A geometric proof of Mostow's rigidity theorem for groups of divergence type

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1. Introduction

The nature of the "boundary map" of a geometric isomorphism between discrete Möbius groups Γ, Γ' acting on upper half n-space $\mathbf{U}^n$, has been studied extensively under various hypotheses on Γ. Without undue elaboration at this point, it comes down to a homeomorphism g of $\mathbf{R}^{n-1}=\partial\mathbf{U}^n$, with the property that for every $A\in\Gamma$, the composition $g\cdot A\cdot g^{-1}$ belongs to Γ' (as Γ' acts on $\mathbf{R}^{n-1}=\partial\mathbf{U}^n$).

The most dramatic result [8] is a special case of what has become known as Mostow's rigidity theorem, and states that if Γ is of finite covolume, and if $n\geqslant 3$, then g is conformal. Ahlfors gave a shortened proof in [3]. Mostow later [9] extended his theorem to the case $n=2$, with startling alternative conclusions: either g is linear fractional, or it is purely singular. In this context, Kuusalo [6] obtained two similar results under weaker hypotheses, though by "singular" he did not mean quite as strongly singular as Mostow. For example, he proved that if $\mathbf{U}^2/\Gamma$ is of class O_{HB} (no bounded nonconstant harmonic functions), then g is absolutely continuous or singular, and if $\mathbf{U}^2/\Gamma$ is of class O_G (no Green's function), then g is linear fractional or singular.

In another direction, three alternative limit sets for the group Γ have been considered, which I will refer to as the topological, horocyclic, and conical limit sets, and denote respectively by Λ_T, Λ_H, Λ_C. Because Γ acts discontinuously in $\mathbf{U}^n$, these sets all lie in $\bar{\mathbf{R}}^{n-1}=\partial\mathbf{U}^n$, and we have the inclusions $\Lambda_T\supseteq\Lambda_H\supseteq\Lambda_C$. The condition that $\bar{\mathbf{R}}^{n-1}\setminus\Lambda_H$ have measure zero corresponds to the class O_{HB}, and the condition that $\bar{\mathbf{R}}^{n-1}\setminus\Lambda_C$ have measure zero corresponds to the class O_G. Groups with the latter property are precisely the same as groups of "divergence type", where this term traditionally refers to groups for which a certain series (3.6) diverges. The present paper has as its main objective, to show that Mostow's rigidity theorem applies to such groups.

In his important paper [11], Sullivan has proved some very general properties of Λ_H in case $n \geqslant 3$. He draws the conclusion that Mostow's rigidity theorem applies to groups for which $\bar{\mathbf{R}}^{n-1} \setminus \Lambda_H$ has measure zero. In light of Sullivan's work, then, part of the present paper is a step backward. However, by paying attention to the case $n=2$, we can obtain Kuusalo's result, and some valuable insight into Mostow's stronger version. The latter, however, is a story in itself [14].

In addition, however, I have attempted to make it geometrically clear what is happening on the boundary, and hopefully to make the theorem accessible to persons interested in quasiconformal mappings and discrete Möbius groups. To this end, I have minimized the use of non-constructive (ergodic) existence criteria. In order to retain reasonable completeness, I have repeated some arguments used by Mostow and/or Sullivan in the simplified versions which are appropriate to this paper.

The major single step in the present argument is an adaptation of a nontrivial but elementary argument that if Λ_C has measure zero, then the series (3.6) converges. This argument was presented by Ahlfors in a recent series of lectures at Minnesota, [2], and is attributed by him to Thurston.

The adaptation is to a method devised many years ago by P. J. Myrberg [10]. However, Myrberg dealt with very special finitely generated groups in the plane, and the present paper does extend his results both with respect to weakening the hypotheses, and raising the dimension.

The paper is organized as follows: after the very general background material of § 2, we discuss the special properties of groups of divergence type in § 3. The main lemma is proved in § 4, after which Myrberg's density theorem is derived in § 5, along with Mostow's density theorem as a nearly topological corollary. Finally, the Rigidity theorem is discussed in § 6.

2. Background

2.1. *Möbius groups.* For our purposes we shall consider the Möbius group $\mathbf{GM}_n$ ($n \geqq 1$) to be generated by the groups

$$\begin{aligned} \mathbf{T}_n &= \{t_a : a \in \mathbf{R}^n\}, & t_a(x) &= x+a, \\ \mathbf{H} &= \{h_\lambda : \lambda > 0\}, & h_\lambda(x) &= \lambda x \quad . \\ \mathbf{\Sigma} &= \{\mathrm{id}, \sigma\}, & \sigma(x) &= x/|x|^2 \end{aligned}$$

The group $\mathbf{GM}_n$ contains the orthogonal group $\mathbf{O}_n$ as a compact subgroup. The reflection principle applies to the generators, and therefore to the entire group.

For a set $E \subseteq \bar{\mathbf{R}}^n = \mathbf{R}^n \cup \{\infty\}$, we will denote by $\mathbf{GM}(E)$ the subgroup of $\mathbf{GM}_n$ which fixes E, and by $\mathbf{GM}|_{\partial E}$ the group of restrictions to ∂E of $\mathbf{GM}(E)$. The two models for E will be $\mathbf{U}^n = \{x \in \mathbf{R}^n : x_n > 0\}$, and $\mathbf{B}^n = \{x \in \mathbf{R}^n : |x| < 1\}$, with respective boundaries $\bar{\mathbf{R}}^{n-1} = \partial \mathbf{U}^n$, and $\mathbf{S}^{n-1} = \partial \mathbf{B}^n$.

For each $f \in \mathbf{GM}_{n-1}$, there exists a unique $\hat{f} \in \mathbf{GM}(\mathbf{U}^n)$, such that with the identification $\bar{\mathbf{R}}^{n-1} = \partial \mathbf{U}^n$, we have $\hat{f}|_{\partial \mathbf{U}^n} = f$. In this way we have an isomorphism $\mathbf{GM}|_{\partial \mathbf{U}^n} = \mathbf{GM}_{n-1} \cong \mathbf{GM}(\mathbf{U}^n)$.

The fixed map $s = \sigma t_{e_n} h_2 \sigma t_{-e_n} \in \mathbf{GM}_n$, explicitly

$$s(x) = \sigma(e_n + 2\sigma(x - e_n)),\ x \in \bar{\mathbf{R}}^n, \quad e_n = (0, 0, \ldots, 0, 1), \tag{2.1}$$

maps $\mathbf{U}^n$ on $\mathbf{B}^n$. The conjugation $f \mapsto sfs^{-1}$ is an isomorphism of $\mathbf{GM}(\mathbf{U}^n)$ onto $\mathbf{GM}(\mathbf{B}^n)$. Under this isomorphism, restriction to $\bar{\mathbf{R}}^{n-1}$ corresponds to restriction to $\mathbf{S}^{n-1}$, so that we also have isomorphisms $\mathbf{GM}(\mathbf{B}^n) \cong \mathbf{GM}_{n-1} \cong \mathbf{GM}|_{\partial \mathbf{B}^n}$. We remark that $s^{-1}|_{\mathbf{S}^{n-1}} : \mathbf{S}^{n-1} \to \bar{\mathbf{R}}^{n-1}$ is the usual stereographic projection.

We obtain the more usual Möbius groups $\mathbf{M}_n$, $\mathbf{M}(\mathbf{U}^n)$, $\mathbf{M}(\mathbf{B}^n)$, by considering only the orientation preserving maps. These may be characterized as those words in letters from $\mathbf{T}_n$, $\mathbf{H}$, $\mathbf{\Sigma}$, in which an even number of appearances of σ occur.

2.2. *Fixpoint analysis*. Given $p, q \in \bar{\mathbf{R}}^n$ $(p \neq q)$, we shall require a fixed map $k_{pq} \in \mathbf{GM}_n$, which carries p, q respectively on 0, ∞, and such that if $p, q \in \partial \mathbf{U}^n$, then $k_{pq} \in \mathbf{GM}(\mathbf{U}^n)$. The formulae are simply

$$k_{pq}^{-1} = t_p \sigma t_{q'} \sigma \quad (q' = \sigma(q-p)),$$
$$k_{p\infty}^{-1} = t_p, \quad k_{\infty q}^{-1} = t_q \sigma.$$

We note that for $p \neq \infty$, k is continuous in p, q, including $q = \infty$.

Given $f \in \mathbf{M}_{n-1}$, consider its extension $\hat{f} \in \mathbf{M}(\mathbf{U}^n)$. According to the Brouwer fixpoint theorem, $\hat{f}$ has a fixpoint in $\mathrm{cl}(\mathbf{U}^n)$. If there is but one fixpoint in all of $\bar{\mathbf{R}}^n$, the map f is classified as *parabolic*. By the reflection principle, the fixpoint must lie on $\partial \mathbf{U}^n$, and hence from the viewpoint of f, there is a single fixpoint in $\bar{\mathbf{R}}^{n-1}$.

If $\hat{f}$ has at least a pair of fixpoints $p, q \in \bar{\mathbf{R}}^n$ $(p \neq \infty)$, then we find that $k_{pq} \hat{f} k_{pq}^{-1}$ fixes 0, ∞, and has the form uh_λ, for some $u \in \mathbf{O}_n$, $\lambda > 0$. The classification is *loxodromic* if $\lambda \neq 1$, *elliptic* if $\lambda = 1$.

In the loxodromic case, there are no other fixpoints, and as in the parabolic case, they lie in $\partial \mathbf{U}^n$. For if they did not, then by the reflection principle, they would be symmetric about $\partial\mathbf{U}^n$. Again by the reflection principle, the map k_{pq} would map $\partial\mathbf{U}^n$ on some sphere S in which $0=k_{pq}(p)$ and $\infty=k_{pq}(q)$ are symmetric. Thus S has center 0, and therefore $uh_\lambda=k_{pq}fk_{pq}^{-1}$ fixes 0, ∞, and S. Hence $\lambda=1$, and f is elliptic. In hindsight then, f has two fixpoints p, q in $\bar{\mathbf{R}}^{n-1}$, and $k_{pq}fk_{pq}^{-1}=uh_\lambda$, $u\in\mathbf{O}_{n-1}, \lambda\neq 1$.

Of the two fixpoints, p is considered *attractive* if $\lambda<1$ and *repulsive* if $\lambda>1$. We shall denote by $\mathscr{L}_{pq}$ the collection of loxodromic Möbius transformations with attractive fixpoint p and repulsive fixpoint q. One arrives at the following summary : if f is a member of a discrete group $\Gamma\subseteq\mathbf{M}(\mathbf{U}^n)$, then one of three conditions holds. Either

(1) f is parabolic with fixpoint on $\partial\mathbf{U}^n$, or
(2) f is loxodromic with fixpoints p, q on $\partial\mathbf{U}^n$, or
(3) f is elliptic of finite order.

The finite order in the elliptic case follows because $k_{pq}^{-1}\mathbf{O}_n k_{pq}$ is compact.

Taking a viewpoint from all of $\bar{\mathbf{R}}^n$, we may consider for any $a\in\bar{\mathbf{R}}^n$, the set $\Gamma(a)^*$ of points $x\in\bar{\mathbf{R}}^n$ for which there exist infinitely many distinct $\gamma_k\in\Gamma$, with $\gamma_k(a)\to x$. The *topological limit set* Λ_T is defined by $\bigcup\Gamma(a)^*: a\in\bar{\mathbf{R}}^n$. Evidently $\Lambda_T\subseteq\partial\mathbf{U}^n$ whenever Γ is a discrete subgroup of $\mathbf{M}(\mathbf{U}^n)$, but we have even more information in this context [2]:

(2.2) $\Gamma(a)^*=\Lambda_T$ for all $a\in\mathbf{U}^n$,

(2.3) $\Gamma(a)^*=\Lambda_T$ for any $a\in\partial\mathbf{U}^n$ which is not a fixed point for the entire group.

Furthermore, following methods of Lehner [7], we easily find:

(2.4) a discrete group with a common fixpoint has, if any, only that point and possibly one other in Λ_T.

Following methods of Hedlund [13] (pp. 121–123) we deduce: that if $\mathscr{L}(\Gamma)\subseteq\partial\mathbf{U}^n\times\partial\mathbf{U}^n$ is the set of loxodromic fixpoint pairs,

$$\mathscr{L}(\Gamma)=\{(p,q):\Gamma\cap\mathscr{L}_{pq}\neq\varnothing\},$$

then

(2.5) $\mathscr{L}(\Gamma)$ is dense in $\Lambda_T\times\Lambda_T$ whenever $\Gamma\subseteq\mathbf{M}(\mathbf{U}^n)$ is discrete with card $\Lambda_T>2$.

We conclude this section with two computational lemmas.

LEMMA 2.1. *If $g_m\in\mathbf{M}_n$, with $g_m(0)\to 0$, $g_m(\infty)\to\infty$, then there exists a subsequence $\{m_k\}$, numbers $\mu_k>0$, and a mapping $u_0\in\mathbf{O}_n$, with*

$$g_{m_k}h_{\mu_k}u_0\to\mathrm{id}\quad(k\to\infty).$$

Proof. Set $p_m=g_m(0)$, $q_m=g_m(\infty)$, $k_m=k_{p_mq_m}$, noting by the continuity of k at $(0,\infty)$ that $k_m\to\mathrm{id}$.

Meanwhile, $k_m g_m$ has fixpoints 0, ∞ and therefore has the form

$$k_m g_m = h_{\lambda_m} v_m \quad (v_m \in \mathbf{O}_n, \lambda_m > 0).$$

By compactness we may assume $v_m \to v_0 \in \mathbf{O}_n$, and because $\mathbf{H}$ and $\mathbf{O}_n$ commute,

$$g_m h_{1/\lambda_m} v_0^{-1} = k_m^{-1} v_m v_0^{-1} \to \text{id}. \qquad \text{Q.E.D.}$$

LEMMA 2.2. *Suppose a linear map g: $\mathbf{R}^n \to \mathbf{R}^n$ has the property that for some $A \in \mathbf{M}_n$, the composition $g \cdot A \cdot g^{-1}$ belongs to $\mathbf{M}^n$. Then either A fixes ∞, or $g \in \mathbf{M}_n$.*

Proof. We introduce the dilatation matrix for any differentiable map f,

$$\mu_f(x) = \frac{f'(x)^T f'(x)}{\det^{2/n} f'(x)} \quad (x \in \mathbf{R}^n).$$

The familiar and elementary properties are

(1) $f \in \mathbf{M}_n$ if and only if $\mu_f(x) \equiv I_n$,

(2) $\dfrac{f'(x)^T \mu_{g \cdot f^{-1}}(f(x)) f'(x)}{\det^{2/n} f'(x)} = \mu_g(x)$.

From these, it follows that $g \cdot f^{-1}$ is conformal if and only if $\mu_f \equiv \mu_g$. Indeed, clearly $\mu_f \equiv \mu_g$ if $\mu_{g \cdot f^{-1}} \equiv I_n$. Conversely, if $\mu_f \equiv \mu_g$, then

$$\frac{f'(x)^T [\mu_{g \cdot f^{-1}}(f(x)) - I_n] f'(x)}{\det^{2/n} f'(x)} = \mu_g(x) - \mu_f(x) = 0,$$

hence $\mu_{g \cdot f^{-1}} \equiv I_n$.

We apply formula (2) to $f = A \in \mathbf{M}_n$, to conclude

$$A'(x)^{-1} \mu_{g \cdot A^{-1}}(A(x)) A'(x) = \mu_g(x). \tag{2.9}$$

We then apply the criterion for conformality of $g \cdot A \cdot g^{-1} = g \cdot (g \cdot A^{-1})^{-1}$ to conclude $\mu_{g \cdot A^{-1}} \equiv \mu_g$. But g is linear, and μ_g is a constant symmetric matrix M of determinant 1. Therefore, from (2.9),

$$A'(x)^{-1} M A'(x) = M,$$

which is to say that M commutes with $A'(x)$ for all x.

Now suppose that $A(\infty) \neq \infty$. Then $A^{-1}(\infty) = q \neq \infty$, hence $A t_q \sigma$ fixes ∞, and A stands in a relationship $A t_q = t_p B \sigma$, where B is linear and conformal, and $p \in \mathbf{R}^n$. Hence

$$A'(x+q) = B\sigma'(x) \quad (x \in \mathbf{R}^n \setminus \{0\}).$$

Now, it follows that

$$MB\sigma'(x) = B\sigma'(x)M \quad (x \neq 0).$$

But M, as a real symmetric matrix, has a real eigenvalue λ, and a maximal invariant eigenspace V associated to λ. Now $(B\sigma'(x))^{-1}V$ is the corresponding eigenspace for $(B\sigma'(x))^{-1}MB\sigma'(x)$, but the latter is again M, and so V is invariant under $B\sigma'(x)$ for every $x\neq 0$. We have in addition the simple formula

$$\sigma'(x)h = h/|x|^2 - 2x\cdot h\,x/|x|^4 \quad (h \in \mathbf{R}^n, x \neq 0).$$

Let n be any normal vector to V. Because $B\sigma'(x)V \subseteq V$, we have

$$O = n\cdot B\sigma'(x)v = n\cdot Bv/|x|^2 - 2n\cdot Bx\,x\cdot v/|x|^4 \quad (v \in V, x \neq 0). \tag{2.10}$$

A choice of $x \in B^{-1}V$ gives $n\cdot Bx=0$, and therefore

$$O = n\cdot Bv \quad (v \in \mathrm{V}), \tag{2.11}$$

and because B is conformal,

$$O = B^{-1}n\cdot v \quad (v \in V). \tag{2.12}$$

Returning to (2.10), we have by continuity and (2.11),

$$O = n\cdot Bx\,x\cdot v \quad (v \in V, x \in \mathbf{R}^n).$$

Choosing $x=B^{-1}n+v$, and using (2.11) and (2.12), we find

$$O = (|n|^2 + n\cdot Bv)(|v|^2 + B^{-1}n\cdot v) = |n|^2|v|^2 \quad (v \in V).$$

It follows that $n=\mathbf{0}$, that V is all of $\mathbf{R}^n$, and that M has only one eigenvalue. Because M is positive definite of determinant one, the eigenvalue must be 1, and $M=I_n$. Thus, either $A(\infty)=\infty$, or $M=I_n$. The latter implies that $g \in \mathbf{M}_n$. Q.E.D.

3. Geometry in $\mathbf{B}^n$

3.1. *Projections, shadows, and eclipses.* The results of § 2 transfer naturally from $\mathbf{M}(\mathbf{U}^n)$ to $\mathbf{M}(\mathbf{B}^n)$, with $\mathbf{S}^{n-1}=\partial\mathbf{B}^n$ taking the role of $\partial\mathbf{U}^n$. Let us now set some notation for use in $\mathbf{B}^n$.

Given $a \in \mathbf{B}^n$, $\varrho>0$, we shall denote by $B_\varrho(a)$ the ball of non-Euclidean center a, and non-Euclidean radius ϱ. We understand by B_ϱ the ball $B_\varrho(0)$.

These balls are of course Euclidean balls as well. We require precise information about the Euclidean center and radius, which we denote respectively by $c(\varrho, a)$ and $r(\varrho, a)$. From the basic formula for non-Euclidean distance,

$$d(0,a) = \log\frac{1+|a|}{1-|a|},$$

we derive the formulae

$$c(\varrho, a) = a\frac{1-\tanh^2 \varrho/2}{1-|a|^2\tanh^2\varrho/2}, \tag{3.1}$$

$$r(\varrho, a) = \frac{(1-|a|^2)\tanh\varrho/2}{1-|a|^2\tanh^2\varrho/2}. \tag{3.2}$$

In particular, there exist positive functions R_0, R_1, such that

$$R_0(\varrho)(1-|a|) \leq r(\varrho, a) \leq R_1(\varrho)(1-|a|). \tag{3.3}$$

Given a point $a \in \mathbf{B}^n$, $a \neq 0$, we shall define the *projection* $\mathrm{pr}\{a\} \subseteq \mathbf{S}^{n-1}$ by $a/|a|$, and the *shadow* $\mathrm{sh}\{a\} \subseteq \mathbf{B}^n \cup \mathbf{S}^{n-1}$ by the closed line segment $[a, \mathrm{pr}\{a\}]$. For brevity, denote the Hausdorff $(n-1)$-measure on $\mathbf{S}^{n-1}$, of the projection of a set $E \subseteq \mathbf{B}^n$, by $\alpha[E]$. Thus,

$$\alpha[E] = \mathcal{H}_{n-1}[\mathrm{pr}\{E\}] \quad (E \subseteq \mathbf{B}^n).$$

In this case, if $0 \in E$, we shall set $\mathrm{pr}\{E\} = \mathbf{S}^{n-1}$, and $\alpha[E] = \omega_{n-1}$. In some earlier writing, α was known as the *centri-angle* [10].

One calculates easily that

$$\frac{\alpha[B_\varrho(a)]}{(1-|a|)^{n-1}} \to \sinh\varrho \quad (|a| \to 1). \tag{3.4}$$

A first consequence is that there exists a function $K(\varrho, t)$ such that

$$\frac{\alpha[B_\varrho(a)]}{\alpha[B_\varrho(b)]} \leq K(\varrho, t) \quad \text{whenever } d(a,b) \leq t. \tag{3.5}$$

For indeed, the ratios are bounded by simple continuity and compactness considerations, as long as a is confined to any compact set $\{|x| \leq 1-\delta\}$. On the other hand, by (3.4), as $|a| \to 1$, the ratios are asymptotically equal to

$$\left(\frac{1-|b|}{1-|a|}\right)^{n-1},$$

whereas

$$\begin{aligned}\left|\log\left(\frac{1-|b|}{1-|a|}\right)\right| &\leqslant \left|\log\left(\frac{1+|a|}{1-|a|}\Big/\frac{1+|b|}{1-|b|}\right)\right| + \log 2\\ &= |d(0,a)-d(0,b)|+\log 2\\ &\leqslant d(a,b)+\log 2 \leqslant t+\log 2.\end{aligned}$$

Next, we introduce the notion of *eclipse*. A set $E\subseteq \mathbf{B}^n$ is eclipsed by a set $F\subseteq \mathbf{B}^n$, if $E\cap \mathrm{sh}\{F\}\neq\emptyset$. The eclipse is *total* if $E\subseteq \mathrm{sh}\{F\}$, *partial* if $E\setminus \mathrm{sh}\{F\}\neq\emptyset$.

3.2. *Groups of Divergence type.* A discrete group $\Gamma\subseteq \mathbf{M}(\mathbf{B}^n)$ is said to be of *convergence* type if the sum

$$\sum_{S\in\Gamma}(1-|S(0)|)^{n-1} \tag{3.6}$$

is finite. In view of (3.4), the condition is equivalent to the finiteness of the sum

$$\sum_{S\in\Gamma}\alpha[S(B_\varrho)] \quad (\varrho>0).$$

Alternatively, Γ is said to be of *divergence* type if the series (3.6) is divergent. This characterization of divergence type in terms of the action of Γ on $\mathbf{B}^n$ is analogous to the condition of *finite covolume*, which is to say that Γ has a fundamental region in $\mathbf{B}^n$ of finite non-Euclidean n-measure.

Both the classifications have intrinsic characterizations. Thus, let $\Gamma\subseteq \mathbf{M}_{n-1}$ be discrete, and using conjugation by stereographic projection and extension from $\mathbf{S}^{n-1}$ to $\mathbf{B}^n$, let Γ correspond to $\tilde{\Gamma}\subseteq \mathbf{M}(\mathbf{B}^n)$.[1] Then $\tilde{\Gamma}$ has finite covolume if and only if the quotient space $\Gamma\setminus\mathbf{M}_{n-1}$ has finite invariant measure, whereas $\tilde{\Gamma}$ is of divergence type if and only if the series

$$\sum_{A\in\Gamma}\exp\{-(n-1)\,\tau(A,\mathrm{id})\}$$

is divergent. Here, τ is the left-invariant distance function in $\mathbf{M}_{n-1}$. We make no use of these relations, but the interested reader may consult [1] for details.

[1] Explicitly, $\tilde{\Gamma}=s\tilde{\Gamma}s^{-1}$, with s as in (2.1).

However, it is important to remark that Γ is of divergence type if Γ has finite covolume. A proof for the case of mappings of $\mathbf{B}^2$ may be found in Tsuji [12]. The proof is easily adapted to any dimension. Kleinian groups, on the other hand, are discrete groups of the second kind. $\mathbf{S}^{n-1}\setminus\Lambda_T$ is open and non-empty, and the present theory says nothing about them.

Ahlfors gives simple proofs [2] of

PROPOSITION A. *If* Γ *is of divergence type, then*
(1) Γ *is of the first kind* ($\Lambda_T=\mathbf{S}^{n-1}$), *and*
(2) Γ *acts ergodically on* $\mathbf{S}^{n-1}$.

The latter condition means that if $\mathscr{E}\subseteq\mathbf{S}^{n-1}$ is invariant in the sense that $S(\mathscr{E})=\mathscr{E}$ for every $S\in\Gamma$, then $\mathscr{H}_{n-1}[\mathscr{E}]$ is either ω_{n-1} or 0. (One assumes that $\mathscr{E}$ is measurable.)

3.3. *Some estimates*. Ahlfors has further calculated explicitly [2] that if $S\in\mathbf{M}(\mathbf{B}^n)$, $S(0)=a\in\mathbf{B}^n$, then

$$\|S^{-1\prime}(x)\|=\frac{1-|a|^2}{|x-a|^2}\quad(|x|=1). \tag{3.7}$$

The norm on the left is the sup-norm. Whenever A is conformal, $A'(x)$ has the form $h_\lambda u$, with $u\in\mathbf{O}_n$ and $\lambda=\lambda(x)>0$. Then $\|A'(x)\|=\lambda(x)$.

Of crucial importance to us is the observation that there exists a positive function $M(\varrho)$, such that if $\xi_1,\xi_2\in\mathbf{S}^{n-1}$ are confined to $S^{-1}(\mathrm{pr}\,\{S(B_\varrho)\})$, then

$$\frac{\|S'(\xi_1)\|^{n-1}}{\|S'(\xi_2)\|^{n-1}}\leq M(\varrho). \tag{3.8}$$

The estimate is straightforward. Set $x_i=S(\xi_i)\in\mathrm{pr}\,\{S(B_\varrho)\}$. Then in view of (3.7), with $a=S(0)$,

$$\frac{\|S'(\xi_1)\|}{\|S'(\xi_2)\|}=\frac{\|S^{-1\prime}(x_2)\|}{\|S^{-1\prime}(x_1)\|}=\frac{|x_1-a|^2}{|x_2-a|^2},$$

whereas $|x-a|$ is maximized in $\mathrm{pr}\,\{B_\varrho(a)\}$ by any x on the relative boundary $\partial\,\mathrm{pr}\,\{B_\varrho(a)\}$ of $\mathrm{pr}\,\{B_\varrho(a)\}$ in $\mathbf{S}^{n-1}$, and minimized by $x=\mathrm{pr}\,\{a\}$.

One calculates by elementary geometry that whenever $x\in\partial\,\mathrm{pr}\,\{B_\varrho(a)\}$, then

$$|x-a|^2=1+|a|^2-2|a|\cos\left\{\sin^{-1}\left(\frac{r(\varrho,a)}{|c(\varrho,a)|}\right)\right\}.$$

In view of the formulae (3.1), (3.2), it easily follows that for $x \in \partial\,\mathrm{pr}\,\{B_\varrho(a)\}$,

$$\frac{|x-a|^2}{|\mathrm{pr}\,\{a\}-a|^2} \to \cosh^2 \varrho \quad (|a| \to 1).$$

Hence any number $M(\varrho)$ larger than $\cosh^{n-1}\varrho$ will suffice in (3.8) for all but a finite number of $S \in \Gamma$, and a suitably larger number will do for all $S \in \Gamma$.

We further require some simple relations among shadows and projections, all of which follow from the fact that two non-Euclidean lines become farther apart as one moves away from their point of intersection, even if their intersection lies on $\mathbf{S}^{n-1}$.

LEMMA 3.1. *If two geodesic rays with common terminal* $\xi \in \mathbf{S}^{n-1}$ *both meet a ball* $B_\varrho(a)$, *and if one meets* $B_\sigma(b)$ *between* ξ *and* $B_\varrho(a)$, *then the other meets* $B_{\sigma+2\varrho}(b)$. *The estimate is sharpened to* $B_{\sigma+\varrho}(b)$ *if one meets a itself.*

LEMMA 3.2. $B_\varrho(a) \subseteq \mathrm{sh}\,\{B_{2\varrho}(b)\}$ *if any radius r meets, in order:* 0, b, $B_\varrho(a)$.

LEMMA 3.3. $B_{\varrho+\tau}(a) \subseteq \mathrm{sh}\,\{B_{\sigma+\tau}(b)\}$ *whenever* $B_\varrho(a) \subseteq \mathrm{sh}\,B_\sigma(b)$.

In case $B=B_\varrho(a)$ is any ball in $\mathbf{B}^n$, we consider the collection $\beta \subseteq \mathbf{B}^n \cup \mathbf{S}^{n-1}$ of all radial segments $[0, \mathrm{pr}\,\{x\}]$, $x \in B$. This set will be known as *the solid angle supporting* B. Any image $S(\beta)$ ($S \in \mathbf{M}(\mathbf{B}^n)$) will be a *solid angle with vertex* $S(0)$. It is geometrically evident that there is a positive function $\alpha_0(\varrho)$, such that

$$\mathscr{H}_{n-1}[\beta \cap \mathbf{S}^{n-1}] \geq \alpha_0(\varrho) \tag{3.9}$$

whenever β is a solid angle containing B_ϱ.

We shall in the following section, have need for the concept of a *half-ball*. By this we understand a set $\varkappa \subseteq \mathbf{B}^n \cup \mathbf{S}^{n-1}$, bounded by a Euclidean sphere orthogonal to $\mathbf{S}^{n-1}$. It is helpful to agree that $\varkappa$ includes its closure in $\mathbf{B}^n$. Thus $\varkappa \cap \mathbf{S}^{n-1}$ is a spherical cap.

The non-Euclidean line in $\mathbf{B}^n$ with endpoints $p, q \in \mathbf{S}^{n-1}$ will be denoted by ℓ_{pq}. We shall also use this notation in $\mathbf{U}^n$, provided $p, q \in \partial \mathbf{U}^n$.

4. The main lemma

4.1. *Preliminaries*. In this section, $n \geq 2$ will be fixed, and we shall shorten the notations $\mathbf{B}^n$, $\mathbf{S}^{n-1}$, $\mathscr{H}_{n-1}$, ω_{n-1} to simply $\mathbf{B}$, $\mathbf{S}$, $\mathscr{H}$, ω. Suppose now that $\Gamma \subseteq \mathbf{M}(\mathbf{B})$ is a discrete group. Fix $T \in \Gamma$, and take any $W \in \Gamma$, $\varrho > 0$. We consider the set $E_\varrho(W) \subseteq \mathbf{S}$, defined by

$$E_\varrho(W) = \mathrm{pr}\,\{WT(B_{2\varrho}) \cap \mathrm{sh}\,\{W(B_{7\varrho})\}\}.$$

For any enumeration $W_1, W_2, \ldots$ of Γ, let

$$\mathscr{E}_\varrho(T) = \limsup E_\varrho(W_k) \quad (k \to \infty).$$

The purpose of this section is to prove the following

MAIN LEMMA. *If $\Gamma \subseteq \mathbf{M}(\mathbf{B})$ is of the first kind, and if there exists $\varrho>0$ and $T \in \Gamma$ with $\mathscr{H}[\mathscr{E}_\varrho(T)]=0$, then Γ is of convergence type.*

To motivate this result, let us define a second set, with T fixed as before:

$\tilde{\mathscr{E}}_\varrho(T) = \{q \in \mathbf{S}$: for every $p \in \mathbf{S}$, $p \neq q$, there exists a sequence $\{W_k\} \subseteq \Gamma$, with ℓ_{pq} meeting $W_k(B_\varrho)$, $W_kT(B_\varrho)$ in order from p to q, and $W_k(B_\varrho) \to q$ as $k \to \infty\}$.

Finally, set $\tilde{\mathscr{E}}_\varrho(\Gamma) = \cap \tilde{\mathscr{E}}_\varrho(T)$: $T \in \Gamma$.

THEOREM 1. *If Γ is of divergence type, then $\mathscr{H}[\tilde{\mathscr{E}}_\varrho(\Gamma)]=\omega$ for every $\varrho>0$.*

Proof of Theorem 1. The set $\tilde{\mathscr{E}}_\varrho(T)$ is invariant, and since Γ is of divergence type, $\mathscr{H}[\tilde{\mathscr{E}}_\varrho(T)]$ is ω or 0, and Γ is of the first kind, by Proposition A, § 3.2.

But we observe that $q \in \mathscr{E}_\varrho(T)$ if and only if q lies in infinitely many of the sets $E_\varrho(W_k)$. This occurs if and only if there exist infinitely many $W \in \Gamma$, with the radius $0q$ meeting in order 0, $W(B_{2\varrho})$, $WT(B_{7\varrho})$. This implies that any line ℓ_{pq}, $p \neq q$, meets, near q, infinitely many balls $W(B_{8\varrho})$, $WT(B_{8\varrho})$, and that $q \in \tilde{\mathscr{E}}_{8\varrho}(T)$. In other words,

$$\mathscr{E}_\varrho(T) \subseteq \tilde{\mathscr{E}}_{8\varrho}(T) \quad (T \in \Gamma, \varrho > 0).$$

Therefore, if $\mathscr{H}[\tilde{\mathscr{E}}_\varrho(T)]<\omega$ for some ϱ, T, it follows that $\mathscr{H}[\tilde{\mathscr{E}}_\varrho(T)]=0$, and therefore $\mathscr{H}[\mathscr{E}_{\varrho/8}(T)]=0$. By the main lemma, it follows that Γ is of convergence type. From this contradiction, we conclude that for every $\varrho>0$, $T \in \Gamma$, we have $\mathscr{H}[\tilde{\mathscr{E}}_\varrho(T)]=\omega$. But since Γ is countable, with each set $\tilde{\mathscr{E}}_\varrho(T)$ having a complement in S of measure zero, it follows that

$$\mathscr{H}[\tilde{\mathscr{E}}_\varrho(\Gamma)] = \mathscr{H}\left[\bigcap_T \tilde{\mathscr{E}}_\varrho(T)\right] = \omega. \qquad \text{Q.E.D.}$$

The proof of the main lemma will be achieved with the help of three preliminary lemmas, throughout which T will be a fixed element of Γ.

LEMMA 4.1. *Suppose that* Γ *is of the first kind. Then for every half ball* $\varkappa$, *there exists* $A\in\Gamma$, *with* $A(B_\varrho)$, $AT(B_\varrho)\subseteq\varkappa$, *and*

$$AT(B_\varrho)\subseteq \operatorname{sh}\{A(B_{2\varrho})\}.$$

Proof. Choose another half ball $\varkappa'$, opposite 0 from $T(B_\varrho)$, with the property that every ray from any point inside $\varkappa'$, through 0, meets $T(B_\varrho)$. By reduction of $\varkappa$, $\varkappa'$ if necessary, we may assume $\varkappa\cap\varkappa'=\emptyset$, $0\notin\varkappa\cup\varkappa'$.

By (2.5), choose $A\in\mathscr{L}_{pq}\cap\Gamma$, $p\in\varkappa\cap\mathbf{S}$, $q\in\varkappa'\cap\mathbf{S}$. By taking sufficiently high powers of A, we may assume that A maps $\mathbf{B}\setminus\varkappa'$ inside $\varkappa$. From $A^{-1}(0)\in\varkappa'$, draw the ray r through 0, which necessarily meets $T(B_\varrho)$. Then $A(r)$ is a radius, meeting in order 0, $A(0)\in A(B_\varrho)$, and $AT(B_\varrho)$, the latter two sets lying in $\varkappa$. By Lemma 3.2, $AT(B_\varrho)\subseteq \operatorname{sh}\{A(B_{2\varrho})\}$. Q.E.D.

LEMMA 4.2. *Suppose* Γ *is of the first kind. Given* $\varrho>0$, *there exists a finite set* $\mathscr{A}\subseteq\Gamma$, *such that for any solid angle* β *containing* B_ϱ, *there exists* $A\in\mathscr{A}$, *with* $A(B_\varrho)$, $AT(B_\varrho)\subseteq\beta$, *and* $AT(B_\varrho)\subseteq\operatorname{sh}\{A(B_{2\varrho})\}$.

Proof. For any half ball $\varkappa$, with $\varkappa\cap B_\varrho=\emptyset$, consider the collection $\mathscr{C}$ of half balls $\varkappa'$ which do not meet $\varkappa\cup B_\varrho$. Set

$$E(\varkappa)=\bigcap \mathbf{B}\cup\mathbf{S}\setminus\varkappa' : \varkappa'\in\mathscr{C}.$$

The set $E(\varkappa)$ is a tubular set containing $\varkappa$ and B_ϱ, and if $\varkappa$ is sufficiently small, then $E(\varkappa)$ meets S in an open, nonempty set $F(\varkappa)\subseteq\mathbf{S}$, opposite $\varkappa$. In this case, that part $G(\varkappa)$ of $E(\varkappa)$ between B_ϱ and $F(\varkappa)$ has the property that if a solid angle β containing B_ϱ has vertex in $G(\varkappa)$, then $\varkappa\subseteq\beta$. Choose a finite collection of half balls $\varkappa$ with this property, whose union covers S. This has the consequence that the regions $G(\varkappa)$ cover $\mathbf{B}\setminus B_\varrho$, and therefore if β is any solid angle containing B_ϱ, its vertex lies in some $G(\varkappa)$. In each $\varkappa$, find $A\in\Gamma$ with the properties of Lemma 4.1. Q.E.D.

LEMMA 4.3. *Given* $\mathscr{A}$ *as in Lemma* 4.2, *there exists a positive constant* $M_0=M_0(\mathscr{A})$, *such that for any* $U\in\Gamma$, *there exists* $W=W(U)\in\Gamma$, *with the properties*

(i) $W(B_\varrho)$, $WT(B_\varrho)\subseteq\operatorname{sh}\{U(B_\varrho)\}$,
(ii) $WT(B_\varrho)\subseteq\operatorname{sh}\{W(B_{4\varrho})\}$,
(iii) $\alpha[U(B_\varrho)]\leq M_0\alpha[WT(B_\varrho)]$.

Proof. Given U, let β be the solid angle supporting $U(B_\varrho)$. Then $U^{-1}(\beta)$ is a solid angle containing B_ϱ, and therefore containing $A(B_\varrho)$ and its companion $AT(B_\varrho)\subseteq \text{sh}\{A(B_{2\varrho})\}$, for some $A\in\mathscr{A}$. Define $W=UA$.

As regards the properties, (i) is obvious from the construction. For (ii), take any radius $r=0q$ meeting $WT(B_\varrho)$. From $U^{-1}(q)$, draw the radius s' to 0, and the geodesic $r'=U^{-1}(r)$. Set $s=U(s')$.

Now r' meets $AT(B_\varrho)$ and B_ϱ, and therefore s' meets $AT(B_{2\varrho})$. Hence s' meets $A(B_{3\varrho})$, s meets $W(B_{3\varrho})$, and r meets $W(B_{4\varrho})$. We have used Lemma 3.1.

Finally, part (iii) is obvious from (3.5), once we note that

$$\begin{aligned} d(WT(0),U(0)) &= d(AT(0),0)\leqslant d(AT(0),A(0))+d(A(0),0)\\ &= d(T(0),0)+d(A(0),0)\\ &\leqslant t_0=\max d(A(0),0)+d(T(0),0)\colon A\in\mathscr{A}. \end{aligned}$$

4.2. *Proof of the Main lemma.* Our objective is to show that

$$\sum_{U\in\Gamma}\alpha[U(B_\varrho)]<\infty, \tag{4.1}$$

and to this end we closely follow Ahlfors–Thurston [2]. In an enumeration $U_1, U_2, \dots$ of Γ, fix $t>t_0$, and choose a new sequence U_{k_1}, $U_{k_2}, \dots$ with the property that $U_{k_0}=\text{id}$, and for $j\geqslant 1$, k_j is the first index k such that $d(U_k(0),\ U_{k_i}(0))>3t$, for $i=0,1,2,\dots,j-1$. The sets

$$\mathscr{F}_j=\{U\in\Gamma\colon d(U(0),\ U_{k_j}(0))\leqslant 3t\}$$

are of constant cardinality $N(t)$, and $\cup\mathscr{F}_j=\Gamma$. By (3.5),

$$\frac{\alpha[U(B_\varrho)]}{\alpha[U_{k_j}(B_\varrho)]}\leqslant K(\varrho,3t)$$

whenever $U\in\mathscr{F}_j$. Hence

$$\begin{aligned} \sum_{U\in\Gamma}\alpha[U(B_\varrho)] &\leqslant \sum_{j=0}^{\infty}\sum_{U\in\mathscr{F}_j}\alpha[U(B_\varrho)]\\ &\leqslant \sum_{j=0}^{\infty}N(t)\,K(\varrho,3t)\,\alpha[U_{k_j}(B_\varrho)], \end{aligned}$$

and so for (4.1), it suffices to prove

$$\sum_{j=0}^{\infty} \alpha[U_{k_j}(B_\varrho)] < \infty.$$

Let us relabel this sequence $U_1, U_2, \dots$, and set $V_j = WT$, where $W = W(U_j)$ as in Lemma 4.3. Now it suffices by Lemma 4.3 (iii), to prove that

$$\sum_{j=0}^{\infty} \alpha[V_j(B_\varrho)] < \infty.$$

We note that the V_j are well spaced. Indeed,

$$\begin{aligned} d(V_j(0), V_i(0)) &\geq d(U_j(0), U_i(0)) - d(U_j(0), V_j(0)) - d(U_i(0), V_i(0)) \\ &> 3t - 2t_0 \geq t. \end{aligned}$$

Henceforth we denote by B_j the ball $V_j(B_\varrho)$, and we shall assume that $t \geq 3\varrho$, so that these balls are disjoint. Following [2], we set up classes of balls:

$$\begin{aligned} I_0 &= \{B_\varrho\} \\ I_1 &= \{B_k : B_k \text{ is not eclipsed by any } B_i\} \\ &\vdots \\ I_m &= \{B_k : B_k \text{ is not eclipsed by any } B_i \notin \textstyle\bigcup_{j<m} B_j\}. \end{aligned}$$

Note that in any class I_m, the shadows $\operatorname{sh}\{B_k\}$ are disjoint. Further, every $B_j \in I_{m+1}$ is eclipsed by some $B_k \in I_m$, for if it were not, it would have been selected in an earlier class. Our immediate objective is to show that

$$\sum_{B_j \in I_{m+1}} \alpha[B_j] \leq \frac{2}{3} \sum_{B_k \in I_m} \alpha[B_k]. \tag{4.2}$$

In this way,

$$\sum_{j=0}^{\infty} \alpha[B_j] = \sum_{m=0}^{\infty} \sum_{B_j \in I_m} \alpha[B_j] \leq \sum_{m=0}^{\infty} \left(\frac{2}{3}\right)^m \omega < \infty.$$

We shall further subdivide each class I_{m+1} into two subclasses:

$$\begin{aligned} I'_{m+1} &= \{B_j \in I_{m+1} : B_j \text{ is partially eclipsed by some } B_k \in I_m\}, \\ I''_{m+1} &= \{B_j \in I_{m+1} : B_j \text{ is totally eclipsed by some } B_k \in I_m\}. \end{aligned}$$

As regards the class I'_{m+1}, fix $B_j = B_\varrho(a_j)$, partially eclipsed by $B_k = B_\varrho(a_k) \in I_m$. This means that there exists $q \in \mathbf{S} \cap \partial \operatorname{pr}\{B_k\}$, such that the radius $0q$ meets ∂B_k at a point b_k, and B_j at a point b_j, in the order 0, b_k, b_j. Then denoting $r_j = r(a_j, \varrho)$, $r_k = r(a_k, \varrho)$, we find

$$\begin{aligned} t \leq d(a_k, a_j) &\leq d(b_k, b_j) + 2\varrho \\ &= d(b_j, 0) - d(b_k, 0) + 2\varrho \\ &\leq d(a_j, 0) - d(a_k, 0) + 4\varrho, \end{aligned}$$

hence

$$d(0, a_j) \geq d(0, a_k) - 4\varrho + t.$$

or

$$\log \frac{1-|a_k|}{1-|a_j|} \geq -4\varrho + t - \log 2,$$

and in view of (3.3),

$$\frac{r_k}{r_j} \geq \frac{1}{2} \frac{R_0(\varrho)}{R_1(\varrho)} e^{t-4\varrho},$$

or

$$r_j \leq C(\varrho)\, e^{-t} r_k \quad \left(C(\varrho) = 2 \frac{R_1(\varrho)}{R_0(\varrho)} e^{4\varrho} \right).$$

This means that the projections $\operatorname{pr}\{B_j\}$, B_j partially eclipsed by $B_k \in I_m$, all lie inside an "annular" region on $\mathbf{S}$, of Euclidean inner radius asymptotic to $r_I = r_k(1 - 2C(\varrho)\, e^{-t})$, and Euclidean outer radius asymptotic to $r_O = r_k(1 + 2C(\varrho)\, e^{-t})$. This "annulus" has total $\mathscr{H}$-area asymptotically proportional to

$$\begin{aligned} r_O^{n-1} - r_I^{n-1} &= r_k^{n-1} [(1 + 2C(\varrho)\, e^{-t})^{n-1} - (1 - 2C(\varrho)\, e^{-t})^{n-1}] \\ &= r_k^{n-1} O(e^{-t}) = \alpha[B_k]\, O(e^{-t}). \end{aligned}$$

We sum this over $B_k \in I_m$, and conclude that for all sufficiently large t, we have the estimate

$$\sum_{B_j \in I'_{m+1}} \alpha[B_j] \leq \frac{1}{3} \sum_{B_k \in I_m} \alpha[B_k] \tag{4.3}$$

As regards the class I''_{m+1}, fix $B_k \in I_m$, and an index set $\tilde{I}_k$ such that for $j \in \tilde{I}_k$, the balls $B_j \in I''_{m+1}$ are all totally eclipsed by B_k. Let $E_k = \text{pr}\{\bigcup B_j : j \in \tilde{I}_k\} = \bigcup \text{pr}\{B_j\} : j \in \tilde{I}_k$, and $F_k = V_k^{-1}(\text{pr}\{B_k\}) \supseteq V_k^{-1}(E_k)$.

Let β be the solid angle supporting B_k, and consider $V_k^{-1}(\beta)$, a solid angle containing B_ϱ. We have the estimate (see (3.8), (3.9))

$$\begin{aligned}
\alpha_0(\varrho)\,\mathscr{H}[E_k] &\leq \mathscr{H}[V_k^{-1}(\beta)\cap S]\int_{V_k^{-1}(E_k)} \|V_k'(\xi)\|^{n-1}\,d\mathscr{H}(\xi) \\
&\leq \mathscr{H}[V_k^{-1}(\text{pr}\{B_k\})]\max_{\xi\in F_k}\|V_k'(\xi)\|^{n-1}\mathscr{H}[V_k^{-1}(E_k)] \\
&\leq \int_{F_k} d\mathscr{H}(\xi)\,M(\varrho)\min_{\xi\in F_k}\|V_k'(\xi)\|^{n-1}\mathscr{H}[V_k^{-1}(E_k)] \\
&\leq \int_{F_k} \|V_k'(\xi)\|^{n-1}d\mathscr{H}(\xi)\,M(\varrho)\,\mathscr{H}[V_k^{-1}(E_k)],
\end{aligned}$$

or finally,

$$\mathscr{H}[E_k] \leq \frac{M(\varrho)}{\alpha_0(\varrho)}\,\alpha[B_k]\,\mathscr{H}[V_k^{-1}(E_k)]. \tag{4.4}$$

Now consider for $j \in \tilde{I}_k$, $V = V_j = WT$, where $W = W(U_j)$. From Lemma 4.3, we know that $V(B_\varrho) \subseteq \text{sh}\, W(B_{4\varrho})$. We now claim that

$$V_k^{-1}(V(B_\varrho)) \subseteq \text{sh}\{V_k^{-1}(W(B_{6\varrho}))\}, \tag{4.5}$$

for if $r' = 0q$ is a radius meeting $V_k^{-1}(V(B_\varrho))$, then $r = V_k(r')$ is the geodesic ray $[V_k(0), V_k(q)]$. Take s as the radius $[0, V_k(q)]$, and take $s' = V_k^{-1}(s)$. Then s meets $V(B_{2\varrho})$, and therefore also meets $W(B_{5\varrho})$. Hence s' meets $V_k^{-1}(W(B_{5\varrho})$, and r' meets $V_k^{-1}W(B_{6\varrho})$.

We also claim that

$$V_k^{-1}(\text{pr}\{B_j\}) \subseteq \text{sh}\{V_k^{-1}V(B_{2\varrho})\}. \tag{4.6}$$

To see this, suppose $q \in \text{pr}\{V(B_\varrho)\}$, and consider $V_k^{-1}(q)$. Let r be the radius $0q$, and s' be the radius $[0, V_k^{-1}(q)]$. Then $s = V_k(s')$ is the geodesic ray $[V_k(0), q]$, and we find as usual that s meets $V(B_{2\varrho})$, and therefore s' meets $V_k^{-1}V(B_{2\varrho})$.

From (4.5) and Lemma 3.3, it follows that

$$V_k^{-1}WT(B_{2\varrho}) = V_k^{-1}V(B_{2\varrho}) \subseteq \text{sh}\{V_k^{-1}W(B_{7\varrho})\},$$

and therefore with (4.6), that

$$E_\varrho[V_k^{-1}W] = \mathrm{pr}\,\{V_k^{-1}V(B_{2\varrho})\} \supseteq V_k^{-1}(\mathrm{pr}\,\{B_j\}). \tag{4.7}$$

It is to be observed that

$$\begin{aligned} d(V_k^{-1}W(0), 0) &= d(W(0), V_k(0)) = d(V_jT^{-1}(0), V_k(0)) \\ &\geq d(V_j(0), V_k(0)) - d(V_j(0), V_jT^{-1}(0)) \\ &\geq t - t_0, \end{aligned}$$

and therefore, we obtain from (4.7), the estimate

$$V_k^{-1}(E_k) = \bigcup_{j\in I_k} V_k^{-1}(\mathrm{pr}\,\{B_j\}) \subseteq \bigcup E_\varrho(A) \colon d(A(0), 0) \geq t - t_0.$$

Upon taking Hausdorff measure, and using the hypothesis that

$$\mathscr{H}[\limsup E_\varrho(A) \colon A \in \Gamma] = 0,$$

we see that

$$\mathscr{H}[V_k^{-1}(E_k)] \to 0 \quad (t \to \infty),$$

and therefore we may select t sufficiently large that

$$\mathscr{H}[V_k^{-1}(E_k)] \leq \frac{\alpha_0(\varrho)}{3M(\varrho)},$$

and with (4.4), that $\mathscr{H}[E_k] \leq \frac{1}{3}\alpha[B_k]$, uniformly for $B_k \in I_m$, $m = 1, 2, 3, \ldots$ The last inequality is now summed in k, to yield

$$\sum_{B_j \in I'_{m+1}} \alpha[B_j] = \sum_{B_k \in I_m} \mathscr{H}[E_k] \leq \frac{1}{3} \sum_{B_k \in I_m} \alpha[B_k],$$

which, along with (4.3), yields (4.2), and completes the proof. Q.E.D.

5. The density theorems of P. J. Myrberg and G. D. Mostow

With $n \geq 2$, referring again to our standard map k_{pq} (see § 2.2), let us suppose that $f \in \mathbf{M}_n$ has $f(p) = s$, $f(q) = t$. Then $k_{st} f k_{pq}^{-1}$ fixes 0, ∞ and has the form

$$k_{st} f k_{pq}^{-1} = h_\lambda u \quad (u \in \mathbf{O}_n, \lambda > 0).$$

This number λ will be denoted by $\lambda_{pq}(f)$.

Fix $p, q \in \mathbf{S}^{n-1}(p \neq q)$, and $s, t \in \mathbf{S}^{n-1}(s \neq t)$, and assume that $\Gamma \subseteq \mathbf{M}(\mathbf{B}^n)$ is of divergence type. Because Γ is also of the first kind, there exist sequences $\{S_m\} \subseteq \Gamma$, $\{T_m\} \subseteq \Gamma$, with

$$S_m(0) \to s, T_m(0) \to t \quad (m \to \infty).$$

Suppose that $q \in \mathscr{E}_\varrho(\Gamma)$. Then for each m, there exists $W_m \in \Gamma$ with the line ℓ_{pq} meeting $W_m(B_\varrho)$ and $W_m S_m^{-1} T_m(B_\varrho)$, and we may require that $W_m(B_\varrho) \to q$. It follows that $S_m(W_m^{-1}(\ell_{pq})$ meets $S_m(B_\varrho)$ and $T_m(B_\varrho)$, and hence $S_m W_m^{-1}(\ell_{pq})$ approaches ℓ_{st}. But it is also geometrically evident that because $W_m(B_\varrho) \to q$, we have

$$\lambda_{pq}(S_m W_m^{-1}) \to 0 \quad (m \to \infty).$$

We have just proved the following approximation theorem, first formulated by P. J. Myrberg [10] in case $n=2$ and Γ is finitely generated.

THEOREM 2. *If* $\Gamma \subseteq \mathbf{M}(\mathbf{B}^n)$ *is of divergence type, then for* a.e. $q \in \mathbf{S}^{n-1}$, *every* $p \in \mathbf{S}^{n-1}(p \neq q)$, *and any line* $\ell \subseteq \mathbf{B}^n$, *there exists a sequence* $\{V_m\} \subseteq \Gamma$, *with* $V_m(\ell_{pq}) \to \ell$, *and with* $\lambda_{pq}(V_m) \to 0$.

Before moving to Mostow's theorems, it is better for technical reasons to transfer to $\mathbf{U}^n$, where the auxiliary maps $k_{pq}(p, q \in \partial \mathbf{U}^n)$ all belong to $\mathbf{M}(\mathbf{U}^n)$. We denote the intersection $\mathbf{M}(\mathbf{U}^n) \cap \mathscr{L}_{pq}$ by $\mathscr{L}_{pq}^{\mathbf{U}}$.

THEOREM 3. *If* $\Gamma \subseteq \mathbf{M}(\mathbf{U}^n)$ *is of divergence type, then for* a.e. $q \in \partial \mathbf{U}^n$, *the set* $\Gamma \mathscr{L}_{qp}^{\mathbf{U}}$ *is dense in* $\mathbf{M}(\mathbf{U}^n)$ *for every* $p \in \partial \mathbf{U}^n \setminus \{q\}$.

Proof. The qualitative and quantitative content of Theorem 2 transfer immediately to $\mathbf{U}^n$. Hence take $q \in \partial \mathbf{U}^n$ with the properties of Theorem 2, and any other $p \in \partial \mathbf{U}^n$. Next, take any $g \in \mathbf{M}(\mathbf{U}^n)$, and set $s=g(p)$, $t=g(q)$, $\lambda_0=\lambda_{pq}(g)$. Next, find a sequence $\{V_m\} \subseteq \Gamma$, with $V_m(\ell_{pq}) \to \ell_{st}$, and $\lambda_m=\lambda_{pq}(V_m) \to 0$. Set $s_m=V_m(p)$, $t_m=V_m(q)$, and $k_m=k_{s_m t_m}$. We find

$$k_m V_m k_{pq}^{-1} = u_m h_{\lambda_m} \quad (\text{some } u_m \in \mathbf{O}_n \cap \mathbf{M}(\mathbf{U}^n),\ \lambda_m > 0),$$

$$k_{st} g k_{pq}^{-1} = u_0 h_{\lambda_0} \quad (\text{some } u_0 \in \mathbf{O}_n \cap \mathbf{M}(\mathbf{U}^n)),$$

hence

$$\begin{aligned} V_m k_{pq}^{-1} &= k_m^{-1} u_m h_{\lambda_m} \\ &= k_m^{-1} k_{st} k_{st}^{-1} u_m h_{\lambda_m} \\ &= k_m^{-1} k_{st} g k_{pq}^{-1} u_0^{-1} h_{\lambda_0}^{-1} u_m h_{\lambda_m}, \end{aligned}$$

or,

$$V_m k_{pq}^{-1} h_{\lambda_0/\lambda_m} u_m^{-1} u_0 k_{pq} = k_m^{-1} k_{st} g.$$

In other words,

$$k_m^{-1} k_{st} g = V_m g_m,$$

where

$$g_m = k_{pq}^{-1} h_{\lambda_0/\lambda_m} u_m^{-1} u_0 k_{pq} \in \mathbf{M}(\mathbf{U}^n),$$

and thus $g_m \in \mathscr{L}_{qp}^{\mathbf{U}}$ as soon as $\lambda_m < \lambda_0$.

On the other hand, with the possible exception $s=\infty$, we have $k_m^{-1} k_{st} \to \mathrm{id}$. In other words, for a.e. $q \in \partial \mathbf{U}^n$, and every $p \in \partial \mathbf{U}^n$ $(p \neq q)$, the closure in $\mathbf{M}(\mathbf{U}^n)$ of $\Gamma \mathscr{L}_{qp}^{\mathbf{U}}$ includes all $g \in \mathbf{M}(\mathbf{U}^n)$ except those with $g(p)=\infty$. But this set has no interior in $\mathbf{M}(\mathbf{U}^n)$, and so the result is established for a.e. q, and every $p \neq q$. Q.E.D

The proof of Theorem 3 is now completed, but looking solely at the statement, and keeping in mind the basic homeomorphic isomorphism between $\mathbf{M}_n$ and $\mathbf{M}(\mathbf{U}^{n+1})$, we have the following:

THEOREM 4. *If $\Gamma \subseteq \mathbf{M}_n$ is a discrete group of divergence type, then for* a.e. $q \in \bar{\mathbf{R}}^n$, *the set $\Gamma \mathscr{L}_{qp}$ is dense in $\mathbf{M}_n$ for every $p \neq q$.*

A final remark for this section: it is implicit in the statements that Theorem 3 applies to the case $n \geq 2$, but that Theorem 4 applies as well to the case $n=1$. I shall refer to the points $q \in \bar{\mathbf{R}}^n$ satisfying the conclusion as "density" points.

6. Mostow's rigidity theorem

I believe that the proper setting for this theorem is $\bar{\mathbf{R}}^n$, rather than $\partial \mathbf{U}^n$ or $\mathbf{S}^n$, as it is often presented. The theorem says that a map g which is compatible with a discrete group Γ of divergence type, is either Möbius or very strange indeed.

I am taking the hypothesis that g is quasiconformal, by which I mean that g is quasisymmetric if $n=1$. It was a significant portion of Mostow's work to show that if $\hat{\Gamma}$ has finite covolume, and if g is the boundary mapping induced by a quasi-isometry $\hat{g}: \mathbf{U}^{n+1}/\hat{\Gamma} \to \mathbf{U}^{n+1}/\hat{\Gamma}$, then g is indeed quasiconformal. Sullivan makes the same assertion in case $\hat{\Gamma}$ has $\partial(\mathbf{U}^{n+1}) \setminus \Lambda_H$ of measure zero, which would cover the case that Γ is of divergence type. In any case, I prefer to set this issue aside.

It will be clear from the argument that the two truly relevant properties of g, in addition to the compatibility, are

(i) Γ must satisfy the conclusions of Theorem 4,

(ii) g must have the property that for every p,q with $q=g(p)$, the collection of mappings $\mathbf{H}t_{-q}gt_p\mathbf{H}$ forms a normal family.

The conclusion would then be that either g is Möbius, or that g is extremely singular. Examples of the later alternative are commonplace in case $n=1$, but I would say it is an open problem to give an example if $n \geq 2$. Such an example will not be found among groups of the special types herein considered, for as mentioned above, g *will be* quasiconformal in these cases.

THEOREM 5. *Suppose that* $\Gamma \subseteq \mathbf{M}_n$ *is a discrete group of divergence type. Suppose that* g: $\mathbf{R}^n \to \mathbf{R}^n$ *is quasiconformal with the property that*

$$g \cdot A \cdot g^{-1} \in \mathbf{M}_n \quad (all\, A \in \Gamma).$$

Then

(i) *if* $n \geq 2$, *we have* $g \in \mathbf{M}_n$, *and*

(ii) *if* $n=1$, *either* $g \in \mathbf{M}_1$, *or* g *is singular.*

Proof. (This part of the argument is similar to Mostow's, cf. [9].) Let us first assume that g has nonsingular total differential (positive, finite derivative if $n=1$) at the density point $x=0$, and that $g(0)=0$. Our normalization g: $\mathbf{R}^n \to \mathbf{R}^n$ means that $g(\infty)=\infty$. By assumption, $\Gamma\mathscr{L}_{0\infty}$ is dense in $\mathbf{M}_n$. Of course $\mathscr{L}_{0\infty} \subseteq \mathbf{HO}_n$, so select $\lambda_m \in (0,1)$, $u_m \in \mathbf{O}_n$, $T_m \in \Gamma$, such that $T_m h_{\lambda_m} u_m \to \mathrm{id}$. We may assume that $u_m \to u_0 \in \mathbf{O}_n$, and because Γ is discrete, that $\lambda_m \to 0$. By hypothesis, we have $B_m \in \mathbf{M}_n$, with

$$g \cdot T_m = B_m \cdot g.$$

For any $x \in \mathbf{R}_n$, we have

$$B_m(g(h_{\lambda_m}(u_m(x)))) = g(T_m(h_{\lambda_m}(u_m(x)))) \to g(x),$$

and testing at $x=0$, ∞, we find $B_m(0)\to g(0)=0$, and $B_m(\infty)\to\infty$. By Lemma 2.1, and passing to a subsequence, we may assume that there exist numbers $\mu_m>0$, and a mapping $v_0\in \mathbf{O}_n$, with

$$B_m h_{\mu_m} v_0 \to \mathrm{id}.$$

Next, set $f_m=h_{\mu_m}^{-1}\cdot g\cdot h_{\lambda_m}$. Then

$$B_m(h_{\mu_m}(f_m(u_m(x)))) = B_m(g(h_{\lambda_m}(u_m(x)))) \to g(x).$$

But the mappings f_m are all $K[g]$-quasiconformal, with $f_m(0)=0$, $f_m(\infty)=\infty$, and hence belong to a normal family, [5].(1) We may assume that they converge uniformly on compact sets of $\mathbf{R}^n\setminus\{0\}$, to a limit function φ, which is either a homeomorphism or constant. However, for $x\neq 0$, we have

$$\begin{aligned} g(x) &= \lim B_m(h_{\mu_m}(f_m(u_m(x)))) \\ &= \lim B_m(h_{\mu_m}(f_m(u_0(x)))) = v_0^{-1}(\varphi(u_0(x))). \end{aligned}$$

In particular, φ is not constant, and $\varphi=v_0\cdot g\cdot u_0^{-1}$. We have established that there exist numbers $\lambda_m\searrow 0$, $\mu_m>0$, such that for all $x\in\mathbf{R}^n\setminus\{0\}$,

$$\frac{1}{\mu_m} g(\lambda_m x) = f_m(x) \to v_0(g(u_0^{-1}(x))).$$

But now g has the total differential at $x=0$, and therefore for any $x\in\mathbf{R}^n\setminus\{0\}$, we have

$$0 \neq g'(0)\,x = \lim \frac{g(\lambda_m x)}{\lambda_m},$$

and therefore we must have the existence of the limit

$$0 \neq \lambda = \lim \lambda_m/\mu_m \quad (m\to\infty),$$

and the identity

$$\lambda g'(0)\,x = v_0(g(u_0^{-1}(x))) \quad (x\in\mathbf{R}^n).$$

We deduce by differentiation at $x=0$, that $\lambda=1$, and in any case, we see that g is linear.

(1) This fact in case $n=1$ can be deduced from the case $n=2$ with the help of the Beurling–Ahlfors extension formula, [4].

According to Lemma 2.2, either ∞ is a fixed point for the entire group, or g is Möbius. The former alternative is excluded by (2.4) and Proposition A(1), § 3.2.

In the general case, suppose that g has nonsingular total differential (positive, finite derivative if $n=1$) at some density point $p \in \mathbf{R}^n$, and set $q=g(p)$. Then the special hypothesis applies to $t_{-q} \cdot g \cdot t_p$ with respect to the group $t_{-p}\Gamma t_p$, and the same conclusion follows for g.

Since the density points and the points of nonsingular total differentiability comprise sets of full measure when $n \geq 2$, it is clear that such points exist in this case. In the case $n=1$, we can only say that *if* f' exists and is positive on a set of *positive measure*, then such p can be found. Otherwise, f is singular. This is Kuusalo's theorem.

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