ELLIPTIC SYSTEMS IN $H_{s,\delta}$ SPACES ON MANIFOLDS WHICH ARE EUCLIDEAN AT INFINITY

BY

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1. Introduction

In the subject of global analysis, there is a wealth of results in the case of a compact manifold which do not depend on the choice of a riemannian structure on the manifold, but in the non-compact case much less is known and moreover the results depend on the choice of a riemannian structure.

In this paper we study elliptic differential systems of order m on non-compact manifolds which are euclidean at infinity, in weighted Sobolev spaces $H_{s,\delta}$. Such a study has been done in weighted Hölder spaces $C_{\beta}^{1,\alpha}$, for equations of order 2 in [4]. On the other hand, M. Cantor has proved [2] closed range and isomorphism theorems for elliptic operators of order m in $\mathbf{R}^n$, in weighted Sobolev spaces $W_{s,\delta}^{p}$, where $p > n/(n-m)$. His paper is based on a work by L. Nirenberg and H. Walker [14] on the null spaces of such operators with continuous coefficients. In the present article we show that this restriction on p is unnecessary. Although we shall treat explicitly only the case $p=2$ which is of special interest since $W_{s,\delta}^{2}=H_{s,\delta}$ is a Hilbert space, the results extend trivially to any $p>1$. The hypotheses on the coefficients which we make, permit the study of nonlinear systems in the same framework.

Our exposition is self-contained, except in as far as it requires knowledge of local elliptic theory and results proved in [14] for operators with continuous coefficients on $\mathbf{R}^n$. The method relies on an improvement, given in § 2, of the imbedding theorem and multiplication lemma for the $W_{s,\delta}^{p}$ spaces. This improvement allows us to have $\delta > -n/p$ instead of $\delta \geqslant 0$ as in [2]. In § 3 some of the elliptic estimates on a compact manifold, with or without boundary, are recalled. In § 4 we extend the elliptic theory on $\mathbf{R}^n$ of [14] to operators with coefficients in the spaces H_{s_k,δ_k}. In § 5 we derive an isomorphism theorem for

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elliptic operators with constant coefficients on $\mathbf{R}^n$. Finally, in § 6, we use these results to study elliptic operators on a euclidean at infinity manifold, deriving theorems on the finite dimensionality of the kernel, the closedness of the range, as well as isomorphism theorems.

Our work finds application to the study of the constraint system in general relativity, extending previous work by M. Cantor [3] and Y. Choquet-Bruhat and J. York [6], as well as to many geometrical and physical problems which have been previously studied on compact manifolds (cf. for example [11]).

2. Properties of the $H_{s,\delta}$ spaces

Let M be a C^∞, connected, manifold with a (positive definite) C^∞ riemannian metric e. Let E be a C^∞ tensor bundle over M. We denote by $\mathcal{D}(E, M)$, or simply by $\mathcal{D}$ when no confusion can arise, the space of C^∞ sections of E with compact support in M, endowed with the Schwartz topology. We denote by $\mathcal{D}'(E, M)$, or simply $\mathcal{D}'$, the dual space of distributions. We denote by D the operator of covariant derivation in the sense of distributions: $D^k f$ is the tensorfield, kth derivative of the tensorfield (section of E) f.

We choose arbitrarily a fixed point O on M and we set

$$\sigma(x) = (1 + (d(O, x))^2)^{1/2}, \tag{2.1}$$

where d is the riemannian distance function on M.

Definition 2.1. C^1_β, $1 \in \mathbf{N}$, $\beta \in \mathbf{R}$, is the Banach space of C^1 sections of E for which the following norm is finite

$$\|f\|_{C^1_\beta} \equiv \sup_{x \in M} \sum_{0 \leqslant k \leqslant 1} \{(\sigma(x))^{\beta+k} |D^k f(x)|\}.$$

Here, $|D^k f(x)|$ denotes the e-norm of the tensor $D^k f$ at x.

Definition 2.2. $H_{s,\delta}$, $s \in \mathbf{N}$, $\delta \in \mathbf{R}$, is the space of all equivalence classes of sections f of E such that for $0 \leqslant k \leqslant s$, $D^k f$ is measurable and $\sigma^{\delta+k}|D^k f|$ is square integrable in the metric e. $H_{s,\delta}$ is a Hilbert space with respect to the inner product

$$(f_1, f_2)_{H_{s,\delta}} = \sum_{0 \leqslant k \leqslant s} \int_M \sigma^{2(\delta+k)} D^k f_1 \cdot D^k f_2 \, d\mu(e),$$

where $D^k f_1 \cdot D^k f_2(x)$ denotes the inner product in the metric e of the tensors $D^k f_1$, $D^k f_2$ at x, and $d\mu(e)$ is the volume element of e. We shall write the norm

$$\|f\|_{H_{s,\delta}} = (f, f)^{1/2}_{H_{s,\delta}} = \left\{ \sum_{0 \leqslant k \leqslant s} \int_M \sigma^{2(\delta+k)} |D^k f|^2 \, d\mu(e) \right\}^{1/2}.$$

The topology of the space $H_{s,\delta}$ does not depend on the choice of the origin point O: different origin points give equivalent norms. If $s \geqslant s_1$, $\delta \geqslant \delta_1$, it obviously holds $H_{s,\delta} \subset H_{s_1,\delta_1}$. Moreover, we have the following:

LEMMA 2.1. *If (M, e) is a complete riemannian manifold and if $s > s_1$, $\delta > \delta_1$, the injection $H_{s,\delta} \to H_{s_1,\delta_1}$ is a compact map.*

Proof. The result is well known when M is $\mathbf{R}^n$ and e the euclidean metric. (Cf. for example [1].)

We shall give here a simple proof that the closed unit ball in $H_{s,\delta}$ is a compact set in H_{s_1,δ_1}, which does not make hypotheses on the sign of δ_1.

We denote by B_R the open ball

$$B_R = \{x \in M \mid d(O, x) < R\}$$

and by χ_R a C^∞ function with support in B_{2R}, equal to 1 in B_R. Let $\{f_n\}$ be a sequence in $H_{s,\delta}$ with $\|f_n\|_{H_{s,\delta}} \leqslant 1$. Since $H_{s,\delta}$ is a Hilbert space, $\{f_n\}$ admits a subsequence, still denoted $\{f_n\}$, which converges weakly to some $f \in H_{s,\delta}$ with $\|f\|_{H_{s,\delta}} \leqslant 1$. Let us set

$$f_n = \chi_R f_n + (1 - \chi_R) f_n.$$

The sequence $\{\chi_R f_n\}$ is bounded in the Sobolev space $H_s(B_{2R})$ as one sees from

$$\|\chi_R f_n\|^2_{H_s(B_{2R})} = \int_{B_{2R}} \sum_{0 \leqslant k \leqslant s} |D^k(\chi_R f_n)|^2 d\mu(e)$$
$$\leqslant c \sum_{0 \leqslant k \leqslant s} \int_{B_{2R}} |D^k f_n|^2 d\mu(e)$$

(c a constant depending only on $\|\chi_R\|_{C^s(B_{2R})}$), which together with

$$\int_{B_{2R}} |D^k f_n|^2 d\mu(e) \leqslant \sup_{B_{2R}} \sigma^{-2(k+\delta)} \int_{B_{2R}} \sigma^{2(k+\delta)} |D^k f_n|^2 d\mu(e)$$

gives

$$\|\chi_R f_n\|_{H_s(B_{2R})} \leqslant c_R \|f_n\|_{H_{s,\delta}}.$$

(M, e) being complete and C^∞, the closure $\bar{B}_{2R}$ of B_{2R} is a compact manifold with C^∞ boundary. Therefore, by the Rellich compactness theorem, $\{\chi_R f_n\}$ admits a subsequence $\{\chi_R f_{n_i}\}$ which converges strongly in $H_{s_1}(B_{2R})$ to some $f_R \in H_s(B_{2R})$. Clearly, $f_R = f$ on B_R. To show that $\{f_{n_i}\}$ converges strongly to f in H_{s_1,δ_1} we write

$$\|f-f_{n_i}\|_{H_{s_1,\delta_1}}=\left\{\int_M \sum_{k=0}^{s_1} \sigma^{2(\delta_1+k)}|D^k(f-f_{n_i})|^2 d\mu(e)\right\}^{1/2}$$
$$\leqslant\left\{\int_{B_R} \sum_{k=0}^{s_1} \sigma^{2(\delta_1+k)}|D^k(f-f_{n_i})|^2 d\mu(e)\right\}^{1/2}$$
$$+R^{\delta_1-\delta}\left\{\int_{M-B_R} \sum_{k=0}^{s_1} \sigma^{2(\delta+k)}|D^k(f-f_{n_i})|^2 d\mu(e)\right\}^{1/2}.$$

The last integral is bounded by

$$\|f\|_{H_{s,\delta}}+\|f_{n_i}\|_{H_{s,\delta}}\leqslant 2.$$

Thus, the last term is bounded by $\varepsilon/2$ if

$$R>(4/\varepsilon)^{1/(\delta-\delta_1)}.$$

When R has been so chosen we take n_i large enough for the first integral to be also less than $\varepsilon/2$, which achieves the proof. □

Remark. The lemma is still true for non-complete manifolds with boundary if $\bar{B}_{2R}$ is compact and has the cone property.

In a similar way we can show

LEMMA 2.2. *If (M, e) is a complete riemannian manifold and if $s>s_1$, $\delta>\delta_1$, for every $\varepsilon>0$ there exists a C such that for every $f\in H_{s,\delta}$:*

$$\|f\|_{H_{s_1,\delta_1}}\leqslant\varepsilon\|f\|_{H_{s,\delta}}+C\|f\|_{H_{0,\delta}}.$$

We now introduce

Definition 2.3. A riemannian manifold (M, e) is euclidean at infinity if there exists a number R_0 such that (1) $M-\bar{B}_{R_0}$ is the union of a finite number of disjoint connected open sets Ω_A, $A=1, ..., p$ diffeomorphic by a diffeomorphism φ_A to $\mathbf{R}^n-\bar{B}$, where $\bar{B}$ is a closed ball of $\mathbf{R}^n$. (2) On each Ω_A, the metric e is the pullback by φ_A of the canonical euclidean metric of $\mathbf{R}^n-\bar{B}$.

Remarks. If (M, e) is euclidean at infinity it is complete if and only if $\bar{B}_{R_0}$ is compact. If (M, e) is complete and euclidean at infinity, the space $\mathcal{D}(E, M)$ is dense in $H_{s,\delta}$ for every $s\in\mathbf{N}$, $\delta\in\mathbf{R}$.

Consider now $\mathbf{R}^n$ together with its euclidean metric and for $0<\varepsilon\leqslant 1$, let φ_ε be the differentiable transformation of $\mathbf{R}^n$ defined by

$$x\mapsto y=\frac{x}{\sigma(x)^{1-\varepsilon}}, \tag{2.2}$$

where

$$\sigma(x) = (1+|x|^2)^{1/2} \tag{2.3}$$

is the function defined for general manifolds by (2.1) (φ_1 is the identity transformation of $\mathbf{R}^n$). The Jacobian matrix of φ_ε is

$$\frac{\partial y^i}{\partial x^j} = \frac{\theta_{ij}}{\sigma^{1-\varepsilon}} \tag{2.4}$$

with

$$\theta_{ij} = \delta_{ij} - (1-\varepsilon)\frac{x^i x^j}{(1+|x|^2)}$$

a quadratic form which is uniformly equivalent to δ_{ij} if $0<\varepsilon\leqslant 1$. The inverse of θ is given by

$$(\theta^{-1})_{ij} = \delta_{ij} + (1-\varepsilon)\frac{x^i x^j}{(1+\varepsilon|x|^2)}.$$

Consider the operator T_ε, acting on functions f on $\mathbf{R}^n$ with values in some vector space V in the following way:

$$T_\varepsilon f = f\circ\varphi_\varepsilon^{-1}. \tag{2.5}$$

T_ε is bijective, it is a linear operator, and satisfies $T_\varepsilon(f_1 f_2) = T_\varepsilon(f_1)\, T_\varepsilon(f_2)$. Thus T_ε is an automorphism of the ring of functions on $\mathbf{R}^n$.

LEMMA 2.3. *For each* $0<\varepsilon\leqslant 1$, T_ε *is an isomorphism:* (a) $C^s_\delta(\mathbf{R}^n)\to C^s_{\delta/\varepsilon}(\mathbf{R}^n)$ *and* (b) $H_{s,\delta}(\mathbf{R}^n)\to H_{s,(\delta+n/2)/\varepsilon-n/2}(\mathbf{R}^n)$, *for every* $s\in\mathbf{N}$, $\delta\in\mathbf{R}$.

Proof. From (2.5) and (2.4) we have:

$$\sigma(x)\frac{\partial f}{\partial x^i}(x) = \sigma(x)\frac{\partial y^j}{\partial x^i}(x)\,\partial\frac{T_\varepsilon f}{\partial y^j}(y)$$
$$= \theta_{ji}(x)\,\sigma^\varepsilon(x)\,\partial\frac{T_\varepsilon f}{\partial y^j}(y).$$

On the other hand from (2.3) and (2.2) we obtain

$$\sigma^\varepsilon(x)\leqslant\sigma(y)\leqslant 2^{1/2}\sigma^\varepsilon(x). \tag{2.6}$$

Thus, since θ_{ij} is uniformly equivalent to δ_{ij}, we conclude that there exist positive constants c_1 and c_2 such that

$$c_1|(\sigma Df)(x)|^2\leqslant|(\sigma DT_\varepsilon f)(y)|^2\leqslant c_2|(\sigma Df)(x)|^2.$$

Similarly we find

$$c_1 \sum_{k=0}^{s} |(\sigma^k D^k f)(x)|^2 \leqslant \sum_{k=0}^{s} |(\sigma^k D^k T_\varepsilon f)(y)|^2 \leqslant c_2 \sum_{k=0}^{s} |(\sigma^k D^k f)(x)|^2$$

(for different constants c_1, c_2). Then (2.6) implies that

$$c_1 \sum_{k=0}^{s} |(\sigma^{\delta+k} D^k f)(x)|^2 \leqslant \sum_{k=0}^{s} |(\sigma^{\delta/\varepsilon+k} D^k T_\varepsilon f)(y|^2 \leqslant c_2 \sum_{k=0}^{s} |\sigma^{\delta+k} D^k f)(x)|^2, \tag{2.7}$$

from which part (a) of the lemma follows at once. To obtain part (b) we note that according to (2.4) the volume elements dy and dx are related by

$$dy = \frac{\det \theta(x)}{\sigma^{n(1-\varepsilon)}(x)}\, dx$$

and therefore by (2.7) and (2.6) there exist positive constants c_1 and c_2 such that

$$c_1 \int_{\mathbf{R}^n} \sum_{k=0}^{s} |(\sigma^{\delta+k} D^k f)(x)|^2 dx \leqslant \int_{\mathbf{R}^n} \sum_{k=0}^{s} (\sigma^{n(1-\varepsilon)/2\varepsilon+\delta/\varepsilon+k} D^k T_\varepsilon f)(y)|^2 dy$$
$$\leqslant c_2 \int_{\mathbf{R}^n} \sum_{k=0}^{s} |(\sigma^{\delta+k} D^k f)(x)|^2 dx. \qquad \square$$

LEMMA 2.4. *If (M, e) is a complete riemannian manifold, euclidean at infinity, the following inclusion holds and is continuous:*

$$H_{s,\delta} \subset C^{s'}_{\delta'}$$

if $s' < s - n/2$, $\delta' < \delta + n/2$.

Proof. If $f \in H_{s,\delta}$ we write

$$f = \chi_R f + (1-\chi_R) f,$$

where $\chi_R \in \mathcal{D}$, supp $\chi_R \subset B_{2R}$, $R > R_0$. The classical result for compact manifolds with C^∞ boundary implies

$$\chi_R f \in C^{s'}_\beta, \quad s' < s - n/2, \quad \beta \text{ arbitrary}.$$

We are then left to the study of the function $(1-\chi_R)f$ whose support is the disjoint union of the Ω_A's. We denote by f_A the function on $\mathbf{R}^n$ (with support the exterior of some open ball) corresponding to $(1-\chi_R)f$ in the diffeomorphism φ_A. We have $f_A \in H_{s,\delta}(\mathbf{R}^n)$; Now since $f \in H_{s,\delta}(\mathbf{R}^n)$ is equivalent to $\sigma^{\delta+k} D^k f \in H_{s-k}(\mathbf{R}^n)$ for $0 \leqslant k \leqslant s$, it is clear that

$$H_{s,\delta}(\mathbf{R}^n) \subset C^{s'}_\delta(\mathbf{R}^n) \quad \text{if} \quad s' < s - n/2. \tag{2.8}$$

To prove the stronger statement we argue as follows: $f\in H_{s,\delta}(\mathbf{R}^n)$ implies by Lemma 2.3 (b)

$$T_\varepsilon f\in H_{s,(\delta+n/2)/\varepsilon-n/2}(\mathbf{R}^n).$$

Therefore, by (2.8)

$$T_\varepsilon f\in C^{s'}_{(\delta+n/2)/\varepsilon-n/2}(\mathbf{R}^n)$$

and by Lemma 2.3 (a)

$$f\in C^{s'}_{\delta+n/2-\varepsilon n/2}(\mathbf{R}^n).$$

Since this holds for every $0<\varepsilon\leqslant 1$, we conclude that $f\in C^{s'}_{\delta'}(\mathbf{R}^n)$ for every $\delta'<\delta+n/2$. Finally, the continuity of the inclusion follows from the fact that T_ε is an isomorphism. □

LEMMA 2.5. *If (M, e) is a complete riemannian manifold euclidean at infinity we have the continuous multiplication property*

$$H_{s_1,\delta_1}\times H_{s_2,\delta_2}\to H_{s,\delta}$$

by

$$(f_1, f_2)\mapsto f_1\otimes f_2$$

if $s_1, s_2\geqslant s$, $s<s_1+s_2-n/2$, $\delta<\delta_1+\delta_2+n/2$.

Proof. As in Lemma 2.4 the statement follows from the corresponding statement on $\mathbf{R}^n$ together with the classical result for compact manifolds with C^∞ boundary. Now since $f\in H_{s,\delta}(\mathbf{R}^n)$ is equivalent to $\sigma^{\delta+k}D^kf\in H_{s-k}(\mathbf{R}^n)$ for $0\leqslant k\leqslant s$, it is clear that we have the continuous multiplication property

$$H_{s_1,\delta_1}(\mathbf{R}^n)\times H_{s_2,\delta_2}(\mathbf{R}^n)\to H_{s,\delta_1+\delta_2}(\mathbf{R}^n) \tag{2.9}$$

if $s_1, s_2\geqslant s$, $s<s_1+s_2-n/2$. To prove the stronger statement we argue as follows: $f_1\in H_{s_1,\delta_1}(\mathbf{R}^n)$, $f_2\in H_{s_2,\delta_2}(\mathbf{R}^n)$ imply by Lemma 2.3 (b)

$$T_\varepsilon(f_1)\in H_{s_1,(\delta_1+n/2)/\varepsilon-n/2}(\mathbf{R}^n),\ T_\varepsilon(f_2)\in H_{s_2,(\delta_2+n/2)/\varepsilon-n/2}(\mathbf{R}^n).$$

Therefore, by (2.9)

$$T_\varepsilon(f_1)\,T_\varepsilon(f_2)=T_\varepsilon(f_1\otimes f_2)\in H_{s,(\delta_1+\delta_2+n)/\varepsilon-n}(\mathbf{R}^n)$$

and again by Lemma 2.3 (b)

$$f_1\otimes f_2\in H_{s,\delta_1+\delta_2+n/2-\varepsilon n/2}(\mathbf{R}^n).$$

Since this holds for every $0<\varepsilon\leqslant 1$, we conclude that $f_1\otimes f_2\in H_{s,\delta}(\mathbf{R}^n)$ for every $\delta<\delta_1+\delta_2+n/2$. Finally, continuity follows from the fact that T_ε is an isomorphism. □

COROLLARY 2.1. *If (M, e) is a complete riemannian manifold euclidean at infinity, $H_{s,\delta}$ is a Banach algebra if $s>n/2$, $\delta>-n/2$.*

3. Elliptic theory on a compact manifold K (with or without boundary)

A linear differential operator of order m on sections of the tensor bundle E over the manifold M is a linear mapping $C^\infty(E, M) \to \mathcal{D}'(F, M)$, with F another tensor bundle over M, which reads

$$Lu \equiv \sum_{k=0}^{m} a_k D^k u$$

where a_k, $0 \leqslant k \leqslant m$, are given tensorfields over M, sections of the tensor bundle $(\otimes TM)^k \otimes E^* \otimes F$ (in local coordinates where $u = (u^{i_1 \dots i_p})$ we have $a_k D^k u = (v^{j_1 \dots j_q})$ with $v^{j_1 \dots j_q} = a^{l_1 \dots l_k, j_1 \dots j_q}_{i_1 \dots i_p} D_{l_1} \dots D_{l_k} u^{i_1 \dots i_p}$). The operator L can be extended from C^∞ sections of E some vector subspace of $\mathcal{D}'(E, M)$, depending on the regularity properties of the coefficients a_k. The system is said to be elliptic if for each $x \in M$ the linear map between the fibers E_x and F_x over x of E and F

$$E_x \to F_x \quad \text{by} \quad u(x) \mapsto a_m (\otimes \xi)^m u(x)$$

is an isomorphism for every covector $\xi \neq 0$ at x.

This hypothesis is expressed in local coordinates by

$$\det (\bar{a}_m \bar{\xi}^m) \neq 0, \quad \text{for all } \bar{\xi} \neq 0,$$

where $\bar{a}_m$ and $\bar{\xi}$ are the representatives of a_m and ξ:

$$(\bar{a}_m \bar{\xi}^m)^I_J = \bar{a}^I_{i_1 \dots i_m J} \bar{\xi}^{i_1} \cdots \bar{\xi}^{i_m}, \quad I, J = 1, \dots, N.$$

Note that this determinant does not depend on the choice of coordinates, since $a_m \xi^m$ is a section of $E \otimes E^*$. The ellipticity condition implies that Nm is even. If

$$\det (\bar{a}_m \bar{\xi}^m) \geqslant \lambda (e(\xi, \xi))^{Nm/2}, \quad \lambda > 0$$

we say that the system admits the ellipticity constant λ.

We shall give here a result for elliptic linear systems on a compact manifold which is not the finest possible as far as the hypotheses on the a_k are concerned, but which is easy to obtain from results proved in the available literature, and is sufficient for application to quasilinear elliptic systems.

THEOREM 3.1. (Gårding [9], Douglis-Nirenberg [7], Morrey [13].) *Let* $\tilde{L}$,

$$\tilde{L}u \equiv a_m D^m u$$

be a linear homogeneous elliptic operator on K, with a_m continuous on K. Then the following estimate holds for every $u \in H_m$

$$\|u\|_{H_m} \leqslant c(\|\tilde{L}u\|_{L^2} + \|u\|_{L^2}),$$

where the constant c depends only on K, the ellipticity constant λ, the C^0 norm of a_m and its modulus of continuity.

From this fundamental theorem we shall deduce the following:

THEOREM 3.2. *Let $L \equiv \sum_{k=0}^{m} a_k D^k$ be a linear elliptic operator on K with coefficients such that $a_k \in H_{s_k}$ with, for $0 \leqslant k \leqslant m$, $s_k \geqslant s - m \geqslant 0$ and*

$$s_k > \frac{n}{2} + k - m + 1.$$

Then the following estimate holds

$$\|u\|_{H_s} \leqslant c\,\{\|Lu\|_{H_{s-m}} + \|u\|_{L^2}\}, \tag{3.1}$$

where the constant c depends only on (K, e) the ellipticity constant λ of a_m and the H_{s_k} norms of the a_k's.

Proof. (1) The hypotheses imply that $a_m \in C^1$, its C^0 norm and modulus of continuity depending only on its H_{s_m} norm. We have

$$a_m D^m u = Lu - \sum_{k=0}^{m-1} a_k D^k u.$$

Thus, by the previous theorem, for every $u \in H_m$

$$\|u\|_{H_m} \leqslant c\left\{\left\| Lu - \sum_{k=0}^{m-1} a_k D^k u \right\|_{L^2} + \|u\|_{L^2}\right\}. \tag{3.2}$$

Since $\|D^k u\|_{H_{m-1-k}} \leqslant \|u\|_{H_{m-1}}$ if $0 \leqslant k \leqslant m-1$, the multiplication lemma gives

$$\|a_k D^k u\|_{L^2} \leqslant c\|a_k\|_{H_{s_k}} \|u\|_{H_{m-1}}, \quad 0 \leqslant k \leqslant m-1, \tag{3.3}$$

if $s_k + m - 1 - k - n/2 > 0$. Also, it is well known that, on a compact manifold, one can choose arbitrarily $\varepsilon > 0$ and find a C such that, for every $u \in H_m$,

$$\|u\|_{H_{m-1}} < \varepsilon\|u\|_{H_m} + C\|u\|_{L^2}. \tag{3.4}$$

We then deduce from (3.2), (3.3) and (3.4) that

$$\|u\|_{H_m} \leqslant c\,\{\|Lu\|_{L^2} + \|u\|_{L^2}\}.$$

(2) When the inequality (3.1) is known for some $s \geqslant m$, one deduces the estimate of order $s+1$. Indeed, differentiating Lu we obtain

$$\sum_{k=0}^{m} a_k D^k(Du) = DLu - \sum_{k=0}^{m} Da_k D^k u - \sum_{k=0}^{m} a_k M_k, \tag{3.5}$$

where

$$M_k = \sum_{i=0}^{k-1} R_i D^{k-1-i} u$$

and the R_i are linear in the ith derivatives of the Riemann tensor of the metric e and come from the commutation of covariant derivatives in (3.5). If we suppose that $u \in H_{s+1}$ and for $0 \leqslant k \leqslant m$: $s_k \geqslant s+1-m$ and $s_k > n/2 + k - m + 1$, then

$$\left\| \sum_{k=0}^{m} Da_k D^k u \right\|_{H_{s-m}} \leqslant c \sum_{k=0}^{m} \|Da_k\|_{H_{s_k-1}} \|D^k u\|_{H_{s-k}}$$
$$\leqslant c \left(\sum_{k=0}^{m} \|a_k\|_{H_{s_k}} \right) \|u\|_{H_s}.$$

Similarly, we obtain

$$\left\| \sum_{k=0}^{m} a_k M_k \right\|_{H_{s-m}} \leqslant c \sum_{k=0}^{m} \|a_k\|_{H_{s_k}} \|M_k\|_{H_{s-k}}$$

and

$$\|M_k\|_{H_{s-k}} \leqslant c \|u\|_{H_{s-1}},$$

where c depends only on (K, e). Applying inequality (3.1) to (3.5) gives then

$$\|Du\|_{H_s} \leqslant c \{ \|DLu\|_{H_{s-m}} + \|u\|_{H_s} \}, \tag{3.6}$$

c depending only on (K, e), the ellipticity constant λ of a_m and H_{s_k} norms the a_k's. Inequality (3.6), together with inequality (3.1) for s, yields inequality (3.1) for $s+1$.

Remark. The hypothesis on s_k can be weakened to

$$s_k > \frac{n}{2} + k - m,$$

if we assume that $\|a_k\|_{H_{s_k}}$ for $0 \leqslant k \leqslant m-1$ and $\|Da_m\|_{H_{s_m-1}}$ are sufficiently small.

4. Elliptic theory on R^n

The following theorem is a particular case of a theorem proved in [14].

THEOREM 4.1 ([14], Theorem 3.1). *Let* L,

$$\tilde{L}u \equiv a_m D^m u$$

be a linear homogeneous elliptic operator on $\mathbf{R}^n$*, such that*

$$a_m - A_m \in C^0_\beta, \quad \beta > 0$$

where $L_\infty \equiv A_m D^m$ *is an elliptic operator with constant coefficients. Then for any real number* δ *there exists a constant c such that for every* $u \in H^{\text{loc}}_m \cap H_{0,\delta}$ *the following estimate holds:*

$$\|u\|_{H_{m,\delta}} \leqslant c\{\|\tilde{L}u\|_{H_{0,\delta+m}} + \|u\|_{H_{0,\delta}}\}$$

Remark. It is clear from the proof that the hypothesis $u \in H_m$ made by Nirenberg-Walker can be replaced by $u \in H^{\text{loc}}_m$.

We deduce from this theorem (see also Cantor [2] in the same spirit):

THEOREM 4.2. *Let Lu be an elliptic differential system on* $\mathbf{R}^n$,

$$Lu \equiv \sum_{k=0}^{m} a_k D^k u,$$

with coefficients such that:

$$a_k \in H_{s_k,\delta_k}, \quad 0 \leqslant k \leqslant m-1$$
$$a_m - A_m \in H_{s_m,\delta_m}$$

with $s_k \geqslant s - m$, $s_k > (n/2) + k - m + 1$, $\delta_k > m - k - (n/2)$, *for* $0 \leqslant k \leqslant m$. *Then for any real number* δ *and any* $s \geqslant m$ *there exists a constant c such that for every* $u \in H^{\text{loc}}_s \cap H_{0,\delta}$ *the following inequality holds*

$$\|u\|_{H_{s,\delta}} \leqslant c\{\|Lu\|_{H_{s-m,\delta+m}} + \|u\|_{H_{0,\delta}}\}. \tag{4.1}$$

Proof. (1) The hypotheses imply by Lemma 2.4 that $a_m - A_m \in C^1_\beta$ for $\delta_m + n/2 > \beta > 0$. Thus we can apply the previous theorem to

$$a_m D^m u = Lu - \sum_{k=0}^{m-1} a_k D^k u,$$

obtaining, for every $u \in H^{\text{loc}}_m \cap H_{0,\delta}$

$$\|u\|_{H_{m,\delta}} \leqslant c\left\{\left\| Lu - \sum_{k=0}^{m-1} a_k D^k u\right\|_{H_{0,\delta+m}} + \|u\|_{H_{0,\delta}}\right\}. \tag{4.2}$$

Choosing a real number δ' such that for every $0 \leqslant k \leqslant m$, $\delta_k - m + k + n/2 > \delta - \delta' > 0$, Lemma 2.5 gives

$$\left\|\sum_{k=0}^{m-1} a_k D^k u\right\|_{H_{0,\delta+m}} \leqslant c\left(\sum_{k=0}^{m-1} \|a_k\|_{H_{s_k,\delta_k}}\right) \|u\|_{H_{m-1,\delta'}}. \tag{4.3}$$

under the above hypothesis on s_k. Also, since $\delta' < \delta$, Lemma 2.2 implies that we can choose arbitrarily $\varepsilon > 0$ and find a C such that, for every $u \in H_{m,\delta}$,

$$\|u\|_{H_{m-1,\delta'}} < \varepsilon \|u\|_{H_{m,\delta}} + C\|u\|_{H_{0,\delta}}. \tag{4.4}$$

We then deduce from (4.2), (4.3) and (4.4) that

$$\|u\|_{H_{m,\delta}} \leqslant c\,\{\|Lu\|_{H_{0,\delta+m}} + \|u\|_{H_{0,\delta}}\}.$$

(2) We now suppose that inequality (4.1) holds for some $s \geqslant m$ and every $\delta \in \mathbf{R}$ and we shall prove it for $s+1$. Applying the estimate (4.1) for s and the real number $\delta+1$ to

$$\sum_{k=0}^{m} a_k D^k(Du) = DLu - \sum_{k=0}^{m} Da_k D^k u$$

we obtain

$$\|Du\|_{H_{s,\delta+1}} \leqslant c\left\{\left\| DLu - \sum_{k=0}^{m} Da_k D^k u \right\|_{H_{s-m,\delta+m+1}} + \|Du\|_{H_{0,\delta+1}}\right\}.$$

Lemma 2.2 implies that under the hypotheses made on s_k and δ_k (where now $s_k \geqslant s+1-m$)

$$\begin{aligned}\left\| \sum_{k=0}^{m} Da_k D^k u\right\|_{H_{s-m,\delta+m+1}} &\leqslant c \sum_{k=0}^{m} \|Da_k\|_{H_{s_k-1,\delta_k+1}} \|D^k u\|_{H_{s-k,\delta+k}} \\ &\leqslant c\left(\sum_{k=0}^{m-1} \|a_k\|_{H_{s_k,\delta_k}} + \|a_m - A_m\|_{H_{s_m,\delta_m}}\right) \|u\|_{H_{s,\delta}}.\end{aligned}$$

Thus we find

$$\|Du\|_{H_{s,\delta+1}} \leqslant c\,\{\|DLu\|_{H_{s-m,\delta+m+1}} + \|u\|_{H_{s,\delta}}\},$$

which, together with the inequality (4.1) supposed for s, δ gives inequality (4.1) for $s+1$, δ.

□

Remark. The hypothesis on s_k can be weakened to

$$s_k > \frac{n}{2} + k - m$$

if we assume that

$$\sum_{k=0}^{m-1} \|a_k\|_{H_{s_k,\delta_k}} + \|a_m - A_m\|_{H_{s_m,\delta_m}}$$

is sufficiently small.

5. Equations with constant coefficients

It is known (cf. John [10]) that the homogeneous elliptic system on $\mathbf{R}^n$ with constant coefficients

$$Lu \equiv A_m D^m u$$

has a fundamental solution of the form

$$\Gamma(x) = \Gamma_0(x) + \Gamma_1(x) \log |x|$$

where Γ_0 and Γ_1 are homogeneous of degree $m-n$ in x, and $\Gamma_1 = 0$ either if n is odd or if $n > m$. We deduce from the existence of such a fundamental solution the following lemmas (adapted from Nirenberg-Walker [14]):

LEMMA 5.1. *The homogeneous elliptic system with constant coefficients*

$$Lu \equiv A_m D^m u = f$$

has at most one solution in $H_m^{\mathrm{loc}} \cap H_{0,\delta}$ *if* $\delta > -n/2$.

Proof. Let $\{\xi_R\}_{0<R<\infty}$ be a collection of C_0^∞ functions on $\mathbf{R}^n$ such that

(i) $0 \leqslant \xi_R(x) \leqslant 1$, $\forall x \in \mathbf{R}^n$; $\xi_R(x) = 1$ for $|x| \leqslant 1$; $\xi_R(x) = 0$ for $|x| \geqslant 2R$.

(ii) there exists a constant c, independent of R, such that

$$|D^k \xi_R(x)| \leqslant cR^{-k}, \quad \forall x \in \mathbf{R}^n,\ 0 \leqslant k \leqslant m.$$

Since Γ is a fundamental solution ($L\Gamma = \delta$) and $\xi_R u$ has compact support we have

$$\xi_R u = \Gamma * L(\xi_R u),$$

the convolution of three distributions, two of them $(L, \xi_R u)$ with compact support being associative (cf. for instance [5]). We may express

$$L(\xi_R u) = \xi_R Lu + \sum_{l=1}^{m} C_l D^l \xi_R D^{m-l} u.$$

Thus

$$\xi_R u = \Gamma * (\xi_R Lu) + \int_{R \leqslant |y| \leqslant 2R} \Gamma(x-y) \sum_{l=1}^{m} C_l (D^l \xi_R D^{m-l} u)(y)\, dy,$$

which if $u \in H_m^{\mathrm{loc}}$ may be rewritten as

$$\xi_R u = \Gamma * (\xi_R Lu) + \sum_{l=1}^{m} (-1)^{m-l} C_l I_l,$$

where

$$I_l = \int_{R \leqslant |y| \leqslant 2R} D^{m-l}[\Gamma(x-y)\, D^l \xi_R(y)]\, u(y)\, dy.$$

By Schwarz' inequality and the properties of Γ and the family $\{\xi_R\}$, I_l is bounded by

$$\begin{aligned} |I_l| &\leqslant \left\{ \int_{R \leqslant |y| \leqslant 2R} \sigma^{-2\delta}(y)\, |D^{m-l}[\Gamma(x-y)\, D^l \xi_R(y)]|^2\, dy \right\}^{1/2} \|u\|_{H_{0,\delta}} \\ &\leqslant c \left\{ \int_{R \leqslant |y| \leqslant 2R} \sigma^{-2\delta}(y) \sum_{k=0}^{m-l} \frac{\log^2 |x-y|}{|x-y|^{2(n-l-k)}} R^{-2(l+k)}\, dy \right\}^{1/2} \|u\|_{H_{0,\delta}} \\ &\leqslant c' R^{-\delta-n/2} \log R \|u\|_{H_{0,\delta}}. \end{aligned}$$

I_l tends to zero when R tends to infinity if $\delta > -n/2$. Thus $Lu = 0$ implies $u = 0$. □

Under the hypotheses of Lemma 5.1, if $|x| < R$ we have:

$$u(x) = \int_{B_{2R}} \Gamma(x-y)\, f(y)\, dy + J(x) + \sum_{l=1}^{m} (-1)^{m-l} C_l I_l(x),$$

where

$$J(x) = \int_{R \leqslant |y| \leqslant 2R} \Gamma(x-y)\, [\xi_R(y) - 1]\, f(y)\, dy.$$

If now we assume $f \in H_{0,\delta+m}$, then J is bounded by

$$\begin{aligned} |J| &\leqslant \left\{ \int_{R \leqslant y \leqslant 2R} \sigma^{-2(\delta+m)}(y)\, |\Gamma(x-y)\, [\xi_R(y) - 1]|^2 dy \right\}^{1/2} \|f\|_{H_{0,\delta+m}} \\ &\leqslant c \left\{ \int_{R \leqslant y \leqslant 2R} \sigma^{-2(\delta+m)}(y) \frac{\log^2 |x-y|}{|x-y|^{2(n-m)}}\, dy \right\}^{1/2} \|f\|_{H_{0,\delta+m}} \\ &\leqslant c' R^{-\delta-n/2} \log R \|f\|_{H_{0,\delta+m}} \end{aligned}$$

and therefore, if $\delta > -n/2$, also tends to zero when R tends to infinity. Hence

$$u(x) = \int_{\mathbf{R}^n} \Gamma(x-y)\, f(y)\, dy, \quad f = Lu. \tag{5.1}$$

LEMMA 5.2. *If $m < n$ and $f \in H_{0,\delta+m}$ with $-n/2 < \delta < -m + n/2$, then every solution u in $H_m^{\text{loc}} \cap H_{0,\delta}$ of*

$$Lu \equiv A_m D^m u = f$$

is also in $H_{m,\delta}$ and there exists a constant c such that

$$\|u\|_{H_{m,\delta}} \leqslant c \|f\|_{H_{0,\delta+m}}. \tag{5.2}$$

Proof. We write $u=u_1+u_2$, with $u_1=\xi_R u$ for some fixed R. Since u_1 has support in $\bar{B}_{2R}$ we have from Theorem 3.1

$$\|u_1\|_{H_m(B_{2R})}\leqslant c\ \|Lu_1\|_{H_0(B_{2R})}+\|u_1\|_{H_0(B_{2R})}\}. \tag{5.3}$$

To estimate u_2 we use (5.1) to write, if $m<n$

$$|x|^{\delta}|u_2(x)|\leqslant c\int_{\mathbf{R}^n}\frac{|y|^{m+\delta}|f_2(y)|}{|x|^{-\delta}|x-y|^{n-m}|y|^{m+\delta}}dy,\quad f_2=Lu_2$$

which gives if $x\in\mathbf{R}^n-B_R$, since also supp $f\subset\mathbf{R}^n-B_R$,

$$\sigma^{\delta}(x)|u_2(x)|\leqslant c\int_{\mathbf{R}^n}\frac{\sigma^{m+\delta}(y)|f_2(y)|}{|x|^{-\delta}|x-y|^{n-m}|y|^{m+\delta}}dy$$

and thus, by Lemma 2.1 of Nirenberg-Walker [14], if $\delta>-n/2$ and $m+\delta<n/2$

$$\|u_2\|_{H_{0,\delta}}\leqslant c\|Lu_2\|_{H_{0,m+\delta}}$$

which in turn by Theorem 4.1 implies

$$\|u_2\|_{H_{m,\delta}}\leqslant c\|Lu_2\|_{H_{0,m+\delta}}. \tag{5.4}$$

On the other hand we can express

$$Lu_1=A_mD^m(\xi_Ru)=\xi_RA_mD^mu+\sum_{k=1}^{m}C_kA_mD^k\xi_RD^{m-k}u.$$

Hence we have

$$\|Lu_1\|_{H_0(B_{2R})}\leqslant c\ \{\|Lu\|_{H_0(B_{2R})}+\|u\|_{H_{m-1}(B_{2R})}\}. \tag{5.5}$$

Analogously we find

$$\|Lu_2\|_{H_{0,m+\delta}}\leqslant c\ \{\|Lu\|_{H_{0,m+\delta}}+\|u\|_{H_{m-1}(B_{2R})}\} \tag{5.6}$$

having used the fact that for functions with support in some fixed bounded set the norms H_s and $H_{s,\delta}$ are equivalent. We conclude from (5.3), (5.4), (5.5), (5.6) that, δ' being chosen arbitrarily,

$$\|u\|_{H_{m,\delta}}\leqslant c\ \{\|Lu\|_{H_{0,m+\delta}}+\|u\|_{H_{m-1,\delta'}}\}. \tag{5.7}$$

Using the fact that, under the hypothesis on δ, L is injective on $H_{m,\delta}$ (Lemma 5.1) and that $H_{m,\delta}$ is compactly imbedded in $H_{m-1,\delta'}$ if $\delta'<\delta$ (Lemma 2.1) we deduce from inequality (5.7) the existence of a constant c such that

$$\|u\|_{H_{m,\delta}}\leqslant c\|Lu\|_{H_{0,m+\delta}}.$$

Indeed, if there is no such c there exists a sequence $\{u_n\}$ with $\|u_n\|_{H_{m,\delta}}=1$ such that $\|Lu_n\|_{H_{0,\delta+m}}$ tends to zero. For $\delta'<\delta$ we have $\|u_n\|_{H_{m-1,\delta'}}\leq 1$. Thus $\{u_n\}$ admits a convergent subsequence in $H_{m-1,\delta'}$ still denoted $\{u_n\}$, and $\|u_n-u_{n'}\|_{H_{m-1,\delta'}}$ is arbitrarily small, as well as $\|L(u_n-u_{n'})\|_{H_{0,\delta+m}}$ by the hypothesis. By inequality (5.7) applied to $u_n-u_{n'}$, the sequence $\{u_n\}$ converges also in $H_{m,\delta}$ to some u, different from zero since $\|u_n\|_{H_{m,\delta}}=1$. This contradicts the injectivity of L.

THEOREM 5.1. *The homogeneous elliptic differential operator with constant coefficients on* $\mathbf{R}^n$:

$$L\equiv A_m D^m$$

$m<n$, *is an isomorphism* $H_{s,\delta}\to H_{s-m,\delta+m}$ *if* $s\geq m$, $-n/2<\delta<-m+n/2$.

Proof. L is injective by Lemma 5.1. We construct a solution of $Lu=f$ when $f\in\mathcal{D}$ by

$$u=\Gamma * f.$$

It is well-known that this solution is C^∞, thus also in H_m^{loc}. We prove it is in $H_{0,\delta}$ by the same computation used to estimate $\|u\|_{H_{0,\delta}}$ in the proof of Lemma 5.2, which shows also that $u\in H_{m,\delta}$. The density of $\mathcal{D}$ in $H_{s-m,\delta+m}$, together with inequality (5.2), completes the proof of the surjectivity of L and the continuity of its inverse.

6. Elliptic linear systems on manifolds euclidean at infinity

We consider now linear differential operators of order m on sections of a tensor bundle E over a riemannian manifold (M, e), euclidean at infinity (definition 2.3),

$$Lu\equiv\sum_{k=0}^{m} a_k D^k u.$$

We shall make the following hypotheses on the coefficients a_k, $0\leq k\leq m$, which are given tensorfields over M, sections of the tensor bundle $(\otimes TM)^k\otimes E^*\otimes F$.

Hypothesis I (regularity). (M, e) is a complete riemannian manifold euclidean at intinity and there exist nonnegative integers s_k and real numbers δ_k such that

$$s_k>\frac{n}{2}+k-m+1,\ \delta_k>m-k-\frac{n}{2},\quad 0\leq k\leq m$$

and

(1) $a_k\in H_{s_k,\delta_k}$, $0\leq k\leq m-1$

(2) $a_m-A_m\in H_{s_m,\delta_m}$,

where A_m is a C^∞ tensorfield on M which is constant ($DA_m=0$) in each neighborhood Ω_A of infinity.

It results from hypothesis I and Lemma 2.5 that for any $\delta\in\mathbf{R}$, L is a continuous linear map

$$L: H_{s,\delta}\to H_{s-m,\delta+m} \quad \text{by} \quad u\mapsto \sum_{k=0}^{m} a_k D^k u$$

if $s_k\geqslant s-m\geqslant 0$.

Also, hypothesis I and Lemma 2.4 imply that there exists a $\beta>0$ such that $a_m-A_m\in C^1_\beta$.

Hypothesis II (ellipticity, cf. § 3). For each $x\in M$ the linear map between the fibers E_x and F_x over x of E and F

$$E_x\to F_x \quad \text{by} \quad u(x)\mapsto a_m(\otimes\xi)^m u(x)$$

is an isomorphism for every covector $\xi\neq 0$ at x.

THEOREM 6.1. *Let (M, e) be a complete riemannian manifold, euclidean at infinity. Suppose that the coefficients of L satisfy hypotheses I and II. Then if $s\geqslant m$, $s_k\geqslant s-m$ and δ is any real number, there exists a positive constant c such that for every $u\in H_s^{\mathrm{loc}}\cap H_{0,\delta}$ with $Lu\in H_{s-m,\delta+m}$ the following estimate holds:*

$$\|u\|_{H_{s,\delta}}\leqslant c\,\{\|Lu\|_{H_{s-m,\delta+m}}+\|u\|_{H_{0,\delta}}\}.$$

Proof. It relies on the elliptic estimates on a compact manifold (§ 3) and on $\mathbf{R}^n$ (§ 4). Let R be a fixed number $R>R_0$ and let χ_R and ψ_A be C^∞ functions on M such that

$$\chi_R(x)=1 \quad \text{for } x\in B_R,\ \chi_R(x)=0 \quad \text{for } x\in M-B_{2R}$$

$$\psi_A(x)=1 \quad \text{if } x\in\Omega_A,\ \psi_A(x)=0 \quad \text{if } x\in\Omega_B,\ B\neq A.$$

For every u we have

$$u=u_R+\sum_{A=1}^{p}u_A,$$

where $u_R=\chi_R u$ has its support in B_{2R} while $u_A=(1-\chi_R)\psi_A u$ has its support in Ω_A. We have

$$\|u\|_{H_{s,\delta}}\leqslant\|u_R\|_{H_{s,\delta}}+\sum_A\|u_A\|_{H_{s,\delta}}. \tag{6.1}$$

(a) By the elliptic theory on a compact manifold with boundary there exists a constant c such that

$$\|u_R\|_{H_s(B_{2R})}\leqslant c\{\|Lu_R\|_{H_{s-m}(B_{2R})}+\|u_R\|_{L^2(B_{2R})}\}. \tag{6.2}$$

On the other hand we have

$$Lu_R = \sum_{k=0}^{m} a_k D^k(\chi_R u) = \chi_R Lu + \sum_{k=1}^{m} \sum_{l=1}^{k} C_l^k a_k D^l \chi_R D^{k-l} u,$$

from which follows

$$\|Lu_R\|_{H_{s-m}(B_{2R})} \leqslant c\,\{\|Lu\|_{H_{s-m}(B_{2R})} + \|u\|_{H_{s-1}(B_{2R})}\}. \tag{6.3}$$

Combining (6.2) and (6.3) and taking into account the fact that on B_{2R}, with R fixed, the H_s and $H_{s,\varrho}$ norms are equivalent for any $\varrho \in \mathbf{R}$, we obtain

$$\|u_R\|_{H_{s,\delta}} \leqslant c\,\{\|Lu\|_{H_{s-m,\delta+m}} + \|u\|_{H_{s-1,\delta'}}\} \tag{6.4}$$

with δ' arbitrary.

(b) Let us denote by $\bar{u}_A$ the image by the diffeomorphism φ_A of u_A; it is a $\mathbf{R}^N$ valued function on $\varphi_A(\Omega_A) = \mathbf{R}^n - \bar{B}$. Let us denote by $\bar{L}$ the linear operator acting on $\bar{u}_A$ and corresponding to L. Applying Theorem 4.2 to $\bar{L}\bar{u}_A$ we obtain

$$\|\bar{u}_A\|_{H_{s,\delta}} \leqslant c\,\{\|\bar{L}\bar{u}_A\|_{H_{s-m,\delta+m}} + \|\bar{u}_A\|_{H_{0,\delta}}\}.$$

The $H_{s,\delta}$ norms defined on M of tensorfields with support in Ω_A are equivalent to the $H_{s,\delta}$ norms defined on $\mathbf{R}^n$ of their images by φ_A. Thus, we have also

$$\|u_A\|_{H_{s,\delta}} \leqslant c\,\{\|Lu_A\|_{H_{s-m,\delta+m}} + \|u_A\|_{H_{0,\delta}}\}. \tag{6.5}$$

On the other hand we can express

$$Lu_A = \sum_{k=0}^{m} a_k D^k((1-\chi_R)\,u) = (1-\chi_R)\,Lu - \sum_{k=1}^{m} \sum_{l=1}^{k} C_k^l a_k D^l \chi_R D^{k-l} u,$$

from which follows

$$\|Lu_A\|_{H_{s-m,\delta+m}} \leqslant c\,\{\|Lu\|_{H_{s-m,\delta+m}} + \|u\|_{H_{s-1}(B_{2R})}\}. \tag{6.6}$$

Combining (6.5) and (6.6) we obtain

$$\|u_A\|_{H_{s,\delta}} \leqslant c\,\{\|Lu\|_{H_{s-m,\delta+m}} + \|u\|_{H_{0,\delta}} + \|u\|_{H_{s-1,\delta'}}\} \tag{6.7}$$

with δ' arbitrary. Finally, (6.7) and (6.4) together with (6.1) imply the result, if we choose $\delta' < \delta$ and use Lemma 2.2.

THEOREM 6.2. *If the operator L satisfies hypotheses I and II, if $s \geqslant m$, $s_k \geqslant s-m$ and if $-n/2 < \delta < -m + n/2$,*(1) *then there exists a constant c such that for every $u \in H_{s,\delta}$*

$$\|u\|_{H_{s,\delta}} \leqslant c\,\{\|Lu\|_{H_{s-m,\delta+m}} + \|u\|_{H_{s-1,\delta'}}\} \tag{6.8}$$

(1) Note that there exists such a $\delta \in \mathbf{R}$ if and only if $n > m$.

with δ' arbitrary. If in addition L is injective, there exists a constant c such that the following inequality also holds:

$$\|u\|_{H_{s,\delta}} \leqslant c\|Lu\|_{H_{s-m,\delta+m}}. \tag{6.9}$$

Proof. Following the argument of Theorem 6.1, we now use Lemma 5.2 to estimate u_A by

$$\|u_A\|_{H_{s,\delta}} \leqslant c\|L_\infty u_A\|_{H_{s-m,\delta+m}} \tag{6.10}$$

where

$$L_\infty u_A \equiv A_m D^m u_A = Lu_A - (a_m - A_m) D^m u_A - \sum_{k=0}^{m-1} a_k D^k u_A.$$

By hypothesis I we have $a_m - A_m \in H_{s_m,\delta_m}$ and $a_k \in H_{s_k,\delta_k}$, $0 \leqslant k \leqslant m-1$, with $\delta_k > m-k-n/2$, $0 \leqslant k \leqslant m$. Thus we have also $a_m - A_m \in H_{s_m,\tilde{\delta}_m}$ and $a_k \in H_{s_k,\tilde{\delta}_k}$, $0 \leqslant k \leqslant m-1$, for some $\tilde{\delta}_k$ such that $\delta_k > \tilde{\delta}_k > m-k-n/2$. Therefore, by Lemma 2.5

$$\left\|(a_m - A_m) D^m u_A + \sum_{k=0}^{m-1} a_k D^k u_A\right\|_{H_{s-m,\delta+m}} \leqslant c\left(\|a_m - A_m\|_{H_{s_m,\tilde{\delta}_m}} + \sum_{k=0}^{m-1}\|a_k\|_{H_{s_k,\tilde{\delta}_k}}\right)\|u_A\|_{H_{s,\delta}}.$$

Now for any $f \in H_{s,\varrho}$ restricted to $M - \bar{B}_R$ and any $\varrho \in \mathbf{R}$, we have if $\varrho' < \varrho$

$$\|f\|_{H_{s,\varrho'}(M-\bar{B}_R)} \leqslant R^{\varrho'-\varrho}\|f\|_{H_{s,\varrho}(M-\bar{B}_R)}.$$

It follows that if we take R large enough we can deduce from (6.10)

$$\|u_A\|_{H_{s,\delta}} \leqslant c\|Lu_A\|_{H_{s-m,\delta+m}},$$

which, together with (6.6), (6.4) and (6.1) gives inequality (6.8) with δ' arbitrary. From this inequality we can prove, by following the argument of Lemma 5.2, that if L is injective inequality (6.9) is also satisfied.

THEOREM 6.3. *If the operator L satisfies hypotheses I and II, if $s \geqslant m$, $s_k \geqslant s-m$ and if $-n/2 < \delta < -m+n/2$ then L maps $H_{s,\delta}$ into $H_{s-m,\delta+m}$ with finite dimensional kernel and closed range.*

Proof. Since L is continuous, ker (L) is a closed subspace of $H_{s,\delta}$. It is finite dimensional if and only if the set $S = \{u \in \ker(L) \mid \|u\|_{H_{s,\delta}} = 1\}$ is compact. It therefore suffices to show that every sequence in the closed subset S contains a subsequence which is Cauchy in $H_{s,\delta}$. Let $\{u_n\}$ be such a sequence. If $\delta' < \delta$ we have $\|u_n\|_{H_{s-1,\delta'}} \leqslant 1$ and since $\|u_n\|_{H_{s,\delta}} = 1$

by Lemma 2.1 there exists a subsequence, still denoted $\{u_n\}$ which is Cauchy in $H_{s-1,\delta'}$. Applying inequality (6.8) of Theorem 6.2 to $u_n - u_{n'}$ we have, considering that $\{u_n\} \subset \ker(L)$

$$\|u_n - u_{n'}\|_{H_{s,\delta}} \leq c\|u_n - u_{n'}\|_{H_{s-1,\delta'}}$$

which shows that $\{u_n\}$ is Cauchy in $H_{s,\delta}$. Hence ker (L) is finite dimensional.

To show that the range of L is closed we note that since ker (L) is finite dimensional we may write $H_{s,\delta} = \ker(L) \otimes W$ with W closed. The operator L is injective on W; therefore, as in Theorem 6.2 there exists a constant c such that

$$\|u\|_{H_{s,\delta}} \leq c\|Lu\|_{H_{s-m,\delta+m}} \tag{6.11}$$

holds for all $u \in W$. Now let $\{f_n\}$ be a sequence in $L(H_{s,\delta})$ such that $f_n \to f \in H_{s-m,\delta+m}$. Let $\{w_n\}$ be the corresponding sequence in W such that $Lw_n = f_n$, $n \in \mathbf{N}$. Applying (6.11) to $w_n - w_{n'}$ we see that $\{w_n\}$ is a Cauchy sequence in W. Hence, since W is closed, $w_n \to w \in W$ and $Lw = f$. Thus the range of L is closed. □

We denote by E_s, $s \geq m$, the set of elliptic linear differential operators $L = \sum_{k=0}^{m} a_k D^k$, whose coefficients satisfy hypotheses I and II, with $s_k \geq s - m$. We define a metric (hence a topology) on E_s by setting

$$d(L, L') = \sup_{0 \leq k \leq m} \|a_k - a'_k\|_{H_{s_k,\delta_k}}.$$

THEOREM 6.4. *Let Lu be an elliptic linear differential system*

$$Lu = \sum_{k=0}^{m} a_k D^k u$$

such that L belongs to a continuous family $L_t \in E_s$, $t \in I = [0, 1]$, $L_1 \equiv L$, of injective operators. Then if L_0 is an isomorphism $H_{s,\delta} \to H_{s-m,\delta+m}$, with

$$-n/2 < \delta < -m + n/2,$$

the same is true for L.

Proof. L_0 being an isomorphism, the same is true of L_t for $t < \varepsilon$. One shows that L_ε, which is by hypothesis injective, is also surjective, by considering that if $f \in H_{s-m,\delta+m}$, there exists, for each $t < \varepsilon$, $u_t \in H_{s,\delta}$ such that $L_t u_t = f$. By Theorem 6.2 we have for every $v \in H_{s,\delta}$ and $t \in I$

$$\|v\|_{H_{s,\delta}} \leq c\|L_t v\|_{H_{s-m,\delta+m}},$$

where the constant c can be chosen independent of t, since I is compact. Thus we have

$$\|u_t\|_{H_{s,\delta}} \leqslant c\|f\|_{H_{s-m,\delta+m}}.$$

Hence if t_n is a sequence of numbers converging to ε the sequence $\{u_{t_n}\}$ is uniformly bounded in $H_{s,\delta}$ and therefore admits a subsequence still denoted $\{u_{t_n}\}$ which converges weakly to some u in $H_{s,\delta}$. The sequence $\{L_{t_n}u_n = f\}$ converges weakly to $L_\varepsilon u$ in $H_{s-m,\delta+m}$. Thus $L_\varepsilon u = f$, and L_ε is surjective. □

In the case that E is the trivial tensor bundle $M \times \mathbf{R}$ and $m=2$, the elliptic linear operator, satisfying the regularity hypothesis I

$$Lu = a_2 \cdot D^2 u + a_1 \cdot Du + a_0,$$

acting on scalar functions u on M, is by the maximum principle (cf. for example [8]) injective on $H_2^{\text{loc}} \cap C^0_{\delta'}$, $\delta' > 0$, if a_1 is bounded and $a_0 \leqslant 0$. Thus if L is a linear differential operator of order 2 acting on scalar functions, $L \in E_s$ and $a_0 \leqslant 0$ implies that L is injective on $H_{s,\delta}$ if $s > n/2+2$, $\delta > -n/2$. Since the subset I_s of operators in E_s having $a_0 \leqslant 0$ is convex, by Theorem 6.4 if there is some $L_0 \in I_s$ which is an isomorphism $H_{s,\delta} \to H_{s-2,\delta+2}$, with $-n/2 < \delta < -2+n/2$, then every $L \in I_s$ is such an isomorphism. We shall in fact show

THEOREM 6.5. *$(\Delta_e)^\mu$ with μ a positive integer less than $n/2$ is an isomorphism: $H_{s,\delta} \to H_{s-2\mu,\delta+2\mu}$ if $s > 2\mu + n/2$ and $-n/2 < \delta < -2\mu + n/2$.*

Proof. (1) Δ_e is by the maximum principle injective on $C^2 \cap C^0_{\delta'}$, $\delta' > 0$, therefore on $H_{s,\delta}$ if $s > n/2+2$, $\delta > -n/2$. If $f \in \mathcal{D}$, $\Delta_e u = f$ has one solution [4] $u \in C^\infty_{n-2}$, thus $u \in H_{s,\delta}$, $\delta < n/2 - 2$. By Theorem 6.2 this solution satisfies the inequality

$$\|u\|_{H_{s,\delta}} \leqslant c\|f\|_{H_{s-2,\delta+2}},$$

which gives, by completion of $\mathcal{D}$ in the $H_{s-2,\delta+2}$ norm the surjectivity of Δ_e on $H_{s-2,\delta+2}$ and the continuity of its inverse.

(2) We then proceed by induction. Assuming the theorem for $\mu = \nu - 1$, we shall demonstrate it for $\mu = \nu$. Let $u \in H_{s,\delta}$ with $s > 2\nu + n/2$, $\delta > -n/2$, satisfy $(\Delta_e)^\nu u \equiv \Delta_e(\Delta_e^{\nu-1} u) = 0$. Since $\Delta_e^{\nu-1} u \in H_{s-2(\nu-1),\delta+2(\nu-1)}$ and $s - 2(\nu-1) > 2 + n/2$, $\delta + 2(\nu-1) > -n/2$, it follows that $\Delta_e^{\nu-1} u = 0$ and therefore, by the assumed injectivity of $\Delta_e^{\nu-1}$, $u=0$. Hence $(\Delta_e)^\nu$ is injective on $H_{s,\delta}$. Let now $f \in H_{s-2\nu,\delta+2\nu}$ with $s > 2\nu + n/2$, $-n/2 < \delta < -2\nu + n/2$. We shall show that there exists a $u \in H_{s,\delta}$ such that $(\Delta_e)^\nu u = f$. We write

$$\Delta_e v = f, \; (\Delta_e)^{\nu-1} u = v.$$

Since $f \in H_{s'-2,\delta'+2}$ with $s' = s - 2(\nu-1) > 2 + n/2$ and $\delta' = \delta + 2(\nu-1)$ satisfying $-n/2 < \delta' < -2 + n/2$ there is a $v \in H_{s',\delta'} = H_{s-2(\nu-1),\delta+2(\nu-1)}$ such that $\Delta_e v = f$. Then the inductive hypothesis gives a $u \in H_{s,\delta}$ such that $(\Delta_e)^{\nu-1}u = v$. Thus $(\Delta_e)^\nu$ is surjective on $H_{s-2\nu,\delta+2\nu}$. Finally, the continuity of the inverse of $(\Delta_e)^\nu$ follows from that of the inverses of $(\Delta_e)^{\nu-1}$ and Δ_e.

We conclude with the following theorem which is a consequence of the preceding ones and is useful in general relativity (a particular case $L = \Delta_e$ on $\mathbf{R}^n$ is treated in [12]).

THEOREM 6.6. *Let L be the linear second order operator on scalar functions over M*

$$L \equiv \Delta_g + f$$

where Δ_g is the laplace operator of an asymptotically euclidean riemannian metric g on (M, e), that is such that

$$g - e \in H_{\sigma,\varrho}, \quad \sigma > n/2 + 1, \; \varrho > -n/2$$

and f is a scalar function, $f \in H_{s_0,\delta_0}$, $\delta_0 > 2 - n/2$, $s_0 > n/2 - 1$, $f \leqslant 0$, then L is an isomorphism $H_{s,\delta} \to H_{s-2,\delta+2}$ if $2 \leqslant s \leqslant 2 + \inf(s_0, \sigma - 1)$, $s > n/2$, $-n/2 < \delta < -2 + n/2$.

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