Every covering of a compact Riemann surface of genus greater than one carries a nontrivial L^2 harmonic differential

by

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This paper contains the proof of the assertion in its title. My motivation for considering this problem was the following. Let M be a compact differentiable manifold and let D be an elliptic differential operator between the spaces of C^∞ sections of two bundles on M. In [4] M. F. Atiyah proves that if index $D>0$, then, for every Galois covering $\tilde{M}\to M$, the operator $\tilde{D}$ induced by D on $\tilde{M}$ has nontrivial L^2 solutions $\tilde{D}u=0$. Atiyah goes on to ask whether the same is true for coverings which are not Galois. His guess is that the answer is negative in general but that the counter-example will be difficult to construct. The simplest situation to consider in this connection is that of a surface and the operator whose index is the Euler characteristic. For infinite coverings, since every L^2 harmonic function would be constant and nonzero constants are not in L^2, Atiyah's question reduces to the question of existence of L^2 harmonic forms of degree one. It is shown here that a counter-example will not be found in this simple setting.

From now on S will denote an oriented, compact surface of genus $g=g(S)>1$ equipped with a smooth Riemannian metric. $\tilde{S}\xrightarrow{\pi}S$ will be an arbitrary (usually infinite) covering of S, and $\Delta=\tilde{\Delta}$ will be the Laplace operator on $\tilde{S}$ with respect to the pull back metric. A differential (exterior form of degree one) on $\tilde{S}$ is harmonic, i.e. $\Delta\omega=0$, and in L^2 if and only if (cf. [15], Theorem 26)

$$d\omega = d*\omega = 0$$

and

$$\int_{\tilde{S}} |\omega|^2 = \int_{\tilde{S}} \omega\wedge *\bar{\omega} < \infty.$$

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Since $*$ is a conformally invariant operation L^2 harmonic differentials are conformal invariants. Moreover every two metrics on S, and hence their pull backs to $\tilde{S}$, are mutually bounded. It follows (cf. [6], § 1) that the existence of L^2 harmonic differentials on $\tilde{S}$ is independent of the choice of metric and conformal structure on S, i.e. is determined by the topology of the covering $\tilde{S}\xrightarrow{\pi}S$.

The paper is organized as follows. In the first section an isoperimetric inequality (cf. (1.2) below) for subsets of $\tilde{S}$ is proved. This inequality was proved by Nishino [13] for the Poincaré metric. The proof presented here is simpler than Nishino's proof and imposes no restriction on curvature. However, Nishino uses a more general notion of covering. Since L^2 harmonic differentials are conformal invariants, Nishino's result is sufficient for proving the theorem of the title.

In § 2 $\tilde{S}$ is replaced by arbitrary Riemannian manifold for which the higher dimensional analog of (1.2) holds. Such manifolds are shown to carry nonconstant positive superharmonic functions. It is well known that a Riemannian manifold admits such functions if and only if the Brownian motion on it is transient, i.e. if a Brownian particle on M tends to infinity with probability one as time approaches infinity. Transience of Brownian motion was proved previously for simply connected manifolds of negative curvature by numerous authors [5], [14], [17], [19]. The result proved here is a generalization of their results since variable curvature and nontrivial fundamental group are allowed. It has a very simple proof, which, however, does not give any more detailed information about asymptotic behavior of Brownian paths (such as existence of angular limits proved in [14]).

In § 3 the existence of L^2 harmonic differentials is proved. It is known from the work of Ahlfors [2] that, in the crucial case when $\tilde{S}$ is planar, L^2 harmonic differentials on $\tilde{S}$ exist if and only if $\tilde{S}$ carries a nonconstant harmonic function u with finite Dirichlet integral, $\int_{\tilde{S}}|du|^2<\infty$. Certain function theoretic consequences of this fact are derived. The most interesting is a new proof of Myrberg's theorem [12] that the logarithmic capacity of the limit set of a nonelementary Kleinian group is positive.

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§ 1. An isoperimetric inequality

In this section $\tilde{S} \xrightarrow{\pi} S$ are as in the introduction. In addition, it will be assumed that $\tilde{S}$ is planar, i.e. that every simple closed curve disconnects. The following theorem was proved in [13] for the metric of constant negative curvature.

THEOREM 1.1. *For every open, relatively compact subset D of $\tilde{S}$ with smooth boundary*

$$A(D) \leqslant \alpha L(\partial D), \tag{1.2}$$

where $A(\cdot)$ stands for the area, $L(\cdot)$ is the length, and $\alpha>0$ is a constant depending only *on S and its metric.*

Proof. According to "Metrisch-topologischer Hauptsatz" of Ahlfors [1]

$$A(S) \max(-\chi(D), 0) \geqslant 2(g-1) A(D) - kL(\partial D), \tag{1.3}$$

where $g=g(S)>1$, $k>0$ is a constant depending only on S and $\chi(\cdot)$ stands for the Euler characteristic. Since $\tilde{S}$ is planar

$$\chi(D) = 2-N,$$

where N is the number of boundary components of D. If $N=1,2$, (1.3) implies the isoperimetric inequality (1.2) so that it is enough to assume $N \geqslant 3$. In this case (1.2) yields

$$A(S) N \geqslant 2(g-1) A(D) - kL(\partial D).$$

Now choose $b>0$ so small that the following conditions are satisfied. Every simple curve γ in the base surface S of length smaller than b is contained in a convex, evenly covered coordinate neighborhood and bounds a disc of area less than $\frac{1}{2}A(S)$.

Assume first that all boundary curves of D have length at least b. Then $N \leqslant L(\partial D)/b$ and (1.2) holds with

$$\alpha = (2g-2)^{-1}\left(k + \frac{A(S)}{b}\right). \tag{1.4}$$

To prove the case of general D list the "short" (i.e. of length smaller than b) curves on $\partial D, \gamma_1, \gamma_2, \ldots, \gamma_m$. These curves can be divided into two classes as follows. Since $\tilde{S}$ is planar $\tilde{S} \setminus \gamma_i$, has two components for every $i=1,2,\ldots,m$. Moreover, since γ_i is "short", one of the components of $\tilde{S} \setminus \gamma_i$ is a disc B_i. Either $B_i \subset D$ or $B_i \subset \tilde{S} \setminus D$. The two possibilities are illustrated in Figure 1.

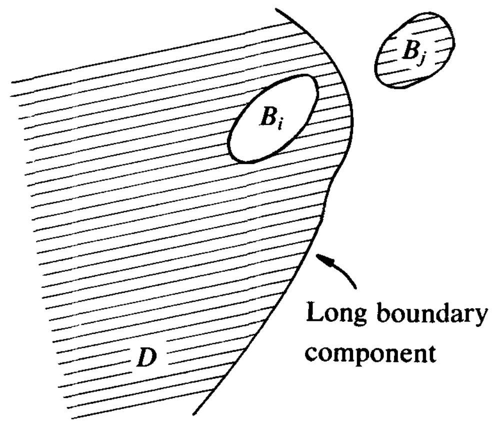

Fig. 1.

Renumber $\{\gamma_i\}_{i=1}^m$ so that $\gamma_1, \gamma_2, \dots, \gamma_k$ bound "holes" $B_i \not\subset D$ and $\gamma_{k+1}, \dots, \gamma_m$ bound small discs $B_j \subset D$. Let $D_1 = D \cup \bigcup_{i=1}^k \bar{B}_i$, i.e. D_1 is D with "all holes plugged". Clearly

$$\frac{L(\partial D_1)}{A(D_1)} \leqslant \frac{L(\partial D)}{A(D)}.$$

Hence it will suffice to prove (1.2) in case $D = D_1$ that is, when $D = D' \cup B_1 \cup B_2 \cup \dots \cup B_m, D'$ has only "long" boundary components and each B_i is a topological disc whose boundary has length smaller than b. The inequality (1.2) holds for D' with α given by (1.4). By the assumption on $b, \pi|\bar{B}_i$ is an isometry so that

$$\frac{L(\gamma_i)}{A(B_i)} = \frac{L(\pi(\gamma_i))}{A(\pi(B_i))}.$$

By Hilfsatz 1 of [1] the isoperimetric ratio

$$\frac{L(\gamma)}{\min(A(S_1), A(S_2))},$$

where γ is a simple smooth curve on S separating S into two components S_1 and S_2, is bounded below by a positive constant h. Hence, since $A(\pi(B_i)) < \frac{1}{2} A(S)$,

$$\frac{L(\gamma_i)}{A(B_i)} \geqslant h \quad \text{for } i = 1, 2, \dots, m.$$

It follows that (1.2) holds for a general D with

$$\alpha = \max\left(\frac{1}{h}, (2g-2)^{-1}\left(k+\frac{A(S)}{b}\right)\right).$$

Clearly the constant α depends on S and its metric but not on $\tilde{S}$ or D.

§ 2. Positive superharmonic functions

Let M be an open, oriented Riemannian manifold of dimension $n \geqslant 2$, such that the analog of (1.2) holds. Namely, assume there exists a constant $h>0$ so that for every relatively compact D of M with smooth boundary

$$\frac{v_{n-1}(\partial D)}{v_n(D)} \geqslant h. \tag{2.1}$$

Here $v_k(\cdot)$ denotes the k-dimensional volume with respect to the Riemannian metric of M.

THEOREM 2.2. *Suppose the manifold M satisfies the conditions above. Then M carries a nonconstant positive superharmonic function. In particular the Brownian motion on M (i.e. the minimal diffusion process associated to the Laplace-Beltrami operator, cf.* [5]) *is transient.*

Proof. The sign convention used here is such that $\Delta = -d^2/dx^2$ in $\mathbf{R}^1$. According to Cheeger [7], the first eigenvalue $\lambda(D)$ for the problem

$$\Delta u = \lambda u \text{ on } D, \quad u|_{\partial D} = 0,$$

satisfies $\lambda(D) \geqslant h^2/4$, for every open, relatively compact set $D \subset M$ with smooth boundary. By Theorem 1 of [9] or Theorem 1 of [17] the equation

$$\Delta u = \lambda u \tag{2.3}$$

has a positive solution for every $\lambda \leqslant h^2/4$. Let $v>0$ be a solution of (2.3) for an arbitrary $\lambda \in (0, h^2/4]$. Then $\Delta v = \lambda v > 0$ so that v is a nonconstant positive superharmonic function. Existence of such functions implies transience (see e.g. [17], Part I where transience is referred to as 0-transience).

§ 3. Functions and differentials on Riemann surfaces

In this section $\tilde{S}$ will be an open oriented Riemannian surface. Later on it will be assumed that $\tilde{S}$ covers a compact surface. If the isoperimetric inequality (1.2) holds for subsets of $\tilde{S}$ with some constant $\alpha>0$, then $\tilde{S}$ carries a nonconstant positive superharmonic function by Theorem 2.2. In the language of classification theory of Riemann surfaces (cf. [3]) this means that $\tilde{S}$ is hyperbolic. The connection between the type problem and the isoperimetric problem has been observed by Ahlfors [1] for simply connected surfaces, but it is new for surfaces $\tilde{S}$ with $\pi_1(\tilde{S})\neq\{e\}$. In case the surface $\tilde{S}$ is planar one can conclude much more.

THEOREM 3.1. *Let $\tilde{S}$ be a planar Riemannian surface for whose subregions the isoperimetric inequality* (1.2) *holds. Then*

(a) *$\tilde{S}$ admits a nonconstant bounded harmonic function,*

(b) *$\tilde{S}$ carries a nonconstant harmonic function with finite Dirichlet integral,*

(c) *The action of $\pi_1(\tilde{S})$ on the unit circle $|z|=1$ in the complex plane (induced by a conformal indentification of the universal covering of $\tilde{S}$ with the unit disc) is not ergodic.*

Proof. $\tilde{S}$ is hyperbolic by Theorem 2.2. For planar surfaces hyperbolicity is equivalent to (a) and (b) by Theorem 7 E, Chapter IV of [3]. Let U be the unit disc in the complex plane and let Γ be the Fuchsian group such that $\Gamma\backslash U$ is conformally equivalent to $\tilde{S}$. The pull back of a nonconstant bounded harmonic function from $\tilde{S}=\Gamma\backslash U$ to U defines a Γ-invariant bounded nonconstant harmonic function on U. By the theorem of Seidel [16] the action of Γ on the unit circle is not ergodic.

Note that if u is a nonconstant harmonic function with finite Dirichlet integral $\int |du|^2$, then $\omega=du$ is a nontrivial L^2 harmonic differential. This obvious remark will be used in the proof of the main result of this paper.

THEOREM 3.2. *Suppose $\tilde{S}\to S$ is an arbitrary covering of a compact Riemann surface S of genus $g(S)>1$. Then $\tilde{S}$ carries a nonzero L^2 harmonic differential, and consequently a nontrivial L^2 abelian differential.*

Proof. If φ is a real harmonic differential, $\omega=\varphi+i{*}\varphi$ is analytic and $|\omega|^2=2\,|\varphi|^2$. Thus it will suffice to prove the existence of an L^2 harmonic differential. If $\tilde{S}$ is not planar this is classical (cf. [3], Chapter V, § 20 E). If $\tilde{S}$ is planar, choose a metric on S compatible with the conformal structure and equip $\tilde{S}$ with the pull back metric. By

Theorem 1.1 and Theorem 3.1 (b) $\tilde{S}$ carries a function u with $du \not\equiv 0$ and $\int_{\tilde{S}} |du|^2 < \infty$. As remarked above $\varphi = du$ is a differential with required properties.

As an application of ideas used in the proof of Theorem 3.2 one can give a new proof of the main theorem of Myrberg [12] (cf. [10] for definitions and terminology concerning Kleinian groups).

THEOREM 3.3. *The logarithmic capacity of the limit set* $\Lambda(\Gamma)$ *of a nonelementary Kleinian group* Γ *is positive.*

Proof. It is well known (cf. [12]) that a nonelementary Kleinian group Γ contains a Schottky group Γ_1 with two generators. Since $\Lambda(\Gamma_1) \subset \Lambda(\Gamma)$ it is enough to prove the theorem for $\Gamma = \Gamma_1$. Let $\Omega = \Omega(\Gamma)$ be the region of discontinuity for Γ. It is quite easy to prove and well known (cf. [10]) that for a Schottky group Γ with two generators Ω is connected, Γ acts properly discontinuously and freely on $\Omega = \tilde{S}$ and $S = \Gamma \setminus \Omega$ is compact of genus two (cf. [10]). Having chosen an appropriate Riemannian metric on S one can apply Theorem 1.1 and Theorem 3.1 (a) to conclude that $\Omega = \tilde{S}$ carries nonconstant bounded harmonic function. It follows from [3], Chapter V, 22 B that $\Lambda = \mathbf{C} \setminus \Omega$ has positive logarithmic capacity.

Remark. Conversely, if $\tilde{S} \xrightarrow{\pi} S$ is a planar covering of a compact Riemann surface of genus $g > 1$, then by a theorem of Maskit [11], there exists a Kleinian group Γ such that the covering $\tilde{S} \xrightarrow{\pi} S$ is conformally equivalent to $U \to \Gamma \setminus U$, where U is a component of $\Omega(\Gamma)$. Myrberg's theorem implies then that $\tilde{S}$ is hyperbolic and carries nonzero L^2 harmonic differentials. Thus, Theorem 3.2 can be regarded as a generalization of Myrberg's theorem to the case where the Kleinian group is not present.

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