Measured foliations and the minimal norm property for quadratic differentials

by

FREDERICK P. GARDINER(1)

City University of New York,
Brooklyn, NY, U.S.A.

Introduction

The objective is to prove two important inequalities of Riemann surface theory and to present them in such a way that one can see their close relationship. The first inequality is the minimal norm property of quadratic differentials first observed by Marden and Strebel [8]. Actually a somewhat different inequality is given. The minimum norm properties have many applications, one of which is a proof of the result of Hubbard and Masur [7] concerning the existence of a unique holomorphic quadratic differential whose horizontal foliation and vertical measure realize a given measure class of measured foliations on a given Riemann surface. This result is given in section 7. The method of proof is quite different from the one in [7].

The second important inequality is the main inequality of Reich and Strebel. It has central importance in Teichmüller theory and can be used to derive the infinitesimal form of Teichmüller's metric [10]. Moreover, this inequality can be used to show that Teichmüller's metric is the integral of its infinitesimal form [6]. A proof of this result by other methods was given by O'Byrne in [9].

Both of these inequalities turn out to be consequences of certain properties of the trajectory structure of a holomorphic quadratic differential. For the purposes of the "main inequality" we are able to bypass some of the theory of trajectories by means of an averaging device used by Teichmüller. This averaging device also appears in the proofs of Teichmüller's theorem given by Bers [4] and Abikoff [1]. But for the minimum norm property in the form given by Marden and Strebel [8], the detailed theory of trajectories seems to be necessary.

(1) Research partially supported by the Research Foundation of CUNY.

§ 1. The trajectory structure of quadratic differentials

Throughout this paper it is assumed that R is a Riemann surface obtained from a compact surface by removing at most a finite number of points. The removed points are called punctures. If g is the genus of R and n is the number of punctures, we will assume $3g-3+n>0$, except when $g=1$ we allow n to be zero. This is enough to ensure that the space of integrable, holomorphic, quadratic differentials on R has positive dimension. By a quadratic differential ψ on R we mean a function defined on each coordinate patch in such a way that the expression $\psi(z)\,dz^2$ is invariant. This means that if z_1 and z_2 are two different local coordinates defined in overlapping open sets in R, then the quadratic differential has two different representations $\psi_1(z_1)$ and $\psi_2(z_2)$ which are related by the equation $\psi_1(z_1)=\psi_2(z_2)\,(dz_2/dz_1)^2$. Unless otherwise specified we will assume ψ is continuous except at a finite number of points. We will not need to consider any weaker regularity properties.

A quadratic differential $\varphi(z)$ is holomorphic if in addition each $\varphi_i(z_i)$ is a holomorphic function of z_i. The order of a zero of φ is the order of the zero of $\varphi_i(z_i)$ and this does not depend on which local coordinate z_i we use. If p in R and $\varphi(p_0)\neq 0$ and z is a local coordinate defined in a neighborhood of p_0 with $z(p_0)=z_0$, we obtain a special kind of local coordinate ζ, called a natural parameter, by letting

$$\zeta(z)=\int_{z_0}^{z}\sqrt{\varphi(z)}\,dz.$$

It is clear that if $\zeta_1(z_1(p))$ and $\zeta_2(z_2(p))$ are two natural parameters coming from φ and defined in overlapping coordinate patches U_1 and U_2, then

$$\zeta_1(z_1(p))=\pm\zeta_2(z_2(p))+(\text{const}) \tag{1}$$

for p in $U_1\cap U_2$.

Notice that $d\zeta^2=\varphi(z)\,dz^2$ for any natural parameter ζ associated with φ. A parametric curve $\gamma\colon I\to S$ is called a horizontal (vertical) trajectory of φ if, given any local coordinate z defined in a patch overlapping the image of γ, the function $\varphi(\gamma(t))$ satisfies $\varphi(\gamma(t))\,\gamma'(t)^2\geq 0$ (≤ 0). This means that in the ζ-plane, where ζ is a natural parameter, the curve $\gamma(t)$ is transformed into a horizontal (vertical) line. Clearly, this notion is independent of which local parameter you take. In fact, if you take two different natural parameters, the transition mapping is of the form expressed by formula (1) and this preserves horizontal and vertical lines.

In an obvious sense, the horizontal and vertical trajectories of φ give two transverse foliations in R in a neighborhood of any nonsingular point of φ. With a slight extension of the notion of transversality, we can also include the singular points. Let φ have a zero of order m at p in R. At any such point there will exist a local coordinate z with $z(p)=0$ such that $\varphi(z)\,dz^2$ takes the form $z^m dz^2$. Let $d\zeta=z^{m/2}dz$. Although for odd integers m, ζ is not a single-valued function of z, for any integer $m\neq -2$ a radial line $t\omega$ (where $t\geq 0$ and $|\omega|=1$) emanating from the origin in the z-plane will be horizontal if $(t\omega)^m\omega^2\geq 0$, that is if $\omega^{m+2}=1$. It turns out that in order for φ to have finite norm, m must be ≥ -1. For the case where $m=1$, the trajectories in the z-plane have the appearance shown in Figure 1.

The directions of the vertical trajectories come from the equation $\omega^{m+2}=-1$. We say that two foliations are transversal at a singular point if they have C^1-topological structure equivalent to the horizontal and vertical trajectories of $z^m dz^2$ for some integer $m\geq -1$.

§ 2. Invariants for quadratic differentials

Any nonconstant, holomorphic, quadratic differential φ on R carries with it several invariants. First of all there is the area element

$$dA_\varphi = |\varphi(z)|\,dx\,dy = d\xi\,d\eta$$

where $\zeta=\xi+i\eta$ is any natural parameter. From this one obtains the norm of φ by letting

$$\|\varphi\| = \iint_R |\varphi(z)|\,dx\,dy\,.$$

Of course the area element and the norm are defined even for nonholomorphic quadratic differentials. If $\|\varphi\|<\infty$ and φ is holomorphic on R, then it is elementary (by switching to polar coordinates) to see that φ can have at most simple poles at the punctures of R.

The expression $ds_\varphi=|\varphi|^{1/2}|dz|$ is a line element. The φ-length of a piecewise differentiable arc γ in R is $l_\varphi(\gamma)=\int_\gamma ds_\varphi$. Away from the singularities of φ and in terms of a natural parameter ζ, one has $ds_\varphi^2=d\xi^2+d\eta^2$, so local geodesics away from singularities are just straight line segments in the ζ-plane. However, at singularities geodesics can have vertices. Although the curvature of the metric ds_φ is not defined at singular points, in an intuitive sense the points where φ is zero contribute to negative curvature. Since we will consider the trajectory structure of quadratic differentials which are holomorphic in R, there is a negative-curvature-like property which forces global

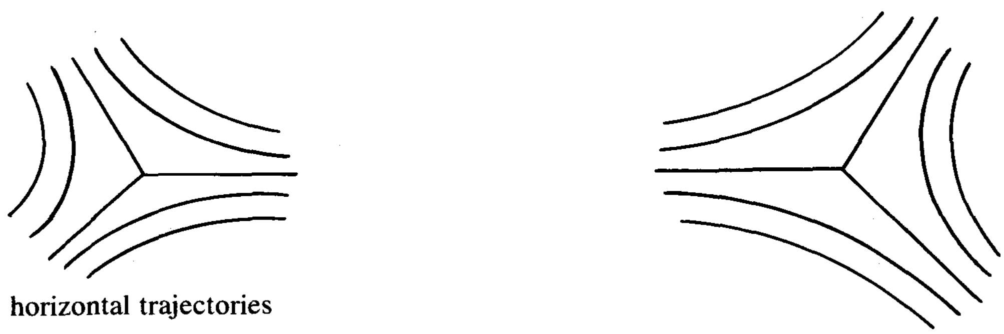

Fig. 1

geodesics to be unique. In our approach, we will not need to prove this, but the fact that φ is holomorphic in R will enter in an essential way.

For our purposes, the most important notion is the height of a curve. By definition, if γ is a differentiable curve on R, its height with respect to φ is given by

$$h_\varphi(\gamma) = \int_\gamma |\operatorname{Im} \sqrt{\varphi(z)}\, dz|. \tag{2}$$

Similarly, its width is given by

$$w_\varphi(\gamma) = \int_\gamma |\operatorname{Re} \sqrt{\varphi(z)}\, dz|. \tag{3}$$

Obviously the φ-length of a curve is greater than or equal to its width and its height.

We call a trajectory critical if, when it is continued in either direction, it meets a singularity of φ. Let b_φ be the subset of R which consists of the union of all critical vertical trajectories. Since there are a finite number of singularities of φ, b_φ consists of finitely many smooth images of intervals, and therefore, b_φ has measure zero. (In the generic case, b_φ is a dense subset of R.)

§ 3. A minimal norm property

Now let $\psi(z)\, dz^2$ be a quadratic differential on R, but not necessarily holomorphic. All we need is sufficient regularity of ψ so that line integrals of the type $h_\psi(\beta) = \int_\beta |\operatorname{Im} \sqrt{\psi(z)}\, dz|$ are well-defined for every vertical segment β of the holomorphic quadratic differential. The following theorem is analogous to but quite different from the minimal norm property given by Marden and Strebel [8]. The proof we follow imitates Teichmüller's averaging trick which is contained in Bers' fundamental paper [4].

THEOREM 1. *Let φ be a holomorphic quadratic differential on R with $\|\varphi\|=\int\int_R|\varphi(z)|\,dx\,dy<\infty$ and for which every noncritical vertical trajectory can be continued indefinitely in both directions. Let $\psi(z)\,dz^2$ be another quadratic (not necessarily holomorphic) differential which is continuous on R. Assume there exists a constant $M>0$ such that for every noncritical vertical segment β, one has $h_\varphi(\beta)\leq h_\psi(\beta)+M$. Then*

$$\|\varphi\| \leq \iint_R |\sqrt{\varphi(z)}|\,|\sqrt{\psi(z)}|\,dx\,dy. \tag{4}$$

Remarks. (1) Any holomorphic quadratic differential φ on a compact Riemann surface or a compact surface with finitely many punctures for which $\|\varphi\|<\infty$ will satisfy the hypothesis on the trajectories of φ.

(2) When we say the noncritical trajectories can be continued indefinitely in both directions, we do not exclude the possibility that they may be closed.

To give the proof we need the following lemma.

LEMMA 1. *Let g be a continuous nonnegative function on R and let g be integrable with respect to the areal element $|\varphi(z)|\,dx\,dy$ induced by the holomorphic quadratic differential φ. Let φ satisfy the hypotheses of Theorem 1 and ζ be a natural parameter for φ. Then the function $h(\zeta)=g(\zeta+i\tau)+g(\zeta-i\tau)$ is well-defined on $R-b_\varphi$ and*

$$\iint_R h(\zeta)\,d\xi\,d\eta = 2\iint_R g(\zeta)\,d\xi\,d\eta. \tag{5}$$

Proof. First note that integrating over R and over $R-b_\varphi$ are equivalent since b_φ is a set of measure zero. Since noncritical trajectories can be continued indefinitely in both directions, after choice of orientation, the expression $g(\zeta+i\tau)$ is well-defined for ζ on any particular noncritical trajectory. Thus $g(\zeta+i\tau)+g(\zeta-i\tau)$ is well-defined on $R-b_\varphi$, independently of the choice of orientation of any particular trajectory. Since the locally defined mapping $\zeta\mapsto\zeta+i\tau$ has Jacobian identically equal to one, the expression $h(\zeta)\,d\xi\,d\eta$ has the same local mass as $2g(\zeta)\,d\xi\,d\eta$ and thus one has (5).

To proceed with the proof of the theorem, let p be a point of $R-b_\varphi$ and define a nonnegative function $g(p)$ by

$$g(p)=\int_{\beta_p} |\mathrm{Im}\sqrt{\psi(s)}\,ds|, \tag{6}$$

where β_p is the vertical segment for φ with height b and midpoint p.

Notice that if an orientation of β_p is selected, then $g(p)$ can be rewritten as

$$g(p)=\int_{-b/2}^{b/2} |\mathrm{Im}\sqrt{\psi(p+it)}\,i\,dt| = \int_0^{b/2} (|\mathrm{Re}\sqrt{\psi(p+it)}|+|\mathrm{Re}\sqrt{\psi(p-it)}|)\,dt$$

and this formula is valid no matter which orientation is selected. Thus, one finds that

$$\iint_R g(p)\,d\xi\,d\eta = \int_0^{b/2}\iint_R |\mathrm{Re}\sqrt{\psi(p+it)}|+|\mathrm{Re}\sqrt{\psi(p-it)}|\,d\xi\,d\eta\,dt.$$

From Lemma 2, the right hand side of this equation becomes

$$2\int_0^{b/2}\iint_R |\mathrm{Re}\sqrt{\psi(\zeta)}|\,d\xi\,d\eta\,dt$$

and thus, we obtain

$$\iint_R g(\zeta)\,d\xi\,d\eta = b\iint_R |\mathrm{Re}\sqrt{\psi(\zeta)}|\,d\xi\,d\eta\,. \tag{7}$$

A word concerning the meaning of the integral on the right hand side of (7) is in order. The variable ζ is assumed to be a natural parameter for the quadratic differential φ. If ζ_1 and ζ_2 are two natural parameters defined in overlapping neighborhoods and, if in terms of these parameters the quadratic differential ψ is represented by ψ_1 and ψ_2, then $\psi_1(\zeta_1)=\psi_2(\zeta_2)(d\zeta_2/d\zeta_1)^2$. Since $d\zeta_2/d\zeta_1=\pm 1$, the expression $|\mathrm{Im}\sqrt{\psi(\zeta)}|\,d\xi\,d\eta$ is defined independently of the choice of natural parameter. However, if one uses a local parameter z (not necessarily natural), the integrand in the right hand side of (7) ceases to be invariant.

We can now proceed to the proof of the theorem. From the hypothesis we know that

$$b \leq \int_{\beta_p} |\mathrm{Re}\sqrt{\psi(z)}\,dz|+M$$

where β_p is the vertical segment of height b with midpoint p. This means that $b-M\leq g(p)$ for all p in $R-b_\varphi$. From (7), this implies

$$(b-M)\iint_R d\xi\, d\eta \leq b \iint_R |\mathrm{Re}\sqrt{\psi(\zeta)}|\, d\xi\, d\eta\,. \tag{8}$$

Dividing both sides by b and taking the limit as b approaches infinity, one obtains

$$\iint_R d\xi\, d\eta \leq \iint_R |\mathrm{Re}\sqrt{\psi(\zeta)}|\, d\xi\, d\eta \leq \iint_R |\sqrt{\psi(\zeta)}|\, d\xi\, d\eta. \tag{9}$$

Notice that $\varphi(\zeta)\equiv 1$ for any natural parameter ζ, so the integrand on the right hand side of (9) may be multiplied by $|\sqrt{\varphi(\zeta)}|$ without changing it. Of course, the purpose of this is to render it invariant under changes of holomorphic local parameters. Then (9) becomes

$$\iint_R |\varphi(z)|\, dx\, dy \leq \iint_R |\mathrm{Re}\sqrt{\psi(\zeta)}|\, d\xi\, d\eta \leq \iint_R |\sqrt{\psi(z)}\sqrt{\varphi(z)}|\, dx\, dy, \tag{10}$$

and this proves the theorem.

THEOREM 2. *Let φ and ψ satisfy the same hypothesis as in Theorem 1. Then*

$$\|\varphi\| \leq \|\psi\| \tag{11}$$

and, if this inequality is an equality, then $\psi(z)\equiv\varphi(z)$.

Proof. Schwarz's inequality gives

$$\iint_R |\sqrt{\psi(z)}\sqrt{\varphi(z)}|\, dx\, dy \leq \|\psi\|^{1/2}\|\varphi\|^{1/2}. \tag{12}$$

Substituting this into (10) and dividing both sides by $\|\varphi\|^{1/2}$ yields (11). Moreover, if you have equality in (11), then (10) and (12) yield

$$\|\varphi\| \leq \iint_R |\sqrt{\psi(z)}\sqrt{\varphi(z)}|\, dx\, dy \leq \|\varphi\|$$

and, when an application of Schwarz's inequality yields equality, the two functions must be multiples of one another. Thus $|\sqrt{\psi(z)}|=c|\sqrt{\varphi(z)}|$. Since (10) is an equality, one has $c=1$. Equality in (10) also forces $\mathrm{Re}\sqrt{\psi(\zeta)}=\pm 1$ for any natural parameter ζ. Since $\varphi(\zeta)=1$ and $|\varphi(\zeta)|=|\psi(\zeta)|$, this obviously forces $\psi(\zeta)=1$, for any natural parameter ζ. Thus $\psi=\varphi$ and the proof is complete.

§4. The main inequality

In order to obtain the "main inequality" of Reich and Strebel from Theorem 1, we need two lemmas. The first says that the height of a noncritical vertical segment is minimum among all homotopic arcs with the same endpoints. The second concerns the extent to which a quasiconformal self-mapping of R which is homotopic to the identity can distort heights.

LEMMA 2. *Let φ be a holomorphic quadratic differential on R and $\tilde{R}$ be the universal covering surface of R. Let β be a differentiable mapping of a closed interval into a vertical trajectory of φ such that a lifting $\tilde{\beta}$ of β is a one-to-one mapping into $\tilde{R}$. Let γ be any differentiable mapping from the same interval into R with the same endpoints as β and which is homotopic to β with fixed endpoints. Then $h_\varphi(\beta)\leq h_\varphi(\gamma)$.*

Proof. Our first step is to lift the curves β and γ and the differential φ to the universal covering surface $\tilde{R}$, where they become $\tilde{\beta}$, $\tilde{\gamma}$ and $\tilde{\varphi}$. Notice that $h_{\tilde{\varphi}}(\tilde{\beta})=h_\varphi(\beta)$ and $h_{\tilde{\varphi}}(\tilde{\gamma})=h_\varphi(\gamma)$. We select the liftings of β and γ in such a way that $\tilde{\beta}$ and $\tilde{\gamma}$ have coinciding initial and terminal points and there is a homotopy connecting $\tilde{\beta}$ and $\tilde{\gamma}$ with fixed endpoints in $\tilde{R}$.

The next step is to replace $\tilde{\gamma}$ by a homotopic curve made up of a chain of vertical and horizontal segments $\tilde{\alpha}_1\tilde{\beta}_1\ldots,\tilde{\alpha}_n\tilde{\beta}_n$ for which $h_\varphi(\Pi_{i=1}^n\tilde{\alpha}_i\tilde{\beta}_i)\leq h_\varphi(\tilde{\gamma})$. To see that this can be done, we cover $\tilde{\gamma}$ by parametric disks parameterized by natural parameters. Then we take a subdivision $\{x_i\}_{i=1}^m$ of the interval such that each $\tilde{\gamma}([x_{i-1},x_i])$ is contained in one parametric disk. Within each disk it is a simple matter to see that $\tilde{\gamma}([x_{i-1},x_i])$ can be replaced by one horizontal and one vertical segment, such that the height of the vertical segment is less than or equal to the height of $\tilde{\gamma}([x_{i-1},x_i])$.

The third step is to observe that we may assume $\tilde{\alpha}_1\tilde{\beta}_1\ldots\tilde{\alpha}_n\tilde{\beta}_n$ has no self-intersection. This is achieved in two stages. First one arranges for the number of points of self-intersection to be finite. The only way they could be infinite is for part of a segment $\tilde{\alpha}_i$ (or $\tilde{\beta}_i$) to coincide with part of segment $\tilde{\alpha}_j$ (or $\tilde{\beta}_j$). If this happens it is clear that $\tilde{\alpha}_j$ or $\tilde{\beta}_j$ may be shifted slightly to the side in one of the parametric disks without losing the homotopy. The second stage prescribes a way of reducing the number of self-intersections by at least one. You move along the path until you come to the first self-intersection point. Then you proceed along the path marking as you go in red until you return to that intersection point. The part marked in red may contain further intersection points. Whether it does or not, you delete from the path the part marked in red.

You have reduced by at least one the number of self-intersection points and you have not lost the homotopy because $\tilde{R}$ is simply connected.

The fourth step is to observe that you may assume $\tilde{\alpha}_1\tilde{\beta}_1, \ldots, \tilde{\alpha}_n\tilde{\beta}_n$ does not intersect $\tilde{\beta}$ except at the two endpoints. Since the inequality to be proved is $h_\varphi(\tilde{\beta}) \leq \Sigma_{i=1}^n h_\varphi(\tilde{\beta}_i)$, one simply deletes the segments where $\tilde{\beta}$ is common with any of the $\tilde{\beta}_i$ and then one proves the inequality between each successive point of intersection.

The fifth and final step is to treat the case where $\tilde{\beta}$ and $\tilde{\alpha}_1\tilde{\beta}_1 \ldots \tilde{\alpha}_n\tilde{\beta}_n$ joined at the two endpoints make up a simple closed curve C. We will do this by defining a measure-preserving injective map from $\tilde{\beta}$ into $\bigcup_{i=1}^n \tilde{\beta}_i$ defined at all but a finite number of points of $\tilde{\beta}$. Given a point on $\tilde{\beta}$, we consider the horizontal trajectory $\tilde{\alpha}$ passing through this point inside the curve C. Since $\tilde{\varphi}$ has only finitely many zeros inside C, by omitting consideration of finitely points of $\tilde{\beta}$ we can assume the horizontal trajectory is noncritical inside of C. It must be a crosscut, by which we mean $\tilde{\alpha}$ is a simple arc with two endpoints on the curve C. We will omit the proof of this rather elementary fact. It depends on the fact that $\tilde{\varphi}$ has no poles. A second important fact is that, while one of its endpoints is on $\tilde{\beta}$, the other endpoint must be on one of the $\tilde{\beta}_j$. If it also were located on $\tilde{\beta}$, then we would have the picture shown in Figure 2.

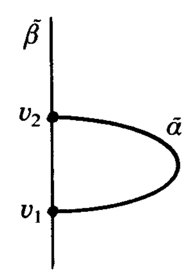

Fig. 2

The crosscut $\tilde{\alpha}$ would have to return to the same side of $\tilde{\beta}$ since it always remains inside the curve C. Let C_1 be the simple closed curve made up of $\tilde{\alpha}$ and the part of $\tilde{\beta}$ lying between the two endpoints of α. Along $\tilde{\alpha}$ and $\tilde{\beta}$ $\arg(\varphi(z)\,dz^2)$ is constant and thus $d(\arg\varphi(z)) = -2d\arg(dz)$. The change in $\arg dz$ along the closed curve C_1 is 2π. The change in $\arg dz$ along $\tilde{\alpha}$ and $\tilde{\beta}$ is π, because the change at each vertex v_1 and v_2 is $\pi/2$. Therefore, the change in $\arg\varphi(z)$ on $\tilde{\alpha}$ and $\tilde{\beta}$ is -2π.

Finally, since φ is nonsingular at v_1 and v_2, the change in $\arg\varphi(z)$ at v_1 and v_2 is π. Hence the total change in $\arg\varphi(z)$ as you move around C_1 is $-\pi$. But the argument principle tells us this must be equal to 2π times the number of zeros minus the number of poles. Since there are no poles we have a contradiction and we conclude that the crosscut $\tilde{\alpha}$ cannot return to $\tilde{\beta}$.

We see that the crosscuts map $\tilde{\beta}$ into $\cup \tilde{\beta}_i$. Since the measure $|d\eta|$ is preserved (where $\zeta=\xi+i\eta$ is a natural parameter) as you move along horizontal trajectories, we have

$$h_\varphi(\tilde{\beta}) \leq \sum h_\varphi(\tilde{\beta}_i) \leq h_\varphi(\gamma).$$

Remark. A better form of Lemma 2 is true for measured foliations (which are defined in section 7). Let $|dv|$ be a measured foliation and let β be any curve which is quasitransversal to the leaves of the foliation $|dv|$. (These leaves are the curves along which $v\equiv$const.) Let γ be any curve with the same endpoints as β and which is homotopic to β. Let $h_v(\gamma)=\int_\gamma |dv|$. Then $h_v(\beta)\leq h_v(\gamma)$. Hence Lemma 2 would follow on letting $|dv|=|\mathrm{Im}\sqrt{\varphi(z)}\,dz|$.

We will not need this stronger form of the lemma. We mention it only to observe that the fact φ is holomorphic is not essential. All that is necessary is that its singularities take a form to which the argument principle can be applied.

LEMMA 3. *Let φ be a holomorphic quadratic differential on R with $\|\varphi\|<\infty$. Let f be a quasiconformal self-mapping of R which is homotopic to the identity. Then there exists a constant M such that for every noncritical vertical segment β of φ, one has*

$$h_\varphi(\beta) \leq h_\varphi(f(\beta))+M.$$

The constant M depends on φ and f but not on β.

Proof. Let $\bar{R}$ be the completion of R with the n punctures added to it. So $\bar{R}$ is compact with no punctures. Since f is quasiconformal, f extends to $\bar{f}$ a quasiconformal self-mapping of $\bar{R}$ and $\bar{f}$ fixes the punctures because f is homotopic to the identity on R. The line element $ds_\varphi=|\varphi|^{1/2}|dz|$ determines a finite-valued metric on $\bar{R}$. To see that the distance from a point in R to a puncture is finite, one observes that φ has at most simple poles and so to find the length of a short arc ending at a puncture, one has to calculate an integral of the form $\int_0^a t^{-1/2}\,dt$ and this clearly converges.

Let f_t be the homotopy connecting f to the identity, so $f_0(p)=p$ and $f_1(p)=f(p)$. Let $l(p)$ be the infimum of the φ-lengths of all curves which go from p to $f(p)$ and which are homotopic with fixed endpoints to the curve $f_t(p)$. Clearly, $l(p)$ is a continuous function on the compact set $\bar{R}$. Let M_1 be the maximum of this function.

Now let β have endpoints p and q. The segment β and the curve which consists of $f_t(p)$ followed by $f(\beta)$ and then followed by $f_{1-t}(q)$ is clearly homotopic to β with fixed endpoints. So by Lemma 2

$$h_\varphi(\beta) \leq h_\varphi(f_t(p)) + h_\varphi(f(\beta)) + h_\varphi(f_{1-t}(q)).$$

Since the φ-length of a curve is greater than its height and the first and third terms in this inequality are bounded by M_1, the lemma is proved if we let $M=2M_1$.

THEOREM 3. *Let φ be a holomorphic quadratic differential on R with $\|\varphi\|<\infty$. Let f be a quasiconformal self-mapping of R which is homotopic to the identity and let $\mu(z)=f_{\bar z}/f_z$ be the Beltrami coefficient of f. Then*

$$\|\varphi\| \leq \iint_R |\varphi(z)| \frac{|1-\mu(z)\,\varphi(z)/|\varphi(z)||^2}{1-|\mu(z)|^2}\,dx\,dy. \tag{13}$$

Proof. As was mentioned in the remarks to Theorem 1, the noncritical vertical trajectories of φ can be continued indefinitely in both directions. Let ψ be defined by

$$\psi(z) = \varphi(f(z))\, f_z^2(z)\,(1-\mu(z)\,\varphi(z)/|\varphi(z)|)^2. \tag{14}$$

We will show that ψ satisfies the hypotheses of Theorem 1. An elementary calculation shows that ψ is a quadratic differential. From Lemma 3, $h_\varphi(\beta)\leq h_\varphi(f(\beta))+M$ for all noncritical vertical segments β. From the definition of h_φ, we have

$$h_\varphi(f(\beta)) = \int_{f(\beta)} |\operatorname{Im}\sqrt{\varphi(f)}\,df|.$$

Since $df=f_z\,dz+f_{\bar z}\,d\bar z=f_z(1+\mu(d\bar z/dz))\,dz$, by introducing $\sqrt{\psi}$ from (14), this last integral becomes

$$h_\varphi(f(\beta)) = \int_\beta \left|\operatorname{Im}\sqrt{\psi(z)}\left(1+\mu\frac{d\bar z}{dz}\right)\left(1-\mu\frac{\varphi}{|\varphi|}\right)^{-1} dz\right|.$$

Since $\varphi(z)\,dz^2<0$ along the vertical segment β, one easily sees that $\varphi/|\varphi|=-d\bar z/dz$ along β. The final result is that $h_\varphi(f(\beta))=h_\psi(\beta)$. Hence from Lemma 3, $h_\varphi(\beta)\leq h_\psi(\beta)+M$ for all vertical segments β. By Theorem 1, this tells us that

$$\|\varphi\| \leq \iint_R |\sqrt{\psi(z)}\,\sqrt{\varphi(z)}|\,dx\,dy. \tag{15}$$

Substituting (14) into (15) yields

$$\|\varphi\| \leq \iint_R |\varphi(f(z))|^{1/2} |f_z| \, |1-\mu\varphi/|\varphi|| \, |\varphi|^{1/2} \, dx \, dy. \tag{16}$$

Introducing a factor of $(1-|\mu|^2)^{1/2}$ into the numerator and denominator of (16) and applying Schwarz's inequality yields

$$\|\varphi\| \leq \left(\iint |\varphi(f(z))| \, |f_z|^2 (1-|\mu|^2) \, dx \, dy\right)^{1/2} \left(\iint |\varphi| \frac{|1-\mu\varphi/|\varphi||^2}{1-|\mu|^2} \, dx \, dy\right)^{1/2}.$$

The first integral on the right hand side of this expression is simply $\|\varphi\|^{1/2}$ and so we have (13).

Remark 1. If, instead of using the stronger inequality (4), one uses the inequality (11), then one obtains

$$\|\varphi\| \leq \iint_R |\varphi(w)| \frac{|1-\mu(z)\,\varphi(z)/|\varphi(z)||^2}{1-|\mu(z)|^2} \, du \, dv$$

where $w=u+iv=f(z)$. This is exactly the same as (12.2) of [4, page 111] and it is enough to prove Teichmüller's uniqueness theorem, but it does not yield (13) which is more useful in Teichmüller theory [6].

§ 5. Minimal norm property again

Given a closed curve γ in R, define $h_\varphi[\gamma]$ to be the infimum of the values of $h_\varphi(\gamma')$ where γ' varies over all closed curves in R freely homotopic to γ. Let $\mathcal{S}$ be the set of all homotopy classes of simple closed curves in R which are not homotopically trivial and not homotopic to a puncture. The following is a slight generalization of a theorem of Marden and Strebel [8].

THEOREM 4. *Let φ be a holomorphic quadratic differential on R and ψ another quadratic differential on R (not necessarily holomorphic). Assume $|\mathrm{Im}\sqrt{\psi(\zeta)}|$ is a bounded function on R as a function of natural parameters ζ. Suppose there is a number $M\geq 0$ such that $h_\varphi[\gamma]\leq h_\psi[\gamma]+M$ for all γ in $\mathcal{S}$. Then*

$$\|\varphi\| \leq \iint_R |\sqrt{\psi}\,\sqrt{\varphi}| \, dx \, dy \leq \|\varphi\|^{1/2} \|\psi\|^{1/2}, \tag{17}$$

and $\|\varphi\|=\|\psi\|$ *only if* $\varphi=\psi$. *The same conclusion remains valid if we assume* $h_\varphi[\gamma]\leq h_\psi[\gamma]+M$ *only for those* γ *which have the special form* $\gamma=\alpha_1\beta_1\alpha_2\beta_2$ *where* α_1 *and* α_2 *are arbitrarily short horizontal trajectories of* φ, β_1 *and* β_2 *are vertical trajectories of* φ *and the path* γ *is quasitransversal to horizontal trajectories of* φ.

Remark. To say that γ is quasitransversal to the horizontal trajectories means γ may move along a horizontal trajectory but if it enters from one side it must exit from the other. The path $\alpha_1\beta_1\alpha_2\beta_2$ is quasitransversal if it makes alternately left and right turns.

We omit the proof as we cannot improve on the method given by Marden and Strebel [8]. We remark however that their proof remains valid with the weaker hypothesis obtained by introducing the number M. Furthermore, the stronger conclusion (17) goes through in the same way we made the step from formula (9) to (10).

§6. Convergence of quadratic differentials

Let Q be the space of integrable holomorphic quadratic differentials on R where R is compact except for a finite number of punctures. As before, $\mathcal{S}$ is the set of homotopy classes of simple closed curves not homotopically trivial and not homotopic to a puncture. Let $\mathbf{R}_+^{\mathcal{S}}$ have the product topology. The following is contained in the papers of Hubbard and Masur [7] and Marden and Strebel [8].

THEOREM 5. *The mapping* $\Phi: Q\to\mathbf{R}_+^{\mathcal{S}}$ *defined by* $\Phi(\varphi)=(h_\varphi[\gamma];\gamma\in\mathcal{S})$ *is injective and bicontinuous onto its image.*

Proof. The fact that Φ is injective follows from the uniqueness part of Theorem 4. To show that it is continuous assume $\|\varphi_n-\varphi\|\to 0$. Let $[\gamma]$ be in $\mathcal{S}$. Then there is a representative γ' of $[\gamma]$ which is transversal to the horizontal foliation coming from φ. So $h_\varphi(\gamma')=h_\varphi[\gamma]$. For large n, γ' will also be transversal to the horizontal foliations of the φ_n. Hence $h_{\varphi_n}[\gamma]=h_{\varphi_n}(\gamma')$ for large n. It is obvious that $h_{\varphi_n}(\gamma')$ converges to $h_\varphi(\gamma')$. Hence $h_{\varphi_n}[\gamma]$ converges to $h_\varphi[\gamma]$ and this shows Φ is continuous.

Now assume φ_n is a sequence of elements of Q and $h_{\varphi_n}[\gamma]$ converges for each γ in $\mathcal{S}$. Then we claim $\|\varphi_n\|$ is bounded. If not, we may take a subsequence φ_{n_k} for which $\|\varphi_{n_k}\|\to\infty$. By forming $g_k=\varphi_{n_k}/\|\varphi_{n_k}\|$ we get a sequence of elements of Q such that $\|g_k\|=1$ and $h_{g_k}[\gamma]\to 0$ for each γ in $\mathcal{S}$. Since Q is finite dimensional we may take a limit point g of the sequence g_k. Then $\|g\|=1$ and $h_g[\gamma]=0$ for all γ. Obviously this is impossible (Theorem 4 would imply that $\|g\|=0$). Hence the original sequence φ_n is bounded and, thus, some subsequence φ_{n_k} of it converges to a limit, say φ_0. By the hypothesis any other convergent subsequence would have to converge to a limit with exactly the same heights and, hence to the same limit. Therefore the sequence φ_n itself converges.

§7. Measured foliations

Let R be a C^1 orientable surface of genus g which is compact except for a finite number n of punctures and assume $3g-3+n>0$. A measured foliation with measure $|dv|$ on R with singularities of order $k_1, \ldots, k_m$ at the points $p_1, \ldots, p_m$ is given by an open cover U_i of $R-\{p_1, \ldots, p_m\}$ and C^1 functions v_i on each U_i such that

(a) $dv_i = \pm dv_j$ on $U_i \cap U_j$.

(b) At each point p_i there is a local C^1-chart (u, v): $V \to \mathbf{R}^2$ such that for $z=u+iv$, $dv_i = \operatorname{Im}(z^{k_i/2} dz)$ on $V \cap U_i$ for some branch of $z^{k_i/2}$ in $U_i \cap V$.

(c) $k_i \geq 0$ at all points p_i in R and $k_j \geq -1$ if p_j is a puncture of R.

The leaves of the foliation are curves along which v is constant. The height of an arc γ is defined analogously to the way it was in (2); $h_v(\gamma) = \int_\gamma |dv|$. Moreover the height of an element $[\gamma]$ of $\mathscr{S}$ is defined by $h_v[\gamma] = \inf h_v(\gamma')$ where the infimum is taken over all $\gamma' \in [\gamma]$. We will denote a measured foliation by the symbol $|dv|$. Two measured foliations $|dv_1|$ and $|dv_2|$ are called measure equivalent if $h_{v_1}[\gamma] = h_{v_2}[\gamma]$ for all γ in $\mathscr{S}$.

THEOREM 6 [7]. *Given a measured foliation $|dv|$ on R and a complex structure on R, there exists a unique holomorphic quadratic differential φ in Q such that the foliation given by the horizontal trajectories of φ and the measure $|\operatorname{Im}\sqrt{\varphi}\, dz|$ is measure equivalent to $|dv|$.*

Proof. The uniqueness is taken care of by Theorem 4 (or the part of Theorem 5 which says that Φ is injective). The existence depends on the following general facts.

(1) Given a measured foliation, there is a measure equivalent foliation which has a transverse measured foliation [7].

(2) The convergence criterion contained in Theorem 5.

(3) The density of Jenkins-Strebel differentials [5, 8].

(4) The existence of Jenkins-Strebel differentials with prescribed heights [11].

(5) The existence of solutions to the Beltrami equation [2].

Before giving the proof we need to explain items (3) and (4).

A Jenkins-Strebel differential φ on R is an element of Q all of whose noncritical horizontal trajectories are closed. The totality of all these noncritical trajectories form an open set in R with at most $3g-3+n$ components and each component is a cylinder (or annulus). Each cylinder has a height which is measured by the vertical measure $|\mathrm{Im}\sqrt{\varphi(z)}\,dz|$. Given any set of m ($\leq 3g-3+n$) elements $[\gamma_1], \ldots, [\gamma_m]$ of $\mathcal{S}$ representable by disjoint simple closed curves and any set of positive numbers $h_1, \ldots, h_m$, there exists a Jenkins-Strebel differential φ with associated of cylinders $A_1, \ldots, A_m$ such that the core curve of each A_i is in the homotopy class of γ_i and such that the height of A_i is h_i. By the core curve of A_i we mean a curve which loops around A_i once. This fact is proved most neatly by Renelt [11].

The second important fact is that Jenkins-Strebel differentials are dense in Q. This was first proved for compact surfaces by Douady and Hubbard [5] and in the case we consider here is proved by Marden and Strebel [8].

Outline of proof of existence part of Theorem 6.

Step 1. Given the measured foliation on R, replace it by a measure equivalent foliation $|dv|$ which possesses a transverse measured foliation $|du|$. We require the mapping which takes the horizontal and vertical trajectories of $z^k dz^2$ in a neighborhood of the origin in the complex plane onto the horizontal trajectories of a singularity of $|dv|$ and $|du|$ to be a C^1-mapping.

Step 2. From the two measured foliations we construct a quadratic differential as follows. Given coordinate patches U_1 and U_2, let u_i and v_i represent u and v in U_1. If u_1 is given, pick the sign of v_1 so that $f_1=u_1+iv_1$ has positive Jacobian in U_1. In the patch U_2 let u_2 be given and select the sign of v_2 so that $f_2=u_2+iv_2$ has positive Jacobian. In $U_1 \cap U_2$ we know that $u_1=\pm u_2+(\text{const})$. Because of the Jacobian condition, we must have $v_1=\pm v_2+(\text{const})$ with the signs occurring in the same order. Thus $f_1=\pm f_2+(\text{const})$.

Step 3. We introduce a complex structure on R and a holomorphic quadratic differential such that the heights of ψ with respect to this complex structure are the same as the heights of $|dv|$. The way to do this is to form the Beltrami differential $\mu(z)\,(d\bar{z}/dz)=(f_i)_{\bar{z}}/(f_i)_z$ where the f_i are the mappings in Step 2. Notice that the sign of μ is unambiguously defined in overlapping patches $U_i \cap U_j$. The condition that the Jacobian is positive and that the transition mappings are C^1 in a neighborhood of singularities ensures that $\|\mu\|_\infty<1$. Hence, the Beltrami equation can be solved [2] and a complex structure introduced on R with respect to which the f_i are holomorphic. Let z_1 be a local parameter for this new complex structure and form

$$\psi(z_1)\,dz_1^2=\left(\frac{\partial f}{\partial z_1}\,dz_1\right)^2.$$

Using the Cauchy-Riemann equations a calculation shows that $\operatorname{Im}\sqrt{\psi(z_1)}\,dz_1=dv$.

Step 4. Take a sequence ψ_n of Jenkins-Strebel differentials on R_μ which converge to ψ. Then construct on R a sequence φ_n of Jenkins-Strebel differentials such that φ_n has the same core curves as ψ_n with the same corresponding heights. Now apply the convergence theorem (Theorem 5). Since the heights of φ_n converge, φ_n converges to a holomorphic quadratic differential φ on R with the same heights that ψ has on R_μ. Finally, observe that since $h_\psi[\gamma]=h_v[\gamma]$ we also have $h_\varphi[\gamma]=h_v[\gamma]$ for all γ in $\mathcal{S}$.

§ 8. The norm functional on Teichmüller space

Let the Riemann surface R together with the measured foliation $|dv|$ be given. As usual, R is compact except for a finite number of punctures. One can pose an extremal problem which is a kind of generalization of extremal module. Let $C(R)$ be the set of all continuous quadratic differentials $\psi\,dz^2$ on R. Let $Q(R)$ be the subset of $C(R)$ consisting of those elements which are holomorphic. Let

$$M[v]=\inf\{\|\psi\|;\text{ where }\psi\in C(R)\text{ and }h_\psi[\gamma]\geq h_v[\gamma]\text{ all }\gamma\text{ in }\mathcal{S}\}. \tag{18}$$

Notice that $M[v]$ depends only on the measure class of the measured foliation $|dv|$ since the dependency on the measured foliation only enters through the numbers $h_v[\gamma]$. Furthermore, $M[v]$ depends on the complex structure for R. This dependency comes through the definition

$$h_\psi[\gamma]=\inf_{\gamma'\sim\gamma}\int_{\gamma'}|\operatorname{Im}\sqrt{\psi(z)}\,dz|$$

since the entity $|\operatorname{Im}\sqrt{\psi(z)}\,dz|$ depends on the complex structure.

THEOREM 7. *Let $|dv|$ be a measured foliation on R. The infimum $M[v]$ in* (18) *is achieved by a unique element φ of $C(R)$. It is the same φ as the unique holomorphic quadratic φ in Theorem* 6 *for which $h_\varphi[\gamma]=h_v[\gamma]$ for all γ in $\mathcal{S}$.*

Proof. Theorem 6 tells us that there is a holomorphic quadratic differential φ for which $h_v[\gamma]=h_\varphi[\gamma]$ for all γ in $\mathcal{S}$ and Theorem 4 tells us that $\|\varphi\|\leq\|\psi\|$ and equality holds only if $\varphi=\psi$.

Now let $[f_1,R_1]$ be a point in the Teichmüller space. This means $f: R\to R_1$ is a quasiconformal mapping and the class $[f_1,R_1]$ consists of all similar pairs (f_2,R_2) with $f_2: R\to R_2$ for which there is a conformal map $c: R_1\to R_2$ such that $f_2^{-1}\circ c\circ f_1$ is homotopic to the identity. The class $[f_1,R_1]$ depends only on $\mu_1=f_{1\bar z}/f_{1z}$ and we say $\mu_1\sim\mu_2$ if (f_1,R_1) and (f_2,R_2) are similar in the above sense. When $\mu=f_{\bar z}/f_z$, we will write R_μ in place of R_1.

The measured foliation $|dv|$ induces by its heights a measure class of measured foliations on all of the surfaces R_μ, by considering the measure $|dv\circ f^{-1}|$. Clearly $h_v(\gamma)=h_{v\circ f^{-1}}(f(\gamma))$. Also it is clear, since $c\circ f_1(\gamma)$ is homotopic to $f_2(\gamma)$, that the induced measure class of measured foliations on R_μ depends only on the Teichmüller class of μ. Therefore, the solution $M_\mu[v]=M[v\circ f^{-1}]$ to the infimum problem (18) on the surface R_μ depends only on the Teichmüller class of μ. So for fixed v, we may consider $M_\mu[v]$ to be a function on Teichmüller space.

LEMMA 4. *$K^{-1}M[v]\leq M_\mu[v]\leq KM[v]$ where $K=(1+\|\mu\|_\infty)/(1-\|\mu\|_\infty)$.*

Proof. Let φ_μ be the unique holomorphic quadratic differential on R_μ for which $h_v[\gamma]=h_{\varphi_\mu}[f(\gamma)]$. We know that $M_\mu[v]=\|\varphi_\mu\|$. Let γ be any loop in R. Then

$$\int_{f(\gamma)}|\operatorname{Im}\sqrt{\varphi_\mu(w)}\,dw| = \int_\gamma |\operatorname{Im}\sqrt{\varphi_\mu(f(z))}f_z(1+\mu(d\bar z/dz))\,dz| \leq (1+k)\, l_{\tilde\varphi}(\gamma)$$

where $\tilde\varphi(z)=\varphi_\mu(f(z))\,f_z^2$ and $l_{\tilde\varphi}(\gamma)$ is the $\tilde\varphi$-length of γ. Since this inequality holds for every path γ, we see that $h_\varphi[\gamma]\leq(1+k)\,l_{\tilde\varphi}[\gamma]$ for all γ. Therefore

$$\|\varphi\| \leq (1+k)^2\iint_R |\varphi_\mu(f(z))|^2\,|f_z|^2\,dx\,dy \leq \frac{(1+k)^2}{1-k^2}\|\varphi_\mu\|$$

and this yields $M[v]\leq KM_\mu[v]$. The opposite inequality follows by applying the same reasoning to the quasiconformal mapping f^{-1}.

Remark. Lemma 4 shows that $M_\mu[v]$ is a continuous function on $T(R)$ since clearly $K^{-1}M_\sigma[v]\leq M_\mu[v]\leq KM_\sigma[v]$ where K is the dilatation of the mapping $f^\sigma\circ(f^\mu)^{-1}$.

THEOREM 8. *$M_\mu[v]$ is a differentiable function on the Teichmüller space $T(R)$ and*

$$\log M_\mu[v] = \log M[v] + 2\,\mathrm{Re}\,\frac{1}{\|\varphi\|}\iint_R \mu\varphi\,dx\,dy + o(\|\mu\|_\infty)$$

where φ is the unique holomorphic quadratic differential on R for which $|\mathrm{Im}\sqrt{\varphi(z)}\,dz|$ realizes the measured foliation with measure $|dv|$.

Proof. Let φ_μ be the unique holomorphic quadratic differential on R_μ with the same heights as $|dv\circ f^{-1}|$ where $f\colon R\to R_\mu$ and $f_{\bar z}/f_z=\mu$. Form the differential

$$\tilde\varphi(z) = \varphi_\mu(f(z))\,f_z^2\left(1-\mu(z)\frac{\varphi(z)}{|\varphi(z)|}\right)^2. \tag{19}$$

$\tilde\varphi$ is a quadratic differential on R and there exists a constant M such that for every vertical segment β on R we have

$$\int_\beta |\mathrm{Im}\sqrt{\varphi(z)}\,dz| \le \int_\beta |\mathrm{Im}\sqrt{\tilde\varphi(z)}\,dz| + M.$$

The reason for this is $\int_\beta|\mathrm{Im}\sqrt{\tilde\varphi(z)}\,dz|=\int_{\tilde\beta}|\mathrm{Im}\sqrt{\varphi_\mu(f)}\,df|$, where $\tilde\beta=f(\beta)$. Take γ on R of the form $\gamma=\alpha_1\beta_1\alpha_2\beta_2$ where α_2 and β_2 may both reduce to a point. By Theorem 4 we may consider only those elements of $\mathscr{S}$ which are of this form where the φ-lengths of α_1 and α_2 are arbitrarily small. Let $\tilde\gamma=f(\gamma)$ and similarly $\tilde\alpha_i=f(\alpha_i)$ and $\tilde\beta_i=f(\beta_i)$ for $i=1$ and 2. The bound on $\|\mu\|_\infty$ forces a bound on the φ_μ-lengths of $\tilde\alpha_1$ and $\tilde\alpha_2$ because of Lemma 4.

Now $h_{\varphi_\mu}[\tilde\gamma]=h_\varphi[\gamma]$ by the definition of φ and φ_μ. Hence $h_\varphi[\gamma]\le\int_{\tilde\gamma}|\mathrm{Im}\sqrt{\varphi_\mu(w)}\,dw|\le\int_{\tilde\beta_1\cup\tilde\beta_2}|\mathrm{Im}\sqrt{\varphi_\mu(w)}\,dw|+M=\int_{\beta_1\cup\beta_2}|\mathrm{Im}\sqrt{\tilde\varphi(z)}\,dz|+M$. Thus $h_\varphi(\beta_1\cup\beta_2)\le h_{\tilde\varphi}(\beta_1\cup\beta_2)+M$. This is true for every pair of vertical segments β_1 and β_2 which, after joining short horizontal segments α_1 and α_2, become closed curves transversal to the horizontal foliation of φ. We conclude that

$$\iint_R |\varphi|\,dx\,dy \le \iint_R |\sqrt{\varphi}|\,|\sqrt{\tilde\varphi}|\,dx\,dy. \tag{20}$$

Upon multiplying the integrand on the right hand side of (20) in numerator and denominator by $|f_z|(1-|\mu|^2)^{1/2}$ and applying Schwarz's inequality (with the term $|\sqrt{\varphi_\mu(f(z))}|\,|f_z|(1-|\mu|^2)^{1/2}$ lumped together), we find that

$$\|\varphi\| \le \|\varphi_\mu\|^{1/2}\left(\iint_R |\varphi(z)|\frac{|1-\mu(\varphi/|\varphi|)|^2}{1-|\mu|^2}\,dx\,dy\right)^{1/2}.$$

Squaring both sides and dividing by $\|\varphi\|\,\|\varphi_\mu\|$ we get

$$\frac{\|\varphi\|}{\|\varphi_\mu\|} \leqslant \frac{1}{\|\varphi\|} \iint |\varphi| \frac{|1-\mu(\varphi/|\varphi|)|^2}{1-|\mu|^2}\,dx\,dy \leqslant 1-\frac{2}{\|\varphi\|}\operatorname{Re}\iint \mu\varphi\,dx\,dy+O(\|\mu\|_\infty^2)$$

and so

$$\log\|\varphi_\mu\| \geqslant \log\|\varphi\|+2\operatorname{Re}\frac{1}{\|\varphi\|}\iint_R \mu\varphi+O(\|\mu\|_\infty^2).$$

To get a reverse inequality we will apply a similar argument to the inverse mapping f_1 of f; $f_1^{-1}\circ f(z)=z$. The Beltrami coefficient μ_1 of f_1 is related to μ by $\mu_1(f(z))=-\mu(z)/\theta$ where $\theta=\bar{f}_{\bar z}/f_z$. Note that $\|\mu\|_\infty=\|\mu_1\|_\infty$. The analogous argument yields

$$\frac{\|\varphi_\mu\|}{\|\varphi\|} \leqslant \frac{1}{\|\varphi_\mu\|}\iint_{R_\mu} |\varphi_\mu| \frac{|1-\mu_1(\varphi_\mu/|\varphi_\mu|)|^2}{1-|\mu_1|^2},$$

and so

$$\log\|\varphi\| \geqslant \log\|\varphi_\mu\|+2\operatorname{Re}\frac{1}{\|\varphi_\mu\|}\iint_{R_\mu} \mu_1\,\varphi_\mu+O(\|\mu\|_\infty^2).$$

The integral on the right hand side of this inequality transforms into

$$-2\operatorname{Re}\frac{1}{\|\varphi_\mu\|}\iint_R \mu(z)\,\varphi_\mu(f(z))\,f_z^2(z)\,dx\,dy.$$

Notice that as long as we assume μ is C^1 on the interior of R, then $\varphi_\mu(f(z))\,f_z^2(z)$ converges uniformly to $\varphi(z)$ as $\|\mu\|_\infty\to 0$. Also Lemma 4 shows that $\|\varphi_\mu\|$ converges to $\|\varphi\|$. The fact that $\varphi_\mu(f(z))\,f_z^2(z)$ converges in L_1-norm to $\|\varphi\|$ follows from the next well-known lemma in functional analysis.

LEMMA 5. *Suppose $f_n(z)$ is a sequence of L_1 functions on a domain D. Suppose $f_n(z)$ converges uniformly on compact subsets of D to $f(z)$ and suppose $\|f_n\|=\iint_D |f_n(z)|\,dx\,dy\to\iint_D |f(z)|\,dx\,dy=\|f\|$. Then $\|f_n-f\|\to 0$.*

References

[1] ABIKOFF, W., *The real analytic theory of Teichmüller space*. Springer Lecture Notes in Mathematics, 820 (1980).

[2] AHLFORS, L. & BERS, L., Riemann's mapping theorem for variable metrics. *Ann. of Math.*, 72 (1960), 385–404.
[3] BERS, L., A new proof of a fundamental inequality for quasiconformal mapping. *J. Analyse Math.*, 36 (1979), 15–30.
[4] — Quasiconformal mappings and Teichmüller's theorem, in *Analytic functions*. Eds. R. Nevanlinna et al., PUP, 1960.
[5] DOUADY, A. & HUBBARD, J., On the density of Strebel differentials. *Invent. Math.*, 30 (1975), 175–179.
[6] GARDINER, F., The Teichmüller-Kobayashi metric for infinite dimensional complex Teichmüller spaces. *Proceedings of the Oaxtepec conference* (1981), Mexico.
[7] HUBBARD, J. & MASUR, H., Quadratic differential and foliations. *Acta Math.*, 142 (1979), 221–274.
[8] MARDEN, A. & STREBEL, K., The heights theorem for quadratic differentials on Riemann surfaces. To appear in *Acta Math.*
[9] O'BYRNE, B., On Finsler geometry and applications to Teichmüller space, in *Advances in the theory of Riemann surfaces*. Ann. of Math. Study 66. Ed. L. V. Ahlfors et al., 1971.
[10] REICH, E. & STREBEL, K., Extremal quasiconformal mappings with given boundary values, in *Contributions to Analysis*, 375–392. Ed. L. V. Ahlfors et al. (New York and London, Academic Press, 1974).
[11] RENELT, H., Konstruktion gewisser quadratischer Differentiale mit Hilfe von Dirichletintegralen. *Math. Nachr.*, 73 (1976), 125–142.

Received April 18, 1983