n$ to ∂D;
3. $D_\varepsilon = \{z \in D: \delta(z, \partial D) > \varepsilon\}$. For small $\varepsilon > 0$, ∂D_ε is C^1;
4. $d\Omega$ is the $(2n-1)$-dimensional surface element;
5. P is the set of poles of f;
6. $d\mu$ is the $(2n-2)$-dimensional surface element with the modification that it counts poles of higher order with multiplicity.

The set of these functions will be denoted by $B = B(D)$.

Fortunately, in general, such functions can not be represented as the quotient of two bounded holomorphic functions, see W. Rudin [5]. Thus, if one wants to investigate the boundary behaviour of a function $f \in B$, the one-variable method will be of no use. In [7] E. M. Stein found one way to settle the problem for the class $N \subset B$, i.e. for holomorphic functions subject to the condition (3). He used (pluri-)subharmonic majorants and maximal functions to prove the existence of nontangential limits in a.e. point of ∂D. It seems, however, that the case of meromorphic functions can not be settled with his ideas. (Although, by a result of H. Skoda (see [6]) on certain pseudoconvex domains every $f \in B$ is the quotient of two functions from N. This solves the problem in special cases.)

As stated above, bounded holomorphic functions in several variables present nothing new compared with the one-variable case. This is true only as long as nontangential convergence is concerned. A. Korányi has discovered the phenomenon that in more than one dimension boundary points can be approached even parabolically from certain directions, and the bounded holomorphic function still has limits under such an approach in a.e. boundary point (see Korányi [2]). This sort of approach was termed "admissible" in [2]. For the exact definition see Chapter 6. The original result (for balls and bounded functions) has been extended to more general domains and larger classes of functions by several authors, finally by E. M. Stein in [7] to domains with C^2 boundary and functions of class N.

The main result of this paper is the proof of the admissible convergence in a.e. boundary point for functions of class B. En route we shall prove the Blaschke condition for the zeros of such a function. The Blaschke condition has been proved for the class N on special domains by several authors, see e.g. P. S. Chee [1] and P. Malliavin [3]. However, for the class B it was not to be found in the literature. Therefore, we shall give the proof here. We would like to express our thanks to the referee for showing us the short proof to be given in Chapter 2.

1. Outline of the proof of the main theorem

Before going into the thick of the proof, it may be useful to give some motivations. This will be done in the one variable case. Thus, let now D be the unit disc $\{z: |z|<1\}$, and consider a holomorphic function on it. We may even suppose that this function $f \not\equiv 0$ is bounded. The claim is that

(5) *to almost no point of the circumference ∂D can the zeros of f accumulate nontangentially.*

That is, the set of those $\zeta \in \partial D$ where the zeros of f can accumulate nontangentially, is of zero measure on D.

This follows easily from the theorems of Fatou and of F. and M. Riesz. Indeed, suppose that the zeros accumulate nontangentially to a $\zeta \in \partial D$. There are two possibilities. Either f has no nontangential limit in ζ, or the nontangential limit is zero. Both can happen only if ζ is in a certain set of zero measure, and we are done.

There is an other way to deduce (5), this time from the Blaschke condition

$$\sum_k (1-|a_k|) < \infty,$$

where $a_1, a_2, \ldots$ denote the zeros of f. To see this, choose a fixed angle $\theta \in (0, \pi/2)$ and consider the set E_θ of those $\zeta \in \partial D$ where the zeros can accumulate "θ-angularly" or in a "θ-Stolz-angle". This means that ζ is an accumulation point of zeros a_k satisfying the inequality $|\arg(1-a_k/\zeta)| < \theta$. For every k, construct now an arc $A_k \subset \partial D$ with middlepoint $a_k/|a_k|$ and of length $|A_k| = 3(1-|a_k|)\operatorname{tg}\theta$. It is easy to see that every $\zeta \in E_\theta$ is contained in infinitely many A_k's. Since

$$\sum_1^\infty |A_k| = \sum_1^\infty 3(1-|a_k|)\operatorname{tg}\theta < \infty,$$

an application of the Borel-Cantelli lemma shows that E_θ is of zero measure. But this, in turn, implies (5).

Thus we observe that both statements

(6) *f has nontangential limits in a.e. point of ∂D*

and

(7) *the zeros a_k of f satisfy* $\sum (1-|a_k|)<\infty$

(separately) imply (5). Would it be too daring to guess that the implication (5) $\Rightarrow$ (6) also holds? Because then it would be possible to deduce Fatou's theorem (or Nevanlinna's theorem) from the Blaschke condition (7) according to the pattern (7) $\Rightarrow$ (5) $\Rightarrow$ (6).

Of course, such a guess *would* be too daring. For even the total lack of zeros does not give us any control whatsoever on the boundary behaviour of a function.

But suppose we substitute (5) by the more general statement

(5′) *for every $w\in\mathbf{C}$, to almost no point of ∂D can the roots of the equation $f(z)=w$ accumulate nontangentially.*

Does (5′) imply (6)?

This is a better question. In fact, it is so good that we were unable to answer it. Instead, we could prove the weaker (5′)+(8) $\Rightarrow$ (6) implication, where (8) stands for the statement

(8) *f has radial limits in a.e. point of ∂D.*

Weak it may be, but this implication is the key in our approach. Here is our scheme (now for a smooth domain $D\subset\mathbf{C}^n$ and a meromorphic function of bounded characteristic f defined on it):

I. First we prove the Blaschke condition, i.e. (4), with P replaced by the set $N_w=\{z: f(z)=w\}$, $w\in\mathbf{C}$ fixed.

II. Then we shall show that the Blaschke condition implies (5′).

III. We shall proceed by showing that in a.e. point of ∂D there is a direction along which f tends to a limit.

IV. Finally, it will be deduced from II. and III. that f has nontangential limits a.e. on ∂D.

Once this accomplished, the whole story will have to be repeated with "nontangential" everywhere replaced by "admissible".

Finally, some words about the notations and terminology. We shall need too many letters to use them consistently. Nevertheless, here are some notations which will be preserved throughout the paper. z and ζ (and Z, $\tilde{z}$ etc.) will be points of $\mathbf{C}^n$ $(n>1)$, ζ mostly a boundary point of a fixed domain $D\subset\mathbf{C}^n$. The complex coordinates of z will be $z_1, \ldots, z_n$; $z_j=x_j+iy_j$. z' will denote the point $(z_2, \ldots, z_n)$. Thus $z=(z_1, z')$. Similar conventions hold for ζ, Z etc. $|z|=(\sum_1^n |z_j|^2)^{\frac{1}{2}}$; if A and B are subsets of $\mathbf{C}^n$, then

$$\delta(A, B)=\inf\{|z-Z|: z\in A, Z\in B\}. \tag{9}$$

$\delta(z, A)=\delta(\{z\}, A)$. grad means real gradient, $\langle ,\rangle$ means real, $\langle\langle ,\rangle\rangle$ means complex scalar product. Thus $\langle\langle z, w\rangle\rangle=\sum z_j\bar{w}_j$; $\langle z, w\rangle=\operatorname{Re}\langle\langle z, w\rangle\rangle$. Parallel and orthogonal will be used in the real sense, unless otherwise stated. μ stands for the $(2n-2)$-dimensional Hausdorff measure. By "zero set of a meromorphic function" we shall understand something more than the mere collection of those points where the function vanishes. Namely we shall count multiple zeros with corresponding multiplicity. This regards integration on and area of zero sets. For instance

$$\mu\{z: h(z)=0\}=\tfrac{1}{2}\mu\{z: h^2(z)=0\}.$$

The same convention holds for the set of poles. Closure of a set will be denoted by a bar. Finally, $K_1, K_2, \ldots$ will denote positive "constants". In every case it will be explained to what extent they are constant. The notation $O(1)$ will also be used.

2. The Blaschke condition

Let $D\subset\mathbf{C}^n$ be a bounded domain with C^2 boundary ∂D. Denote by $\Delta(z)$ the distance $\delta(z, \partial D)$ of z to ∂D; let

$$D_\varepsilon=\{z\in D: \Delta(z)>\varepsilon\},$$

Denote by $\nu(z)$ the exterior unit normal vector to $\partial D_{\Delta(z)}$ in z. For $z\in D$ near enough to ∂D this makes sense, since then $\partial D_{\Delta(z)}$ will be of class C^1, Moreover, if ε_0 is small, the mapping

$$\partial D\times[0, \varepsilon_0]\ni(\zeta, \varepsilon)\mapsto\zeta-\varepsilon\nu(\zeta)\in\bar{D}\setminus D_{\varepsilon_0}$$

is a C^1-diffeomorphism. For the sake of simplicity we shall suppose that $\varepsilon_0=1$. Then ∂D_ε is C^1 for $0<\varepsilon\leqslant 1$. Let $d\Omega_\varepsilon$ denote the surface element on ∂D_ε, $d\Omega_0=d\Omega$.

THEOREM 1. *With the above notations and assumptions concerning D, suppose that $f \not\equiv 0$ is a meromorphic function of bounded characteristic on D, i.e.*

$$\int_{\partial D_\varepsilon} \log^+ |f(z)| \, d\Omega_\varepsilon(z) = O(1) \qquad (0 < \varepsilon \leqslant 1) \tag{10}$$

and

$$\int_P \Delta(z) \, d\mu(z) < \infty, \tag{11}$$

where P is the set of poles of f. Let $N = \{z \in D: f(z) = 0\}$. Then

$$\int_N \Delta(z) \, d\mu(z) < \infty. \tag{12}$$

Proof. The proof is based on Green's formula for subharmonic functions. Let σ_0 and σ_1 be the nonnegative measures associated to the zero set and pole set of f in D, so that we have in the sense of distribution theory in D

$$\Delta \log |f| = \sigma_0 - \sigma_1.$$

Let V_j be a subharmonic function in D such that $\Delta V_j = \sigma_j$ for $j = 0, 1$, and let H be the harmonic function in D defined by

$$H = \log |f| - V_0 + V_1.$$

Let be respectively G_ε and P_ε the Green function and the Poisson kernel associated to the open set D_ε and a fixed point $z^0 \in D$, which is neither pole nor zero of f.

Then Green's formula gives

$$\int_{\partial D_\varepsilon} P_\varepsilon V_j \, d\Omega_\varepsilon = \int_{D_\varepsilon} G_\varepsilon \, d\sigma_j + V_j(z^0), \tag{13}$$

$$\int_{\partial D_\varepsilon} P_\varepsilon H \, d\Omega = H(z^0).$$

Therefore we obtain

$$\int_{\partial D_\varepsilon} P_\varepsilon \log |f| \, d\Omega_\varepsilon = \int_{D_\varepsilon} G_\varepsilon \, d\sigma_0 - \int_{D_\varepsilon} G_\varepsilon \, d\sigma_1 + \log |f(z^0)|. \tag{14}$$

But G_ε increases to G_0, and for all $\varepsilon > 0$ we have

$$\int_{D_\varepsilon} G_\varepsilon \, d\sigma_1 \leqslant \int_D G \, d\sigma_1 = K_1 < \infty \tag{15}$$

(using the definition of class B). On the other hand

$$\int_{\partial D_\varepsilon} P_\varepsilon \log|f|\,d\Omega_\varepsilon \leqslant \int_{\partial D_\varepsilon} P_\varepsilon \log^+|f|\,d\Omega_\varepsilon \leqslant K_2 \int_{\partial D_\varepsilon} \log^+|f|\,d\Omega_\varepsilon \tag{16}$$

because ∂D is of class C^2. Then the estimates (14), (15) and (16) imply

$$\int_D G_0\,d\sigma_0 = \lim_{\varepsilon\to 0} \int_{D_\varepsilon} G_\varepsilon\,d\sigma_0 < \infty,$$

which is equivalent to (12).

3. Boundary behaviour of the zero set

In this chapter we shall use the notations and assumptions of Theorem 1.

Definition. $\zeta \in \partial D$ is a nontangential (or angular) accumulation point of a set $S \subset D$ if there is a cone $C_\alpha(\zeta)$ with vertex ζ, axis $-\nu(\zeta)$, and aperture $\alpha < \pi$ so that ζ is an accumulation point of $S \cap C_\alpha(\zeta)$.

LEMMA 1. *The nontangential accumulation points of $N = \{z \in D: f(z) = 0\}$ constitute a set of zero measure on ∂D.*

COROLLARY. *The same holds if N is replaced by $N_w = \{z \in D: f(z) = w\}$ $(w \in \mathbf{C})$, or by P.*

Proof of the corollary. If $f(z)$ is of bounded characteristic, so is $f(z) - w$ and $1/f(z)$ (this latter in view of Theorem 1 and its proof).

For the proof of the lemma we shall need the following result, first proved by P. Lelong and H. Rutishauser (see L. I. Ronkin [5] p. 370):

PROPOSITION 1. *If a function is meromorphic in a ball of radius r in $\mathbf{C}^n$ and vanishes at the centre of the ball, then the $(2n-2)$-dimensional area of its zero set is at least $\omega_{2n-2} r^{2n-2}$ (where ω_k is the volume of the unit ball in the Euclidean k-space). In other words, the area of the zero set is minimal for linear functions.*

Proof of Lemma 1. The measure of a set X on ∂D ($(2n-1)$-dimensional Hausdorff measure) will be denoted by $\Omega(X)$. For $\zeta \in \partial D$, $C_\alpha(\zeta)$ will mean the cone described in the above definition.

It will clearly be sufficient to prove that given an $\alpha \in (0, \pi)$, the set

$$E = E_\alpha = \{\zeta \in \partial D: \zeta \in \overline{C_\alpha(\zeta) \cap N}\}$$

is of zero measure. Fix α; from now on in this proof all constants K_7, K_8, ... may depend beside D and f also on α.

Consider two points $\zeta\in\partial D$ and $z\in D$. If $z\in C_\alpha(\zeta)$ is near enough to ζ, then ζ will be contained in the ball $B(z)$ of radius $K_7\Delta(z)$ around z. Introduce the sets

$$B_k=\bigcup\{B(z)\colon z\in N,\ 2^{-k+1}>\Delta(z)\geqslant 2^{-k}\}.$$

By the preceding remark, points of E will be contained in B_k for infinitely many $k\in\mathbf{N}$. We are going to show that

$$\sum_{k=1}^{\infty}\Omega(B_k\cap\partial D)<\infty; \tag{17}$$

once this established, the Borel-Cantelli lemma will guarantee that E is indeed of measure zero.

Choose a $k\geqslant 2$. Around points of the set

$$N^k=\{z\in N;\ 2^{-k+1}>\Delta(z)\geqslant 2^{-k}\}$$

construct balls $B'(z)$ of radius 2^{-k-1} and select a maximal disjoint system of these balls: $B'(z^1)$, $B'(z^2)$, ... If these selected balls are replaced by twice as big concentric balls, the resulting system will cover the whole N^k. Therefore if they are replaced by K_8 times as big concentric balls, these enlarged balls will cover B_k (provided that K_8 is sufficiently large). Hence

$$\Omega(B_k\cap\partial D)\leqslant K_9\sum_j\Omega(B'(z^j)\cap\partial D). \tag{18}$$

Obviously

$$\Omega(B'(z^j)\cap\partial D)\leqslant K_{10}2^{-k(2n-1)}.$$

On the other hand, by Proposition 1 above

$$\mu(N\cap B'(z^j))\geqslant K_{11}2^{-k(2n-2)}.$$

Therefore

$$\Omega(B'(z^j)\cap\partial D)\leqslant\frac{K_{10}}{K_{11}}\mu(N\cap B'(z^j))\cdot 2^{-k}\leqslant K_{12}\int_{N\cap B'(z^j)}\Delta(z)\,d\mu(z).$$

Since the $B'(z^j)$:s are disjoint, (18) yields

$$\Omega(B_k\cap\partial D)\leqslant K_9K_{12}\sum_j\int_{N\cap B'(z^j)}\Delta(z)\,d\mu(z)\leqslant K_9K_{12}\int_{N^{k-1}\cup N^k\cup N^{k+1}}\Delta(z)\,d\mu(z)$$

so that

$$\sum_{k=2}^{\infty} \Omega(B_k) \leqslant 3K_9 K_{12} \int_N \Delta(z)\, d\mu(z);$$

and this, as we have already seen, proves the lemma.

4. Limit from one direction

Our plan is to exhibit directions in a.e. $\zeta \in \partial D$ along which our f has a limit. This will be done in the most natural way. We shall restrict f to parallel complex lines and then we shall show that almost all of these restrictions are of bounded characteristic. Once this done, the corresponding one variable theorem will prove our claim.

Again we shall use the notations and assumptions of Theorem 1.

LEMMA 2. *Let $Z \in \mathbf{C}^n$. Then almost every complex line passing through Z is not tangent to ∂D.*

Proof. The whole set-up will be considered in $\mathbf{CP}_n$ rather than in $\mathbf{C}^n$. Then it can be assumed that Z is the ideal point $(1:0:...:0) \in \mathbf{CP}_n$, so that the non-ideal complex lines through Z are characterized by a system of equations

$$z_j = c_j \quad (j = 2, ..., n). \tag{19}$$

($z_1, ..., z_n$ are the coordinate functions in $\mathbf{C}^n \subset \mathbf{CP}_n$.)

Define a C^2-function $h: \mathbf{C}^n \cap \partial D \to \mathbf{C}^{n-1}$ by

$$h(\zeta_1, ..., \zeta_n) = (\zeta_2, ..., \zeta_n).$$

If the line (19) is tangent to ∂D in $\zeta \in \mathbf{C}^n \cap \partial D$, then ζ is a critical point of h. Therefore $(c_2, ..., c_n)$ is a critical value of h, so that by Sard's lemma the points $(c_2, ..., c_n)$ for which (19) is tangent to ∂D constitute a set of zero measure. Since almost every line is non-ideal, the claim follows.

LEMMA 3. *In a.e. $\zeta \in \partial D$ there is a direction along which f tends to a finite limit.*

Proof. It will be sufficient to show that every $Z \in \partial D$ has a neighbourhood U such that the statement of the lemma holds for a.e. $\zeta \in U \cap \partial D$. Thus, pick out a point $Z \in \partial D$. For the sake of simplicity let it be the origin. By the preceding lemma, there is a complex line through 0 which is not tangent to ∂D. We shall assume that this line is the z_1 axis.

There is a positive ε such that the complex lines

$$\{z: z_j = c_j \quad (j=2, \ldots, n)\}$$

are not tangent to ∂D if $|c| \leqslant \varepsilon$. We are going to show that for almost every small z' ($|z'| < \varepsilon$) the function

$$g(t) = f(t, z')$$

is of bounded characteristic on $D_{z'} = \{t: (t, z') \in D\} \subset \mathbf{C}$.

Observe that the boundary $\Gamma_{z'}$ of $D_{z'}$ is C^2 and $\delta(t, \Gamma_{z'})/\Delta(t, z')$ is bounded. By virtue of Fubini's theorem, the Blaschke-condition for the poles of f then implies for a.e. z'

$$\sum_{g(t)=0} \delta(t, \Gamma_{z'}) < \infty. \tag{20}$$

We still have to exhibit curves along which $\log^+|g(t)|$ is small in the mean. To this end, let

$$\Gamma_{z'}^j = \{t \in \mathbf{C}: (t, z') \in \partial D_{1/j}\}.$$

If $|z'| \leqslant \varepsilon$ and j is big enough, $\Gamma_{z'}^j$ is a C^2-curve, which tends to $\Gamma_{z'}$ (in the C^2-topology) as $j \to \infty$.

Notice that

$$\int_{|z'| \leqslant \varepsilon} \int_{\Gamma_{z'}^j} \log^+|f(t, z')|\,|dt|\, dx_2\, dy_2 \ldots dx_n\, dy_n \leqslant \int_{\partial D_{1/j}} \log^+|f(z)|\, d\Omega_{1/j}(z) \leqslant K_{13}. \tag{21}$$

Choose a large positive number M. From (21) it follows that for a fixed j the set

$$S_j = \left\{z': |z'| \leqslant \varepsilon, \int_{\Gamma_{z'}^j} \log^+|f(t, z')|\,|dt| > M\right\}$$

is of ($(2n-2)$-dimensional) measure less than K_{13}/M. Therefore the measure of the set

$$\left\{z': |z'| \leqslant \varepsilon, \int_{\Gamma_{z'}^j} \log^+|f(t, z')|\,|dt| \leqslant M \quad \text{is satisfied only for finitely many } j \in \mathbf{N}\right\} = \bigcup_{k=1}^{\infty} \bigcap_{j=k}^{\infty} S_j$$

is likewise at most K_{13}/M. Since M was arbitrary, we conclude that for a.e. z', $|z'| \leqslant \varepsilon$, there is a sequence $j_1 < \ldots < j_k < \ldots$ such that

$$\int_{\Gamma_{z'}^{j_k}} \log^+|g(t)|\,|dt| = O(1) \quad (k \to \infty). \tag{22}$$

Now, although $D_{z'}$ is not necessarily simply connected, it can be shown that if (20) and (22) hold, $g(t)$ is the quotient of two bounded holomorphic functions. Therefore it has nontangential limits in a.e. point of $\Gamma_{z'}$, whence Lemma 3 follows.

5. Nontangential limits

We shall use the notations of Chapter 3.

Definition. A function g, defined on D, has nontangential limit in a boundary point $\zeta \in \partial D$ if for every α $(0<\alpha<\pi)$ $g(z)$ tends to a finite limit as z tends to ζ in $C_\alpha(\zeta) \cap D$.

THEOREM 2. *Under the assumptions of Theorem 1, f has nontangential limits in almost every boundary point of D.*

Proof. Let $\{w_1, w_2, ...\}$ be a countable dense subset of $\mathbf{C}$. Consider a point $\zeta \in \partial D$ which is not an angular accumulation point of any N_{w_p} and for which there is a direction ν such that the finite limit

$$\lim_{\varepsilon \to +0} f(\zeta + \varepsilon \nu)$$

exists. By the corollary to Lemma 1 and by Lemma 3, almost every point of ∂D is such. We are going to show that f has a nontangential limit in ζ.

Suppose this were not true. Then there would be an $\alpha < \pi$ and a sequence

$$\{z^j : j=1, 2, ...\} \subset C_\alpha(\zeta) \cap D, \quad z^j \to \zeta \quad (j \to \infty)$$

such that

$$\lim_{j \to \infty} f(z^j) = B \neq A = \lim_{\varepsilon \to +0} f(\xi + \varepsilon \nu).$$

Here B is allowed to be ∞. For the sake of simplicity we shall assume $\zeta = 0$ and $\nu = (1, 0, ..., 0)$. Choose a β with $\alpha < \beta < \pi$, and let

$$G = \{z \in C_\beta(0): |z| < 2\}.$$

If β was near enough to π, then $(1, 0, ..., 0) \in G$. For sufficiently large j the functions $f_j(Z) = f(Z|z^j|)$ are defined on G and do not assume the distinct values w_1, w_2, w_3 there. Thus they constitute a normal family, and we can select a subsequence f_{j_k} which is uniformly convergent on compact subsets of G. The limit function g is not identically ∞, since $g(1, 0, 0, ..., 0) = A$. On the other hand $g \not\equiv A$, because the set $\{z^j/|z^j| : j=1, 2, ...\}$ is relatively compact in G and

$$f_j(z^j/|z^j|) \to B \neq A.$$

Therefore the meromorphic function g is not constant, whence its range is open. In particular, this range contains a w_p. By a theorem of Hurwitz, the functions f_{j_k} also assume the value w_p in G if j_k is large enough. Hence f assumes w_p in $C_\beta(0)$ however near to the vertex, a contradiction. This proves our theorem.

6. Admissible approach

If α is a positive number and q is a nondegenerate real quadratic form on $\mathbf{R}^{2n-2} \cong \mathbf{C}^{n-1}$, we define a domain

$$B_{\alpha,q} = \{z \in \mathbf{C}^n: \alpha |\operatorname{Im} z_1| < |\operatorname{Re} z_1|, \ q(z') < \operatorname{Re} z_1\}$$

and call it a standard parabolic cone. If q is positive definite, $B_{\alpha,q}$ certainly has a parabolic character; for general q it looks rather hyperbolic—nevertheless, the same name will be used in all cases. Domains congruent to standard parabolic cones will be called parabolic cones. Here "congruent" means $\mathbf{C}$-linear (not just $\mathbf{R}$-linear) congruency.

Returning to the notations of Theorem 1, choose a $\zeta \in \partial D$. Fix a motion of $\mathbf{C}^n$ which transforms the origin to ζ and the ray $\{(t, 0, ..., 0): t \geqslant 0\}$ to the ray $\{\zeta - t\nu(\zeta): t \geqslant 0\}$. The image of $B_{\alpha,q}$ under this transformation will be denoted by $E_{\alpha,q}(\zeta)$. Of course $E_{\alpha,q}(\zeta)$ depends on the transformation that brought 0 to ζ. Therefore we shall select for every ζ one such transformation and these transformations will be kept fixed throughout the rest of this paper. Then it will be unambiguous to speak about $E_{\alpha,q}(\zeta)$. A parabolic cone $E_{\alpha,q}(\zeta)$ will be called inscribed, if ζ has a neighbourhood U such that $U \cap E_{\alpha,q}(\zeta) \subset D$; it will be called thoroughly inscribed if there is an $\alpha' < \alpha$, $q' < q$ such that $E_{\alpha',q'} \supset E_{\alpha,q}$ is inscribed. Here $q' < q$ means that $q - q'$ is positive definite.

Definition. f has an admissible limit in $\zeta \in \partial D$ if for every thoroughly inscribed parabolic cone $E_{\alpha,q}(\zeta)$, $f(z)$ tends to a finite limit as z goes to ζ in $E_{\alpha,q}(\zeta)$.

Definition. $\zeta \in \partial D$ is an admissible accumulation point of a set $S \subset D$ if there is a thoroughly inscribed $E_{\alpha,q}(\zeta)$ such that ζ is an accumulation point of the set $S \cap E_{\alpha,q}(\zeta)$.

Thus, in the definition of admissible limit (resp. accumulation) the thoroughly inscribed parabolic cones play the same role as the cones played in the definition of angular limit (resp. accumulation).

In [7] E. M. Stein gave a similar definition of admissible convergence. However, instead of thoroughly inscribed parabolic cones $E_{\alpha,q}(\zeta)$ he uses domains of type

$$A_\beta(\zeta) = \{z \in D: |\langle\langle z - \zeta, \nu(\zeta)\rangle\rangle| < (1+\beta)\delta_\zeta(z), \ |z-\zeta|^2 < \beta\delta_\zeta(z)\},$$

where $\beta > 0$ and $\delta_\zeta(z)$ is the minimum of the distances of z to ∂D and of z to the tangent plane in ζ. I.e. $\delta_\zeta(z) = \min\{\Delta(z), |\langle z-\zeta, \nu(\zeta)\rangle|\}$.

If D is convex, the two definitions are equivalent, for every $A_\beta(\zeta)$ is contained in some thoroughly inscribed $E_{\alpha,q}(\zeta)$ and vice versa (at least near to ζ). However, for a general D only the first half of the above statement is true. Thus our admissible domains are larger than the domains $A_\beta(\zeta)$; therefore if in a point the admissible limit exist according to our definition, then it exists also in the sense of [7].

We shall briefly sketch why every $A_\beta(\zeta)$ is contained in some $E_{\alpha,q}(\zeta)$. To this end we shall suppose that $\zeta = 0$, $\nu(\zeta) = (-1, 0, ..., 0)$ and $E_{\alpha,q}(0) = B_{\alpha,q}$. Let

$$\{z: \operatorname{Re} z_1 = Q_0\,(\operatorname{Im} z_1, z')\}$$

be the osculating quadric of ∂D at 0, $q_0(z') = Q_0(0, z')$. If $Q < Q_0$ is another quadratic form, then

$$C_Q = \{z: \operatorname{Re} z_1 > Q\,(\operatorname{Im} z_1, z')\}$$

contains D (at least a part of D near to 0); furthermore, it is easy to see that $B_{\alpha,q}$ is thoroughly inscribed if and only if $q > q_0$.

Suppose now that $z \in A_\beta(0)$, i.e.

$$|z_1| < (1+\beta)\delta_0(z), \quad |z|^2 < \beta\delta_0(z),$$

where $\delta_0(z) = \min\{\Delta(z), |\operatorname{Re} z_1|\}$. From this it follows at once that

$$|\operatorname{Im} z_1|/(1+\beta) < |\operatorname{Re} z_1|. \tag{23}$$

On the other hand

$$\Delta(z) \leqslant \delta(z, \partial C_Q) \sim \operatorname{Re} z_1 - Q\,(\operatorname{Im} z_1, z'),$$

as $z \in D$ goes to 0. Hence for $z \in A_\beta(0)$ near to the origin we have

$$|z|^2 < \beta\Delta(z) < 2\beta\,(\operatorname{Re} z_1 - Q\,(\operatorname{Im} z_1, z')),$$

whence $Q\,(\operatorname{Im} z_1, z') + |z'|^2/(2\beta) < \operatorname{Re} z_1$. In view of $Q\,(\operatorname{Im} z_1, z') = Q(0, z') + o(1)|\operatorname{Re} z_1|$ as $z \to 0$, it follows that

$$(1+o(1))\,(Q(0, z') + |z'|^2/(2\beta)) < \operatorname{Re} z_1.$$

Choosing $Q < Q_0$ very near to Q_0, the left-hand side here will be $> q(z') > q_0(z')$; thus $q(z') < \operatorname{Re} z_1$. This latter inequality together with (23) implies $A_\beta(0) \subset B_{\alpha,q}$ with $\alpha = (1+\beta)^{-1}$.

Q.E.D.

At the same time, as we have already remarked, $B_{\alpha,q}$ is not necessarily contained in some $A_\beta(0)$. For suppose that with the previous notations q_0 is indefinite. Choose a $q > q_0$ also indefinite, and let w' be such that $q(w') < 0$. Then

$$\overline{B_{\alpha,q}} \cap \{z\colon \operatorname{Re} z_1 = 0\} \ni (0, w').$$

However, for any $\beta > 0$

$$\overline{A_\beta(0)} \cap \{z\colon \operatorname{Re} z_1 = 0\} = \{0\}.$$

This proves that no $A_\beta(0)$ can contain $B_{\alpha,q}$ (not even the part of $B_{\alpha,q}$ near to 0).

In establishing the existence of an admissible limit a.e. on ∂D, the bulk of the work will be the investigation of the boundary behaviour of the zero set. We shall prove that the admissible accumulation points of the zeros constitute a set of zero measure on ∂D. For the proof we shall need an eccentric version of Proposition 1. In this version we shall consider a function meromorphic on a domain which is small in one direction and alongated in other directions; and the function will be known to vanish at some point and not to vanish at certain other points. In the next chapter we shall estimate the area of the zero set of such a function (see Lemma 6).

7. Quantitative properties of analytic sets

Our starting point will be the following result of I. L. Ronkin (see [4], p. 371). Let g be analytic in the unit ball of $\mathbf{C}^n$. Suppose $g(0) \neq 0$, and introduce the counting functions

$$N(r) = \frac{1}{(2n-2)\,\omega_{2n-2}} \int_{\{z:\, g(z)=0,\, |z|<r\}} \left(\frac{1}{|z|^{2n-2}} - \frac{1}{r^{2n-2}} \right) d\mu(z),$$

$$N_1(r) = \frac{1}{2\pi} \sum_{\{t:\, g(t,0,\ldots,0)=0,\, |t|<r\}} \log \frac{r}{|t|}$$

for $0 \le r \le 1$. (In the notation of [4], $N_g(r)$ and $N_g(r, \varkappa)$ with $\varkappa = (1, 0, \ldots, 0)$.) According to [4], for every $\delta > 1$

$$N_1(1/\delta) \le K_{14} \frac{\delta^{2n}}{(\delta^2 - 1)^{n-1}} N(1), \tag{24}$$

with a constant $K_{14} > 0$ depending only on n.

A sequence of easy consequences follows.

LEMMA 4. *If g is holomorphic in the unit ball of $\mathbf{C}^n$ and vanishes here in a point z^0 then*

$$\mu\{z\colon g(z) = 0\} > K_{15}(1 - |z^0|)^{n-1}, \tag{25}$$

with $K_{15} > 0$ depending only on n.

Proof. We may suppose that z^0 is the zero nearest to the origin and also that $z^0=(|z^0|, 0, ..., 0)$. If $|z^0|\leqslant\frac{1}{2}$ then (25) follows immediately from Proposition 1. Otherwise (24) can be applied with $\delta=2/(1+|z^0|)$:

$$(1-|z^0|)\,\mu\{z: g(z)=0\}\geqslant K_{16}\int_{\{z:\,g(z)=0\}}(|z|^{2-2n}-1)\,d\mu(z)=K_{17}\cdot N(1)$$
$$\geqslant K_{18}(1-|z^0|)^{n-1}N_1\left(\frac{1+|z_0|}{2}\right)$$
$$\geqslant\frac{K_{18}}{2\pi}(1-|z^0|)^{n-1}\log\frac{1+|z^0|}{2|z^0|}>K_{15}(1-|z^0|)^n.$$

Remark. If g is holomorphic in the ball $B=\{z\in\mathbf{C}^n\colon |z-z^1|<R\}$ instead of the unit ball, and vanishes in $z^0\in B$ then it follows that

$$\mu\{z: g(z)=0\}>K_{15}R^{n-1}\delta(z^0,\partial B)^{n-1}.$$

LEMMA 5. *Let h be holomorphic in the domain*

$$G=\{z\in\mathbf{C}^n\colon |z_j|<1\quad (j=1,...,n),\quad \operatorname{Re}z_1>0\}.$$

Suppose that $h(a, 0, ..., 0)=0$ with some a, $0<a<\frac{1}{2}$, but h has no zero in $\{z\colon \operatorname{Re}z_1\geqslant 10^4a\}$. Then

$$\mu\{z: h(z)=0\}>K_{19},$$

where $K_{19}>0$ depends only on n.

Proof. If a is not very small, say $a\geqslant 10^{-9}$, then the claim follows from Proposition 1. Suppose therefore that $a<10^{-9}$, and consider the ball

$$B=\left\{z\in\mathbf{C}^n\colon \left|z-\left(\frac{1}{10^5a}, 0, \dots, 0\right)\right|<\frac{1}{10^5a}\right\}.$$

It intersects the hyperplane $\{z\colon \operatorname{Re}z_1=10^4a\}$ in a ball of radius

$$\sqrt{10^4a\left(\frac{2}{10^5a}-10^4a\right)}<1;$$

in particular, this intersection is contained in G. More generally, $\{z\in B\colon \operatorname{Re}z_1\leqslant 10^4a\}\subset G$. Choose now two domains B_1, B_2 such that $B_1\cup B_2=B$ and

$$\{z\colon z\in B,\ \operatorname{Re}z_1\leqslant 10^4a\}\subset B_1\subset G,$$
$$\{z\colon z\in B,\ \operatorname{Re}z_1\geqslant 10^4a\}\subset B_2,$$

and h does not vanish on $B_1 \cap B_2$. It is possible to find B_1, B_2 with these properties, because h is nowhere zero on the set $\{z \in B: \operatorname{Re} z_1 = 10^4 a\}$.

$B = B_1 \cup B_2$ is a convex domain, so that the multiplicative Cousin problem with the data $h_{21}(z) = h(z)$ $(z \in B_1 \cap B_2)$ can be solved. There are nonvanishing h_j:s on B_j such that $h_{21} = h_2/h_1$ on $B_1 \cap B_2$. Then

$$g(z) = \begin{cases} h_1(z) \cdot h(z) & z \in B_1 \\ h_2(z) & z \in B_2 \end{cases}$$

is holomorphic on B and its zero set is contained in that of h. By the remark following the previous lemma,

$$\mu\{z: h(z) = 0\} \geqslant \mu\{z: g(z) = 0\} > K_{15}(10^5 a)^{-(n-1)} a^{n-1} = K_{19}.$$

The final result of this chapter is

LEMMA 6. *Let $w \in \mathbf{C}^n$, v a unit vector in $\mathbf{C}^n$, $0 < b, c$, $2b \leqslant c$;*

$$A = \{z \in \mathbf{C}^n: |\langle\langle z-w, v\rangle\rangle| < b, |z-w| < c\}.$$

Let g be a meromorphic function on A vanishing at w. Denote by N_g that connected component of the zero set of g which contains w. If

$$\bar{N}_g \cap \{z \in A: \langle z-w, v\rangle = \tfrac{2}{3}b\} = \emptyset \tag{26}$$

then

$$\mu(N_g) > K_{20}\, c^{2n-2}$$

with $K_{20} > 0$ depending only on n.

Proof. It will be convenient to assume that $v = (1, 0, ..., 0)$ and $w = 10^{-2}b$. Then $\langle\langle z-w, v\rangle\rangle = z_1 - 10^{-2}b$. Since A is convex, the multiplicative Cousin problem is solvable on it. From this we infer two facts. First, g can be represented as the quotient of two locally coprime holomorphic functions; therefore it will be enough to prove the lemma for a holomorphic g. Similarly, it can also be assumed that the zero set of g is connected, and thus coincides with N_g. Then for any zero z of g we have $\operatorname{Re} z_1 < b(\tfrac{2}{3} + 10^{-2})$.

Consider the domain

$$T = \{z \in \mathbf{C}^n: |z| < c/2, \quad 0 < \operatorname{Im} z_1 < \operatorname{Re} z_1\}.$$

We claim that there is a $\tilde{g}$ holomorphic on T such that $\{z \in T: \tilde{g}(z) = 0\} = N_g \cap T$. Indeed,

$$\{z \in T: \operatorname{Re} z_1 - 10^{-2}b \leqslant \tfrac{2}{3}b\} \subset A,$$

and g does not vanish on the set

$$\{z \in T: \operatorname{Re} z_1 - 10^{-2} b = \tfrac{2}{3} b\};$$

so that (again solving a Cousin-II problem on T) it is possible to construct a $\tilde{g}$ with the required properties.

Next we define a biholomorphic mapping $z \to \zeta$ from T given by

$$\zeta_1 = 10 z_1^2 / c^2, \quad \zeta_k = 10 n z_k / c \quad (k = 2, \ldots, n).$$

Let $h(\zeta) = \tilde{g}(z)$. Then h will be defined on a superset of

$$G = \{\zeta \in \mathbf{C}^n: |\zeta_j| < 1 \quad (j = 1, \ldots, n), \ \operatorname{Re} \zeta_1 > 0\}.$$

Putting $a = 10^{-3} b^2 / c^2$, $h(a, 0, \ldots, 0) = 0$, but $h(\zeta)$ does not vanish if $\operatorname{Re} \zeta_1 \geqslant 10^4 a$. Furthermore

$$|d\zeta_1| = 20 |z_1| \, |dz_1| / c^2 \leqslant 10 n |dz_1| / c$$
$$|d\zeta_k| = 10 n |dz_k| / c \quad (k = 2, \ldots, n).$$

Applying Lemma 5 we obtain the desired result:

$$\mu(N_g) \geqslant \mu(\{z \in T: \tilde{g}(z) = 0\}) \geqslant \left(\frac{c}{10n}\right)^{2n-2} \mu(\{\zeta \in G: h(\zeta) = 0\}) > K_{20} c^{2n-2}.$$

8. Admissible behaviour of the zero set

In this chapter we shall return to our domain D and the function f on it. We shall adopt the notations and assumptions of Theorem 1 and Chapter 6.

LEMMA 7. *The set of admissible accumulation points of N constitute a set of zero measure on ∂D.*

In the course of the proof domains of type

$$A(z; b, c) = \{w \in \mathbf{C}^n: |\langle\langle w - z, \nu(z) \rangle\rangle| < b, \ |w - z| < c\}$$

will play an important role ($z \in D \setminus D_1$, $0 < b$, $0 < c$). They will be called admissible balls. First we shall record two properties of these "balls".

LEMMA 8. *If $b, c > 0$ are chosen so that b/c^2 is big enough then the following holds:*

(a) *If $\Delta(z) \geqslant \Delta(z')$ and*

$$A(z; b\Delta(z), c\Delta(z)^{\frac{1}{2}}) \cap A(z'; b\Delta(z'), c\Delta(z')^{\frac{1}{2}}) \neq \emptyset \tag{27}$$

then

$$A(z; 4b\Delta(z), 4c\Delta(z)^{\frac{1}{2}}) \supset A(z'; b\Delta(z'), c\Delta(z')^{\frac{1}{2}}).$$

(b) *If* $\Delta(z) \geqslant \frac{5}{4}\Delta(z')$ *and* (27) *holds then*

$$A(z; 100b\Delta(z), \; 100c\Delta(z)^{\frac{1}{2}}) \supset A(z'; 100b\Delta(z'), \; 100c\Delta(z')^{\frac{1}{2}}).$$

Proof. We shall prove only (b), since (a) can be treated in a similar manner. Thus let

$$w \in A(z; b\Delta(z), c\Delta(z)^{\frac{1}{2}}) \cap A(z'; b\Delta(z'), c\Delta(z')^{\frac{1}{2}}),$$

and suppose that

$$Z \in A(z'; 100b\Delta(z'), \; 100c\Delta(z')^{\frac{1}{2}}).$$

Then

(28) $$|z-Z| \leqslant |z-w| + |w-z'| + |z'-Z| < c\Delta(z)^{\frac{1}{2}} + c\Delta(z')^{\frac{1}{2}} + 100c\Delta(z')^{\frac{1}{2}} < 100c\Delta(z)^{\frac{1}{2}}.$$

Furthermore $|z-z'| < 2c\Delta(z)^{\frac{1}{2}}$, hence

$$|\nu(z) - \nu(z')| < K_{21} c\Delta(z)^{\frac{1}{2}},$$

where K_{21} depends only on D. Therefore

$$\begin{aligned}
|\langle\langle Z-z, \nu(z)\rangle\rangle| &\leqslant |\langle\langle Z-w, \nu(z)\rangle\rangle| + |\langle\langle w-z, \nu(z)\rangle\rangle| \\
&\leqslant |\langle\langle Z-w, \nu(z')\rangle\rangle| + |\langle\langle Z-w, \nu(z')-\nu(z)\rangle\rangle| + b\Delta(z) \\
&\leqslant |\langle\langle Z-z', \nu(z')\rangle\rangle| + |\langle\langle w-z', \nu(z')\rangle\rangle| + 100K_{21}c^2\Delta(z) + b\Delta(z) \\
&\leqslant \{(101 \cdot \tfrac{4}{5} + 1)b + 100K_{21}c^2\}\Delta(z) < 100b\Delta(z)
\end{aligned}$$

if b/c^2 is big enough. This latter and (28) prove (b) of the lemma.

Proof of Lemma 7. Choose positive numbers $\eta_1, \eta_2, \eta_3, \eta_4$, all of them less than 1, and for every $\zeta \in \partial D$ pick out thoroughly inscribed parabolic cones $E_{\alpha,q}(\zeta)$ (here α and q may depend on ζ) such that

1. α and the moduli of the eigenvalues of q lie between η_1 and $1/\eta_2$;

2. Denoting by $U(\zeta, \eta_3)$ the η_3 neighbourhood of ζ, we should have $U(\zeta, \eta_3) \cap E_{\alpha',q'} \subset D$ with $\alpha' = \alpha - \eta_4$ and $q'(z') = q(z') - 2\eta_4|z'|^2$.

These conditions mean that the parabolic cones $E_{\alpha,q}(\zeta)$ are "uniformly" thoroughly inscribed. The lemma would follow if we could prove that for any choice of $\eta_1, ..., \eta_4$, the set

$$\Theta = \{\zeta \in \partial D : \zeta \in \overline{E_{\alpha,q} \cap N}\}$$

is of zero measure on ∂D. Fix therefore $\eta_1, ..., \eta_4$, and the parabolic cones $E_{\alpha,q}(\zeta)$ with them. From now on in this proof all constants $K_{22}, ...$ may depend beside D and f on $\eta_1, ..., \eta_4$ as well. We shall show that Θ is indeed of measure zero.

Step 1. We claim that there are constants K_{22} and K_{23} with the following property: If $z \in D$ is near to $\zeta \in \partial D$ and $z \in E_{\alpha,q}(\zeta)$, then

$$\zeta \in A(z;\, K_{22}\Delta(z),\, K_{23}\Delta(z)^{\frac{1}{2}}).$$

To prove this, we may suppose that $\zeta = 0$ and $\nu(\zeta) = (-1, 0, ..., 0)$, furthermore, that $E_{\alpha,q}(\zeta) = B_{\alpha,q}$. Let $Q\,(\operatorname{Im} z_1, z')$ be such a quadratic form that $Q(0, z') = q(z') - \eta_4 |z'|^2$, and putting

$$C_Q = \{z\colon \operatorname{Re} z_1 > Q\,(\operatorname{Im} z_1, z')\},$$

we have $C_Q \cap U(0, \eta_3) \subset D$. Then for z as in the claim,

$$\Delta(z) \geqslant \delta(z, \partial C_Q) \sim \operatorname{Re} z_1 - Q\,(\operatorname{Im} z_1, z') > (1 + o(1))\operatorname{Re} z_1 - Q(0, z') > \eta_4 |z'|^2/2,$$

as $z \to 0$. Thus $|z'|^2 < 2\Delta(z)/\eta_4$, if z is near to 0. Hence $|Q(0, z')| < K_{24}\Delta(z)$, and $|\operatorname{Re} z_1| < K_{25}\Delta(z)$.

Consequently

$$|z - \zeta| = |z| < K_{26}\Delta(z)^{\frac{1}{2}}, \tag{29}$$

whence $|\nu(z) - \nu(\zeta)| < K_{27}\Delta(z)^{\frac{1}{2}}$, and

$$|\langle\langle \zeta - z, \nu(z)\rangle\rangle| \leqslant |\langle\langle z, \nu(z) - \nu(\zeta)\rangle\rangle| + |\langle\langle z, \nu(\zeta)\rangle\rangle| < K_{26}K_{27}\Delta(z) + K_{25}\Delta(z). \tag{30}$$

(29) and (30) prove our claim.

Choose therefore K_{22}, K_{23} accordingly. It can even be assumed that K_{22}/K_{23}^2 is so large that Lemma 8 holds with $b = 10^6 K_{22}$, $c = 10^6 K_{23}$ (and therefore with any $b = \xi K_{22}$, $c = \xi K_{23}$ if $0 < \xi < 10^6$). Let

$$A(z) = A(z;\, K_{22}\Delta(z),\, K_{23}\Delta(z)^{\frac{1}{2}}),$$

and

$$\xi A(z) = A(z;\, \xi K_{22}\Delta(z),\, \xi K_{23}\Delta(z)^{\frac{1}{2}}),$$

where $\xi > 0$.

By what has just been proved,

$$\Theta \subset \bigcap_{\varepsilon > 0} \bigcup \{A(z) \cap \partial D\colon z \in N \setminus D_\varepsilon\}. \tag{31}$$

We shall show that the right hand side in (31) is of zero measure on ∂D. To this end, let d_k be a sequence of positive numbers, $d_1 < 1$, $d_{k+1} < 10^{-1} d_k$. Define

$$V_k = \bigcup \{A(z): z \in \bar{N},\ d_{k+1} < \Delta(z) \leqslant d_k\},$$
$$W_k = \bigcup \{A(z): z \in \bar{N},\ \Delta(z) = d_k\}.$$

Then the right hand side of (31) is

$$\limsup_{k\to\infty} V_k \cap \partial D = \bigcap_{m=1}^{\infty} \bigcup_{k>m} V_k \cap \partial D. \tag{32}$$

If we could prove that

$$\sum_{k=1}^{\infty} \Omega(V_k \cap \partial D) < \infty \tag{33}$$

then, by the Borel-Cantelli lemma it would follow that (32) is of zero measure. Next we shall prove that

$$\sum_{k=1}^{\infty} \Omega(V_k \cap \partial D) \leqslant K_{28} \sum_{k=1}^{\infty} \Omega(W_k \cap \partial D) + K_{29}, \tag{34}$$

and then that

$$\sum_{k=1}^{\infty} \Omega(W_k \cap \partial D) < \infty, \tag{35}$$

if $d_k \to 0$ very rapidly.

Step 2. (Proof of (34).) Fix a large k and let

$$S = \{A(z): z \in \bar{N},\ d_{k+1} < \Delta(z) \leqslant d_k\}.$$

We shall select a maximal disjoint subsystem $T = \{A(z^1), A(z^2), \ldots\}$ of S in the following way. Let $A(z^1) \in S$ be such that $\Delta(z^1)$ is maximal, $A(z^2) \in S$ be such the $A(z^1) \cap A(z^2) = \emptyset$ and again $\Delta(z^2)$ is as large as it can be, and so on. If $z^1, z^2, \ldots, z^{j-1}$ are already chosen, let $A(z^j) \in S$ be disjoint from all $A(z^i)$ $(i<j)$, and $\Delta(z^j)$ be maximal among the possible candidates. In this way a finite sequence $\{z^j: j \in J\}$ is obtained. By maximality, for any $A(z) \in S$ there is a $j \in J$ such that $A(z) \cap A(z') \neq \emptyset$. Therefore by Lemma 10.

$$V_k \subset \bigcup \{4A(z^j): j \in J\},$$

so that

$$\Omega(V_k \cap \partial D) \leqslant K_{30} \sum_{j \in J} \Omega(A(z^j) \cap \partial D). \tag{36}$$

To estimate the right hand side, we shall connect it via Lemma 6 with $\int \Delta(z)\,d\mu(z)$, the integration being taken over $\{z \in N: d_{k+1} < \Delta(z) \leqslant d_k\}$.

Let

$$A'(z^j) = A(z^j, \Delta(z^j)/2, \gamma\Delta(z^j)^{\frac{1}{2}})$$

where $0 < \gamma < 1$ is chosen so that $\delta(A'(z^j), \partial D) > \Delta(z^j)/4$. Thus γ depends only on D. Clearly $A'(z^j) \subset A(z^j)$. If condition (26) of Lemma 6 is met with $A = A'(z^j)$, $\nu = -\nu(z^j)$, $w = z^j$, $g = f$, then by this lemma

$$\int_{A'(z^j)\cap N} \Delta(z)\,d\mu(z) \geqslant \tfrac{1}{4}\Delta(z^j)\,\mu(A'(z^j)\cap N) > K_{31}\,\Delta(z^j)^n > K_{32}\,\Omega(A(z^j)\cap \partial D). \tag{37}$$

(The fact that z^j may be an indeterminate point instead of being a zero does not affect the validity of the lemma.)

This is exactly the type of estimate we need. However, there is no reason to believe that the said condition is met for all $j \in J$. Therefore we shall select certain admissible balls $A'(z^j)$ for which we shall know that Lemma 6 can be applied.

Take first a z^j such that $\Delta(z^j) \leqslant \frac{4}{5}d_k$. It is easy to see that Lemma 6 can be applied in $A'(z^j)$ (always with $A = A'(z^j)$, $\nu = -\nu(z^j)$, $w = z^j$, $g = f$) unless there is a $z \in \bar{N} \cap A'(z^j)$ such that $\Delta(z) = \frac{5}{4}\Delta(z^j)$. Indeed, if there is no such a z, the connected component of $N \cap A'(z^j)$ containing z^j will be a subset of $\{z \in A'(z^j): \Delta(z) < \frac{5}{4}\Delta(z^j)\}$, thus will be disjoint from $\{z: \langle z - z^j, -\nu(z^j)\rangle = \frac{1}{3}\Delta(z^j)\}$ (at least for d_k small enough). Suppose therefore that there is such a z and see what this assumption implies.

Of course $\Delta(z) \leqslant d_k$, so that $A(z) \in S$; however $A(z)$ and $A(z^j)$ are not disjoint, thus $A(z)$ was left out from the maximal disjoint system T. This could happen only because there was an $h \in J$ such that $\Delta(z^h) \geqslant \Delta(z) = \frac{5}{4}\Delta(z^j)$ and

$$A(z^h) \cap A(z) \neq \emptyset.$$

By Lemma 8

$$4A(z^h) \supset A(z) \ni z,$$

in particular

$$4A(z^h) \cap 4A(z^j) \neq \emptyset.$$

Again by Lemma 8

$$400A(z^h) \supset 400A(z^j). \tag{38}$$

Take now a z^j with $d_k > \Delta(z^j) \geqslant \frac{4}{5}d_k$. Again, condition (26) of Lemma 6 is met in $A'(z^j)$ unless there is a $z \in \bar{N} \cap A'(z^j)$ such that $\Delta(z) = d_k$. Suppose that, on the contrary, there is such a z. Then $A(z^j) \cap A(z) \neq \emptyset$, thus $A(z) \notin T$. Therefore there is an $i \in J$ such that $\Delta(z^i) = d_k$ and $A(z^i) \cap A(z) \neq \emptyset$. Hence

$$4A(z^i) \supset A(z) \ni z \in A(z^j),$$

thus

$$400A(z^i) \cap 400A(z^j) \neq \emptyset.$$

By Lemma 8

$$1600A(z^i) \supset 400A(z^j). \tag{39}$$

Motivated by all these, call an index $j \in J$ irrelevant if $\Delta(z^j) < d_k$ and either there is an $h \in J$ such that $\Delta(z^j) < \Delta(z^h)$ and (38) holds or there is an $i \in J$ such that $\Delta(z^i) = d_k$ and (39) holds. Otherwise j will be called relevant; their set will be denoted J_0.

Relevant indices are important to us by two reasons. First, by what has just been demonstrated, for a relevant index j such that $\Delta(z^j) < d_k$, Lemma 6 can be applied to yield (37). Secondly, for any $j \in J$ there is a relevant $i \in J_0$ such that (39) holds. Therefore

$$\begin{aligned} V_k \cap \partial D &\subset \bigcup \{\partial D \cap 1600A(z^i) \colon i \in J_0\} \\ &= \bigcup \{\partial D \cap 1600A(z^j) \colon j \in J_0,\ \Delta(z^j) < d_k\} \cup \bigcup \{\partial D \cap 1600A(z^i) \colon i \in J_0,\ \Delta(z^i) = d_k\}. \end{aligned}$$

The sets $A(z^i)$ being disjoint, the second union is of measure less than $K_{28}\Omega(W_k \cap \partial D)$. For the members of the first union (37) holds; considering that the sets $A'(z^j)$ are disjoint as well, we obtain

$$\begin{aligned} &\Omega(\bigcup \{\partial D \cap 1600A(z^j) \colon j \in J_0,\ \Delta(z^j) < d_k\}) \\ &\quad \leqslant K_{33} \sum_{j \in J_0,\, \Delta(z^j) < d_k} \Delta(z^j)^n \leqslant K_{34} \sum_{j \in J_0} \int_{A'(z^j) \cap N} \Delta(z)\, d\mu(z) \leqslant K_{34} \int_{N \cap D_{d_{k+2}} \setminus D_{d_{k-1}}} \Delta(z)\, d\mu(z). \end{aligned}$$

All added up

$$\Omega(V_k \cap \partial D) \leqslant K_{29}\Omega(W_k \cap \partial D) + K_{34} \int_{N \cap D_{d_{k+2}} \setminus D_{d_{k-1}}} \Delta(z)\, d\mu(z),$$

which, on account of the Blaschke condition, implies (34).

Step 3. Let now the sequence d_k be defined by

$$\log d_{k+1} = \gamma d_k^{-1/2} \log d_k.$$

(γ was the constant figuring in the definition of $A'(z)$.)

Fix again k, and for $d_k \leqslant \Delta(z) \leqslant d_{k-1}$ define

$$\xi \tilde{A}(z) = A\left(z;\, K_{22}\,\xi\Delta(z),\, K_{23}\,\xi\Delta(z)^{1/2}\frac{\log \Delta(z)}{\log d_k}\right),$$

$$\tilde{A}'(z) = A\left(z;\, \Delta(z)/2,\, \gamma\Delta(z)^{1/2}\frac{\log \Delta(z)}{\log d_k}\right).$$

It is easy to check that an analoguous result to Lemma 8 holds: if $0<\xi<10^6$ and $\xi\tilde{A}(z) \cap \xi\tilde{A}(z') \neq \emptyset$, then

(a) $4\xi\tilde{A}(z) \supset \xi\tilde{A}(z')$ if $\Delta(z) \geqslant \Delta(z')$;

(b) $100\xi\tilde{A}(z) \supset 100\xi\tilde{A}(z')$ if $\Delta(z) \geqslant \frac{5}{4}\Delta(z')$.

Putting $\tilde{A}(z) = 1\tilde{A}(z)$, we have

$$W_k = \bigcup\{\tilde{A}(z) \colon z \in \bar{N},\ \Delta(z) = d_k\}.$$

Let furthermore

$$X_k = \bigcup\{\tilde{A}(z) \colon z \in \bar{N},\ \Delta(z) = d_{k-1}\}.$$

These formulae may seem to imply $X_{k+1} = W_k$ but this is not the case. It should namely be remembered that the definition of $\tilde{A}(z)$ depends on k.

It is easy to see that $\sum \Omega(X_k \cap \partial D) < \infty$. Indeed, denoting by $B(z)$ the ball of radius $(K_{22}+K_{23}/\gamma)\Delta(z)$ around z,

$$X_k \subset Y_k = \{B(z) \colon z \in \bar{N},\ \Delta(z) = d_{k-1}\},$$

and in the proof of Lemma 1 it has already been shown that $\sum \Omega(Y_k \cap \partial D) < \infty$. Thus in order to prove (35) it will be sufficient to prove

$$\sum_{k=2}^{\infty} \Omega(W_k \cap \partial D) \leqslant K_{35} \sum_{k=2}^{\infty} \Omega(X_k \cap \partial D) + K_{36}. \tag{40}$$

Step 4. (Proof of (40).) (Along the same lines as the proof of (34).) For fixed k let

$$\tilde{S} = \{\tilde{A}(z) \colon z \in N,\ d_k \leqslant \Delta(z) \leqslant d_{k-1}\}.$$

Select a maximal disjoint subfamily $\tilde{T} = \{\tilde{A}(z^j) \colon j \in J\}$ of $\tilde{S}$ as in Step 2, i.e. if $\tilde{A}(z^i)$ is already chosen for $i<j$, let $\tilde{A}(z^j)$ be disjoint from all $\tilde{A}(z^i)$ $(i<j)$, and if there are several candidates, choose one with maximal $\Delta(z^j)$.

Again, call an index $j \in J$ irrelevant if there is an $h \in J$ such that $\Delta(z^h) > \Delta(z^j)$ and

$$400\tilde{A}(z^h) \supset 400\tilde{A}(z^j)$$

or there is an $i \in J$ such that $\Delta(z^i) = d_{k-1}$ and

$$1600\tilde{A}(z^i) \supset 400\tilde{A}(z^j). \tag{41}$$

The relevant indices constitute the set J_0; for every $j \in J$ there is a relevant i such that (41) holds. Furthermore, if $i \in J_0$ and $\Delta(z^i) < d_k$, then Lemma 6 can be applied with the cast $A = \tilde{A}'(z^i)$, $\nu = -\nu(z^i)$, $w = z^i$, $g = f$ to yield

$$\int_{\tilde{A}'(z^i)\cap N} \Delta(z)\, d\mu(z) \geqslant \frac{\Delta(z^i)}{4}\, \mu(\tilde{A}'(z^i) \cap N) > K_{34}\, \Delta(z^i)^n \left(\frac{\log \Delta(z^i)}{\log d_k}\right)^{n-1} > K_{38}\, \Omega(\tilde{A}(z^i) \cap \partial D).$$

Thus

$$\begin{aligned}\Omega(W_k \cap \partial D) &\leqslant K_{39} \sum_{i \in J_0} \Omega(\tilde{A}(z^i) \cap \partial D)\\ &\leqslant K_{35}\, \Omega(X_k \cap \partial D) + K_{39} \sum_{i \in J_0,\, \Delta(z^i) < d_{k-1}} \Omega(\tilde{A}(z^i) \cap \partial D)\\ &\leqslant K_{35}\, \Omega(X_k \cap \partial D) + K_{40} \sum_{i \in J_0} \int_{\tilde{A}'(z^i)\cap N} \Delta(z)\, d\mu(z)\\ &\leqslant K_{35}\, \Omega(X_k \cap \partial D) + K_{40} \int_{N \cap D_{d_{k+1}} \setminus D_{d_{k-2}}} \Delta(z)\, d\mu(z).\end{aligned}$$

In view of the Blaschke condition this implies (40). The proof of Lemma 7 is complete.

9. Admissible limit

Again, we shall use the notations of Theorem 1 and Chapter 6.

THEOREM 3. *Under the assumptions of Theorem 1, f has a (finite) admissible limit in almost every point of ∂D.*

Proof. Of course, Lemma 7 remains true if N is replaced by $N_w = \{z \in D\colon f(z) = w\}$, where w is any complex number. Choose a countable dense set of complex numbers $\{w_1, w_2, \ldots\}$, and consider a point $\zeta \in \partial D$ which is not an admissible accumulation point of any N_{w_p}, and where f has a nontangential limit. Almost all points of ∂D are such. We are going to show that in ζ the admissible limit exists.

Suppose that there is no admissible limit. Then there are two thoroughly inscribed parabolic cones $E_{\alpha,q}(\zeta)$ and $E_{\alpha',q'}(\zeta)$ such that $\alpha > \alpha'$, $q > q'$, and a sequence $\{z^{(k)}\} \subset E_{\alpha,q}(\zeta)$, $z^{(k)} \to \zeta$, such that

$$\lim_{k\to\infty} f(z^{(k)}) = L \neq L' = \lim_{\varepsilon \to +0} f(\zeta - \varepsilon\nu(\zeta)).$$

L may eventually be infinite.

For the sake of simplicity we shall assume that $\zeta=0$, $\nu(\zeta)=(-1,0,\ldots,0)$, $E_{\alpha,q}(\zeta)=B_{\alpha,q}$, $E_{\alpha',q'}(\zeta)=B_{\alpha',q'}$. Choose $a_k>0$ so that putting

$$Z^{(k)}=(z_1^{(k)}/a_k^2,\, z_2^{(k)}/a_k,\, \ldots,\, z_n^{(k)}/a_k)$$

we have $|Z^{(k)}|=1$. Then $a_k\to 0$ as $k\to\infty$. Define

$$f_k(Z)=f_k(Z_1,\ldots,Z_n)=f(a_k^2Z_1,\, a_kZ_2,\ldots,a_kZ_n).$$

Observe that the transformation

$$(Z_1,Z_2,\ldots,Z_n)\mapsto(a_k^2Z_1,\, a_kZ_2,\ldots,a_kZ_n)$$

leaves standard parabolic cones invariant. Thus, for large k, $f_k(Z)$ will be defined for $Z\in B_{\alpha',q'}\cap G$, where G is the ball of radius 2 around the origin. Moreover, again for k large, f_k will not assume the distinct values w_1, w_2, w_3 on $B_{\alpha',q'}\cap G$. So $\{f_k\}$ is a normal family and there is a subsequence f_{k_s} converging to a limit function g uniformly on compact subsets of $B_{\alpha',q'}\cap G$.

Now

$$g(1,0,\ldots,0)=\lim_{k\to 0} f(a_k^2,0,\ldots,0)=L',$$

but g is not constant, since the set $\{Z^{(k)}\colon k\in\mathbf{N}\}$ is relatively compact in $B_{\alpha',q'}\cap G$, and $f_k(Z^k)\to L\neq L'$. Therefore the range of g is open, and so it contains some w_p. By Hurwitz's theorem f_k must assume w_p if k is large enough; in other words, f must assume w_p in $B_{\alpha',q'}$, however near to $\zeta=0$. This contradicts the assumption that ζ was not an admissible accumulation point of the set N_{w_p}, and this contradiction proves the theorem.

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Received December 15, 1978
Received in revised form May 15, 1979