ON COMPACT KÄHLER MANIFOLDS OF NONNEGATIVE BISECTIONAL CURVATURE, II

BY

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This paper is a sequel to the preceding paper [HSW]. This study of compact Kähler manifolds of nonnegative bisectional curvature was inspired by the recent solution of the Frankel conjecture by S. Mori ([M]) in a general algebraic setting, and subsequently by Siu and Yau ([SY]) in the special context of Kähler geometry. With the case of positive bisectional curvature out of the way, a general understanding of the case of nonnegative bisectional curvature is naturally the next order of business. For complex surfaces, the work of Howard and Smyth ([HS]) achieves a complete classification. In higher dimensions, the main conclusion of these two papers is that the study of compact Kähler manifolds of nonnegative bisectional curvature can be essentially reduced to the special case where simple connectivity and the isomorphism $H^2(M, \mathbf{Z}) \cong \mathbf{Z}$ are in addition assumed (the theorem of [HSW] and Theorem C below), and that with a mild positivity assumption these two desirable properties would follow in any case (Theorem E below). We begin by listing the main results; their proofs will be given in subsequent sections.

THEOREM A. *Let M be an n-dimensional compact Kähler manifold with nonnegative Ricci curvature. If the maximum rank of the Ricci tensor on M is $n-k$, then:*

(A) $h^{p,0}(M)=0$ *for* $p=k+1, \ldots, n$ *($h^{p,q}(M)$ denotes the dimension of the space of harmonic (p, q)-forms).*

(B) $h^{1,0}(M) \leq k$, *and* $h^{1,0}(M)=0$ *iff* $\pi_1(M)$ *is finite.*

(C) *If in addition the bisectional curvature is nonnegative, then* $h^{1,0}(M)=k$.

For the next theorem, recall from [Wu2] that a covariant Hermitian tensor is *quasi-positive* iff it is positive definite at one point and positive semi-definite everywhere; the

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Chern class of a holomorphic line bundle is *quasi-positive* iff it contains a quasi-positive real (1,1)-form. (Throughout this paper, the cohomology ring over **C** is identified with the deRham cohomology ring.)

THEOREM B. *A compact Kähler manifold M satisfying any one of the following conditions is a simply connected algebraic manifold without any nonzero holomorphic q-form for $q \geq 1$:*

(A) *The first Chern class $c_1(M)$ is quasi-positive.*

(B) *M has nonnegative bisectional curvature and $h^{1,0}(M)=0$.*

(C) *M has nonnegative bisectional curvature and $h^{1,1}(M)=1$.*

The fact that M is algebraic if $c_1(M)$ is quasi-positive is a special case of a theorem of Riemenschneider ([R]); the fact that M is simply connected if $c_1(M)$ is quasi-positive generalizes a theorem of Kobayashi ([K]) who assumed that $c_1(M)$ is positive. Theorem B together with the theorem of [HSW] imply that if the first Chern class of a compact Kähler manifold of nonnegative bisectional curvature is quasi-positive, then the manifold possesses an Einstein–Kähler metric.

THEOREM C. *Let M_1, M_2 be simply connected compact manifolds and let $M \equiv M_1 \times M_2$. Then every Kähler metric with nonnegative bisectional curvature on M is a product of Kähler metrics on M_1 and M_2. In particular, M possesses a Kähler metric of nonnegative bisectional curvature iff each of M_1 and M_2 does.*

The assumption of simple-connectivity in the preceding theorem is necessary because there are many flat complex tori which are biholomorphic but not isometric to a product of tori. This isometric splitting phenomenon is formally analogous to a theorem of Paul Yang ([Y]) on compact Kähler manifolds of negative bisectional curvature, but the underlying reasons are entirely different. In fact, the proof of Theorem C touches on the cancellation problem for complex manifolds (cf. [Br]), but we managed to bypass this difficult question by systematically exploiting the Kähler assumption on the metric. The proof of Theorem C also shows that if a product of simply connected compact complex manifolds is biholomorphic to a Hermitian symmetric space, then so is each factor. The next theorem uses a few technical concepts: a holomorphic line bundle is *quasi-positive* iff its Chern class is; an *exceptional analytic set* in a complex space is understood in the sense of the well-known work of Grauert ([G]); *blowing-up* and *blowing-down* along a submanifold are understood in the usual sense of quadratic transforms (cf. [GrH], pp. 602–608).

THEOREM D. *Let M be a simply-connected Kähler manifold of nonnegative bisectional curvature. Then:*

(A) *Every quasi-positive holomorphic line bundle on M is positive.*

(B) *Every holomorphic line bundle on M defined by a nonzero effective divisor is positive.*

(C) *M cannot be blown-down along a submanifold.*

(D) *If $H^2(M, \mathbf{Z}) \cong \mathbf{Z}$ and $\dim M \neq 3$, then M has no exceptional nonsingular hypersurfaces.*

Conditions (A)–(C) above are usually false and in general quite elusive; it is therefore remarkable that they should hold for an identifiable sub-collection of complex manifolds. We conjecture that compact Kähler manifolds of nonnegative bisectional curvature (simply-connected or not) are *minimal* in the sense that if M is such a manifold and $\tau: M \to M'$ is a holomorphic map into a compact complex manifold M' such that τ is biholomorphic outside an analytic subset of M, then τ is globally biholomorphic on M.

Because of Theorem C and the theorem of [HSW], it is of interest to determine which of the compact Kähler manifolds of nonnegative bisectional curvature satisfy $\pi_1(M) = 1$ and $H^2(M, \mathbf{Z}) \cong \mathbf{Z}$. The following is a general criterion in terms of the positivity of a certain exterior power of the holomorphic tangent bundle. $\wedge^k TM$ (the kth exterior power of the holomorphic tangent bundle TM) is said to be *quasi-positive* iff its curvature (with respect to the metric induced by the Kähler metric) is everywhere nonnegative and positive at a point. In greater detail, let $R_{xy}w$ be the curvature tensor of the Kähler metric and let $\{e_1, ..., e_n, Je_1, ..., Je_n\}$ be an orthonormal basis of the tangent space M_m at $m \in M$; then the quasi-positivity of $\wedge^k TM$ means that for all $m \in M$, for all $x \in M_m$ and for all orthonormal bases $\{e_j, Je_j\}$ of M_m, $\sum_{i=1}^k \langle R_{e_i Je_i} x, Jx \rangle \geq 0$, and that there exists an $m \in M$ at which strict inequality holds whenever $x \neq 0$ (cf. [KW] and [S] for more details). With this definition, TM being quasi-positive means exactly that M has *quasi-positive bisectional curvature*, i.e., all bisectional curvatures are nonnegative everywhere and are all positive at a point (cf. [Gr]).

THEOREM E. *Let M be a compact Kähler manifold with nonnegative bisectional curvature and suppose*

$$\begin{cases} \wedge^m TM \quad \textit{is quasi-positive if} \quad \dim M = 2m, \\ \wedge^{m+1} TM \quad \textit{is quasi-positive if} \quad \dim M = 2m+1. \end{cases}$$

Then M is simply connected and $H^2(M, \mathbf{Z}) \cong \mathbf{Z}$.

Simple examples, e.g., $P_m\mathbf{C} \times P_m\mathbf{C}$ and $P_m\mathbf{C} \times P_{m+1}\mathbf{C}$, show that the theorem is optimal with respect to the positivity assumption. The proof of the theorem itself is nothing more than a refinement of the arguments of M. Berger, Bishop, Goldberg and Kobayashi ([BG], [GK]).

As mentioned earlier, the theorem of [HSW] and Theorem C above together suggest a more detailed study of compact Kähler manifolds M of nonnegative bisectional curvature which further satisfy $\pi_1(M)=1$ and $H^2(M, \mathbf{Z})\cong\mathbf{Z}$. For these manifolds, the results of [M] and [SY] point to the strong possibility that the free part of $H_2(M, \mathbf{Z})$ should be integral multiples of a rational curve in M. To be more precise, this would be the case if one could show that, under the above assumptions, every energy minimizing map from the Riemann sphere into M is either holomorphic or anti-holomorphic (see Proposition 1 of [SY]). Assuming this for the moment, then the method of [SY] would prove a more general statement: *a compact Kähler manifold of quasi-positive bisectional curvature is biholomorphic to complex projective space;* such a result would seem inaccessible to purely algebraic methods. Returning to those M with $\pi_1(M)=1$ and $H^2(M, \mathbf{Z})\cong\mathbf{Z}$, one *conjectures* (with a bit of wishful thinking to be sure) that they are all biholomorphic to irreducible Hermitian symmetric spaces; furthermore, the rank of the symmetric space should be related to the least integer k such that $\wedge^k TM$ is quasi-positive. In connection with the latter, note the discussion at the beginning of Section 4 below. A more modest conjecture is that these manifolds are at least rational algebraic manifolds.

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Section 1

This section supplies the proofs of Theorems A and B. We shall assume the preliminary material in Section 1 of [HSW].

Proof of Theorem A. If the Ricci tensor of M has maximum rank $n-k$, then the universal covering of M is holomorphically isometric to $M'\times\mathbf{C}^l$ with $l\leqslant k$, where M' is a compact Kähler manifold whose Ricci tensor has maximum rank $n-k$. Consider the finite covering $M^*\to M$ guaranteed by the Cheeger–Gromoll theorem such that M^* is diffeomorphic to $M_\#\times T^l$, where T^l is an l-dimensional complex torus and $M_\#$ is a compact Kähler manifold whose universal covering is M'. Since $M_\#$ has a finite fundamental group, $\dim H^1(M^*, \mathbf{R})=\dim H^1(T^l, \mathbf{R})=2l\leqslant 2k$. Both M^* and M being oriented, Hodge theory implies that $\dim H^1(M, \mathbf{R})=\dim H^1(M^*, \mathbf{R})\leqslant 2k$. Hence $h^{1,0}(M)=\frac{1}{2}\dim H^1(M, \mathbf{R})\leqslant k$. Moreover, with the same notation, $h^{1,0}(M)=0\Leftrightarrow\dim H^1(M, \mathbf{R})=0\Leftrightarrow\dim H^1(M^*, \mathbf{R})=0\Leftrightarrow l=0\Leftrightarrow$ the universal covering of M is the compact manifold M'; this is equivalent to the finiteness of $\pi_1(M)$. Thus part (B) is proved. If in addition the bisectional curvature is nonnegative, then the theorem of [HSW] implies that $l=k$. The preceding inequalities then become equalities and (C) immediately follows. To prove part (A), i.e., the assertion concerning $h^{p,0}(M)$, simply note that it is a consequence of Corollary 3 of [KW] together with the remark in [Wu2] that it suffices to have the positivity of $n-k$ eigenvalues at one point if all the eigenvalues of the Ricci tensor are everywhere nonnegative. Q.E.D.

Proof of Theorem B. We first recall a weak form of the Atiyah–Singer fixed point theorem ([AS], (4.6)): If M is a compact complex manifold and G is a finite group of (holomorphic) automorphisms acting on M, then for each $g \in G$, there is a cohomology class θ_g of M such that

$$\sum_j (-1)^j \operatorname{trace}\,(g \,|\, H^j(M, O)) = \theta_g[M^g], \tag{1}$$

where O denotes the structure sheaf of M and M^g denotes the fixed point set of g in M. We have the following simple consequence.

LEMMA 1. *Let M be a compact complex manifold without any nonzero holomorphic q-forms for $q \geq 1$, and let G be a finite group of automorphisms of M. Then every element of G has a fixed point.*

Proof. Indeed, since $H^j(M, O)$ is the space of all holomorphic j-forms, the left side of (1) reduces to 1 by hypothesis. Thus M^g is never empty for each $g \in G$. (Note: this argument is basically not different from the one using the Hirzebruch proportionality principle, cf. e.g. [K], Lemma 1.) Q.E.D.

We now prove part (A) of Theorem B. Let $\varphi \in c_1(M)$ be a quasi-positive real (1,1)-form. By Yau's solution of Calabi's conjecture ([Ya]), there is a Kähler metric H on M whose Ricci form is φ. Thus H has quasi-positive Ricci curvature. By Theorem A, $h^{q,0}(M)=0$ for $q \geq 1$, and $\pi_1(M)$ is finite. The finite group $\pi_1(M)$ then acts as a group of automorphisms on the compact universal covering manifold M' of M. Relative to the pull-back of H to M', M' also has quasi-positive Ricci curvature. Thus also $h^{q,0}(M')=0$ for $q \geq 1$ and by Lemma 1, each element of $\pi_1(M)$ must have a fixed point in M. This is possible only if $\pi_1(M)$ reduces to the identity, i.e., M is itself simply connected. Finally it was already pointed out that M is algebraic because of Riemenschneider's theorem ([R]; see also the remark at the end of [Wu2]).

To prove part (B), let $h^{1,0}(M)=0$ and let the bisectional curvature of M be nonnegative. Since the first Betti number is zero, Corollary 1 of [HSW] shows that M is simply connected; now the theorem of [HSW] shows that M is algebraic and is without nonzero holomorphic q-forms for $q \geq 1$ (part (B) of that theorem).

Finally to prove part (C), let M have nonnegative bisectional curvature and let $h^{1,1}(M)=1$. Suppose $\dim M=n$. If $n=1$, then M is biholomorphic to $P_1\mathbf{C}$ by the Gauss–Bonnet theorem and the classification of compact Riemann surfaces. The theorem is then obvious in this case. Let now $n>1$. If bisectional curvature is identically zero, then so is the curvature tensor. Thus there exist an n-dimensional complex torus T and a finite covering $\pi: T \to M$ (cf. [Wo], Chapter 3). Then from Hodge theory, $h^{1,1}(M)=h^{1,1}(T)=\binom{2n}{2}>1$ (binomial coefficient), a contradiction. Thus the bisectional curvature is positive somewhere, and so is the Ricci form φ. If ω is the Kähler form of M, then $\varphi \wedge \omega^{n-1}$ is everywhere nonnegative and is positive somewhere. Hence $\int_M \varphi \wedge \omega^{n-1}>0$. On the other hand, since $h^{1,1}(M)=1$, the harmonic component of φ is equal to $c\omega$ for some $c \in \mathbf{R}$. Thus $\int_M (c\omega) \wedge \omega^{n-1}=\int_M \varphi \wedge \omega^{n-1}$, which is positive. This proves $c>0$, so that φ is cohomologous to the positive form $c\omega$. The first Chern class of M is therefore positive, and part (C) is now a consequence of part (A). Q.E.D.

Section 2

This section proves Theorem C. We begin with a useful lemma.

LEMMA 2. *Let $\varphi: M_1 \to M_2$ be a holomorphic map between n-dimensional compact Kähler manifolds M_1 and M_2 whose Kähler forms ω_1 and ω_2 satisfy $\int_{M_1} \omega_1^n = \int_{M_2} \omega_2^n$ and $\varphi^*[\omega_2]=[\omega_1]$ ($[\omega_i]$ denotes the cohomology class of ω_i). Then φ is biholomorphic.*

Proof. Suppose φ is everywhere *degenerate*, i.e., the Jacobian determinant $\mathfrak{J}\varphi$ of φ is everywhere zero, then $\varphi^*\omega_2^n=0$ so that $\varphi^*[\omega_2^n]=0$. This contradicts $\varphi^*[\omega_2^n]=[\omega_1^n]\neq 0$. Thus $\mathfrak{J}\varphi$ is zero at most on a hypersurface J (possibly disconnected). Since M_1 and M_2 are both compact and φ is orientation preserving, a standard argument shows that φ must be onto. It remains to show φ is injective. We may assume $\int_{M_1} \omega_1^n = \int_{M_2} \omega_2^n = 1$ so that $[\omega_1^n]$ and $[\omega_2^n]$ are the fundamental cohomology classes of M_1 and M_2 respectively. Since $\varphi^*[\omega_2^n]=[\omega_1^n]$, the topological degree of φ is equal to 1. Since φ is orientation preserving, Sard's theorem plus standard algebraic topology show that for all regular values y of φ in M_2 (i.e., $\forall y \in M_2-\varphi(J)$), $\varphi^{-1}(y)$ consists of exactly one point. (This can also be proved without any algebraic topology by invoking Lemma 2.12 of [Wu1].) Thus the restriction $\varphi: M_1 - \varphi^{-1}(\varphi(J)) \to M_2-\varphi(J)$ is a biholomorphic mapping. Note that since $\varphi(J)$ is a subvariety of M_2 (proper mapping theorem), $\varphi^{-1}(\varphi(J))$ is a proper subvariety of M_1 so that $M_1-\varphi^{-1}(\varphi(J))$ is open dense in M_1.

If J is nonempty, we shall deduce a contradiction. Thus let J be a hypersurface in M_1. Observe that $\varphi(J)$ is also of pure dimension $n-1$ in M_2 because $\varphi^*[\omega_2]=[\omega_1]$ implies $\varphi^*[\omega_2^{n-1}]=[\omega_1^{n-1}]$ which implies that for each branch J' of J,

$$\int_{\varphi(J')} \omega_2^{n-1} = \varphi^*[\omega_2^{n-1}](J') = [\omega_1^{n-1}](J') = \int_{J'} \omega_1^{n-1} \neq 0.$$

Thus $\varphi\colon J \to \varphi(J)$ is a surjective holomorphic mapping between complex spaces of pure dimension $n-1$. Let J^s and $\varphi(J)^s$ be the singular points of the spaces J and $\varphi(J)$ and let $J_1 \equiv J - J^s - \varphi^{-1}(\varphi(J)^s)$, $\varphi(J)_2 \equiv \varphi(J) - \varphi(J)^s$. Then the restriction map $\varphi_\#\colon J_1 \to \varphi(J)_2$ is a holomorphic mapping between complex manifolds of dimension $n-1$, and is furthermore nondegenerate somewhere because $\varphi\colon J \to \varphi(J)$ is surjective to begin with. Let $x \in J_1$ be a point at which the Jacobian determinant of $\varphi_\#$ is nonzero, and let $y = \varphi_\#(x)$. Now choose coordinate functions $\{z_1, \ldots, z_{n-1}, w\}$ of M_1 centered at x and coordinate functions $\{u_1, \ldots, u_{n-1}, v\}$ of M_2 centered at y such that locally $J_1 = \{w=0\}$ and $\varphi(J)_2 = \{v=0\}$. Relative to these coordinate functions, let the components of $\varphi_\#$ be $\{f_1(z,w), \ldots, f_{n-1}(z,w), g(z,w)\}$, where we have written $z \equiv (z_1, \ldots, z_{n-1})$. Now along J (and near x), the Jacobian determinant of $\{f_1, \ldots, f_{n-1}, g\}$ is zero. Thus at all points of the form $(z, 0)$, we have

$$\det \left[\begin{array}{c|c} \dfrac{\partial f}{\partial z} & \begin{matrix} * \\ \vdots \\ * \end{matrix} \\ \hline 0 \ldots 0 & \dfrac{\partial g}{\partial w} \end{array} \right] = 0,$$

where the row of zeros is due to the fact that $d\varphi((\partial/\partial z_i)(0)) = \sum_j a_{ij}(\partial/\partial u_j)(0)$ for $i=1, \ldots, n-1$. Since the Jacobian determinant of $\varphi_\#$ is nonzero along J_1 near x, $\det[(\partial f/\partial z)(z,0)] \neq 0$ for all z in a small neighborhood B of 0. Hence $(\partial g/\partial w)(z,0) \equiv 0$ for all z in B. From the well-known fact in one complex variable, we deduce the existence of an integer k, $k \geqslant 2$, such that for each fixed z in B, the mapping $w \mapsto g(z,w)$ is k to 1. This implies that for some open set W of M_1 containing x, $\varphi\colon W - J \to \varphi(W) - \varphi(J)$ is a k to 1 map. This contradicts the fact established above that φ is biholomorphic on a dense open subset of M_1. Thus J must be empty and φ is biholomorphic on all of M_1. Q.E.D.

Proof of Theorem C. Given $M = M_1 \times M_2$, where $\dim M_\alpha \geqslant 1$ for $\alpha = 1, 2$. Suppose M has a Kähler metric of nonnegative bisectional curvature. Since M is compact and simply connected, M splits holomorphically and isometrically into $Q_1 \times \ldots \times Q_s$, where each Q_i is a simply connected compact Kähler manifold such that $H^2(Q_i, \mathbf{Z}) \cong \mathbf{Z}$ for each i. We may assume that the Kähler form ω_i of Q_i is the positive generator of $H^2(Q_i, \mathbf{Z})$. Let $\pi_i: M \to Q_i$ be the natural projection; $\pi_i^* \omega_i$ will henceforth be abbreviated to ω_i. Thus

$$H^2(M, \mathbf{R}) = \mathbf{R}[\omega_1] \oplus \dots \oplus \mathbf{R}[\omega_s]. \tag{2}$$

Let $p_\alpha: M \to M_\alpha$ be the natural projection for $\alpha = 1, 2$. Then

$$H^2(M, \mathbf{R}) = p_1^* H^2(M_1, \mathbf{R}) \oplus p_2^* H^2(M_2, \mathbf{R}). \tag{3}$$

After re-numbering if necessary, we may assume that for an integer u, $1 \leqslant u \leqslant s$,

$$p_1^* H^2(M, \mathbf{R}) \subset \mathbf{R}[\omega_1] \oplus \dots \oplus \mathbf{R}[\omega_u]. \tag{4}$$

This expression may be assumed to be *irredundant* in the sense that if any one factor $\mathbf{Z}[\omega_j]$ $(1 \leqslant j \leqslant u)$ is deleted from (4), the inclusion will no longer be valid. Let $\dim M_\alpha = m(\alpha)$ for $\alpha = 1, 2$ and let $\dim Q_i = q(i)$ for $i = 1, \dots, S$. We claim that $q(1) + \dots + q(u) = m(1)$. Indeed, let Ω_1 be the Kähler form of some Kähler metric on M_1 and let $p_1^*[\Omega_1] = \sum_{j=1}^u a_j[\omega_j]$, where each $q_j \in \mathbf{R}$. Raising both sides to the power $m(1)$, we get

$$0 \neq p_1^*[\Omega_1^{m(1)}] = \left(\sum_{j=1}^u a_j[\omega_j]\right)^{m(1)},$$

where the left inequality is due to the injectivity of p_1^* on cohomology. At the same time, $\omega_i^{q(i)+1} = 0$ for each $i = 1, \dots, s$, so that

$$\left(\sum_{j=1}^u a_j[\omega_j]\right)^{q(1)+\dots+q(u)+1} = 0.$$

Thus $m(1) \leqslant q(1) + \dots + q(u)$. Suppose strict inequality holds. An elementary vector space argument using the irredundancy of (4) shows that there exists an $[\eta] \in p_1^* H^2(M_1, \mathbf{R})$ such that $[\eta] = \sum_{j=1}^u b_j[\omega_j]$ and each $b_j \neq 0$. Then

$$[\eta^{q(1)+\dots+q(u)}] = (\sum_j b_j[\omega_j])^{q(1)+\dots+q(u)} = c_0 \left[\prod_{j=1}^{j=u} b_j^{q(j)} \omega_j^{q(j)}\right] \neq 0,$$

where the product $\prod_{j=1}^{j=u}$ denotes exterior product and c_0 denotes the product of binomial coefficients:

$$c_0 = \begin{pmatrix} \sum_{j=1}^{u} q(j) \\ \sum_{j=1}^{u-1} q(j) \end{pmatrix} \begin{pmatrix} \sum_{j=1}^{u-1} q(j) \\ \sum_{j=1}^{u-2} q(j) \end{pmatrix} \cdots \begin{pmatrix} q(1) + q(2) \\ q(1) \end{pmatrix}. \tag{5}$$

Since η is a form on M_1 and dim $M_1 = m(1)$, from the hypothesis that $m(1) < q(1) + \ldots + q(u)$ we deduce $\eta^{q(1)+\ldots+q(u)} = 0$. This is a contradiction. Thus $m(1) = q(1) + \ldots + q(u)$ and $u < s$.

Similarly, there is a subset $\{\alpha(1), \ldots, \alpha(k)\}$ of $\{1, \ldots, s\}$ such that

$$p_2^* H^2(M_2, \mathbf{R}) \subset \mathbf{R}[\omega_{\alpha(1)}] \oplus \ldots \oplus \mathbf{R}[\omega_{\alpha(k)}], \tag{6}$$

and such that this inclusion is irredundant. In the same way, we can prove $m(2) = q(\alpha(1)) + \ldots + q(\alpha(k))$. We now claim that in fact $\{\alpha(1), \ldots, \alpha(k)\} = \{u+1, \ldots, s\}$, so that

$$p_2^* H^2(M_2, \mathbf{R}) \subset \mathbf{R}[\omega_{u+1}] \oplus \ldots \oplus \mathbf{R}[\omega_s]. \tag{7}$$

To prove this, suppose $\alpha(1) \in \{1, \ldots, u\}$; for definiteness, let $\alpha(1) = 1$. Then

$$\begin{aligned} q(1) + \ldots + q(s) = m(1) + m(2) &= \{q(1) + \ldots + q(u)\} + \{q(1) + q(\alpha(2)) + \ldots + q(\alpha(k))\} \\ &\Rightarrow q(u+1) + \ldots + q(s) = q(1) + q(\alpha(2)) + \ldots + q(\alpha(k)) \\ &\Rightarrow \{\alpha(2), \ldots, \alpha(k)\} \neq \{u+1, \ldots, s\}. \end{aligned}$$

From (4) and (6), we conclude

$$p_1^* H^2(M_1, \mathbf{R}) \oplus p_2^* H^2(M_2, \mathbf{R}) \subsetneqq \mathbf{R}[\omega_1] \oplus \ldots \oplus \mathbf{R}[\omega_s],$$

which contradicts (3). Similarly no $\alpha(i)$ can belong to $\{1, \ldots, u\}$. Therefore $\{\alpha(1), \ldots, \alpha(k)\} \subset \{u+1, \ldots, s\}$, which proves (7). Comparing (4), (7) and (2), (3), we obtain:

$$\begin{aligned} p_1^* H^2(M_1, \mathbf{R}) &= \mathbf{R}[\omega_1] \oplus \ldots \oplus \mathbf{R}[\omega_u], \\ p_2^* H^2(M_2, \mathbf{R}) &= \mathbf{R}[\omega_{u+1}] \oplus \ldots \oplus \mathbf{R}[\omega_s]. \end{aligned} \tag{8}$$

Now fix $y_0 \in M_2$, and define $i_1: M_1 \to M_1 \times M_2 \equiv M$ by $i_1(x) = (x, y_0)$. Also let $\pi': M \equiv Q_1 \times \ldots \times Q_s \to Q_1 \times \ldots \times Q_u$ be the natural projection $(\pi_1, \ldots, \pi_u)$. Let $\varphi: M_1 \to Q_1 \times \ldots \times Q_u$ be the composite of the following maps:

$$M_1 \xrightarrow{i_1} M_1 \times M_2 \equiv M \equiv Q_1 \times \ldots \times Q_s \xrightarrow{\pi'} Q_1 \times \ldots \times Q_u.$$

Similarly, let $\psi: Q_1 \times \ldots \times Q_u \to M_1$ be the composite of the following maps:

$$Q_1 \times \ldots \times Q_u \xrightarrow{j'} Q_1 \times \ldots \times Q_s \equiv M \equiv M_1 \times M_2 \xrightarrow{p_1} M_1$$

where j' is defined by fixing $q_j \in Q_j$ $(j = u+1, \ldots, s)$ and let $j'(z_1, \ldots, z_u) = (z_1, \ldots, z_u, q_{u+1}, \ldots, q_s)$. We claim $\psi\varphi: M_1 \to M_1$ is a holomorphic automorphism. To see this, let Ω be the Kähler form of some Kähler metric on M_1 and let $p_1^*[\Omega]=\sum_{j=1}^u a_j[\omega_j]$, where $a_j\in\mathbf{R}$ for all $j=1, ..., u$ (see (8)). Raising both sides to the power $m(1)=q(1)+...+q(u)$, we get

$$p_1^*[\Omega^{m(1)}]=c_0[a_1^{q(1)}\omega_1^{q(1)}\wedge \dots \wedge a_u^{q(u)}\omega_u^{q(u)}],$$

where c_0 is as in (5). Since $\Omega^{m(1)}\neq 0$ and since p_1^* is injective on cohomology, the left side is nonzero. This implies $a_j\neq 0$ for each j. Now Ω is a positive form on M_1 and p_1 is a holomorphic map, so $p_1^*\Omega$ is positive semi-definite on M. Since each ω_j is positive semi-definite on M, we deduce from $[p_1^*\Omega]=[\sum_j a_j\omega_j]$ that each $a_j\geqslant 0$. Hence $a_j>0$ $\forall j$. It follows that each $a_j\omega_j$ is the Kähler form of some Kähler metric on Q_j and that $\omega\equiv a_1\omega_1+...+a_u\omega_u$ is the Kähler form of some Kähler metric on $Q_1\times ...\times Q_u$. Observe also that the composite mapping $M_1\xrightarrow{i_1} M_1\times M_2\xrightarrow{p_1} M_1$ given by $x\mapsto(x, y_0)\mapsto x$ is the identity so that $i_1^*p_1^*=H^2(M_1,\mathbf{R})\to H^2(M_1,\mathbf{Z})$ is the identity. In view of (8), $i_1^*[\omega]=i_1^*p_1^*[\Omega]=[\Omega]$. Combining these remarks we see that $\varphi^*[\omega]=[\Omega]$. In an entirely analogous manner, $\psi^*[\Omega]=[\omega]$. Altogether, $(\psi\varphi)^*[\Omega]=[\Omega]$. By Lemma 2, $\psi\varphi$ is biholomorphic as claimed. Consequently φ: $M_1\to Q_1\times...\times Q_u$ is an injective holomorphic mapping between compact complex manifolds of the same dimension. Since φ is automatically onto under the circumstance, φ is itself biholomorphic.

With the same argument, one shows that M_2 and $Q_{u+1}\times...\times Q_s$ are biholomorphic. Since M is holomorphically isometric to $(Q_1\times...\times Q_u)\times(Q_{u+1}\times...\times Q_s)$, M is also holomorphically isometric to $M_1\times M_2$. Q.E.D.

Section 3

We now prove Theorem D. Since M is simply connected by assumption, the theorem of [HSW] implies that M is holomorphically isometric to $Q_1\times...\times Q_s$ where the notation will be as in the proof of Theorem C in the preceding section. In particular, we recall

$$H^2(M,\mathbf{R}) = \mathbf{R}[\omega_1]\oplus ... \oplus\mathbf{R}[\omega_s]. \tag{2}$$

Proof of (A). Let L be a quasi-positive line bundle on M and let $c_1(L)=\sum_{i=1}^s a_i[\omega_i]$, $a_i\in\mathbf{R}$. Let Φ be a quasi-positive real (1, 1)-form representing $c_1(L)$, and suppose Φ is positive definite at $x=(x_1, ..., x_s)\in Q_1\times...\times Q_s\equiv M$. Since M is algebraic by [HSW], so is each Q_i. Let C be a curve in Q_1 through x_1; we may identify C with $(C, x_2, ..., x_s)\subset M$. Then

$$0<\int_C\Phi=\int_C c_1(L)=\int_C a_1\omega_1=a_1\cdot(\text{volume of } C).$$

This shows $a_1>0$. Similarly $a_i>0$ $\forall i$. Thus $c_1(L)=[\sum_i a_i\omega_i]$ and $c_1(L)$ is positive. Q.E.D.

Proof of (B). Let H be a hypersurface in M; it defines a holomorphic bundle whose Chern class we simply denote by $c_1(H)$. It suffices to show that $c_1(H)$ contains a positive real (1,1)-form. Using (2) again, we write $c_1(H)=\sum_{i=1}^{s} a_i[\omega_i]$, $a_i \in \mathbf{R}$. To prove $a_i>0$ $\forall i$, let h_i be a hypersurface in Q_i whose homology class $[h_i]$ is the Poincaré dual of ω_i in Q_i, and let

$$H_i \equiv Q_1 \times \ldots \times Q_{i-1} \times h_i \times Q_{i+1} \times \ldots \times Q_s.$$

Then H_i is a hypersurface in M whose homology class $[H_i]$ is the Poincaré dual of ω_i in M. From (2) we obtain

$$H_{2n-2}(M, \mathbf{R}) = \mathbf{R}[H_1] \oplus \ldots \oplus \mathbf{R}[H_s],$$

where $n=\dim M$. Since $c_1(H)$ is the Poincaré dual of $[H]$, we also have

$$[H] = \sum_{i=1}^{s} a_i[H_i].$$

Let C be a curve in Q_1 passing through a smooth point $x_1 \in h_1$ such that C has no component lying in h_1; such a curve can be constructed by standard procedures. Then the intersection number $[C]\circ[h_1]$ must be positive because the subvarieties C and h_1 satisfy $C \cap h_1 \neq \varnothing$ (cf. [GrH], pp. 63–64). If we identify C with the curve $(C, x_2, \ldots, x_s)$ of M, where each x_j $(j \geqslant 2)$ is an arbitrary but fixed point in Q_j, then for the same reason, $[C]\circ[H]>0$. Thus

$$0 < [C]\circ[H] = [C]\circ[\sum_i a_i H_i] = a_1[C]\circ[h_1].$$

Therefore $a_1>0$. The same proof now shows $a_j>0$ for each j. Q.E.D.

We wish to extract a more general statement from the preceding proof. Consider the following condition on a compact complex manifold N:

Each nonzero effective divisor in N defines a positive holomorphic line bundle in N. (9)

Now a moment's reflection shows that the reasoning in the proof of (B) also proves

LEMMA 3. *If N is the direct product of a finite number of algebraic manifolds $N_1 \times \ldots \times N_s$ and $H^2(N_i, \mathbf{R}) \cong \mathbf{R}$ for each i, then N has property* (9).

The importance of the condition (9) stems partly from the following theorem of Grauert [G]: If a nonsingular hypersurface N in a complex manifold M has the property (9), then N is an exceptional analytic set iff N has a negative normal bundle.

Proof of (D). By assumption, $H^2(M, \mathbf{R}) \cong \mathbf{R}$. Let N be a nonsingular hypersurface. First assume $\dim M \geqslant 4$. By part (B), the line bundle defined by N has positive Chern class. The Lefschetz hyperplane theorem (cf. [Bo]) then implies that $H^2(N, \mathbf{Z}) \cong H^2(M, \mathbf{Z}) \cong \mathbf{Z}$. By Lemma 3, N has property (9). If N is exceptional, the above-cited theorem of Grauert would imply that N has a negative normal bundle $\mathcal{N}$; this means $c_1(\mathcal{N})$ has a representative real (1,1)-form ξ_1 which is negative definite. On the other hand, $\mathcal{N}$ is a quotient bundle of the restriction of the holomorphic tangent bundle TM to N. Since the bisectional curvature is nonnegative, TM has nonnegative curvature in the sense of Griffiths [Gr] and hence its quotient bundle $\mathcal{N}$ also has nonnegative curvature. Thus $c_1(\mathcal{N})$ can also be represented by a real (1,1)-form ξ_2 which is positive semi-definite. However, on an algebraic manifold this situation of representing the same cohomology class $c_1(\mathcal{N})$ by both a negative definite form ξ_1 and a positive semidefinite form ξ_2 is impossible: take any curve C on M, then

$$0 \leqslant \int_C \xi_2 = \int_C c_1(\mathcal{N}_N) = \int_C \xi_1 < 0.$$

Thus N is not exceptional.

If M has dimension 2, then every nonsingular hypersurface N in M is a Riemann surface which obviously satisfies (9). Thus the preceding proof also applies. Q.E.D.

Remark. Assertion (D) of Theorem D is still expected to be valid when $\dim M = 3$, but the preceding proof completely breaks down in this case. For instance, let $M = P_3\mathbf{C}$; then any nonsingular cubic surface N in $P_3\mathbf{C}$ contains six exceptional curves which of course never give rise to positive line bundles on N ([GrH], p. 480 ff.). Thus there is no way to apply the above theorem of Grauert to conclude that N is not exceptional.

Proof of (C). Let E be a nonsingular divisor of M which can be blown down to a nonsingular submanifold S of dimension k, $0 \leqslant k \leqslant \dim M - 2$. Then E is fibred over S with fibre $P \equiv P_{n-r-1}\mathbf{C}$. The normal bundle $\mathcal{N}$ of E restricted to each fibre P is just the universal (tautological) line bundle of P (cf. [GrH], p. 607). Write L for the homology class in P defined by a complex line. From the well-known facts about the universal bundle, we have $\int_L c(\mathcal{N}) = -1$. On the other hand, $\mathcal{N}$ is a quotient bundle of the nonnegative holomorphic vector bundle $TM|_E$ and is hence nonnegative; therefore $\int_L c(\mathcal{N}) \geqslant 0$. Contradiction. Q.E.D.

Section 4

We finally prove Theorem E. Before giving the proof, we mention an open problem. Because of the theorem of Mori ([M]) we now know that among all compact irreducible Hermitian symmetric spaces, only complex projective spaces have a positive tangent bundle. However, every compact irreducible Hermitian symmetric space S is known to be simply connected, satisfies $H^2(S, \mathbf{Z}) \cong \mathbf{Z}$, and has positive $\wedge^n TS$ ($n = \dim S$ and TS is the holomorphic tangent bundle of S). The open problem is to determine the smallest such integer k in terms of the rank of S, especially when S is a complex Grassmannian.

Proof of Theorem E. First observe that for all integers k such that $1 \leqslant k \leqslant n-1$ ($n = \dim M$),

$$\wedge^k TM \text{ quasi-positive} \Rightarrow \wedge^{k+1} TM \text{ quasi-positive.} \tag{10}$$

By assumption and (10), M has a quasi-positive $\wedge^n TM$ and hence quasi-positive Ricci curvature. By part (A) of Theorem B, M is simply connected and $h^{2,0}(M) = 0$. If we can prove $h^{1,1}(M) = 1$, then $H^2(M, \mathbf{R}) \cong \mathbf{R}$. The universal coefficient theorem for cohomology then implies $H^2(M, \mathbf{Z}) \cong \mathbf{Z}$.

To show $h^{1,1}(M) = 1$, it suffices to show every harmonic (1,1)-form ξ is a constant multiple of the Kähler form ω. Let $\dim M = 2m$ and let $\wedge^m TM$ be quasi-positive (the case where $\dim M = 2m+1$ can be handled the same way). Notation and assumption as in ($*$) of Section 1 in [HSW], we consider the restriction of the function $F(\xi)$ to an open subset W of M where the curvature of $\wedge^m TM$ is positive. Recall that for nonnegative bisectional curvature, we always have $F(\xi) \equiv 0$ and ξ is parallel (Lemma 1 of [HSW]). Fix an $x \in W$ and it will be understood that the following discussion takes place entirely in the tangent space M_x. We claim: $\xi_{11^*} = \xi_{22^*} = \dots = \xi_{(2m)(2m)^*}$. To prove $\xi_{11^*} = \xi_{22^*}$, for instance, if $R_{11^*22^*} > 0$, then the desired equality would follow directly from $F(\xi) = 0$ (see ($*$) of [HSW]). Thus assume $R_{11^*22^*} = 0$. Let S_1 be the subset of $\{3, 4, \dots, 2m\}$ such that $j \in S_1$ iff $R_{11^*jj^*} > 0$. S_1 has at least m elements for the reason that, if not, then $\{3, \dots, 2m\} - S_1$ would contain at least $m-1$ elements which may be assumed to be $\{m+2, \dots, 2m\}$ for definiteness. Then by the definition of S_1,

$$R_{11^*22^*} + R_{11^*(m+2)(m+2)^*} + \dots + R_{11^*(2m)(2m)^*} = 0,$$

and this contradicts the positivity of $\wedge^m TM$ at x. Similarly if S_2 is the subset of $\{3, 4, \dots, 2m\}$ such that $j \in S_2$ if $R_{22^*jj^*} > 0$, then S_2 also has at least m elements. Since $\{3, 4, \dots, 2m\}$ contains only $2m-2$ elements, $S_1 \cap S_2 \neq \varnothing$; let $i \in S_1 \cap S_2$. Then $R_{11^*ii^*} > 0$ and $R_{22^*ii^*} > 0$. Again it follows from $F(\xi) = 0$ that these inequalities force the equalities $\xi_{11^*} = \xi_{ii^*}$ and $\xi_{22^*} = \xi_{11^*}$. Thus $\xi_{ii^*} = \xi_{22^*}$. In the same way, $\xi_{11^*} = \xi_{jj^*}$ for all $j = 2, \dots, 2m$.

Now let $\alpha(x)$ be the common value of $\{\xi_{11^*}, \dots, \xi_{(2m)(2m)^*}\}$ at x, $x \in W$. Then $\xi = \alpha\omega$ on W, where ω is the Kähler form. Since both ξ and ω are parallel, α must be a constant. With this α, consider the globally defined parallel form $(\xi - \alpha\omega)$ on M; $(\xi - \alpha\omega)$ vanishes on W, so being parallel, it vanishes identically. Thus $\xi = \alpha\omega$ on M. Q.E.D.

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