Counterexamples to a conjecture of Grothendieck

by

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In his thesis ([7] II. p. 136) and in his fundamental paper ([6] p. 74), Grothendieck formulated the following conjecture: If two Banach spaces X and Y are such that their injective and projective tensor products $X\check{\otimes}Y$ and $X\hat{\otimes}Y$ coïncide, then either X or Y must be finite dimensional. The aim of this paper is to give a counterexample.

We will exhibit a separable infinite dimensional Banach space X such that $X\check{\otimes}X=X\hat{\otimes}X$, both algebraically and topologically. The space X is of cotype 2 as well as its dual. Moreover, the natural map from $X^*\hat{\otimes}X$ into $X^*\check{\otimes}X$ is surjective, but it is not injective, since X fails the approximation property (in short the A.P.); equivalently, every operator on X which is a uniform limit of finite rank operators is nuclear. This implies that there are (roughly) "very few" operators on X of finite rank and of small norm. For instance, there is a number $\delta>0$ such that, for any finite dimensional subspace E of X and for any projection $P: X\to E$, we have

$$\|P\|\geqslant\delta(\dim E)^{1/2}.$$

Therefore, if $\{P_n\}$ is a sequence of finite rank projections on X, then $\|P_n\|$ must tend to infinity if the rank of P_n tends to infinity. A fortiori, the space X can contain uniformly complemented l_p^n's for *no p* such that $1\leqslant p\leqslant\infty$, so that we have also a negative answer to a question of Lindenstrauss [13].

Finally, since X is not isomorphic to a Hilbert space, although X and X^* are both of cotype 2 we also answer negatively a question raised by Maurey in [17] (as well as question 5.3 in [4]). Moreover, our example shows that the A.P. cannot be removed from the assumptions of the factorization theorem of [23].

In the last ten years, under the impulse of [14], several significant steps were taken towards the solution of Grothendieck's conjecture; besides [22] and [23], the results of the papers [17], [10] and [1] play an important rôle (directly or indirectly) in our construction. During the same period, Grothendieck's conjecture was established under various additional assumptions. It was proved in [9] for spaces with local unconditional structure, in [2] for spaces not containing l_1^n's uniformly, and in [23], [25] for spaces with a basis or with the bounded approximation property (in short B.A.P.).

The paper is organized as follows: section 1 contains the basic preliminaries and the various results that are used. We have tried to clarify the relationship between the extension property of our examples and the usual notions introduced in the literature. Section 2 contains the construction. The announced examples are produced in section 3. Finally, in section 4, we exhibit a Banach space Z which is weakly sequentially complete, of cotype 2, verifies Grothendieck's theorem, but is such that $L^1/H^1 \hat{\otimes} Z$ contains c_0 and therefore fails all these properties (yet shared by L^1/H^1 and Z). This answers negatively several questions raised by various authors. We also discuss in section 4 some open problems related to our work.

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§ 1. Preliminary results

We start by recalling some notations and terminology. An operator $u: X \to Y$ between Banach spaces is called p-absolutely summing $(0<p<\infty)$ if there is a constant λ such that, for any finite sequence (x_i) in X we have

$$\sum \|u(x_i)\|^p \leq \lambda^p \sup \left\{ \sum |\xi(x_i)|^p \,\middle|\, \xi \in X^*,\ \|\xi\| \leq 1 \right\}.$$

Let $\pi_p(u)$ be the smallest constant λ for which this inequality holds. We will denote as usual $\Pi_p(X, Y)$ the space of all p-absolutely summing operators from X into Y, and $B(X, Y)$ the space of all bounded operators from X into Y. For more information on p-summing operators, cf. [21]. We will say that two Banach spaces X and Y are λ-isomorphic if there exists an isomorphism $T: X \to Y$ such that $\|T\|\,\|T^{-1}\| \leq \lambda$.

We will denote $(e_i)_{i \in \mathbb{N}}$ (resp. $((e_i)_{i \leq n})$ the canonical basis of the space l_2 (resp. l_2^n).

Let us briefly recall how an inductive limit of Banach spaces is defined. Let E_n be a sequence of Banach spaces given together with a sequence of isometric imbeddings $j_n: E_n \to E_{n+1}$. Then, the inductive limit X can be defined as follows. We consider the subspace of ΠE_n formed by all the sequences $(x_n)_{n \in \mathbb{N}}$ such that $j_n(x_n) = x_{n+1}$ for all n

sufficiently large. We equip this space with the semi-norm $\|(x_n)\|=\lim_{n\to\infty}\|x_n\|$. Let $\mathscr{X}$ be the normed space obtained after passing to the quotient by the kernel of this semi-norm. The space X is then defined as the completion of the space $\mathscr{X}$. It is clear that there is a system of isometric injections $J_n: E_n \to X$ such that $J_{n+1} j_n = J_n$ for all n, and moreover if $X_n = J_n(E_n)$, then the union $\bigcup_n X_n$ is dense in X.

Therefore, this construction shows that, in practice, we may always do as if the spaces E_n were an increasing sequence of spaces of some larger space, and we then identify X simply with $\overline{\bigcup E_n}$.

Definition 1.1. We will say that a Banach space Z verifies Grothendieck's theorem if every operator from Z into a Hilbert space is 1-absolutely summing.

In [6] (see also [14]), Grothendieck proved that the preceding property holds for $Z=L_1$.

The following facts are well known.

PROPOSITION 1.2. *For a Banach space Z, the following assertions are equivalent:*

(i) *Z verifies Grothendieck's theorem.*

(ii) *There is a constant c such that, for any integer n and any $v: Z \to l_2^n$, we have $\pi_1(v) \le c\|v\|$.*

(iii) *Every operator from Z^* into an L_1 space is 2-absolutely summing.*

Proof. (i)$\Leftrightarrow$(ii) is obvious. (i)$\Leftrightarrow$(iii) follows from a well known duality argument.

Remark 1.3. Assume that Z is a dual space, say $Z=X^*$. Then, it is easy to see that (ii) above is equivalent to

(ii)* There is a constant c such that, for any n and any u: $l_2^n \to X$, we have

$$\pi_1(u^*) \le c\|u\|.$$

Indeed, by the local reflexivity principle (cf. e.g. [21] § 9.2.3), we have the following isometric identifications:

$$B(l_2^n, X)^{**} \cong B(l_2^n, X^{**}) \cong B(X^*, l_2^n).$$

Therefore, any $v: X^* \to l_2^n$ is the weak-$*$ limit of a net of operators $u_i^*: X^* \to l_2^n$ which are weak-$*$ continuous and such that $\|u_i^*\| \le \|v\|$. This shows that (ii)*$\Rightarrow$(ii). The converse is trivial.

Let us introduce some notations to be used throughout the paper. We denote by D

the set $\{-1,1\}^{\mathbf{N}}$, by $\mathscr{B}$ its Borel σ-algebra and by μ the normalized Haar measure on D. We will denote by $\mathscr{B}_n$ the σ-algebra generated by the first n coordinates on D.

We will denote by R the closed linear span in $L_1(D,\mu)$ of the functions $\{\varepsilon_n | n \in \mathbf{N}\}$.

We also need to introduce a standard probability space $(\Omega, \mathscr{A}, \mathbf{P})$, on which is defined a sequence $\{g_n | n \in \mathbf{N}\}$ of independent, identically distributed Gaussian variables, normalized in $L_2(d\mathbf{P})$. We will denote by G the closed linear span in $L_1(d\mathbf{P})$ of the functions $\{g_n | n \in \mathbf{N}\}$.

It is well known that R (resp. G) is isomorphic to l_2.

We will denote by $\alpha: l_2 \to R$ (resp. $\beta: l_2 \to G$) the isomorphism which maps e_n into ε_n (resp. into g_n).

Let $r: R \to L_1(\mu)$ (resp. $\gamma: G \to L_1(\Omega, \mathbf{P})$) be the natural injection. We will use the fact that $r^*: L_\infty(\mu) \to R^*$ (resp. $\gamma^*: L_\infty \to G^*$) is a 2-absolutely summing operator.

This fact is an easy consequence of the equivalence of the L_1 and L_2 norms on R (resp. G). Indeed, let $\tilde{r}: R \to L_2(\mu)$ be the injection of R into $L_2(\mu)$; then we have $r^* = (\tilde{r})^* J$ where $J: L_\infty(\mu) \to L_2(\mu)$ is the natural injection. Hence $\pi_2(r^*) \leq \|\tilde{r}\|$.

The proof for γ^* is similar.

The following notion will be used repeatedly

Definition 1.4. A Banach space Z is called of cotype q $(2 \leq q < \infty)$ if there is a constant c such that, for any finite sequence (z_i) in Z, we have

$$\left(\sum \|z_i\|^q\right)^{1/q} \leq c\mathbf{E}\left\|\sum \varepsilon_i z_i\right\|.$$

We will denote by $c_q(Z)$ the smallest constant c with this property. For more details on this notion, we refer to [18].

Remark. We have chosen to use the first moment of $\|\Sigma\, \varepsilon_i z_i\|$ in definition 1.4. By a well known result of Kahane (cf. [15], Vol. II, p. 74) we may use a pth moment for any finite p (in particular for $p=2$), this leads to the same notion and the related constants are equivalent.

Remark 1.5. (i) Let X be a Banach space and let n be a fixed integer. Let $(x_i)_{i \leq n}$ be an element of X^n. We introduce the following norm on X^n:

$$|(x_i)_{i \leq n}| = \inf\{\|\tilde{u}\|\} \tag{1.1}$$

where the infimum runs over all $\tilde{u}: L_1(\mu) \to X$ such that $\tilde{u}(\varepsilon_i) = x_i$.

We will denote by $\mathscr{X}$ the space X^n equipped with this norm. We claim that $\mathscr{X}^*$ can be identified isometrically with X^{*n} equipped with the norm defined by

$$|(\xi_i)|_* = \left\| \sum_1^n \varepsilon_i \xi_i \right\|_{L_1(X^*)},$$

for $(\xi_i)_{i\leq n}$ in X^{*n}.

Indeed, we have clearly $|(x_i)_{i\leq n}|=\inf\|\tilde{u}\|$ where the infimum runs over all $\tilde{u}: L_1(\mu)\to X$ such that $\tilde{u}(\varepsilon_i)=x_i$ for all $i\leq n$ and $\tilde{u}=0$ on the orthogonal of $L^2(\mathcal{B}_n)$.

It is well known that to such an operator $\tilde{u}: L_1(\mu)\to X$, we may associate a function $\Phi: D\to X$ depending only on $\{\varepsilon_1, \dots, \varepsilon_n\}$ and such that

$$\tilde{u}(\varphi) = \int \varphi\Phi\, d\mu, \quad \forall \varphi \in L_1(\mu).$$

We have $\|\tilde{u}\|=\|\Phi\|_{L^\infty(\mu, X)}$, and $\tilde{u}(\varepsilon_i)=x_i$ implies that we have

$$\Phi = \sum_{i=1}^n \varepsilon_i x_i + \Psi$$

where $\Psi: D\to X$ is such that $\int \varepsilon_i \Psi\, d\mu=0$ for all $i\leq n$.

The correspondence $\tilde{u}\to\Phi$ is clearly bijective, so that we deduce

$$|(x_i)_{i\leq n}| = \inf \left\| \sum_{i=1}^n \varepsilon_i x_i + \Psi \right\|_{L_\infty(X)} \tag{1.2}$$

where the infimum runs over all Ψ as above.

Our original claim then becomes clear since (1.2) means that $\mathcal{X}$ is isometric with the quotient $L_\infty(D, \mathcal{B}_n, \mu; X)/N$ where N is the subspace of all Ψ as above.

It follows that $\mathcal{X}^*$ can be identified with $N^\perp \subset L_1(D, \mathcal{B}_n, \mu; X^*)$ and this proves the above claim since

$$N^\perp = \left\{ \sum_{i=1}^n \varepsilon_i \xi_i \,\middle|\, (\xi_i)_{i\leq n} \in X^{*n} \right\}.$$

(ii) We have an analogous fact with G in the place of R. Let $\mathcal{Y}$ be the space X^n equipped with the norm

$$|\,\|(x_i)_{i\leq n}\|\,| = \inf\{\|\tilde{u}\|\} \tag{1.3}$$

where the infimum runs over all operators $\tilde{u}: L_1(\Omega, \mathbf{P})\to X$ such that $\tilde{u}(g_i)=x_i$ for all $i\leq n$. Then $\mathcal{Y}^*$ can be identified with X^{*n} equipped with the norm

$$|\,\|(\xi_i)_{i\leq n}\|\,| = \left\| \sum g_i \xi_i \right\|_{L_1(\Omega, \mathbf{P}; X^*)}$$

The proof is entirely similar.

The next result is known, it shows that R or G play an equivalent rôle in all the sequel.

PROPOSITION 1.6. (i) *Let Z be a Banach space of cotype $q<\infty$. Then there is a constant λ (depending only on q and $c_q(Z)$ such that, for any $z_1, \ldots, z_n$ in Z we have*

$$\mathbf{E}\left\|\sum g_i z_i\right\| \leq \lambda \mathbf{E}\left\|\sum \varepsilon_i z_i\right\|.$$

(ii) *On the other hand, without any assumption on Z, we have*

$$\lambda' \mathbf{E}\left\|\sum \varepsilon_i z_i\right\| \leq \mathbf{E}\left\|\sum g_i z_i\right\|$$

where $\lambda' = \mathbf{E}|g_1| > 0$.

The reader is referred to [18] p. 68 and to [27] exposé III.

Remark 1.7. It follows immediately from Proposition 1.6 that if X^* is of cotype q for some $q<\infty$, then the norms considered on X^n in (1.1) and (1.3) are equivalent with equivalence constants independent of n.

PROPOSITION 1.8. *Let $C \geq 1$ be a constant. The following properties of a Banach space X are equivalent:*

(i) *X^* is of cotype 2 with $c_2(X^*) \leq C$.*

(ii) *For any sequence (x_i), with only finitely many non zero elements of X, the operator $u: R \to X$ defined by $u(\varepsilon_i) = x_i$ admits an extension $\tilde{u}: L_1(\mu) \to X$ such that $\tilde{u}|_R = u$ and $\|\tilde{u}\| \leq C(\Sigma \|x_i\|^2)^{1/2}$.*

Proof. Let n be a fixed integer. Recall that $l_2^n(X)$ is the space X^n equipped with the norm $(\Sigma_1^n \|x_i\|^2)^{1/2}$ for $(x_i)_{i \leq n}$ in X^n. We use the notations of remark 1.5.

Let $U_n: l_2^n(X) \to \mathscr{X}$ be the operator corresponding to the identity on X^n. Then remark 1.5 shows that Proposition 1.8 reduces to the classical identity $\|U_n\| = \|U_n^*\|$ for each integer n. Q.E.D.

The preceding result has a Gaussian analogue, as follows.

PROPOSITION 1.9. *Let $C \geq 1$ be a constant. The following properties of Banach space X are equivalent:*

(i) *For any finite sequence* (ξ_i) *in* X^*, *we have* $(\Sigma\|\xi_i\|^2)^{1/2} \leq C\mathbf{E}\|\Sigma g_i \xi_i\|$.

(ii) *For any finite rank operator* $u: l_2 \to X$, *there is an operator* $\tilde{u}: L_1(\Omega, \mathbf{P}) \to X$ *extending* u *in the sense that* $\tilde{u}|_G = u\beta^{-1}$ *and* $\|\tilde{u}\| \leq C\pi_2(u^*)$.

Proof. By the rotational invariance of the Gaussian canonical measures, it is easy to see (for details, cf. e.g. [28] exposé no. X, Proposition 2.2) that (i) is equivalent to

(i)* For any $(\xi_1, \ldots, \xi_n)$ in X^*, we have $\pi_2(w) \leq C\mathbf{E}\|\Sigma g_i \xi_i\|$ where $w: l_2^n \to X^*$ is the operator defined by $w(e_i) = \xi_i$ for all i.

Let $(x_i)_{i \leq n}$ be an element of X^n.

Now let $\mathscr{X}$ be the space X^n equipped with the norm

$$\|(x_i)_{i \leq n}\| = \pi_2(v^*)$$

where $v: l_2^n \to X$ is the operator defined by $v(e_i) = x_i$ for all i.

It is well known and easy to check that $\mathscr{X}^*$ can be identified isometrically with $\Pi_2(l_2^n, X^*)$.

Therefore, the equivalence of (i)* and (ii) follows again by an easy duality argument using remark 1.5 (ii). Q.E.D.

We recall a result due to Maurey which we will invoke several times (cf. [16], p. 116, or [17]).

THEOREM 1.10. *Let* X *be a Banach space of cotype* 2, *then any operator* u *from a* $C(K)$ *space into* X *is* 2*-absolutely summing. Morever, there is a constant* C *(depending only on* $c_2(X)$*) such that* $\pi_2(u) \leq C\|u\|$. *Also, for any* Y, *we have* $\Pi_2(X, Y) = \Pi_1(X, Y)$ *and there is a constant* c *such that* $\pi_1(v) \leq c\pi_2(v)$ *for any* $v: X \to Y$.

The following result will be used in the next section.

PROPOSITION 1.11. *The following properties of a Banach space* X *are equivalent:*

(i) X^* *is of cotype* 2 *and verifies Grothendieck's theorem.*

(ii) *There is a constant* C *such that every finite rank operator* $u: R \to X$ *admits an extension* $\tilde{u}: L_1(\mu) \to X$ *such that* $\|\tilde{u}\| \leq C\|u\|$.

(iii) *There is a constant* C *such that every finite rank operator* $u: G \to X$ *admits an extension* $\tilde{u}: L_1(\Omega, \mathbf{P}) \to X$ *such that* $\|\tilde{u}\| \leq C\|u\|$.

Proof. We first observe that each property (ii) or (iii) implies that X^* does not contain l_∞^n's uniformly, and therefore (cf. [18] p. 68) that X^* is of cotype q for some $q < \infty$.

The equivalence of (ii) and (iii) is then an obvious consequence of remark 1.7.

We now prove that (i) implies (iii).

If X^* is of cotype 2, we know by Proposition 1.9 (and Proposition 1.6) that there is a constant C' such that any $u: G \to X$ of finite rank extends to an operator $\tilde{u}: L_1(\Omega, \mathbf{P}) \to X$ with $\|\tilde{u}\| \leq C' \pi_2(u^*)$.

By Proposition 1.2, if X^* verifies Grothendieck's theorem, we have

$$\pi_2(u^*) \leq C'' \|u\|$$

for some constant C'', therefore

$$\|\tilde{u}\| \leq C' C'' \|u\|$$

and this shows that (i) implies (iii).

It remains to show that (ii) implies (i).

Assume that X verifies (ii) and let $u: R \to X$ be a finite rank operator. Then, we can factor u as $u = \tilde{u} r$ where $r: R \to L_1(\mu)$ is the natural injection and $\tilde{u}: L_1(\mu) \to X$ verifies $\|\tilde{u}\| \leq C\|u\|$.

Therefore, we have $u^* = r^* \tilde{u}^*$, so that

$$\pi_2(u^*) \leq \pi_2(r^*) \|\tilde{u}\| \leq C_1 \|u\|$$

where $C_1 = C \pi_2(r^*)$.

By Maurey's theorem (cf. Theorem 1.10), since X^* is of cotype 2, we have $\pi_1(u^*) \leq C_2 \pi_2(u^*)$ for some constant C_2 so that

$$\pi_1(u^*) \leq C_1 C_2 \|u\|.$$

By remark 1.3, this shows that X^* verifies Grothendieck's theorem. It is clearly of cotype 2 by Proposition 1.8, so that the proof of (ii)$\Rightarrow$(i) is complete. Q.E.D.

Remark. It is not hard to check that Proposition 1.11 remains valid if we replace X^* by X in (i) and X by X^* in (ii) and (iii).

Remark. It is also easy to check that the second property in Proposition 1.11 implies that for any bounded operator $u: R \to X$ there is an extension $\tilde{u}: L_1(\mu) \to X^{**}$ such that $\|\tilde{u}\| \leq C\|u\|$. Therefore, if X is complemented in its bidual (for example if X is reflexive), then property (ii) above holds for any bounded operator $u: R \to X$.

We will use the following remarkable result of J. Bourgain concerning the space L^1/H^1. We denote simply by L_1 the L_1-space relative to the circle group, and by H^1 the subspace of L_1 spanned by all the functions $\{e^{int} | n \geq 0\}$.

J. Bourgain proved in [1] that the space L^1/H^1 is of cotype 2 and verifies Grothendieck's theorem. Let $Q: L_1 \to L^1/H^1$ be the quotient map. By a routine argument, it can be shown that Bourgain's theorem has the following consequence.

COROLLARY 1.12. *There is an absolute constant b such that for any finite sequence $x_1, \ldots, x_n$ in L^1/H^1, there is a sequence $\tilde{x}_1, \ldots, \tilde{x}_n$ in L^1 such that: $Q(\tilde{x}_i)=x_i$ for all i and*

$$\mathbf{E}\left\|\sum \varepsilon_i \tilde{x}_i\right\| \leq b\mathbf{E}\left\|\sum \varepsilon_i x_i\right\|.$$

For the convenience of the reader, we sketch the argument.

Proof. Consider $v: l_2^n \to L^1/H^1$ defined by $v(e_i)=x_i$ for all $i \leq n$. Since L^1/H^1 if of cotype 2, we have

$$\pi_2(v) \leq C_1 \mathbf{E}\left\|\sum g_i x_i\right\| \tag{1.4}$$

for some constant C_1 (see e.g. [28], exposé X, for more details).

Therefore, by Pietsch's factorization theorem (cf. [21] p. 232) we can factorize v as $v=uw$ with $w: l_2^n \to l_2^n$ and $u: l_2^n \to L^1/H^1$ such that

$$\pi_2(w)\|u\| \leq \pi_2(v). \tag{1.5}$$

We claim that u can be "lifted" through L_1, that is to say that there exists an operator $\tilde{u}: l_2^n \to L_1$ such that $Q\tilde{u}=u$ and $\|\tilde{u}\| \leq C_2\|u\|$ for some constant C_2.

Indeed, let A be the disc algebra, considered as a subspace of $C(\mathbf{T})$. Consider the operator $T: A \to l_2^n$ defined by $T(f)=(\langle x_i, f\rangle)_{i \leq n}$. By Proposition 1.2 (iii) (or by Corollary 3 in [1]). There is a constant C' such that $\pi_2(T) \leq C'\|T\|$. Therefore, T extends to an operator $\tilde{T}: C(\mathbf{T}) \to l_2^n$ with $\|\tilde{T}\| \leq C'\|T\|$.

Let μ_k be the measures on T defined by $\tilde{T}^*(e_k)=\mu_k$. Let $f_1, \ldots, f_n$ be the absolutely continuous parts of these measures. Then the operator $\tilde{u}: l_2^n \to L_1$ such that $\tilde{u}(e_i)=f_i$ verifies $\|\tilde{u}\| \leq \|\tilde{T}^*\| \leq C'\|T\|=C'\|u\|$ and $Q\tilde{u}=u$.

Finally, we set $\tilde{x}_i=\tilde{u}w(e_i)$.

We have then

$$\begin{aligned}\mathbf{E}\left\|\sum \varepsilon_i \tilde{x}_i\right\| &\leq \|\tilde{u}\|\,\mathbf{E}\left\|\sum \varepsilon_i w(e_i)\right\| \\ &\leq \|\tilde{u}\|\left(\sum \|w(e_i)\|^2\right)^{1/2} \qquad (cont.)\end{aligned}$$

$$\begin{aligned}
&\leqslant \|\tilde{u}\|\, \pi_2(w)\\
&\leqslant C'\|u\|\, \pi_2(w)\\
&\leqslant C'\pi_2(v) \quad \text{by (1.5)}\\
&\leqslant C'C_1\, \mathbf{E}\left\|\sum g_i x_i\right\| \quad \text{by (1.4).}
\end{aligned}$$

Therefore, by Proposition 1.6, the sequence $(\tilde{x}_i)$ verifies the announced inequality.
Q.E.D.

§ 2. The main step

We will use the following result from [24].

PROPOSITION 2.1. *Let S be a closed subspace of a Banach space B. Let $\sigma: B \to B/S$ be the canonical surjection.*

Assume that S is α-isomorphic to a Hilbert space, for some $\alpha \geqslant 1$. Then there exists a constant $\lambda = \lambda(\alpha)$ depending only on α such that

$$\left.\begin{array}{l}\textit{For any finite sequence } (z_1, \dots, z_n) \textit{ in } B/S, \textit{ there is a sequence}\\ (\tilde{z}_1, \dots, \tilde{z}_n) \textit{ in } B \textit{ such that } \sigma(\tilde{z}_i) = z_i \textit{ for all } i \textit{ and } \mathbf{E}\|\Sigma\, \varepsilon_i \tilde{z}_i\| \leqslant \lambda \mathbf{E}\|\Sigma\, \varepsilon_i z_i\|.\end{array}\right\} \quad (2.1)$$

By a well known inequality of Kahane, the norms of a series of the form $\Sigma\, \varepsilon_i \tilde{z}_i$ in $L_2(\mu; B)$ and in $L_1(\mu; B)$ are equivalent (cf. e.g. [15], Vol. II, p. 74). Therefore, the preceding statement is a particular case of the main result of [24].

The above proposition holds under the weaker assumption that S is K-convex with $K(S) \leqslant \alpha$; we will not use this fact, see [24] for details. In the sequel, we will denote $K(S, B)$ the smallest constant λ for which (2.1) holds.

Let S be a reflexive subspace of L_1. Then S satisfies (2.1) for some constant λ (cf. [24]). Actually, it is known that the quotient L_1/S verifies Grothendieck's theorem. This was proved in 1976 independently in [22] and [10]. In his paper [10], Kisliakov used the following construction which is essential in the sequel: Let S be a subspace of a Banach space B and let $u: S \to E$ be an operator with values in some Banach space E. Then, we can find a Banach space E_1 containing E isometrically, such that the operator u "extends" to an operator $\tilde{u}: B \to E_1$ with the same norm, and moreover such that the spaces E_1/E and B/S are isometric.

Roughly, this means that the space E_1 extends E in the same way as B extends S. Kisliakov used the fact that if B/S and E are of some finite cotype, then the same is true for E_1. In this paper, we will use a more precise analysis of the constants involved in this phenomenon.

The following theorem and its corollary are the main results of this section. They are the crucial step on which is based the construction of the announced examples.

THEOREM 2.2. *Let $2 \leq q < \infty$. Let S be a closed subspace of a Banach space B. We assume that B is of cotype q and that S verifies* (2.1). *Then, for any operator u from S into a Banach space E, there is a Banach space E_1 and an isometric imbedding $j: E \to E_1$ with the following properties:*

(i) *There is an operator $\tilde{u}: B \to E_1$ such that $\tilde{u}|_S = ju$ and $\|\tilde{u}\| \leq K_1 \|u\|$, where $K_1 = 2(K(S,B)+1)$.*

(ii) *$c_q(E_1) \leq \max\{c_q(E), K_2\}$ where $K_2 = 2K(S,B)\, c_q(B)$.*

Proof. We first assume that $\|u\| \leq \eta$ where $\eta = (2(K(S,B)+1))^{-1} < 1$.

Let F be the space $B \oplus E$ equipped with the norm

$$\|(b,e)\| = \|b\| + \|e\| \quad \text{for all } b \text{ in } B \text{ and } e \text{ in } E.$$

We denote by $\sigma: B \to B/S$ the quotient map.

Let N be the subspace of F defined by

$$N = \{(s, -u(s)) \mid s \in S\}.$$

The space E_1 will be the quotient space F/N. We will denote by $\pi: F \to F/N$ the quotient map. We claim that the space E_1 has the desired properties. We define $j: E \to E_1$ and $\tilde{u}: B \to E_1$ as follows:

$$j(e) = \pi((0,e)) \quad \text{for all } e \text{ in } E$$

$$\tilde{u}(x) = \pi((x,0)) \quad \text{for all } x \text{ in } B.$$

The first properties are easy to check (exactly as in [10]): For e in E we have

$$\|j(e)\| = \inf\{\|s\| + \|e - u(s)\| \mid s \in S\} = \|e\| \quad \text{since } \|u(s)\| \leq \|s\|.$$

This shows that j is an isometric imbedding. Concerning $\tilde{u}$, we have clearly $\|\tilde{u}\| \leq 1$ and also, for all s in S,

$$\tilde{u}(s) = \pi((s,0)) = \pi((0, u(s)) = ju(s).$$

This shows that $\tilde{u}|_S = ju$.

We now pass to the proof of the second point which is the crucial one. Let $(f_1, \dots, f_n)$ be a finite sequence in $E_1 = F/N$ such that

$$\mathbf{E}\left\| \sum \varepsilon_k f_k \right\| < 1. \tag{2.2}$$

We can find y_k in B and e'_k in E such that

$$f_k = \pi((y_k, e'_k)).$$

Applying (2.1), we may find elements x_k in B such that $\sigma(x_k) = \sigma(y_k)$ and

$$\mathbf{E}\left\| \sum \varepsilon_k x_k \right\| \leqslant K\mathbf{E}\left\| \sum \varepsilon_k \sigma(y_k) \right\|, \tag{2.3}$$

with $K = K(S, B)$.

By adjusting e'_k, we may as well replace y_k by x_k; indeed, we have obviously

$$f_k = \pi((x_k, e_k)) \quad \text{with } e_k = e'_k - u(x_k - y_k).$$

We now develop (2.2). By definition of the quotient norm of $E_1 = F/N$, we can find for each ε in $\{-1, +1\}^n$ an element $s(\varepsilon)$ in S such that

$$\mathbf{E}\left\{ \left\| \sum_1^n \varepsilon_k x_k + s(\varepsilon) \right\| + \left\| \sum \varepsilon_k e_k - u(s(\varepsilon)) \right\| \right\} < 1. \tag{2.4}$$

where the expectation is meant with respect to ε in $\{-1, +1\}^n$ equipped with the uniform probability.

Let us set for simplicity

$$a = \mathbf{E}\left\| \sum \varepsilon_k x_k + s(\varepsilon) \right\| \quad \text{and} \quad b = \mathbf{E}\left\| \sum \varepsilon_k e_k - u(s(\varepsilon)) \right\|.$$

By (2.4), we have $a + b < 1$.

By (2.3), we have

$$\mathbf{E}\left\| \sum \varepsilon_k x_k \right\| \leqslant K\mathbf{E}\left\| \sum \varepsilon_k \sigma(x_k) \right\| \leqslant K\mathbf{E}\left\| \sigma\left(\sum \varepsilon_k x_k + s(\varepsilon) \right) \right\| \leqslant Ka. \tag{2.5}$$

By the triangle inequality, this implies $\mathbf{E}\|s(\varepsilon)\| \leqslant (1+K)\,a$.

Now, since $\|u\| \leqslant \eta$, this implies (again by the triangle inequality)

$$\mathbf{E}\left\|\sum \varepsilon_k e_k\right\| \leqslant b+\mathbf{E}\|u(s(\varepsilon))\| \leqslant b+\eta(1+K)\, a. \tag{2.6}$$

From (2.5) and (2.6), we deduce respectively

$$\left(\sum \|x_k\|^q\right)^{1/q} \leqslant Kac_q(B) \quad \text{and} \quad \left(\sum \|e_k\|^q\right)^{1/q} \leqslant c_q(E)\,(b+\eta(1+K)\,a) = c_q(E)\left(b+\frac{a}{2}\right).$$

Since $\|f_k\| \leqslant \|x_k\|+\|e_k\|$, this yields

$$\left(\sum \|f_k\|^q\right)^{1/q} \leqslant Kac_q(B)+c_q(E)\left(b+\frac{a}{2}\right)$$

and since $a+b<1$, this last expression is less than

$$\max\{c_q(E),\ 2Kc_q(B)\}.$$

By homogeneity, this means that

$$c_q(E_1) \leqslant \max\{c_q(E), K_2\} \quad \text{as announced.}$$

In conclusion, if $u\colon S \to E$ is now an arbitrary operator, we apply the preceding construction to the operator $v=u(\eta\|u\|^{-1})$, this yields a space E_1 and $\tilde{v}\colon B \to E_1$, extending v and such that $\|\tilde{v}\| \leqslant 1$. It is easy to check that $\tilde{u}=\|u\|\,\eta^{-1}\tilde{v}$ has all the required properties. Q.E.D.

Remark. Let f be an element of E_1; assume that $f=\pi(x, e')$ for some x in B and some e' in E. It is easy to check that

$$\|\sigma(x)\|_{B/S} = \inf\{\|f+j(e)\| \mid e \in E\}.$$

Therefore $E_1/j(E)$ is isometric to B/S, cf. [10].

COROLLARY 2.3. *Let* $2 \leqslant q < \infty$.

Let $\{B_i \mid i \in I\}$ *be a family of Banach spaces. For each* i, *let* S_i *be a closed subspace of* B_i.

We assume that

$$c_q = \sup\{c_q(B_i) \mid i \in I\} < \infty$$

$$K = \sup\{K(S_i, B_i) \mid i \in I\} < \infty.$$

Let E be a Banach space and let $u_i: S_i \to E$ be a family of operators. Then there is a Banach space E_1, and an isometric imbedding $j: E \to E_1$ with the following properties:

(i) *For each i in I, there is an operator $\tilde{u}_i: B_i \to E_1$ such that $\tilde{u}_i|_{S_i} = ju_i$ and $\|\tilde{u}_i\| \le K_1 \|u_i\|$ where $K_1 = 2(1+K)$.*

(ii) *$c_q(E_1) \le \max\{c_q(E), K_2\}$, where $K_2 = 2Kc_q$.*

Proof. We denote by $l_1\{B_i\}$ the Banach space of all families $(x_i)_{i\in I}$ in $\Pi_{i\in I} B_i$ such that $\Sigma_{i\in I}\|x_i\| < \infty$, equipped with its usual norm.

We let $B = l_1\{B_i\}$ and $S = l_1\{S_i\}$. We will consider S as a subspace of B.

We denote by $u: S \to E$ the operator defined by

$$u((s_i)_{i\in I}) = \sum_{i\in I} u_i(s_i) \quad \text{for all } (s_i)_{i\in I} \text{ in } S.$$

By homogeneity, we may clearly assume that $\|u_i\| = 1$ for all i in I. (Otherwise, we replace u_i by $v_i = u_i\|u_i\|^{-1}$ and we then set $\tilde{u}_i = \|u_i\|\,\tilde{v}_i$.) With this assumption, we have $\|u\| = 1$.

Let us denote by $\lambda_i: S_i \to S$ (resp. $\tilde{\lambda}_i: B_i \to B$) the canonical isometric injection of S_i into the subspace of S (resp. B) spanned by the elements $(x_i)_{i\in I}$ for which $x_j = 0$ for all $j \neq i$. We have clearly $u\lambda_i = u_i$.

On the other hand, it is easy to check that

$$K(S, B) = \sup\{K(S_i, B_i) \mid i \in I\} = K < \infty$$

and

$$c_q(B) = \sup\{c_q(B_i) \mid i \in I\} = c_q < \infty.$$

Therefore, by Theorem 2.2, we can find a space E_1, an isometric injection $j: E \to E_1$, and an operator $\tilde{u}: B \to E_1$ verifying the conclusions of Theorem 2.2. If we set $\tilde{u}_i = \tilde{u}\tilde{\lambda}_i$, we find $\tilde{u}_i|_{S_i} = \tilde{u}|_S \lambda_i = ju\lambda_i = ju_i$ and $\|\tilde{u}_i\| \le \|\tilde{u}\| \le K_1$. This concludes the proof of Corollary 2.3.

The crucial point in the preceding statement is that we can iterate the construction as many times as we wish without spoiling the estimate (ii). Indeed, let E_1 be the space obtained in Corollary 2.3, and suppose that we are given a family of operators $w_i: S_i \to E_1$.

If we apply again the preceding corollary to this family, then we find a space E_2, an isometric imbedding $j_1: E_1 \to E_2$ and operators $\tilde{w}_i: B_i \to E_2$ such that $\tilde{w}_i|_{S_i} = j_1 w_i$ and $\|\tilde{w}_i\| \le K_1\|w_i\|$.

Moreover, we have $j_1 \tilde{u}_i|_{S_i} = j_1\, j u_i$ so that we have preserved the extension property of E_1 relative to the family (u_i).

Finally, we have

$$c_q(E_2) \leq \max \{c_q(E_1), K_2\} \leq \max \{c_q(E), K_2\};$$

therefore, we have not spoiled either the second estimate in Corollary 2.3. If we continue this iterative process, we can obtain

THEOREM 2.4. *Let q, $\{S_i, B_i\}$, K_1 and K_2 be as in Corollary* 2.3.

For any Banach space E of cotype q, we can find a Banach space X of cotype q, containing E isometrically and possessing the following properties:

(i) *For any i in I and any $\varepsilon > 0$, every finite rank operator $v: S_i \to X$ admits an extension $\tilde{v}: B_i \to X$ such that $\tilde{v}|_{S_i} = v$ and $\|\tilde{v}\| \leq K_1 \|v\| (1+\varepsilon)$.*

(ii) $c_q(X) \leq \max \{c_q(E), K_2\}$.

Moreover, if I is countable and if E as well as each of the spaces B_i and S_i^* is separable, then we can find a separable space X with the above properties.

Proof. Let F_i be the set of all finite rank operators from S_i into E. We will apply the preceding corollary to the family $(S_i, B_i)_{i \in I'}$ where I' is the disjoint union of the sets $\{F_i | i \in I\}$, and where $\{u_l | l \in F_i\}$ is the collection of all finite rank operators from S_i into E. When l is in F_i, the spaces S_l and B_l are just taken identical to S_i and B_i.

By the preceding corollary, we can find E_1 and an isometric imbedding $j: E \to E_1$ such that properties (i) and (ii) are satisfied. If we now repeat the construction with E_1 in the place of E, we find E_2 and an isometric imbedding $j_1: E_1 \to E_2$ with the same properties. Continuing further, we construct inductively a sequence of Banach spaces $E_1, E_2, \ldots, E_n, \ldots$ and isometric imbeddings $j_n: E_n \to E_{n+1}$ with the following properties:

(2.7) For each i in I and each finite rank operator $v: S_i \to E_n$, there is an operator $\tilde{v}: B_i \to E_{n+1}$ such that $\tilde{v}|_{S_i} = j_n v$ and $\|\tilde{v}\| \leq K_1 \|v\|$.

$$c_q(E_{n+1}) \leq \max \{c_q(E_n), K_1\} \leq \max \{c_q(E), K_1\}. \tag{2.8}$$

Finally, we let X be the inductive limit of the system $\{E_n, j_n\}$ as indicated in § 1. (We identify X with the closure of $\cup E_n$.) By (2.8), we have clearly $c_q(X) \leq \max \{c_q(E), K_1\}$.

Let us denote by $J_n: E_n \to X$ the natural injection so that $J_n = J_{n+1}\, j_n$.

Let i be a fixed index in I. Since $\cup E_n$ is dense in X, the finite rank operators $v: S_i \to X$ with range in $\cup E_n$ (or equivalently in E_n for some integer n) are dense in the space of all finite rank operators from S_i into X. Therefore, it is enough to check the first assertion of Theorem 2.4 for such operators.

Now let $v: S_i \to X$ be a finite rank operator with range included in E_n for some integer n, so that we have $v = J_n u$ for some $u: S_i \to E_n$; by (2.7), there is an operator $\tilde{u}: B_i \to E_{n+1}$ such that $\tilde{u}|_{S_i} = j_n u$ and $\|\tilde{u}\| \leq K_1 \|u\| = K_1 \|v\|$.

We may then set $\tilde{v} = J_{n+1} \tilde{u}$, so that $\|\tilde{v}\| \leq K_1 \|v\|$ and $\tilde{v}|_{S_i} = v$.

Finally if E and the spaces B_i and S_i^* are separable and if I is countable, we can consider at each step a dense countable set of finite rank operators and we obtain finally a separable X as above. This concludes the proof.

Remark. Actually, Corollary 2.3 is not really needed. We can deduce directly Theorem 2.4 (or Corollary 2.3) from Theorem 2.2 by transfinite induction.

Remark 2.5. For future reference, we consider here again the separable case, as in the last assertion of Theorem 2.4. Let L be a Banach space such that L^* is separable, and let $P: S \to L$ be a fixed operator from a certain subspace S of a separable Banach space B. We assume that $K(S, B) \leq K$ and $c_q(B) \leq c_q$. Then, we can find a *separable* space X satisfying the conclusions of Theorem 2.4, and possessing moreover the following property:

For each $\varepsilon > 0$, and for each operator $v: L \to X$ of finite rank, there is an operator $\tilde{T}: B \to X$ such that $\tilde{T}|_S = vP$ and $\|\tilde{T}\| \leq K_1(1+\varepsilon)\|vP\|$. The proof is an obvious modification of the preceding argument for Theorem 2.4.

§ 3. The counterexamples

In this section, we present the counterexamples to the conjecture of Grothendieck mentioned earlier.

Let $u: X \to Y$ be an operator between Banach spaces; then if u factors through a Hilbert space, its "norm of factorization" $\gamma_2(u)$ is defined as

$$\gamma_2(u) = \inf\{\|v\| \, \|w\| : u = vw, \; w: X \to H, \; v: H \to Y\}$$

where the infimum runs over all possible factorizations of u through a Hilbert space H.

We will use the following result from [23].

THEOREM 3.1. *Let X, Y be Banach spaces such that X^* and Y are both of cotype 2. Then there is a constant $C_{X,Y}$ (depending only on $c_2(X)$ and $c_2(Y)$ such that, for any finite rank operator $u: X \to Y$, we have*

$$\gamma_2(u) \leq C_{X,Y} \|u\|. \tag{3.1}$$

More generally, every operator $u: X \to Y$, which is approximable uniformly on every compact subset of X by finite rank operators, factors through a Hilbert space and verifies (3.1).

Let us recall the definitions of the injective and projective tensor products. Let X and Y be two Banach spaces. Then any element u of $X \otimes Y$ defines a finite rank operator from X^* into Y. We denote $\|u\|_\vee$ the operator norm of this associated operator.

We denote $\|u\|_\wedge = \inf\{\Sigma \|x_i\| \|y_i\|\}$ where the infimum is over all possible representations $u = \Sigma_{i=1}^n x_i \otimes y_i$ of the tensor u. Finally, we denote by $X \check{\otimes} Y$ (resp. $X \hat{\otimes} Y$) the completion of the space $X \otimes Y$ with respect to the norm $\| \ \|_\vee$ (resp. $\| \ \|_\wedge$).

This is the injective (resp. projective) tensor product of X and Y according to the terminology of [6].

We can now state our main result:

THEOREM 3.2. *Any Banach space E of cotype 2 can be imbedded isometrically into a Banach space X such that:*

(a) *X and X^* are both of cotype 2 and both verify Grothendieck's theorem.*

(b) *$X \hat{\otimes} X$ and $X \check{\otimes} X$ are identical.*

Moreover, if E is separable, we can find a separable space X with these properties.

Remark 3.3. We will apply Theorem 2.4 with a specific family $\{(S_1, B_1), (S_2, B_2), (S_3, B_3)\}$. The space B_2 will be simply $L_1(\mu)$ and S_1 will be R. The space B_2 will be L_1 (over the circle group) and S_2 will be H^1. Finally B_3 will be L^1/H_1. We will denote ξ_n the element of L^1/H^1 corresponding to the function $e^{-i3^n t}$ (so that $\xi_n = Q(e^{-i3^n t})$, where $Q: L^1 \to L^1/H^1$ is the quotient map). The space S_3 will be the subspace of $B_3 = L^1/H^1$ spanned by the sequence $\{\xi_n | n \in \mathrm{N}\}$. It is known that S_3 is isomorphic to l_2 (cf. e.g. [20] § 3).

Let $k: l_2 \to B_3$ be the map defined by $k(e_n) = \xi_n$ which establishes an isomorphism between l_2 and $S_3 = k(l_2)$. We will use the fact that $k^*: H^\infty \to l_2$ admits the factorization $k^* = PJ$ where $J: H^\infty \to H^1$ is the natural injection and where $P: H^1 \to l_2$ is the operator defined by $Pf = (\hat{f}(3^n))_{n \geq 0}$ for all f in H^1. By a classical theorem of Paley (see [20] § 3),

P is a bounded operator. This factorization of k is used in Wojtaszczyk's proof of Grothendieck's theorem (cf. [20] § 3). Finally, we observe that the assumptions of Theorem 2.4 are satisfied: It is well known that B_1 and B_2 are of cotype 2; by Bourgain's theorem [1], the same is true for B_3, and by Corollary 1.12 and Theorem 2.1 each of the three couples (S_i, B_i) $(i=1, 2, 3)$ possesses property (2.1) of Theorem 2.1. We are now in a position to apply Theorem 2.4.

Proof of Theorem 3.2. By Theorem 2.4 E can be isometrically imbedded into a space X of cotype 2 and verifying the extension property with respect to the spaces $\{(S_i, B_i)\}$ $(i=1, 2, 3)$ described above. We first observe that, by Proposition 1.11, X^* is of cotype 2 and verifies Grothendieck's theorem. Let us show that X also does. Let $v: X \to l_2^n$ be an operator. We will estimate $\pi_1(v)$. By definition of $\pi_1(v)$, we have

$$\pi_1(v) = \sup \{\pi_1(vw)\} \tag{3.2}$$

where the supremum runs over all operators $w: c_0 \to X$ with $\|w\| \leq 1$. Since X is of cotype 2, we have by Maurey's theorem (cf. Theorem 1.10) $\pi_2(w) \leq C\|w\| \leq C$ for some constant C. By Pietsch's theorem (cf. [14] Proposition 3.1) we can factor w as $w_2 w_1$ with $w_1: c_0 \to l_2$ and $w_2: l_2 \to X$ such that

$$\pi_2(w_1)\|w_2\| \leq \pi_2(w) \leq C. \tag{3.3}$$

Since $\pi_2(vw_2)$ is nothing but the Hilbert–Schmidt norm,

$$\pi_2((vw_2)^*) = \pi_2(vw_2).$$

Therefore, we have

$$\pi_2(vw_2) \leq \pi_2(w_2^*)\|v\|$$

and since X^* verifies Grothendieck's theorem, we have

$$\pi_2(w_2^*) \leq C'\|w_2\| \quad \text{for some constant } C'.$$

Using the fact that the composition of two 2-absolutely summing operators is 1-absolutely summing, we obtain finally

$$\begin{aligned}\pi_1(vw) = \pi_1(vw_2 w_1) &\leq \pi_2(vw_2)\pi_2(w_1) \quad \text{(cf. [21] p. 286)}\\ &\leq C'\|w_2\|\,\|v\|\,\pi_2(w_1)\\ &\leq CC'\|v\| \quad \text{by (3.3).}\end{aligned}$$

This means, by (3.2), that $\pi_1(v)\leq CC'\|v\|$, which implies (cf. Proposition 1.2) that X verifies Grothendieck's theorem. This proves the first part.

We turn now to part (b).

Let u be an element of $X\otimes X$. We will show that $\|u\|_\wedge$ and $\|u\|_\vee$ are equivalent. The tensor u defines a finite rank operator denoted again (albeit abusively) $u: X^*\to X$.

Since X is of cotype 2, by Theorem 3.1, there is a constant C such that $\gamma_2(u)\leq C\|u\|$. Therefore, we can decompose u as $u=u_2u_1$ where $u_1: X^*\to l_2$ and $u_2: l_2\to X$ are finite rank operators such that $\|u_1\|\,\|u_2\|\leq C\|u\|$. Moreover, since u is weak-$*$ continuous, we can assume that the same is true for u_1. Therefore, we have $u_1=v^*$ for some operator $v: l_2\to X$ of finite rank, say equal to n. We may as well assume that $v(e_i)=0$ for all $i>n$. We now use the extension property of X with respect to (S_3, B_3). This yields an operator $\tilde{v}: L^1/H^1\to X$ such that $\|\tilde{v}\|\leq K\|v\|$ for some constant K and $\tilde{v}|_{S_3}=vk^{-1}|_{S_3}$ or equivalently $\tilde{v}k=v$.

It is well known that, for each integer n, there is a *finite rank* operator $T_n: L^1/H^1\to L^1/H^1$ which is the identity on the span of $\{\xi_1,\dots,\xi_n\}$, vanishes on the span of $\{\xi_i|i>n\}$, and verifies $\|T_n\|\leq C'$ for some constant C' independent of n. (For instance, a Riesz product argument yields this, cf. [29] p. 247.)

We have then obviously:

$$v=\tilde{v}T_nk,$$

so that $u_1=k^*w$ with $w=(\tilde{v}T_n)^*$.

Note that

$$\|w\|\leq C'\|\tilde{v}\|\leq KC'\|v\|=KC'\|u_1\|. \tag{3.4}$$

By the factorization of k^* described above, we have

$$u_1=PJw,\quad \text{hence } u=(u_2P)Jw.$$

If we now apply the extension property to $T=u_2P: H^1\to X$, we obtain an operator $\tilde{T}: L^1\to X$ such that $\|\tilde{T}\|\leq K\|T\|$ for some constant K and $\tilde{T}|_{H^1}=T$.

Let us denote by $\tilde{J}: L_\infty\to L_1$ and by $i: H^\infty\to L_\infty$ the natural injections (relative to the torus).

We have finally obtained $u=\tilde{T}\tilde{J}iw$. This factorization can be summarized in the following commutative diagram.

$$\begin{tikzcd}
L_\infty \arrow[r, "\tilde{J}"] & L_1 \arrow[ddr, "\tilde{T}"] & \\
H^\infty \arrow[u, hook, "i"] \arrow[r, "J"] & H^1 \arrow[u, hook] \arrow[d, "P"] \arrow[dr, "T"] & \\
X^* \arrow[u, "w"] \arrow[r, "u_1"] & l_2 \arrow[r, "u_2"] & X
\end{tikzcd}$$

The operator $\tilde{T}\tilde{J}i$ is clearly integral with integral norm

$$\begin{aligned} I_1(\tilde{T}\tilde{J}i) &\leq \|\tilde{T}\|\,\|i\| = \|\tilde{T}\| \leq K\|T\| \\ &\leq K'\|u_2\| \quad \text{for some constant } K'. \end{aligned}$$

Since w is of finite rank, $E=w(X^*)$ is finite dimensional. We can rewrite w as $w=h\bar{w}$, where $h\colon E\to H^\infty$ is the natural injection, and where $\bar{w}\colon X^*\to E$ is the restriction (relative to the range) of w. Note that, since w is weak-$*$ continuous, the same is true for $\bar{w}$.

Consider now the operator $A=\tilde{T}\tilde{J}ih\colon E\to X$. By (3.5), we have $I_1(A)\leq K'\|u_2\|$, but since E is finite dimensional the nuclear and the integral norm of A coincide; hence, this operator A can be identified with a tensor (denoted again A) in $E^*\otimes X$ such that

$$\|A\|_\wedge = I_1(A) \leq K'\|u_2\| \quad \text{(cf. e.g. [21] p. 102).}$$

Now, since $\bar{w}$ is weak-$*$ continuous, the tensor u, associated to the composition $u=A\bar{w}$, in $X\otimes X$ must verify

$$\begin{aligned} \|u\|_\wedge \leq \|A\|_\wedge \|\bar{w}\|_\vee &\leq K'\|u_2\|\,\|w\| \\ &\leq KK'C'\|u_2\|\,\|u_1\| \quad \text{by (3.4)} \\ &\leq KK'CC'\|u\|. \end{aligned}$$

This proves that $\|\ \|_\vee$ and $\|\ \|_\wedge$ are equivalent on $X\otimes X$, therefore the completed tensor products $X\check{\otimes}X$ and $X\hat{\otimes}X$ must be identical and their norms are equivalent.

Finally, the separability of X can be justified using remark 2.5, applied to $P\colon H^1\to l_2$.

Remark. By a similar argument, we can show that any space E of cotype q can be imbedded into a space X of cotype q such that X^* is of cotype 2 and verifies Grothendieck's theorem.

Remark 3.4. Let X be a space with property (a) of Theorem 3.2. Let $T: X \to X$ be a finite rank operator. Then, by Theorem 3.1, $\gamma_2(T) \leq C\|T\|$ for some constant C, and since X verifies Grothendieck's theorem $\pi_2(T) \leq C'\gamma_2(T)$ for some constant C'. Therefore, $\pi_2(T) \leq CC'\|T\|$. As a consequence, the eigenvalues of T, denoted by $(\lambda_j(T))_{j\geq 1}$, must verify:

$$\left(\sum |\lambda_j(T)|^2\right)^{1/2} \leq \pi_2(T) \leq CC'\|T\|$$

(cf. [21] § 27.4.6).

In particular, if $P: X \to X$ is a projection, we have $\pi_2(P) \geq (\operatorname{rank} P)^{1/2}$ so that $\|P\| \geq (1/CC')(\operatorname{rank} P)^{1/2}$.

This leads to the following application.

COROLLARY 3.5. *There is an infinite dimensional Banach space X and a positive δ such that, for any finite rank projection $P: X \to X$, we have*

$$\|P\| \geq \delta(\operatorname{rank} P)^{1/2}.$$

In particular, this space X cannot contain uniformly complemented l_p^n's for any p, $1 \leq p \leq \infty$; this space is a counterexample to a question of Lindenstrauss [13].

It is worthwhile to recall that if E is an n dimensional subspace of an arbitrary Banach space X, then there exists a projection $P: X \to E$ with $\|P\| \leq \sqrt{n}$. Therefore, the space described in the preceding corollary exhibits an extreme behaviour with respect to finite rank projections; in this space, the general upper bound which we just recalled is also a lower bound.

Remark 3.6. Take $E = l_1$ in Theorem 3.2, then clearly X cannot be isomorphic to a Hilbert space although it is of cotype 2 as well as its dual. This yields a counterexample to a conjecture of Maurey [17]. Actually, except in the finite dimensional case, any space X verifying Theorem 3.2 must fail the approximation property. Otherwise, the identity on X would be approximable, uniformly on compact subsets of X, by finite rank operators and (by Theorem 3.1) X would have to be isomorphic to a Hilbert space, which is impossible because of (a) or (b). Therefore, such a space X also shows that the approximability assumption of Theorem 3.1 cannot be dropped.

We recall that an operator $u: X \to Y$ is called nuclear if there are sequences (x_n^*) and (y_n) in X^* and Y respectively, such that $\Sigma_1^\infty \|x_n^*\| \|y_n\| < \infty$ and

(3.5) $u(x) = \Sigma_1^\infty x_n^*(x) y_n$ for all x in X. The nuclear norm of u, denoted $N(u)$, is defined as the infimum of $\Sigma_1^\infty \|x_n^*\| \|y_n\|$ over all possible representations of the form (3.5). We denote by $N(X, Y)$ the Banach space of all nuclear operators from X into Y. Obviously, there is a natural surjection of $X^*\hat{\otimes} Y$ into $N(X, Y)$ but in general this map is *not* injective. Indeed, Grothendieck showed that X has the approximation property if and only if the surjection from $X^*\hat{\otimes}X$ into $N(X,X)$ is injective. It is not the case in the next result (the first example of a space failing the A.P. goes back to Enflo [5]).

THEOREM 3.7. *Let X be a Banach space verifying property* (a) *in Theorem* 3.2. *Then, there is a constant K such that, any finite rank operator $v: X \to X$ verifies $N(v) \leq K\|v\|$. (Equivalently the map from $X^*\hat{\otimes}X$ into $X^*\check{\otimes}X$ is onto.)*

To prove this result, we will need a dual version of Theorem 3.1, which is also proved in [23]. Let X, Y be Banach spaces and let v: $Y \to X$ be a finite rank operator. The γ_2^* norm of v is defined as

$$\gamma_2^*(v) = \inf\left\{\left(\sum_1^n \|y_i^*\|^2\right)^{1/2}\left(\sum_1^n \|x_i\|^2\right)^{1/2}\right\} \tag{3.6}$$

where the infimum runs over all finite sets (y_i^*) and (x_i) in Y^* and X such that

$$|\langle x^*, v(y)\rangle| \leq \left\{\sum_1^n |y_i^*(y)|^2 \sum_1^n |x^*(x_i)|^2\right\}^{1/2}$$

for all x^* and y in X^* and Y respectively.

This is the definition chosen in [23]. (Note however, that there is a mis-print in [23], which makes the definition of γ_2^* unintelligible: line (2.15) in [23] should be erased.)

Equivalently, we can clearly define $\gamma_2^*(v)$ as equal to (3.6) where the infimum runs over all (y_i^*) and (x_i) for which we have

$$v = \sum_{i,j=1}^n a_{ij} y_i^* \otimes x_j$$

for some matrix (a_{ij}) corresponding to an operator $a: l_2^n \to l_2^n$ with $\|a\| \leq 1$.

In some sense, the γ_2^*-norm can be considered as the dual norm to the γ_2-norm (cf. [12] or [21] for details).

We also use a different reformulation of (3.6). Indeed, it is possible to show that

$$\gamma_2^*(v) = \inf\{\pi_2(A)\,\pi_2(B^*)\}$$

where the infimum runs over all possible factorization $v=BA$ of v. For more details on this, cf. [21] § 19.1.8, p. 264.

The "dual" version of Theorem 3.1 is the following (cf. [23]).

THEOREM 3.8. *Let X, Y be as in Theorem 3.1. Then there is a constant K depending only on $c_2(X^*)$ and $c_2(Y)$ such that, for any finite rank operator $v: Y\to X$, we have $N(v)\leq K\gamma_2^*(v)$.*

Remark. It is worthwhile to emphasize that $N(v)$ *cannot* be replaced by $\|v\|_\wedge$ in the preceding result.

Proof of Theorem 3.7. Let $v:X\to X$ be a finite rank operator. Then, by Theorem 3.1, we can write $v=BA$ with $A:X\to l_2$ and $B:l_2\to X$ such that $\|A\|\,\|B\|\leq C\|v\|$ for some constant C depending only on X. Since we assume that X and X^* both verify Grothendieck's theorem, we have

$$\pi_2(A)\leq C'\|A\| \quad \text{and} \quad \pi_2(B^*)\leq C''\|B\|$$

for some constants C' and C''. (Recall that $\pi_2(u)\leq\pi_1(u)$ for any operator u.)

Therefore, $\gamma_2^*(v)\leq CC'C''\|v\|$.

Finally, by Theorem 3.8, we obtain $N(v)\leq KCC'C''\|v\|$, which concludes the proof of Theorem 3.7.

Remark. Let X be the Banach space associated to $E=l_1$ in Theorem 3.2. Then $B(X,l_1)=\Pi_1(X,l_1)$. Indeed, if $u:X\to l_1$ is a finite rank operator and if $j:l_1\to X$ is an isometric imbedding, we have by Theorem 3.7: $\pi_1(u)\leq N(ju)\leq K\|ju\|=K\|u\|$ for some constant K.

This is the first known (infinite dimensional) space with this property.

Remark. In this remark, we use the terminology and the notations of [24]. It is easy to see, using [25], that a Banach space B verifies property (2.1) for *any* subspace $S\subset B$ if and only if B is K-convex. Therefore, in that case, we may use in Theorem 2.4 the family of *all* subspaces S of B.

For instance we can use $B=L^p$, for $1<p<2$, and we obtain in this way a Banach space X verifying the conclusions of Theorem 3.2 and verifying in addition:

(iii) There is a constant C, such that for any space Y and any finite rank operator $u: Y\to X$, we have $I_p(u)\leq C\pi_p(u)$, where $I_p(u)$ is the p-integral norm of u.

Remark. S. V. Kisliakov observed (private communication) that the results of this section can be proved without appealing to Bourgain's results in [1], by making a different choice of (S_2, B_2) and (S_3, B_3) than the above. We only briefly sketch Kisliakov's ideas (the reader will find similar ideas exploited in [11]). We denote by L_p^n the space $\mathbf{R}^n$ equipped with the norm

$$\|(x_i)_{i\leq n}\| = \left(\frac{1}{n}\sum |x_i|^p\right)^{1/p} \quad \text{for } (x_i) \text{ in } \mathbf{R}^n.$$

By a variant of a construction originally due to Kashin (see [11]) it is possible to prove that, for each integer n, the space L_2^n can be decomposed into three mutually orthogonal parts $S_n^1 \oplus S_n^2 \oplus S_n^3$ with the following properties:

(i) The norms of L_2^n and L_1^n are uniformly equivalent (with constants independent of n) on $S_n^1 \oplus S_n^2$ and $S_n^2 \oplus S_n^3$.

(ii) The dimension of S_n^j tends to infinity with n ($j=1,2,3$).

If we apply Theorem 2.4 to (S_1, B_1) as before and, in addition, to the family $S_n^2 \subset L_1^n/S_n^3$, $S_n^2 \subset L_1^n/S_n^1$ ($n=1,2,\ldots$) we obtain Theorem 3.2.

Remark. Incidentally, the space (S_1, B_1) was used mainly to have a convenient way to see that the space X constructed in Theorem 3.2 (or in the preceding remark) is of cotype 2, but we could have derived this fact from the other extension properties of X. On the other hand, the first part of Theorem 3.2 can be obtained using *only* (S_1, B_1). The results of [1] are really used only to prove part (b).

§ 4. Further remarks and problems

In this section, we present some examples (simpler than those of section 3) showing that the projective tensor product $X \hat{\otimes} Y$ does not inherit the "good" properties of X and Y, such as weak sequential completeness, not containing c_0, or cotype.

THEOREM 4.1. *There is a weakly sequentially complete (in short* w.s.c.*) separable Banach space Z of cotype* 2 *and such that $L^1/H^1 \hat{\otimes} Z$ contains c_0.*

Remark 4.2. Since L^1/H^1 is of cotype 2 [1], it does not contain c_0, and moreover (cf. [19]) it is w.s.c. Therefore, the preceding theorem shows that these properties are not stable by the projective tensor product. Note also that $L^1/H^1 \hat{\otimes} Z$ is of cotype q for no finite q since it contains c_0. This answers several questions raised in the literature, cf. [8], [3], p. 258.

Proof of Theorem 4.1. We use the notations introduced in remark 3.3. The operator $k^*: H^\infty \to l_2$ factors as $k^* = PJ$ where $J: H^\infty \to H^1$ is the natural injection. Recall that we denote by $\tilde{J}: L_\infty \to L_1$ and by $i: H^\infty \to L_\infty$ the natural injections (the underlying measure space being the circle).

We can apply Theorem 2.2 to the operator $P: H^1 \to l_2$. This shows that there is a separable Banach space Z of cotype 2, an isometric imbedding $j: l_2 \to Z$ and a bounded operator $\tilde{P}: L_1 \to Z$ such that $\tilde{P}|_{H^1} = jP$. Let $z_n = j(e_n)$, and let $U = \tilde{P}\tilde{J}i = jk^*$.

The operator U is clearly integral, and we have

$$U\varphi = \sum_{n=1}^{\infty} \xi_n(\varphi)\, z_n$$

for all φ in H^∞. We claim that the sequence $\xi_n \otimes z_n$ is equivalent to the unit vector basis of c_0 in $L^1/H^1 \hat{\otimes} Z$.

By a well known result (cf. e.g. [29], p. 247), there is a constant D such that for any scalar sequence (ω_k) with only finitely many non zero terms, there is a finite rank operator $T: L^1/H^1 \to L^1/H^1$ such that $T\xi_k = \omega_k \xi_k$ for all k, and $\|T\| \leq D \sup |\omega_k|$.

Since T is of finite rank, the composition $UT^*: H^\infty \to Z$ verifies

$N(UT^*) \leq \|T\|\, I_1(U)$, and more precisely we have:

$$\left\| \sum_k \omega_k \xi_k \otimes z_k \right\|_{L^1/H^1 \hat{\otimes} Z} \leq C \sup |\omega_k| \tag{4.1}$$

where $C = DI_1(U)$, for all (ω_k) as above.

On the other hand, let $V = \Sigma\, \omega_k\, \xi_k \otimes z_k$; we have $V(e^{i3^k t}) = \omega_k z_k$, so that $|\omega_k| \leq \|V\|$ for all k.

This yields

$$\sup |\omega_k| \leq \left\| \sum \omega_k \xi_k \otimes z_k \right\|_{L^1/H^1 \check{\otimes} Z}; \tag{4.2}$$

and, since $\|V\| \leq N(V)$, this proves the claim.

It remains to show that Z is w.s.c. Let $Y = j(l_2)$. Then, Y is reflexive (since it is a Hilbert space), and Z/Y is, by construction, isometric to L^1/H^1 (cf. the remark after Theorem 2.2). Since L^1/H^1 is w.s.c. (cf. [19]), so is Z/Y. Let us show that the same is true for Z. Let (z_n) be a weak Cauchy sequence in Z, converging weak-$*$ to an element z in Z^{**}. We have to show that z is in Z.

Let $\pi: Z \to Z/Y$ be the quotient map. Observe that $(Z/Y)^{**}=Z^{**}/Y^{**}=Z^{**}/Y$. Since Z/Y is w.s.c., the sequence $\pi(z_n)$ converges weakly in Z/Y to some element $\tilde{z}$ in Z/Y; on the other hand, $\pi(z_n)=\pi^{**}(z_n)$ must tend in the weak-$*$ to topology to $\pi^{**}(z)$. Therefore $\pi^{**}(z)=\tilde{z}$ so that z must belong to $Z+Y$ which is included in Z. This completes the proof that Z is w.s.c. Actually, G. Godefroy (personal communication) proved that if Y is a w.s.c. subspace of a Banach space Z and if Z/Y is w.s.c., then Z is w.s.c.

Remark 4.3. Actually, for any norm α on $L^1/H^1 \otimes Z$ such that $\|\cdot\|_\vee \leq \alpha \leq \|\cdot\|_\wedge$, the completed tensor product $L^1/H^1 \hat{\otimes}_\alpha Z$ contains c_0. This follows immediately from (4.1) and (4.2). Therefore, not only the projective tensor product, but any other reasonable tensor product fails the stability properties already mentioned.

Remark 4.4. (i) A close look at the proof of Theorem 2.2 shows that the space Z is nothing else but the quotient $L^1/\ker P$, where $P: H^1 \to l_2$ is the operator defined in remark 3.3.

(ii) Kisliakov observed (private communication) that the space Z verifies Grothendieck's theorem. Therefore, Theorem 4.1 yields two spaces verifying Grothendieck's theorem, but such that their projective tensor product does not. Actually, this also follows from Theorems 3.2 or 3.7, but the space Z is simpler. This answers problem 1 in [22].

(iii) J. Bourgain (private communication) was able to show that there is a Banach space X (related to the examples of section 3) with the Radon-Nikodym property (in short R.N.P.), and such that $X \hat{\otimes} X$ contains c_0, and consequently fails the R.N.P. For more examples of this sort, cf. [30].

Finally, we mention several open problems. (In the sequel, we implicitly consider only infinite dimensional spaces.)

Problem 4.5. Does there exist a *reflexive* Banach space X with property (a) or (b) (or both) of Theorem 3.2.

Actually, there is no known example of a reflexive space verifying Grothendieck's theorem. It is also not known whether there is a reflexive Banach space which does not contain uniformly complemented l_p^n's for any p (we conjecture that there are such spaces).

Note however that, if $E=l_2$ in Theorem 3.2, then the space X cannot be reflexive since X^* must contain l_1; indeed, the quotient map from X^* onto l_2 must be 1-summing, but it is not compact, therefore (cf. [26] added in proof) X^* must contain l_1. (I am grateful to H. P. Rosenthal for helpful conversations on this and related questions.)

Remark 4.6. Let X be a space verifying property (b) in Theorem 3.2. By duality, this implies that every *bounded* operator $u: X \to X^*$ is integral (incidentally, this shows that X and X^* are not isomorphic). If we could obtain a reflexive space X (or merely such that X^* has the R.N.P.) then, every *bounded* operator from X into X^* would be nuclear. It is apparently still unknown whether there are infinite dimensional spaces X and Y such that every compact (or every bounded) operator from X into Y is nuclear.

Remark 4.7. It is our feeling that the counterexamples to the Lindenstrauss question [13] presented in section 3 are intimately connected with the absence of approximation property. More precisely, we conjecture that every space X with the B.A.P. (or simply with the A.P.) contains uniformly complemented l_p^n's for some p. The results of [25] strongly support this conjecture. In [25], a positive answer is given if X is of cotype 2 or only of cotype q for any $q>2$. Moreover, (cf. [25]), any space which is of cotype 2 as well as its dual and is not a Hilbert space, cannot be imbedded into any space with the B.A.P. not containing l_∞^n's uniformly.

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