An example of a weakly hyperbolic Cauchy problem not well posed in C^∞

by

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§ 1. Introduction

In this paper we deal with the Cauchy problem

$$u_{tt}-a(t)\,u_{xx}=0, \quad u(x,0)=\varphi(x), \quad u_t(x,0)=\psi(x), \tag{1}$$

where $0\leqslant t\leqslant T, x\in \mathbf{R}$ and $a(t)$ is a C^∞ function on the interval $[0,T]$ satisfying the assumption

$$a(t)\geqslant 0. \tag{2}$$

Our purpose is to show that (1) may be *not well posed* in the class $\mathscr{E}(\mathbf{R}_x)$ of the C^∞ functions, contrary to what occurs when $a(t)\geqslant\lambda>0$.

More precisely, we shall construct a C^∞ function $a(t)$, strictly positive on $[0,\varrho[$ and identically null on $[\varrho,+\infty[$ where ϱ is a given positive number, and two C^∞ functions $\varphi(x)$ and $\psi(x)$ in such a way that (1) has no solution in the class of distributions on $\mathbf{R}_x\times[0,T[$ as $T>\varrho$.

By virtue of the strict positivity of the coefficient $a(t)$ for $t<\varrho$, this problem has a C^∞ solution on $[0,\varrho[\times\mathbf{R}_x$, which is the *unique* solution in the class $C^1([0,\varrho[,\mathscr{D}'(\mathbf{R}_x))$. However, this solution cannot be continued as distribution on any strip $]0,\varrho+\varepsilon[\times\mathbf{R}_x$, $\forall\varepsilon>0$. In particular it does not belong to $C([0,\varrho],\mathscr{D}'(\mathbf{R}_x))$.

Let us recall that problem (1) is said to be *well posed* in some class $\mathscr{F}(\mathbf{R}_x)$ of *real* functions or distributions (or analytic functionals) if, for any φ and ψ in $\mathscr{F}(\mathbf{R}_x)$, it admits a unique solution u in $C^1([0,T],\mathscr{F}(\mathbf{R}_x))$ and the mapping $(\varphi,\psi)\mapsto u$ is continuous.

The equation $u_{tt}-a(t)\,u_{xx}=0$ is called *hyperbolic* (see Mizohata, [2]) when the corresponding Cauchy problem is well posed in $\mathscr{E}(\mathbf{R}_x)$.

It is well known that (1) is well posed in $\mathscr{E}(\mathbf{R}_x)$, or in $\mathscr{D}'(\mathbf{R}_x)$, provided that (2) is reinforced by the assumption $a(t)\geqslant\lambda>0$, while it is well posed in the class $\mathscr{A}(\mathbf{R}_x)$ of the real analytic functions under assumption (2) alone (see [1], where these results are proved under the weakest assumptions of regularity on the coefficients).

The example of the present paper, whose structure resembles the counterexamples of [1], points out that the assumption (2) by itself does not assure that (1) is well posed in $\mathscr{E}(\mathbf{R}_x)$. It would be interesting to find what additional kinds of assumptions on $a(t)$ are needed in order that the problem be well posed.

Let us finally emphasize that several necessary conditions, such as the Levi's condition, are known for the hyperbolicity of the equation $u_{tt}-a(t)\,u_{xx}+b(t)\,u_x=0$, but they are all fulfilled when $b(t)$ is dentically zero.

§ 2. Preliminaries

The initial data $\varphi(x)$ and $\psi(x)$ of our example will be periodic odd functions, so the solution $u(x, t)$ will admit a Fourier expansion such as

$$u(x,t)=\sum_{h=1}^{\infty} v_h(t)\sin(hx), \quad t\in[0,T]. \tag{3}$$

The following well known facts will be used.

PROPOSITION 1. *A sufficient condition in order that* (3) *defines* $u\in C([0,T],\mathscr{E}(\mathbf{R}_x))$ *is*

$$\sup_t\,(|v_h(t)|+|v_h'(t)|)\leqslant C_p\,h^{-p}, \quad \forall p, \forall h. \tag{4}$$

PROPOSITION 2. *A necessary condition in order that* (3) *defines* $u\in C[(0,T],\mathscr{D}'(\mathbf{R}_x))$ *is*

$$\sup_t\,|v_h(t)|\leqslant Mh^p, \quad \forall h, \tag{5}$$

for some M and p.

When $u(x,t)$ is a solution of (1), then the v_h's solve the ordinary equations

$$v''+h^2a(t)\,v=0. \tag{6}$$

Therefore, estimates as (4) or (5) can be obtained by studying the non-negative funcitonal (*the energy-functional*)

$$E_{h,v}(t) = h^2 a(t)\, v^2(t) + v'^2(t).$$

By differentiation of $E_{h,v}(t)$ with respect to t, we can easily obtain the energy estimate

$$E_{h,v}(t_2) \leq E_{h,v}(t_1) \exp \left| \int_{t_1}^{t_2} \frac{|a'(t)|}{a(t)}\, dt \right|, \tag{7}$$

for t_1, t_2 in $[0, T]$.

Later, we shall use the following special case of (7):

PROPOSITION 3. *Let us assume that in some interval $[0, T_*]$, with $T_* \leq T$, the function $a(t)$ is strictly positive and admits just v points of local minimum and v points of local maximum. Let us denote by $\lambda_1, \ldots, \lambda_v$ and by $\Lambda_1, \ldots, \Lambda_v$ the values of $a(t)$ at these points. Finally assume that $a(t)$ is decreasing near the extreme points 0 and T_*.*

Then for any solution of (6) *the following estimate holds*

$$E_{h,v}(t) \leq E_{h,v}(T_*) \frac{a(0)}{a(T_*)} \left(\frac{\Lambda_1 \cdot \ldots \cdot \Lambda_v}{\lambda_1 \cdot \ldots \cdot \lambda_v} \right)^2, \quad \forall t \leq T_*. \tag{8}$$

§ 3. Construction of the coefficient

The coefficient $a(t)$ of the example will be a C^∞ function on the real half-axis $t \geq 0$, strictly positive on $[0, \varrho[$ and identically zero on $[\varrho, +\infty[$, where ϱ is a fixed positive number.

In order to define $a(t)$ we fix two sequences $\{\varrho_k\}$ and $\{\delta_k\}$ of real numbers decreasing to zero and a sequence $\{v_k\}$ of integer numbers increasing to ∞ $(k = 1, 2, \ldots)$. These sequences will be defined later (see § 5) in a suitable way. In particular $\{\varrho_k\}$ will verify the condition

$$\sum_{k=1}^{\infty} \varrho_k = \varrho. \tag{9}$$

Correspondently, we effect the following subdivision of the interval $[0, \varrho[$

$$[0, \varrho[\, = \bigcup_{k=1}^{\infty} J_k$$

where $J_k = [t_k - \varrho_k/2, t_k + \varrho_k/2[$ and $t_k = \varrho_1 + \ldots + \varrho_{k-1} + \varrho_k/2$.

Moreover, we consider the function $a_0(t)\equiv 2\delta_1$ and the functions

$$a_k(t) = \delta_k \alpha\left(2\nu_k \pi \frac{t-t_k}{\varrho_k}\right) \quad (k \geq 1) \tag{10}$$

where

$$\alpha(\tau) = 1-\tfrac{4}{10}\sin 2\tau - \tfrac{1}{100}(1-\cos 2\tau)^2. \tag{11}$$

Let us remark that α is π-periodic and

$$\tfrac{1}{2} \leq \alpha(\tau) \leq 2. \tag{12}$$

Disregarding any request of regularity, we could take as coefficient $a(t)$ of our example the piece-wise regular function $\tilde{a}(t)$ which coincides with $a_k(t)$ on the interval J_k $(k=1,2,\ldots)$.

But we wish for a C^∞ coefficient, so we must modify $\tilde{a}(t)$ near its points of discontinuity $\{t_k-\varrho_k/2\}$. More precisely, we shall modify $\tilde{a}(t)$ in a very small right neighborhood of $t_k-\varrho_k/2$, namely in the interval

$$\tilde{I}_k = \left[t_k - \frac{\varrho_k}{2}, t_k - \frac{\varrho_k}{2} + \frac{\varrho_k}{8\nu_k}\right[.$$

To do this, we consider the subdivision

$$J_k = \tilde{I}_k \cup I_k$$

setting

$$I_k = \left[t_k - \frac{\varrho_k}{2} + \frac{\varrho_k}{8\nu_k}, t_k + \frac{\varrho_k}{2}\right],$$

and the functions

$$b_k(t) = \beta\left(8\nu_k \frac{t-(t_k-\varrho_k/2)}{\varrho_k}\right) \tag{13}$$

where $\beta(\tau)$ is a C^∞ function, increasing on $[0, 1]$ and such that $\beta\equiv 0$ for $\tau\leq 0$ and $\beta\equiv 1$ for $\tau\geq 1$.

We then define

$$a(t) = \begin{cases} a_k(t)\, b_k(t) + a_{k-1}(t)\,(1-b_k(t)), & \text{if } t \in J_k, \\ 0, \quad \text{if } t \geq \varrho. \end{cases} \tag{14}$$

It is immediately seen that $a(t)>0$ on $[0,\varrho[$, so $a(t)$ verifies (2) on $[0,+\infty[$.

Now, in view of Proposition 3, we look for the local minima and maxima of $a(t)$ on the intervals $J_1, J_2, \ldots$ which form $[0,\varrho[$, keeping in mind that $J_k=\tilde{I}_k \cup I_k$.

On I_k, $a(t)$ coincide with $a_k(t)$, i.e. $a(t)=\delta_k \alpha(\tau)$, where $\tau=2\nu_k\pi(t-t_k)/\varrho_k$. The variable τ runs on the interval $[-\nu_k\pi+\pi/4, \nu_k\pi[$ as $t\in I_k$, and the π-periodic function $\alpha(\tau)$ (cf. (11)) has just one point of minimum and one point of maximum in $[0,\pi]$, both these points lying in $]\pi/4,\pi]$. Taking (12) into account, we then arrive to the following conclusion.

At the interior of I_k, $a(t)$ has just $2\nu_k$ points of local minimum and $2\nu_k$ points of local maximum, with respective values $\geqslant\delta_k/2$ and $\leqslant 2\delta_k$; moreover $a(t)$ is decreasing near the endpoints of I_k. (15)

As for the connecting intervals $\tilde{I}_k$, instead of studying the oscillations of $a(t)$ we shall arrange the parameters δ_k in such a way that

$$a(t) \textit{ is decreasing on } \tilde{I}_k. \tag{16}$$

To this end, let us consider the expression

$$a' = b_k a_k' + b_k'(a_k - a_{k-1}) + (1-b_k)\, a_{k-1}',$$

obtained from (14) by derivation.

At the interior of $\tilde{I}_k$, the functions b_k, $1-b_k$ and b_k' are positive, while a_k and a_{k-1} are decreasing (since $\alpha(\tau)$ is decreasing on $[0,\pi/4]$), a_0 is constant. Henceforth (16) will be true if only $a_k(t)\leqslant a_{k-1}(t)$ and, in particular, if

$$2\delta_k \leqslant \frac{\delta_{k-1}}{2}. \tag{17}$$

Going back to the regularity of $a(t)$, it is immediate that $a(t)$ is C^∞ on $[0,\varrho[$. Moreover $a(t)$ tends to zero as $t\to\varrho^-$, since $\{\delta_k\}\to 0$. Hence $a(t)$ is continuous on $[0,+\infty[$.

It remains to verify that $a(t)$ is C^∞ near the point $t=\varrho$, i.e. that every derivative $a^{(r)}(t)$ of $a(t)$ tends to zero as $t\to\varrho^-$. To this purpose, we must add some supplementary conditions on δ_k, ϱ_k and ν_k.

In fact, (14) gives

$$a^{(r)} = \sum_{s=0}^{r} \binom{r}{s} b_k^{(r-s)}(a_k^{(s)} - a_{k-1}^{(s)}) + a_{k-1}^{(r)}, \quad \text{on } J_k. \tag{18}$$

On the other side, (10) and (13) give

$$|a_k^{(r)}| \leqslant \delta_k A_r \left(\frac{\nu_k}{\varrho_k}\right)^r \tag{19}$$

and

$$|b_k^{(r)}| \leqslant B_r \left(\frac{\nu_k}{\varrho_k}\right)^r, \tag{20}$$

where $\{A_r\}$ and $\{B_r\}$ are two increasing sequences such that $A_r \geqslant (2\pi)^r |\alpha^{(r)}(\tau)|$ and $B_r \geqslant 8^r |\beta^{(r)}(\tau)|$, $\forall \tau$.

Introducing (19) and (20) in (18), and taking into account that $\{\nu_k/\varrho_k\}$ is increasing, while $\{\delta_k\}$ is decreasing, we get the estimate

$$|a^{(r)}(t)| \leqslant 3 \cdot 2^r A_r B_r \left(\frac{\nu_k}{\varrho_k}\right)^r \delta_{k-1}. \quad \forall t \in J_k.$$

Hence, a sufficient condition for the C^∞-regularity of $a(t)$ on $[0, +\infty[$ is

$$\delta_k \left(\frac{\nu_{k+1}}{\varrho_{k+1}}\right)^r \to 0, \quad \text{as } k \to \infty, \ \forall r. \tag{21}$$

§ 4. Construction of the initial data

We intend to find two C^∞ functions $\varphi(x)$ and $\psi(x)$ such that Cauchy problem (1) with coefficient $a(t)$ defined in § 3 and with initial data φ, ψ, has *no solution* u in $\mathscr{D}'$ for $t > \varrho$. Of course this involve some further conditions on the sequences $\{\delta_k\}$, $\{\varrho_k\}$ and $\{\nu_k\}$.

We do not define φ and ψ *directly*, but we shall construct a *particular* solution $u(x, t)$ of the equation $u_{tt} - a(t) u_{xx} = 0$ on $\mathbf{R}_x \times [0, \varrho[$, in such a way that

$$u \in C^1([0, \varrho[, \mathscr{E}(\mathbf{R}_x)) \tag{22}$$

and

$$u \notin C([0, \varrho], \mathscr{D}'(\mathbf{R}_x)). \tag{23}$$

Afterwards, we shall take $\varphi(x) = u(x, 0)$ and $\psi(x) = u_t(x, 0)$, observing that these are two C^∞ functions by (22).

Consequently, the problem (1) with $T \geqslant \varrho$ will not admit any solution in $C([0, T], \mathscr{D}'(\mathbf{R}_x))$. Otherwise, such a solution should coincide with $u(x, t)$ for $t < \varrho$, in contradiction with (23) (indeed $a(t)$ is bounded away from zero on each interval $[0, \varrho-\varepsilon[$, $\forall \varepsilon>0$, and hence (1) is uniquely solvable in $C([0, \varrho-\varepsilon[, \mathscr{D}'(\mathbf{R}_x))$.

Hence, let us look for a solution $u(x, t)$ of $u_{tt}-a(t)\, u_{xx}=0$, satisfying (22) and (23). This solution will have the form

$$u(x, t) = \sum_{h=1}^{\infty} v_h(t) \sin(hx), \tag{24}$$

where the v_h's are real functions satisfying (6).

Since $a(t)=\delta_k\, \alpha(2\nu_k \pi(t-t_k)/\varrho_k)$ on I_k, we are led to write

$$v_h(t) = w_{h,k}(\tau), \quad \text{for } t \in I_k,$$

where $\tau=2\nu_k\pi(t-t_k)/\varrho_k$.

Therefore $w_{h,k}$ will solve the equation

$$w''+\left(\frac{h^2\varrho_k^2}{4\nu_k^2\pi^2}\delta_k\right)\alpha(\tau)\, w = 0. \quad \tau \in \left[-\nu_k\pi+\frac{\pi}{4}, \nu_k\pi\right[. \tag{25}$$

Now the equation

$$w''+\alpha(\tau)\, w = 0, \tag{26}$$

with $\alpha(\tau)$ given by (11), admits a solution $\tilde{w}$, which we can write down explicitly, such that the sup of $|\tilde{w}(\tau)|$ on $\{0\leq\tau\leq\bar{\tau}\}$ has an *exponential growth* as $\bar{\tau}\to\infty$. In fact a solution of (26) is

$$\tilde{w}(\tau) = \sin\tau\cdot\exp\left[\tfrac{1}{10}\left(\tau-\tfrac{1}{2}\sin 2r\right)\right]. \tag{27}$$

So, in order to reduce (25) to (26), we must consider merely those values of h for which $h^2\varrho_k^2\delta_k(4\nu_k^2\pi^2)^{-1}=1$, assuming at the same time that

$$h_k \equiv \frac{2\nu_k\pi}{\varrho_k}\frac{1}{\sqrt{\delta_k}} \quad \text{is an integer, } \forall k. \tag{28}$$

More precisely, we define the terms $v_h(t)$ of the Fourier series (24) by taking v_{h_k} equal to the solution of

$$v''+h_k^2 a(t)\, v = 0,\ t>0, \quad v(t_k)=0, \quad v'(t_k)=1, \tag{29}$$

and

$$v_h(t)\equiv 0 \quad \text{if } h \notin \{h_k\}. \tag{30}$$

Hence

$$v_{h_k}(t) \equiv \frac{\varrho_k}{2\nu_k \pi} \tilde{w}\left(2\nu_k \pi \frac{t-t_k}{\varrho_k}\right), \quad \text{for } t \in I_k, \tag{31}$$

where $\tilde{w}(\tau)$ is the function (27).

Let us now prove that for $t<\varrho$ the Fourier series $\Sigma\, v_h(t) \sin(hx)$ converges to some function $u \in C^1([0,\varrho[, \mathscr{E}(\mathbf{R}_x))$, under suitable conditions on $\{\delta_k\}, \{\varrho_k\}, \{\nu_k\}$.

Since the v_h's are solutions of (6), it will be sufficient to prove (cf. Propositon 1) that, for any $\varepsilon>0$ and $t \in [0, \varrho-\varepsilon]$,

$$|v_{h_k}(t)| \leqslant C_p h_k^{-p}, \quad \forall p, \forall k. \tag{32}$$

As a matter of fact it will suffice to prove (32) for any $t \in [0, t_k']$, where t_k' denote the first point of I_k, i.e. $t_k' = t_k - \varrho_k/2 + \varrho_k/(8\nu_k)$ (indeed $\{t_k'\} \to \varrho^-$ as $k \to \infty$). This can be done by using the estimate of the energy

$$E_k(t) = h_k^2\, a(t)\, v_{h_k}^2 + v'^2_{h_k}$$

furnished by Proposition 3.

Indeed by (8), (15) and (16), we get

$$E_k(t) \leqslant E_k(t_k') \frac{4\delta_1}{\delta_k} \exp\left[2(\nu_1 + \ldots + \nu_{k-1}) \log 4\right], \quad \forall t \leqslant t_k'. \tag{33}$$

On the other side by the explicit expression of v_{h_k} on I_k (see (31)) we get

$$E_k(t_k') = C \exp\left(-\nu_k \frac{\pi}{5}\right) \tag{34}$$

where C is a constant.

Moreover $a(t) \geqslant \delta_k/2$ on $[0, t_k']$, so by (28) and (9) we have $h_k^2 a(t) \geqslant 1/\varrho^2$ for $t \leqslant t_k'$. Henceforth

$$|v_{h_k}(t)| \leqslant \frac{1}{\varrho} \left[E_k(t)\right]^{1/2}. \tag{35}$$

Summing up, (33), (34) and (35) give the estimate

$$|v_{h_k}(t)| \leqslant \frac{2C}{\varrho} \exp\left(-\nu_k \frac{\pi}{10}\right) \left(\frac{\delta_1}{\delta_k}\right)^{1/2} \exp\left[(\nu_1 + \ldots + \nu_{k-1}) \log 4\right]$$

which implies (32) for $t \leq t'_k$, provided that

$$-\nu_k \frac{\pi}{10} + \frac{1}{2} \log \frac{1}{\delta_k} + (\nu_1 + \ldots + \nu_{k-1}) \log 4 \leq -p \log h_k + C'_p, \quad \forall p, \forall k. \tag{36}$$

Taking into account the defintion (28) of h_k and the assumption $\{\nu_k\} \to \infty$, we see that (36) is satisfied if only

$$-\frac{\pi}{11} \nu_k + (\nu_1 + \ldots + \nu_{k-1}) \log 4 + \frac{p+1}{2} \log \frac{1}{\delta_k} + p \log \frac{1}{\varrho_k} \leq C''_p, \quad \forall p, \forall k. \tag{37}$$

In conclusion, we have constructed a solution $u(x, t)$ of (1), which satisfies (22) if the parameters ϱ_k, δ_k and ν_k satisfy (37).

We shall now prove that, without any other condition on ϱ_k, δ_k, ν_k, the solution u satisfies also (23).

Indeed, by the explicit expression of v_{h_k} on I_k (see (31)), we have

$$|v_{h_k}(t''_k)|\, h_k^{-p} = C' \frac{\varrho_k}{\nu_k} \exp\left(\nu_k \frac{\pi}{10}\right) \left(\frac{2\nu_k \pi}{\varrho_k \sqrt{\delta_k}}\right)^{-p},$$

where $t''_k = t_k + \varrho_k/2 - \varrho_k/(8\nu_k)$.

Hence, (37) implies that $|v_{h_k}(t''_k)|\, h_k^{-p}$ tends to infinity as $k \to \infty$, for any p, thus (5) is violated. By Proposition 2, we then conclude that u satisfies (23).

§ 5. Choice of the parameters

In constructing the coefficient $a(t)$ and the solution $u(x, t)$, we have been forced to make on the sequences $\{\varrho_k\}$, $\{\delta_k\}$ and $\{\nu_k\}$ several conditions, namely (9), (17), (21), (28) and (37).

To prove that these conditions are not contradictory, we give a concrete choice of ϱ_k, δ_k, ν_k, i.e.

$$\begin{cases} \varrho_k = \varrho \cdot 2^{-k} \\ \nu_k = 8^k \\ \delta_k = \pi^2 \left[\exp\left(\frac{8^k}{k}\right)\right]^{-2} \end{cases} \tag{38}$$

where $[\cdot]$ denotes the integer part.

Summarizing, we have proved the following result:

THEOREM 1. *Let ϱ be a positive number and $\{\varrho_k\}$, $\{\delta_k\}$, $\{\nu_k\}$ be defined by* (38). *Let $a(t)$ be the function defined by* (14) *and $\varphi(x)=u(x,0)$, $\psi(x)=u_t(x,0)$, where $u(x,t)=\Sigma_1^\infty v_h(t)\sin(hx)$ and the v_h's are defined by* (29), (30).

Then $a(t)$ is C^∞ on $[0,+\infty[$ and $a(t)\equiv 0$ for $t\geqslant\varrho$, while φ and ψ are C^∞ functions on $\mathbf{R}_x$.

However Cauchy problem (1) *has no solution u in the class $C^1([0,T],\mathscr{D}'(\mathbf{R}_x))$, if $T\geqslant\varrho$.*

§ 6. Concluding remarks

Remark 1. By using suitably Proposition 3, one could prove that problem (1) is well posed in $\mathscr{E}(\mathbf{R}_x)$ whenever the coefficient $a(t)$ is a non-negative function having only a *finite number of oscillations* on $[0,T]$, in particular when $a(t)$ is *analytic*.

We do not know any examples of non-negative $a(t)$, whose graph has infinitely many oscillations touching the t-axis, such that the corresponding Cauchy problem is well posed.

Remark 2. As a matter of fact, the coefficient $a(t)$ constructed in § 3 is more than C^∞-regular if only the mollifier function $\beta(\tau)$ appearing in (13) is taken sufficiently regular. More precisely, $\beta(\tau)$ can be chosen in any Gevrey class (but not analytic), so $a(t)$ turns out to be a Gevrey function on $[0,+\infty[$

Remark 3. The coefficient $a(t)$ can be modified on $[0,\varrho[$ in such a way that it becomes analytic on $[0,\varrho[$ (and C^∞ on $[0,+\infty[$) and neverthless the conclusion of Theorem 1 is still true. To do this, let us consider an analytic function $a_0(t)$ on $[0,\varrho[$ such that $|a^{(r)}(t)-a_0^{(r)}(t)|\leqslant\varepsilon_k$ for $t\in[\varrho-1/k,\varrho-1/(k+1)]$ and $r\leqslant k$, where $\{\varepsilon_k\}$ is a sequence fast enough decreasing to zero (cf. [3]). Afterwards, let us compare the solution u of problem (1) with the solution u_0 of the analogous problem with the same initial data but with coefficient $a_0(t)$ (instead of $a(t)$). If $\{\varepsilon_k\}$ is suitably chosen, we can conclude that u_0 still verifies (23).

Remark 4. The solution $u(x,t)$ of (1) constructed in § 4, is a C^∞ function on the strip $[0,\varrho[\times\mathbf{R}_x$ which does not belong to $C([0,\varrho],\mathscr{D}'(\mathbf{R}_x))$.

As a matter of fact, u does not even belong to $\mathscr{D}'(]0,\varrho+\varepsilon[\times\mathbf{R}_x)$, $\forall\varepsilon>0$, i.e. it cannot be continued beyond the line $t=\varrho$ as a distribution.

However, one can observe that $u(x,t)$ does not disappear at all after the instant $t=\varrho$. Indeed any Cauchy problem as (1), whose coefficients depend only on t, is well posed in the space of the *periodic real analytic functionals* (see [1]).

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