Characterizations of almost surely continuous p-stable random Fourier series and strongly stationary processes

by

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Introduction

In 1973, X. Fernique showed that Dudley's "metric entropy" sufficient condition for the a.s. continuity of sample paths of Gaussian processes, is also necessary when the processes are stationary ([6], [7], [10]). In this paper we extend the Dudley–Fernique theorem to strongly stationary p-stable processes, $1<p\leq 2$.

Let G be a locally compact Abelian group with dual group Γ. We say that a real (resp. complex) random process $(X(t))_{t\in G}$ is a *strongly stationary p-stable process*, $0<p\leq 2$, if there exists a finite positive Radon measure m on Γ such that for all $t_1,\dots,t_n\in G$ and real (resp. complex) numbers $\alpha_1,\dots,\alpha_n$ we have

$$E\exp i\operatorname{Re}\sum_{j=1}^{n}\bar{\alpha}_jX(t_j)=\exp-\int_\Gamma\left|\sum_{j=1}^{n}\bar{\alpha}_j\gamma(t_j)\right|^p dm(\gamma).$$

We associate with $(X(t))_{t\in G}$ a pseudo-metric d_X on G defined by

$$d_X(s,t)=\left(\int_\Gamma|\gamma(s)-\gamma(t)|^p\,m(d\gamma)\right)^{1/p},\quad \forall s,\ t\in G. \tag{0.1}$$

Let K be a fixed compact neighborhood of the unit element of G. Let $N(K,d_X;\varepsilon)$ denote the smallest number of open balls of radius ε, in the pseudo-metric d_X, which cover K. We will always assume that K is metrizable. We can now state our main result.

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THEOREM A. *Let* $1<p\leq 2$ *and* q *be the conjugate of* p, *i.e.* $1/p+1/q=1$. *Let* $(X(t))_{t\in G}$ *be a strongly stationary* p*-stable process. Then* $(X(t))_{t\in K}$ *has a version with a.s. continuous sample paths if and only if*

$$J_q(d_X)=\int_0^\infty (\log N(K,\, d_X;\varepsilon))^{1/q}\,d\varepsilon<\infty. \tag{0.2}$$

Moreover, there exist constants $\alpha_p(K)>0$ *and* $\beta_p(K)$ *depending only on* p *and* K *such that*

$$\begin{aligned}\alpha_p(K)\,\{J_q(d_X)+m(\Gamma)^{1/p}\}&\leq\left\{\sup_{c>0}c^pP\left\{\sup_{t\in K}|X(t)|>c\right\}\right\}^{1/p}\\ &\leq\beta_p(K)\,\{J_q(d_X)+m(\Gamma)^{1/p}\}.\end{aligned}\tag{0.3}$$

When $p=1$

$$J_\infty(d_X)=\int_0^\infty \log^+\log N(K,\,d_X;\varepsilon)\,d\varepsilon<\infty \tag{0.4}$$

is a necessary condition for $(X(t))_{t\in K}$ *to have a version with continuous sample paths and a lower bound of the form* (0.3), *with* $J_q(d_X)$ *replaced by* $J_\infty(d_X)$ *can be obtained.*

We have not been able to determine if (0.4) is also a sufficient condition for $(X(t))_{t\in K}$ to have a version with continuous sample paths when $p=1$. The case $p<1$ is trivial since in this case the mere fact that m is a finite measure insures that the process $(X(t))_{t\in G}$ has a.s. continuous paths.

In the particular case that m is a discrete measure, it is easy to see that the process must be of the form

$$X(t;\omega)=\sum_{\gamma\in\Gamma}a_\gamma\,\tilde{\theta}_\gamma\gamma(t),\quad t\in G \tag{0.5}$$

where $\{a_\gamma\}_{\gamma\in\Gamma}$ are complex numbers satisfying $\Sigma_{\gamma\in\Gamma}|a_\gamma|^p<\infty$ and where $\{\tilde{\theta}_\gamma\}$ are i.i.d. complex valued p-stable random variables, i.e. $\tilde{\theta}_\gamma$ satisfies

$$E\exp i\operatorname{Re}\bar{z}\tilde{\theta}_\gamma=\exp-|z|^p,\quad \forall z\in\mathbf{C}.$$

Thus $(X(t;\omega))_{t\in G}$ is a random Fourier series. In this case the pseudo-metric $d_X(s,t)$ is

$$d_X(s,t)=\left(\sum_{\gamma\in\Gamma}|a_\gamma|^p\,|\gamma(s)-\gamma(t)|^p\right)^{1/p},\quad \forall s,\,t\in G. \tag{0.6}$$

Note that $\theta_\gamma=\operatorname{Re}\tilde{\theta}_\gamma$ is the ordinary canonical real valued p-stable random variable, i.e. $E\exp it\theta=\exp-|t|^p$, $\forall t\in R$ and the above results apply also to the random Fourier series $\Sigma_{\gamma\in\Gamma}a_\gamma\theta_\gamma\gamma(t)$, $t\in G$.

We are interested in the general question of the almost sure continuity or, equivalently, of the uniform convergence a.s. of random Fourier series with independent coefficients. That is, let $\{\xi_\gamma\}_{\gamma\in\Gamma}$ be independent symmetric, real or complex valued random variables defined on a probability space $(\Omega,\mathscr{A},P)$, $\omega\in\Omega$. When does

$$Y(t;\omega)=\sum_{\gamma\in\Gamma}a_\gamma\,\xi_\gamma(\omega)\,\gamma(t),\quad t\in K, \tag{0.7}$$

have a.s. continuous sample paths? In [21], under the conditions

$$\sup_{\gamma\in\Gamma}E|\xi_\gamma|^2<\infty\quad\text{and}\quad\inf_{\gamma\in\Gamma}E|\xi_\gamma|>0$$

we showed that $(Y(t;\omega))_{t\in K}$ is a.s. continuous if and only if

$$J_2(d)=\int_0^\infty(\log N(K,d;\varepsilon))^{1/2}d\varepsilon<\infty$$

where

$$d(s,t)=\left(\sum_{\gamma\in\Gamma}|a_\gamma|^2|\gamma(s)-\gamma(t)|^2\right)^{1/2},\quad\forall s,t\in G\,.$$

Theorem A enables us to extend this result to random variables $\{\xi_\gamma\}_{\gamma\in\Gamma}$ which do not have finite second moments.

THEOREM B. *Consider* $(Y(t))_{t\in K}$ *as given in* (0.7).

(i) *Assume that for* $1<p<2$

$$P\{|\xi_\gamma|>c\}\leqslant c^{-p},\quad\forall\gamma\in\Gamma,\ \forall c>0.$$

Then $J_q(d_Y)<\infty$ *is sufficient for the a.s. continuity of* $(Y(t))_{t\in K}$, *where* J_q *is defined in* (0.2) *and*

$$d_Y(s,t)=\left(\sum_{\gamma\in\Gamma}|a_\gamma|^p|\gamma(s)-\gamma(t)|^p\right)^{1/p},\quad\forall s,\ t\in G.$$

Moreover, we can find a constant $\Delta_p(K)$ *depending only on p and K, such that*

$$\left\{\sup_{c>0} c^p P\left\{\sup_{s,t\in K} |Y(t)-Y(s)|>c\right\}\right\}^{1/p} \leqslant \Delta_p(K)\left\{J_q(d_Y)+\left(\sum_{\gamma\in\Gamma}|a_\gamma|^p\right)^{1/p}\right\}. \tag{0.8}$$

(ii) *Assume that there exists a* $c_0>0$ *and* $\delta>0$ *such that*

$$P\{|\xi_\gamma|>c\}\geqslant \delta c^{-p}, \quad \forall\gamma\in\Gamma,\ \forall c\geqslant c_0.$$

Then for $1<p\leqslant 2$, $J_q(d_Y)<\infty$ *is necessary for the a.s. continuity of* $(Y(t))_{t\in K}$ *and when* $p=1$, $J_\infty(d_Y)<\infty$ *is necessary for the a.s. continuity of* $(Y(t))_{t\in K}$, *where* J_∞ *is defined in* (0.4).

When G is a compact group we can replace α_p and β_p in (0.3) by absolute constants independent of G.

THEOREM C. *Let G be a compact Abelian group. Consider* $(X(t))_{t\in G}$ *as defined in* (0.5). *Let* $\|X\|=\sup_{t\in G}|\Sigma_{\gamma\in\Gamma} a_\gamma \tilde\theta_\gamma \gamma(t)|$. *Then for* $1<p\leqslant 2$ *and* d_X *as in* (0.6),

$$\frac{1}{\beta_p}E\|X\| \leqslant \int_0^\infty (\log N(G,d_X;\varepsilon))^{1/q}\,d\varepsilon+|a_0| \leqslant \beta_p E\|X\|, \tag{0.9}$$

where a_0 *is the coefficient of the character* $\gamma(t)=1$, $\forall t\in G$, *and* β_p *is a constant depending only on p and independent of the group G. (Inequalities* (0.9) *are also valid with* $\operatorname{Re}\tilde\theta_\gamma$ *replacing* $\tilde\theta_\gamma$.)

It follows from (0.3) that if X_1 and X_2 are two strongly stationary p-stable processes such that

$$d_{X_1}(s,t)\leqslant d_{X_2}(s,t), \quad \forall s,\ t\in K,$$

then the a.s. continuity of $(X_2(t))_{t\in K}$ implies that of $(X_1(t))_{t\in K}$. (This result is made even more evident when G is compact and $K=G$ by considering (0.9).) This is a rather surprising result since examples (cf. [8]) have shown that in general such variations of Slepian's lemma can not be extended from Gaussian to p-stable processes.

In [21] we expressed the results referred to prior to Theorem B in terms of the non-decreasing rearrangement of the metric d. Theorems A, B and C can be stated in an analogous fashion. Note that $d_X(s,t)$ given in (0.1) satisfies $d_X(s,t)=d_X(0,t-s)$. Let $\sigma_X(t-s)=d_X(0,t-s)$ and consider $\sigma_X(u)$, $u\in K+K$, where $K+K=\{s+t: s\in K, t\in K\}$. Let μ denote Haar measure on G, normalized so that $\mu(K+K)=1$. For $\varepsilon>0$ let

$$\mu_{\sigma_X}(\varepsilon)=\mu(\{x\in K+K:\sigma_X(x)<\varepsilon\}) \tag{0.10}$$

and

$$\overline{\sigma_X(u)} = \sup\{\varepsilon > 0 : \mu_{\sigma_X}(\varepsilon) < u\}. \tag{0.11}$$

The function $\overline{\sigma_X(u)}$ is non-decreasing on $[0, 1]$ and is called the non-decreasing rearrangement of $\sigma_X(u)$. By (3.38) there exist constants $b(p)$ depending only on p and $D_p(K)$ depending only on p and K such that for $1 \leq p \leq 2$

$$\begin{aligned} D_p(K)^{-1}[m(\Gamma)^{1/p} + J_q(d_X)] &\leq m(\Gamma)^{1/p} + I_p(\sigma_X) \\ &\leq D_p(K)\,[m(\Gamma)^{1/p} + J_q(d_X)] \end{aligned} \tag{0.12}$$

where we define

$$I_p(\sigma_X) = \int_0^1 \frac{\overline{\sigma_X(u)}}{u\left(\log \dfrac{b(p)}{u}\right)^{1/p}}\, du, \tag{0.13}$$

and set $J_q(d_X) = J_\infty(d_X)$ when $p = 1$. Furthermore, when G is compact and $G = K$ then $D_p(G)$ can be taken independent of G and when $(X(t))_{t \in K}$ is a random Fourier series, $m(\Gamma) = \Sigma_{\gamma \in \Gamma} |a_\gamma|^p$. Therefore, (0.3), (0.8) and (0.9) can be written with $I_p(\sigma_X)$ replacing $J_q(d_X)$.

In Section 1 we clarify what we mean by strongly stationary p-stable random processes and give a very useful representation for p-stable processes which was first shown to us by the authors of [17]. Section 2 is devoted to necessary conditons for continuity. Lemma 2.1 enables us to take all known necessary conditions for the a.s. continuity of Gaussian processes and obtain related necessary conditions for the a.s. continuity of p-stable processes. Not only do we extend Fernique's result, as we have mentioned in Theorems A, B and C, but we can also extend Sudakov's result which applies to non-stationary processes. As an application of the methods developed in Section 2 we show, in Theorem 2.12, that a contraction from a finite subset of L^p into a Hilbert space has an extension with a relatively small norm to a mapping from L^p to H.

In Section 3 we consider sufficient conditions for a.s. continuity. Theorem 3.3 is a rather surprising result about the weak l^p norm of sequences of independent random variables that seems to be of independent interest. It is used in Corollary 3.5 to obtain a generalization of Daniels' theorem on the empirical distribution function to the case when the random variables are not identically distributed, (see also Remark 3.6). In Section 4 we indicate how the results mentioned in this introduction can be obtained from the results of Sections 1, 2 and 3. Finally, in Section 5 we apply these results to harmonic analysis following [25], [26] and Chapter 6 of [21].

1. Representations of stable processes

A real valued random variable θ will be called p-stable of parameter σ if $\forall t \in \mathbf{R}$

$$E \exp i\theta t = \exp -\sigma^p |t|^p \tag{1.1}$$

and a complex valued random variable $\tilde{\theta}$ will be called p-stable of parameter σ if for all $z \in \mathbf{C}$

$$\begin{aligned} E \exp i \operatorname{Re}(\bar{z}\tilde{\theta}) &= E \exp i (\operatorname{Re} zU + \operatorname{Im} zV) \\ &= \exp -\sigma^p |z|^p \end{aligned} \tag{1.2}$$

where we write $\tilde{\theta} = U + iV$, U, V real. By definition these variables are symmetric and in what follows we will only consider symmetric p-stable random variables and refer to them simply as p-stable. (In general stable random variables need not be symmetric cf. [9], Chapter XVII, § 4.)

It is well known that a real valued p-stable random variable can be written as a product of two independent random variables one of which is Gaussian cf. [9], Chapter VI, § 2h. This observation plays an important rôle in our work and also clarifies the relationship between real and complex valued p-stable random variables.

LEMMA 1.1. *Let $\theta(\tilde{\theta})$ be a real (resp. complex) valued p-stable random variable of parameter σ and let g, g' be independent normal random variables with mean zero and variance σ^2. There exists a positive random variable $\eta(p)$ independent of g and g' such that*

$$\theta \overset{\mathcal{D}}{=} \eta(p) \cdot g \tag{1.3}$$

$$\tilde{\theta} \overset{\mathcal{D}}{=} \eta(p)(g + ig') \tag{1.4}$$

where "$\overset{\mathcal{D}}{=}$" denotes "equal in distribution".

Proof. As is well known for each $0 < p \leq 1$ the function $\lambda \mapsto e^{-\lambda^p}$ is completely monotone on $\mathbf{R}^+$. Therefore there exists a random variable which we will denote by $\nu(p)$ such that

$$E \exp -\lambda \nu(p) = \exp -\lambda^p, \quad \forall \lambda \geq 0. \tag{1.5}$$

For $0 < p \leq 2$ we take $\eta(p) = (2\nu(p/2))^{1/2}$. It is easy to check, by taking Fourier transforms, that (1.3) and (1.4) hold.

Note that $\eta(2)=\sqrt{2}$ so in the case $p=2$ we get the standard definition of a real and complex valued normal random variable. However, only for $p=2$, are the real and imaginary parts of $\tilde{\theta}$ independent.

We recall that for $0<p<2$ a real valued p-stable random variable of parameter σ satisfies

$$\lim_{\lambda\to\infty} \lambda^p P\{|\theta|>\lambda\} = (c(p)\sigma)^p \quad \text{where} \quad c(p)^{-1} = \left[\int_0^\infty \frac{\sin v}{v^p}\,dv\right]^{-1/p}. \tag{1.6}$$

(This result is contained in [9], Chapter XVII, § 4, however in this special case one can show directly that

$$\begin{aligned}\sigma^p &= \lim_{t\to 0} t^{-p}(1-\varphi(t)) = \lim_{t\to 0} -2t^{-p}\int_0^\infty (1-\cos\lambda t)\,d(1-F(\lambda))\\ &= \lim_{t\to 0} 2\int_0^\infty \sin v[t^{-p}(1-F(v/t))]\,dv = (c(p)\sigma)^p \int_0^\infty \frac{\sin v}{v^p}\,dv\end{aligned}$$

where φ is the characteristic function and F the distribution function of θ.)

It also follows that for $\tilde{\theta}$ a complex valued p-stable random variable of parameter σ

$$\lim_{\lambda\to\infty} \lambda^p P(|\tilde{\theta}|>\lambda) = (c(p)\,r(p)\,\sigma)^p \tag{1.7}$$

where $r(p)=(E|g^2+g'^2|^{p/2}/E|g|^p)^{1/p}$ and g, g' are i.i.d. real valued mean zero normal random variables.

It follows from (1.1) and (1.2) that if $\theta_1,\dots,\theta_n$ are i.i.d. p-stable real (resp. complex) valued random variables, and if $\lambda_1,\dots,\lambda_n$ are real (resp. complex) coefficients, then

$$\sum_{i=1}^n \theta_i\lambda_i \overset{\mathcal{D}}{=} \theta_1\left(\sum_{i=1}^n |\lambda_i|^p\right)^{1/p}. \tag{1.8}$$

Since by (1.6) and (1.7) we know that $E|\theta_1|^r<\infty$ for each $r<p$ we have

$$E\left|\sum_{i=1}^n \theta_i\lambda_i\right|^r = (E|\theta_1|^r)\left(\sum_{i=1}^n |\lambda_i|^p\right)^{r/p} \tag{1.9}$$

and

$$\sup_{\lambda>0} \lambda^p P\left\{\left|\sum_{i=1}^n \theta_i\lambda_i\right|>\lambda\right\} = \sum_{i=1}^n |\lambda_i|^p \sup_{\lambda>0}\lambda^p P\{|\theta_1|>\lambda\}. \tag{1.10}$$

Moreover, if θ is a real (resp. complex) valued p-stable random variable of parameter σ, then

$$(E|\theta|^r)^{1/r} = \sigma\delta(r,p) \quad (= \sigma\delta'(r,p)) \tag{1.11}$$

where $\delta(r,p)$, $(\delta'(r,p))$ depends only on r and p.

Let T be a set. We will denote by $\mathbf{R}^{(T)}$ (resp. $\mathbf{C}^{(T)}$) the space of all finitely supported families $(\alpha(t))_{t\in T}$ of real (resp. complex) numbers. Let $0<p\leq 2$, we will say that a real (resp. complex) stochastic process is p-stable if there exists a positive measure m on $\mathbf{R}^T$ (resp. $\mathbf{C}^T$) equipped with the cylindrical σ-algebra such that $\forall \alpha \in \mathbf{R}^{(T)}$

$$E \exp i \sum_{t\in T} \alpha(t) X(t) = \exp - \int \left| \sum_{t\in T} \alpha(t)\beta(t)\right|^p dm(\beta), \tag{1.12}$$

(resp. $\forall \alpha \in \mathbf{C}^{(T)}$)

$$E \exp i \operatorname{Re}\left[\sum_{t\in T} \overline{\alpha(t)}\, X(t)\right] = \exp - \int \left|\sum_{t\in T} \overline{\alpha(t)}\, \beta(t)\right|^p dm(\beta). \tag{1.13}$$

Any measure m as above will be called a spectral measure of the process $(X(t))_{t\in T}$. Clearly if $(X(t))_{t\in T}$ is a p-stable process, as defined above, then $\forall \alpha \in \mathbf{R}^{(T)}$ (resp. $\mathbf{C}^{(T)}$) $\Sigma_{t\in T}\alpha(t)X(t)$ is a real (resp. complex) valued p-stable random variable. This is, perhaps, the more usual definition of p-stable processes.

Now let T be a locally compact Abelian group G with dual group Γ. Γ is called the character group of G, i.e. $\gamma\in\Gamma$ is a continuous complex valued function such that $\forall s$, $t\in G$, $|\gamma(t)|=1$ and $\gamma(s)\gamma(t)=\gamma(s+t)$. A real (resp. complex) valued p-stable process $(X(t))_{t\in G}$ will be called *strongly stationary* if it admits a representation as in (1.12) (resp. (1.13)) where the spectral measure m is a finite positive Radon measure supported on Γ.

A strongly stationary process is stationary, since for $(X(t))_{t\in T}$ real and $t_1,\dots,t_n\in T$ and $t_1+s,\dots,t_n+s\in T$, $E \exp i\, \Sigma_{j=1}^n \alpha_j X(t_j+s) = E\exp i\,\Sigma_{j=1}^n \alpha_j X(t_j)$. This follows from (1.12). In the complex case we say that $(X(t))_{t\in T}$ is stationary if for every $t_1,\dots,t_n\in T$ the $2n$ dimensional real sequence $(\operatorname{Re}X(t_1), \operatorname{Im}X(t_1),\dots,\operatorname{Re}X(t_n), \operatorname{Im}X(t_n))$ is stationary. By (1.13) we see that in the complex case also a strongly stationary p-stable process is stationary. Conversely a stationary Gaussian process is strongly stationary but we will show later in this section that there are stationary p-stable processes which are not strongly stationary.

Let $(X(t))_{t\in T}$ be a real (resp. complex) valued p-stable process. It follows from (1.12) (resp. (1.13)) that

$$X(s)-X(t) \overset{\mathscr{D}}{=} \left(\int |\beta(s)-\beta(t)|^p \, dm(\beta)\right)^{1/p} \theta \tag{1.14}$$

(resp. (1.14) with $\tilde{\theta}$ replacing θ) where θ is a real (resp. $\tilde{\theta}$ is a complex) valued p-stable random variable of parameter 1. We define a pseudo-metric for these processes by

$$d_X(s,t) = \left(\int |\beta(s)-\beta(t)|^p \, dm(\beta)\right)^{1/p}, \quad \forall s,\ t\in T. \tag{1.15}$$

Clearly, $d_X(s,t)$ is a pseudo-metric and by (1.14) and (1.11)

$$(E|X(s)-X(t)|^r)^{1/r} = \delta(r,p)\, d_X(s,t) \quad (\text{resp. } \delta'(r,p)\, d_X(s,t)) \tag{1.15'}$$

for $0<r<p$. Note that in both the real and complex case $d_X(s,t)$ is the parameter of $X(s)-X(t)$.

We shall now give some examples of strongly stationary processes. Let $(\tilde{\theta}_\gamma)_{\gamma\in\Gamma}$ be an i.i.d. collection of complex valued p-stable random variables with parameter equal to 1. Let $(a_\gamma)_{\gamma\in\Gamma}$ be a family of real or complex coefficients satisfying $\Sigma_{\gamma\in\Gamma}|a_\gamma|^p<\infty$. The process

$$X(t) = \sum_{\gamma\in\Gamma} a_\gamma\, \tilde{\theta}_\gamma\, \gamma(t), \quad t\in G \tag{1.16}$$

is a strongly stationary, complex valued, p-stable random process on G with spectral measure

$$m = \sum_{\gamma\in\Gamma} |a_\gamma|^p\, \delta_\gamma, \tag{1.17}$$

where δ_γ is the unit point mass at $\gamma\in\Gamma$. For this process we have

$$d_X(s,t) = \left(\sum_{\gamma\in\Gamma} |a_\gamma|^p\, |\gamma(t-s)-1|^p\right)^{1/p}. \tag{1.18}$$

The process $\operatorname{Re} X(t)$ is a strongly stationary real valued p-stable process. One can also use (1.12) directly to see that for $(a_\gamma)_{\gamma\in\Gamma}$ real

$$Y(t) = \sum a_\gamma\, \eta_\gamma (g_\gamma \operatorname{Re}\gamma + g'_\gamma \operatorname{Im}\gamma) \tag{1.19}$$

where $\{\eta_\gamma\}$ are i.i.d. copies of η given in Lemma 1.1 and $\{g_\gamma\}$, $\{g'_\gamma\}$ are i.i.d. (and independent of each other) normal random variables with mean zero and variance 1. In this case the spectral measure m and pseudo-metric $d_X(s,t)$ are the same as in (1.17) and (1.18). For $p=2$, $X(t)$ and $Y(t)$ agree with the definitions of complex and real valued Gaussian random Fourier series (except for a constant multiple) considered in [21]. Also, as in the Gaussian case, one can obtain strongly stationary p-stable random processes with continuous spectral measure m by the usual procedure of approximating by integrals with respect to step functions and passing to the limit. In the complex case such processes are of the form

$$Z=\int_\Gamma \gamma M(d\gamma) \tag{1.20}$$

where M is an independently scattered complex valued random p-stable measure. As in the discrete case the real and imaginary parts of M are not independent. Indeed, for $A\in\Gamma$,

$$M(A)\stackrel{\mathscr{D}}{=}\eta(p)\,(G(A)+iG'(A))$$

where $G(A)$ and $G'(A)$ are independent Gaussian random variables with mean zero and variance $m(A)^{2/p}$. For such processes

$$d_Z(s,t)=\left(\int|\gamma(s)-\gamma(t)|^p m(d\gamma)\right)^{1/p}. \tag{1.21}$$

In the real case a process similar to (1.20) can be obtained as an extension of (1.19).

Lemma 1.1 can be extended to processes. To do this we need to introduce some facts on sums of i.i.d. exponentially distributed random variables. Let X be a positive real valued random variable satisfying $P(X>\lambda)=e^{-\lambda}$. Let $\{X_k\}$ be i.i.d. copies of X and define

$$\Gamma_j=X_1+X_2+\ldots+X_j. \tag{1.22}$$

By [9], p. 10

$$P[\Gamma_j<\lambda]=\int_0^\lambda \frac{x^{j-1}}{(j-1)!}e^{-x}\,dx \tag{1.23}$$

and for $r<pj$

$$E[(\Gamma_j)^{-r/p}]=\frac{\Gamma(j-r/p)}{\Gamma(j)}\sim j^{-r/p}. \tag{1.24}$$

The next lemma will be used frequently in what follows.

LEMMA 1.2. *Let* $p>1$, *then*

$$\frac{c_1}{p-1} \leqslant \alpha_p = E \sup_{j\geqslant 1} \left(\frac{j}{\Gamma_j}\right)^{1/p} \leqslant \frac{c_2}{p-1} \tag{1.25}$$

where c_1 *and* c_2 *are constants independent of* p.

Proof. Let $v\geqslant 2$. Then by (1.23) and the fact that $j^j/e^j j!\leqslant 1$ we have

$$P\left[\Gamma_j < \frac{j}{ev^p}\right] \leqslant \int_0^{j/ev^p} \frac{x^{j-1}}{(j-1)!}\,dx \leqslant v^{-pj}.$$

Thus

$$\begin{aligned} E\sup_{j\leqslant 1}\left(\frac{j}{\Gamma_j}\right)^{1/p} &\leqslant 2e^{1/p}+\sum_{j=1}^{\infty}\int_{2e^{1/p}}^{\infty} P\left[\left(\frac{j}{\Gamma_j}\right)^{1/p}>\lambda\right]d\lambda \\ &= e^{1/p}\left[2+\sum_{j=1}^{\infty}\int_2^{\infty}\frac{1}{v^{pj}}\,dv\right] \leqslant e^{1/p}\,\frac{2p}{p-1}, \end{aligned} \tag{1.26}$$

where we substitute $\lambda=e^{1/p}v$ in the integral in (1.26). To obtain the lower bound we use

$$E(\Gamma_j)^{-1/p} \geqslant \int_0^j \frac{x^{j-1-1/p}}{(j-1)!}e^{-x}\,dx \geqslant e^{-j}\int_0^j \frac{x^{j-1-1/p}}{(j-1)!}\,dx.$$

As an alternate proof of the right side of (1.25) one can show by martingale arguments that $P(\sup_j (j/\Gamma_j)>c)\leqslant 1/c$.

The next lemma relates the Γ_j to the order statistics of the uniform distribution. For a proof see [4], Proposition 13.15.

LEMMA 1.3. *Let* $\{U_j\}_{j=1,\dots,n}$ *be i.i.d. copies of a random variable uniformly distributed on* $[0,1]$. *Let* $\{U_j^*\}_{j=1,\dots,n}$ *denote the non-decreasing rearrangement of* $\{U_j\}_{j=1,\dots,n}$. *Then for* Γ_j *defined in* (1.22) *we have*

$$\left\{\frac{\Gamma_j}{\Gamma_{n+1}}\right\}_{j=1,\dots,n} \overset{\mathcal{D}}{=} \{U_j^*\}_{j=1,\dots,n}.$$

Our object is to show that p-stable processes can be represented as mixtures of Gaussian processes. To do this we use a representation of p-stable processes which was pointed out to us by the authors of [17]. The next lemma is implied in [17] and is included in [18], in which there are additional references and a discussion of the history of this result. The simple proof that we give was shown to us by J. Zinn.

LEMMA 1.4. *Let v be a symmetric real valued random variable with $E|v|^p<\infty$ and let $\{v_j\}$ be i.i.d. copies of v. Then for $0<p<2$*

$$X=\sum_{j=1}^{\infty}(\Gamma_j)^{-1/p}v_j \tag{1.27}$$

is a symmetric p-stable random variable and

$$\lim_{\lambda\to\infty}\lambda^p P(|X|>\lambda)=E|v|^p. \tag{1.28}$$

If $0<p<1$ and $v\geq 0$ then the expression in (1.27) is a positive p-stable random variable and (1.28) remains valid.

Proof. We first consider the case when v is symmetric, $0<p<2$. Let $\xi>0$ be a random variable satisfying

$$G(t)=P(\xi>t)=t^{-p},\quad t\geq 1. \tag{1.29}$$

Let $\{\xi_j\}$ be i.i.d. copies of ξ such that $\{\xi_j\}$ and $\{v_j\}$ are independent of each other. Consider

$$\sum_{j=1}^{n}\xi_j v_j=\sum_{j=1}^{n}G^{-1}(G(\xi_j))v_j\stackrel{\mathscr{D}}{=}\sum_{j=1}^{n}G^{-1}(U_j)v_j \tag{1.30}$$

where $\{U_j\}_{j=1,\dots,n}$ are as given in Lemma 1.3. It is well known and easy to check that because $\{U_j\}$ and $\{v_j\}$ are independent and $\{v_j\}$ is i.i.d.,

$$\sum_{j=1}^{n}G^{-1}(U_j)v_j\stackrel{\mathscr{D}}{=}\sum_{j=1}^{n}G^{-1}(U_j^*)v_j$$

where $\{U_j^*\}_{j=1,\dots,n}$ is defined in Lemma 1.3. Using Lemma 1.3, (1.29) and (1.30) we get

$$n^{-1/p}\sum_{j=1}^{n}\xi_j v_j\stackrel{\mathscr{D}}{=}\left(\frac{\Gamma_{n+1}}{n}\right)^{1/p}\sum_{j=1}^{n}(\Gamma_j)^{-1/p}v_j. \tag{1.31}$$

We also have by (1.29) that

$$\lim_{t\to\infty}t^pP(|\xi v|>t)=E|v|^p;$$

therefore it follows from [9], Chapter XVII, § 5 that ξv is in the domain of attraction of a p-stable random variable θ (using the norming constants $n^{1/p}$). Furthermore by the argument used to obtain $c(p)$ following (1.6) we see that θ has parameter $\beta(p)=(E|v|^p/c(p))^{1/p}$. Since $\lim_{n\to\infty}(\Gamma_{n+1}/n)^{1/p}=1$ a.s. by the strong law of large numbers, we get from (1.31) that (1.27) is also equal to θ in distribution. Actually (1.27) converges a.s. To see this fix each realization of $\{\Gamma_j\}$ and use the Three Series Theorem on the resulting sum of independent random variables.

The same proof works when $v\geq 0$ since, for $0<p<1$, (1.31) remains valid.

We proceed to develop the representation of p-stable processes. Let $(X(t))_{t\in T}$ be a real (resp. complex) valued p-stable stochastic process admitting (see (1.12) and (1.13)) a finite spectral measure m. Let M^p be the total mass of m (i.e. $M\equiv m(\mathbf{R}^T)^{1/p}$ (resp. $M\equiv m(\mathbf{C}^T)^{1/p}$) and let ν be a renormalization of m so that ν is a probability measure, i.e. $\nu=M^{-p}m$. Let $\{Y_j\}$ be a sequence of i.i.d. $\mathbf{R}^T$ (resp. $\mathbf{C}^T$) valued random variables with probability distribution ν. Let $\{\Gamma_j\}$ be as defined in (1.22). Let $\{\varepsilon_j\}$ be a Rademacher sequence (i.e. an i.i.d. sequence of symmetric random variables each one taking on the values ± 1); let $\{g_j\}$ be an i.i.d. sequence of mean zero Gaussian real (resp. complex) valued random variables normalized so that

$$E|g_j|^p = 1, \quad [\text{resp. } E|\operatorname{Re} g_j|^p = 1], \tag{1.32}$$

and let $\{w_j\}$ be equal to $\{\alpha\, e^{i\omega_j}\}$ where $\{\omega_j\}$ are i.i.d. random variables each one uniformly distributed on $[0,2\pi]$ and $\alpha=((2\pi)^{-1}\int_0^{2\pi}|\cos u|^p\,du)^{-1/p}$, i.e. $\{w_j\}$ is a normalized Steinhaus sequence. We assume that all the sequences $\{Y_j\}$, $\{\Gamma_j\}$, $\{\varepsilon_j\}$, $\{g_j\}$, and $\{w_j\}$ are independent of the others.

PROPOSITION 1.5. *Let $(X(t))_{t\in T}$ be a real (resp. complex) valued p-stable process as defined above. Then for* $0<p<2$

$$V(t) = c(p)\,M\sum_{j=1}^{\infty}(\Gamma_j)^{-1/p}g_j\,Y_j(t), \tag{1.33}$$

where $\{g_j\}$ is real (resp. complex), is equal in distribution to $(X(t))_{t\in T}$. If $(X(t))_{t\in T}$ is complex then for $0<p<2$

$$W(t) = c(p)\,M\sum_{j=1}^{\infty}(\Gamma_j)^{-1/p}w_j\,Y_j(t), \quad t\in T, \tag{1.34}$$

is equal in distribution to $(X(t))_{t\in T}$. If $(X(t))_{t\in T}$ is real then for $0<p<2$

$$Z(t) = c(p)\,M \sum_{j=1}^{\infty} (\Gamma_j)^{-1/p} \varepsilon_j\, Y_j(t), \quad t \in T, \tag{1.34'}$$

is equal in distribution to $(X(t))_{t\in T}$.

Proof. The proof follows immediately from Lemma 1.4. Let $(X(t))_{t\in T}$ be real. Consider (1.33) with $\{g_j\}$ real. By hypothesis, for $t_1, \ldots, t_n \in T$, and $\alpha_1, \ldots, \alpha_n$ real

$$E \exp i \sum_{j=1}^{n} \alpha_j X(t_j) = \exp - \int \left| \sum_{j=1}^{n} \alpha_j \beta(t_j) \right|^p dm(\beta).$$

By the proof of Lemma 1.4 we have that $\Sigma_{j=1}^n \alpha_j V(t_j)$ is a real valued p-stable random variable with parameter $[M^p E|\Sigma_{j=1}^n \alpha_j Y(t_j)|^p]^{1/p}$. Therefore we have by (1.1)

$$\begin{aligned} E \exp i \sum_{j=1}^{n} \alpha_j V(t_j) &= \exp - M^p E \left| \sum_{j=1}^{n} \alpha_j Y(t_j) \right|^p \\ &= \exp - \int \left| \sum_{j=1}^{n} \alpha_j Y(t_j) \right|^p dm(Y). \end{aligned}$$

Therefore the finite joint distributions of $(X(t))_{t\in T}$ and $(V(t))_{t\in T}$ agree and this is what is meant by saying that the two processes agree in distribution. The proof for $(W(t))_{t\in T}$ and for $(V(t))_{t\in T}$ in the complex case is entirely similar since $E|\mathrm{Re}\, w_1|^p = E|\mathrm{Re}\, g_1|^p = 1$ for g_1 complex.

We use Proposition 1.5 to justify the following lemma which has probably been observed elsewhere.

LEMMA 1.6. (a) *Let T be an index set. Let $(X(t))_{t\in T}$ be a p-stable real (resp. complex) valued random process,* $0<p<2$, *admitting a finite spectral measure m. Then we can find probability spaces* $(\Omega, \mathscr{A}, P)$ and $(\Omega', \mathscr{A}', P')$ *and a real (resp. complex) valued stochastic process $(\mathscr{X}(t))_{t\in T}$ defined on* $(\Omega, \mathscr{A}, P)\times(\Omega', \mathscr{A}', P')$ *such that:*

(i) *The processes $(\mathscr{X}(t))_{t\in T}$ and $(X(t))_{t\in T}$ have the same distribution.*

(ii) *For each fixed $\omega \in \Omega$, the random process $(\mathscr{X}(t; \omega, \cdot))_{t\in T}$ is a real (resp. complex) valued Gaussian process.*

(b) *Moreover, if T is a locally compact Abelian group G and if $(X(t))_{t\in T}$ is strongly stationary, then we can find $(\mathscr{X}(t))_{t\in T}$ as above verifying the additional condition: For each fixed $\omega \in \Omega$, the process $(\mathscr{X}(t; \omega, \cdot))_{t\in T}$ is a stationary real (resp. complex) valued Gaussian process.*

In other words any p-stable process (resp. any strongly stationary p-stable process) with a finite spectral measure has a version which is a "mixture" of Gaussian (resp. stationary Gaussian) processes.

Proof. This all follows immediately from Proposition 1.5. Let $\{\Gamma_j\}$, $\{Y_j\}$ and $\{g_j\}$ be as in Proposition 1.5. We can assume that $\{Y_j\}$ and $\{\Gamma_j\}$ are defined on a probability space $(\Omega, \mathcal{A}, P)$ and that $\{g_j\}$ is defined on a probability space $(\Omega', \mathcal{A}', P')$. By Proposition 1.5 the process

$$\mathscr{X}(t, \omega, \omega') = c(p)M\sum_{j=1}^{\infty}(\Gamma_j(\omega))^{-1/p}\,Y_j(t, \omega)\,g_j(\omega') \tag{1.35}$$

satisfies (a) parts (i) and (ii). Furthermore, if $(X(t))_{t\in T}$ is strongly stationary then $\nu=M^{-p}m$ (see remarks preceding Proposition 1.5) is concentrated on the characters of G. Therefore, for ω fixed, $\{Y_j(t, \omega)\}$ is a sequence of characters on G so that $(\mathscr{X}(t; \omega, \cdot))_{t\in T}$ is stationary for each fixed ω. (Actually, $\mathscr{X}(t; \omega, \cdot)$ is a Gaussian random Fourier series of the form (1.16) with $p=2$.) This settles both the real and complex case. Note however that in (b) if $(X(t))_{t\in T}$ is a real valued strongly stationary p-stable process we still use complex valued normal variables $\{g_j\}$ and replace (1.35) by

$$\mathscr{X}(t; \omega, \omega') = c(p)M\sum_{j=1}^{\infty}(\Gamma_j(\omega))^{-1/p}\,\mathrm{Re}\,Y_j(t, \omega)\,g_j(\omega'). \tag{1.36}$$

With this definition it is clear that, for ω fixed, $(\mathscr{X}(t; \omega, \cdot))_{t\in T}$ is a stationary Gaussian process of the form (1.9) with $p=2$.

Remark 1.7. We will show that $J_q(d_X)<\infty$ is not a sufficient condition for continuity of stationary p-stable processes. In the process of doing this we will exhibit stationary p-stable processes which are not strongly stationary. For simplicity we will consider the real case. The complex case is entirely similar. It is clear from (1.12) that a real valued p-stable process $(X(t))_{t\in T}$ is stationary iff the measure m is stationary in L^p, i.e. iff $\int|\Sigma_{t\in T}\,\overline{\alpha(t)}\,\beta(t+\tau)|^p\,dm(\beta)$ does not depend on τ. (Incidentally, by (1.15) and (1.15′) applied to $\Sigma_{t\in T}\alpha(t)X(t+\tau)$ instead of $X(t)-X(s)$ we see that for $0<p<2$, a p-stable process $(X(t))_{t\in T}$ is stationary iff it is stationary in L^q for some (equivalently all) $q<p$. Compare this to the fact that a Gaussian process is stationary iff it is stationary in L^2.)

We shall consider the following example. Let G be a compact group and let $(Y(t))_{t\in T}$ be a stationary, mean zero, Gaussian process on G. Let $(Y_j(t))_{t\in T}$ be i.i.d. copies of Y and consider, for $1<p<2$,

$$X(t)=c(p)\sum_{j=1}^{\infty}(\Gamma_j)^{-1/p}Y_j(t),\quad t\in G. \tag{1.37}$$

Thus $X(t)$ is a p-stable process (it is precisely of the form (1.34′)) and it is stationary since the expression $\int|\Sigma_{j=1}^{n}\alpha_j\beta(t_j+\tau)|^p\,dm(\beta)$, in this case, is exactly $E|\Sigma_{j=1}^{n}\alpha_j\,Y(t_j+\tau)|^p$. Let $\|\ \|_\infty$ denote the sup-norm on G. It follows from Lemma 1.2 and (1.24) that

$$c_p'E\|Y\|_\infty\leqslant E\|X\|_\infty\leqslant c_pE\|Y\|_\infty, \tag{1.38}$$

for constants c_p and $c_p'>0$. By (1.15)

$$d_X(s,t)=(E|Y(s)-Y(t)|^p)^{1/p}=\delta_p(E|Y(s)-Y(t)|^2)^{1/2}\equiv d_Y(s,t)$$

where the middle equality is a well known property of Gaussian random variables. This enables us to see that $(X(t))_{t\in G}$, although stationary, is not strongly stationary, since by the Dudley–Fernique theorem $(Y(t))_{t\in G}$ has a version which is a.s. bounded (or continuous) iff

$$\int_0^{\infty}(\log N(G,d_Y;\varepsilon))^{1/2}\,d\varepsilon<\infty \tag{1.39}$$

whereas by Theorem A, if $(X(t))_{t\in G}$ were strongly stationary then it would have a continuous version iff

$$\int_0^{\infty}(\log N(G,d_X;\varepsilon))^{1/q}\,d\varepsilon<\infty \tag{1.40}$$

where $1/q+1/p=1$. It is easy to see using Remark 4.3 and by § 1, Chapter VII, [21] that one can take for Y certain random Fourier series with decreasing coefficients such that (1.40) holds but (1.39) does not. This shows two things. That $(X(t))_{t\in G}$ constructed as in (1.37) with this Y is not strongly stationary and that $J_q(d_X)<\infty$ is not a sufficient condition for a stationary p-stable process to have a version with continuous paths.

Finally, as another application of the representations of stable processes given in Lemma 1.4 and Proposition 1.5 we give an alternate proof of Lemma 1.1 (in the real case). By Lemma 1.4 a real valued p-stable random variable, $0<p<2$, can be represented as

$$\sum_{j=1}^{\infty}(\Gamma_j)^{-1/p}g_j\overset{\mathcal{D}}{=}\left(\sum_{j=1}^{\infty}(\Gamma_j)^{-2/p}\right)^{1/2}g \tag{1.41}$$

where $\{g_j\}$ are i.i.d. copies of g as given in Lemma 1.1. Furthermore, by checking the Laplace transforms of (1.41) we see that

$$\nu(p/2) \overset{\mathscr{D}}{=} \lambda_p' \sum_{j=1}^{\infty} (\Gamma_j)^{-2/p}$$

for some constant λ_p' depending only on p.

Remark 1.8. In some cases we can obtain Lemma 1.6 directly without having to appeal to the representation given in Proposition 1.5. Let $\theta_1, \ldots, \theta_n$ be i.i.d. real valued p-stable random variables with parameter 1 and let $x_1, \ldots, x_n$ be in $\mathbf{R}^T$. Then the process

$$X(t) = \sum_{i=1}^{n} \theta_i\, x_i(t), \quad t \in T \tag{1.43}$$

is clearly p-stable with spectral measure

$$m = \sum_{i=1}^{n} \delta_{x_i}.$$

Now let $(\eta_1, \ldots, \eta_n)$ be i.i.d. random variables each with the same distribution as $\eta(p)$ (see Lemma 1.1) and let $(g_1, \ldots, g_n)$ be i.i.d. real Gaussian random variables where g_1 has variance 1. We may as well assume that $(\eta_1, \ldots, \eta_n)$ is defined on some space $(\Omega, \mathscr{A}, P)$ while $(g_1, \ldots, g_n)$ is defined on $(\Omega', \mathscr{A}', P')$. Then the process

$$\mathscr{X}(t; \omega, \omega') = \sum_{i=1}^{n} g_i(\omega')\, \eta_i(\omega)\, x_i(t), \quad t \in T$$

has the same distribution as $(X(t))_{t \in T}$ since by (1.2) or (1.5), $(\theta_i)_{i \leqslant n}$ has the same distribution as $(\eta_i g_i)_{i \leqslant n}$. The complex case is entirely similar except that we use (1.4).

This same remark applies to infinite sums of the form (1.43) provided we know that the sum converges a.s. for each $t \in T$. Therefore, it applies to random Fourier series such as (1.16) and, in fact, we have already shown this for real p-stable random Fourier series in (1.19).

We will use the following notation with regard to metric entropy. Let (T, d) be a complex space equipped with a pseudo-metric d. We will denote by $N(T, d; \varepsilon)$ the smallest number of open balls of radius ε in the pseudo-metric d which covers T. We introduce the function $\sigma(T, d; n)$ which is defined, for each integer n, by

$$\sigma(T,d;n)=\inf\{\varepsilon>0\,|\,N(T,d;\varepsilon)\leqslant n\}. \tag{1.44}$$

Let p, q be such that $1<p$, $q<\infty$ and $1/p+1/q=1$. It is easy to check that

$$\int_0^\infty (\log N(T,d;\varepsilon))^{1/q}\,d\varepsilon<\infty \quad \text{iff} \quad \sum_{n=2}^\infty \frac{\sigma(T,d;n)}{n(\log n)^{1/p}}<\infty. \tag{1.45}$$

Actually, a simple calculation shows that

$$\begin{aligned}\int_0^\infty (\log N(T,d;\varepsilon))^{1/q}\,d\varepsilon &= \sum_{n=1}^\infty (\log n)^{1/q}[\sigma(T,d;n-1)-\sigma(T,d;n)]\\ &= \sum_{n=1}^\infty \sigma(T,d;n)\,[(\log(n+1))^{1/q}-(\log n)^{1/q}].\end{aligned}$$

Therefore, there exist constants $a_p>0$ and b_p depending only on p such that if $1<p<\infty$

$$\begin{aligned}a_p\sum_{n=1}^\infty \frac{\sigma(T,d;n)}{n(\log(n+1))^{1/p}} &\leqslant \int_0^\infty (\log N(T,d;\varepsilon))^{1/q}\,d\varepsilon\\ &\leqslant b_p\sum_{n=1}^\infty \frac{\sigma(T,d;n)}{n(\log(n+1))^{1/p}}.\end{aligned} \tag{1.46}$$

Similarly, it follows that

$$\int_0^\infty \log^+\log N(T,d;\varepsilon)\,d\varepsilon<\infty \quad \text{iff} \quad \sum_{n=1}^\infty \frac{\sigma(T,d;n)}{n\log(n+1)}<\infty \tag{1.47}$$

and an inequality such as (1.46) can also be obtained. This result is used in the case $p=1$. We will use the following notation

$$J_q(d)=\int_0^\infty (\log N(T,d;\varepsilon))^{1/q}\log N(T,d;\varepsilon))^{1/q}\,d\varepsilon, \quad 2\leqslant q<\infty, \tag{1.48}$$

$$J_\infty(d)=\int_0^\infty \log^+\log N(T,d;\varepsilon)\,d\varepsilon. \tag{1.49}$$

Remark 1.9. In (1.47) if d is translation invariant, i.e. s, $t\in T$ and $s+\tau$, $t+\tau\in T$ imply $d(s+\tau,t+\tau)=d(s,t)$, the function $\sigma(T,d;n)$ is equivalent to $\overline{\sigma(1/n)}$ where $\overline{\sigma(u)}$ is the non-decreasing rearrangement of $d(t+u,t)$. This function plays a major rôle in [21]. In this paper we find it useful to consider $\sigma(T,d;n)$ since we also obtain results when d is not translation invariant.

2. Lower bounds for p-stable processes

Let $(\Omega, \mathscr{A}, P)$ and $(\Omega', \mathscr{A}', P')$ be probability spaces and let $(\mathscr{X}(t))_{t\in T}$ be a p-stable real or complex valued stochastic process on $(\Omega, \mathscr{A}, P)\times(\Omega', \mathscr{A}', P')$ satisfying (ii) in Lemma 1.6.

For all s, t in T we denote $d(s, t)$ the parameter of the p-stable process $\mathscr{X}(t)-\mathscr{X}(s)$, so that we have (in the real case)

$$E\exp i\lambda(\mathscr{X}(t)-\mathscr{X}(s)) = \exp -d^p(s, t)\,|\lambda|^p, \quad \forall\lambda\in\mathbf{R}, \tag{2.1}$$

and also, by (1.15′), $\forall r<p$

$$(E|\mathscr{X}(t)-\mathscr{X}(s)|^r)^{1/r} = \delta(r, p)\, d(s, t)$$

where $\delta(r, p)>0$ depends only on r and p. This shows that d is a pseudo-metric on T.

We also introduce the "random pseudo-metric" d_ω, defined for each $\omega\in\Omega$ as follows: for each s and t in T, we will denote by $d_\omega(t, s)$ the parameter of the Gaussian variable $\mathscr{X}(t;\omega,\cdot)-\mathscr{X}(s;\omega,\cdot)$. In the real (resp. complex) case this is simply

$$d_\omega(s, t) = (\tfrac{1}{2}E_{\omega'}|\mathscr{X}(t)-\mathscr{X}(s)|^2)^{1/2},$$

$$(\text{resp. } d_\omega(s, t) = \tfrac{1}{2}(E_{\omega'}|\mathscr{X}(t)-\mathscr{X}(s)|^2)^{1/2}).$$

Now, since $(\mathscr{X}(t;\omega,\cdot))_{t\in T}$ is a Gaussian process we have, in the real case,

$$E_{\omega'}\exp i\lambda(\mathscr{X}(t;\omega,\omega')-\mathscr{X}(s;\omega,\omega')) = \exp -|\lambda|^2 d_\omega^2(s, t), \quad \forall\lambda\in\mathbf{R}. \tag{2.2}$$

To simplify notation we write

$$\sigma(n) = \sigma(T, d; n)$$

and

$$\sigma_\omega(n) = \sigma(T, d_\omega; n)$$

(see (1.44)).

The next lemma is the crucial result of this section.

LEMMA 2.1. *There is a subset $\Omega_0\subset\Omega$ with $P(\Omega_0)>1/2$, such that for each $\omega\in\Omega_0$, we have:*

$$\sigma_\omega(n) \geq \beta(p)\frac{\sigma(n)}{(\log(n+1))^{1/p-1/2}}, \quad \forall n\geq 1, \tag{2.3}$$

where $\beta(p)>0$ is a constant depending only on p and $0<p<2$.

The proof of Lemma 2.1 is based on the following simple estimate:

LEMMA 2.2. *With the above notation we have for each s and t in T and all $\varepsilon>0$,*

$$P\{\omega\in\Omega: d_\omega(s,t)\leqslant \varepsilon d(s,t)\}\leqslant \exp-\frac{2}{\alpha}\left(\frac{p}{2}\right)^{\alpha/p}\frac{1}{\varepsilon^\alpha}, \tag{2.4}$$

where $\alpha=2p/(2-p)$, $0<p<2$, so that $1/\alpha=1/p-1/2$.

Proof of Lemma 2.2. Let $A=\{\omega\in\Omega: d_\omega(s,t)\leqslant\varepsilon d(s,t)\}$. By (2.1) and (2.2) we have for each $\lambda\in\mathbf{R}$

$$E_\omega \exp-\lambda^2 d_\omega^2(s,t)=\exp-|\lambda|^p d^p(s,t).$$

Setting $x=\lambda^2 d^2(s,t)$, this is

$$E_\omega \exp-x\left(\frac{d_\omega^2(s,t)}{d^2(s,t)}\right)=\exp-x^{p/2},\quad x\geqslant 0. \tag{2.5}$$

Therefore by an exponential Chebysev inequality applied to $x(d_\omega^2(s,t)/d^2(s,t))$ we get

$$P(A)\leqslant \inf_{x\geqslant 0}\exp-(x^{p/2}-\varepsilon^2 x).$$

This expression is minimized by $x=(p/2\varepsilon^2)^{2/(2-p)}$ and we get (2.4). The same proof works in the complex case when (2.1) and (2.2) are suitably modified.

Remark 2.3. Note that for a p-stable process we see by (2.5) that $(d_\omega(s,t)/d(s,t))^2\overset{\mathcal{D}}{=}\nu(p/2)$ where $\nu(p/2)$ was defined in Lemma 1.1. The distribution function of $\nu(p/2)$ is known, see [9] Chapter XVII, § 6.

Proof of Lemma 2.1. Fix an integer n and let $\delta>0$ be such that $\sigma(n)>\delta$. This means that

$$N(T,d;\delta)>n. \tag{2.6}$$

We claim that there exist at least $n+1$ elements $t_1,\dots,t_{n+1}$ in T such that

$$d(t_i,t_k)\geqslant\delta,\quad \forall 1\leqslant i\neq k\leqslant n+1. \tag{2.7}$$

To see this let $M(\delta)$ be the maximal number of points say $t_1,\dots,t_{M(\delta)}$ in T such that $d(t_i,t_k)\geqslant\delta$, $1\leqslant i\neq k\leqslant M(\delta)$, (these points are not necessarily unique). Then by the maximality of $M(\delta)$, T is covered by $M(\delta)$ open balls of radius δ centered at $t_1,\dots,t_{M(\delta)}$. Thus $N(T,d;\delta)\leqslant M(\delta)$ and so by (2.6) we get $M(\delta)\geqslant n+1$.

By Lemma 2.2 we know that there is a constant K depending only on p such that for all $1 \leqslant i \neq k \leqslant n+1$ and all $\varepsilon > 0$,

$$P\{\omega \in \Omega \,|\, d_\omega(t_i, t_k) \leqslant \varepsilon d(t_i, t_k)\} \leqslant \exp - \frac{K}{\varepsilon^\alpha}. \tag{2.8}$$

This clearly implies by (2.7) that for all $1 \leqslant i \neq k \leqslant n+1$ and all $\varepsilon > 0$,

$$P\{\omega \in \Omega \,|\, d_\omega(t_i, t_k) \leqslant \varepsilon d\} \leqslant \exp - \frac{K}{\varepsilon^\alpha}.$$

Obviously, this implies

$$P\left\{\omega \in \Omega \,\middle|\, \inf_{1 \leqslant i \neq k \leqslant n+1} d_\omega(t_i, t_k) < \varepsilon d\right\} \leqslant n^2 \exp - \frac{K}{\varepsilon^\alpha}. \tag{2.9}$$

Suppose there exists an ω such that $\sigma_\omega(n) < \varepsilon\delta/2$. This means that the set T can be covered by n open balls of radius $\varepsilon\delta/2$ in the d_ω pseudo-metric. Obviously, we can find two distinct points amongst $\{t_1, \ldots, t_{n+1}\}$ lying in the same ball of radius $\varepsilon\delta/2$. Therefore,

$$\inf_{1 \leqslant i \neq k \leqslant n+1} d_\omega(t_i, t_k) < \varepsilon\delta\,.$$

Thus we have the inclusion

$$\left\{\omega \,\middle|\, \sigma_\omega(n) < \frac{\varepsilon\delta}{2}\right\} \subset \left\{\omega \,\middle|\, \inf_{1 \leqslant i \neq k \leqslant n+1} d_\omega(t_i, t_k) < \varepsilon\delta\right\},$$

and so, by (2.9)

$$P\left\{\omega \,\middle|\, \sigma_\omega(n) < \frac{\varepsilon\delta}{2}\right\} \leqslant n^2 \exp - \frac{K}{\varepsilon^\alpha},$$

and finally, since $\delta < \sigma(n)$ is arbitrary,

$$P\left\{\omega \,\middle|\, \sigma_\omega(n) < \frac{\varepsilon\sigma(n)}{2}\right\} \leqslant n^2 \exp - \frac{K}{\varepsilon^\alpha}. \tag{2.10}$$

By (2.10) we have that for each $\beta > 0$ and each integer $n \geqslant 1$

$$\begin{aligned} P\left\{\omega \,\middle|\, \sigma_\omega(n) < \frac{\beta\sigma(n)}{2(\log(n+1))^{1/\alpha}}\right\} &\leqslant n^2 \exp - \frac{K}{\beta^\alpha} \log(n+1) \\ &\leqslant n^2 (n+1)^{-K/\beta^\alpha}. \end{aligned}$$

Therefore,

$$P\left(\bigcup_{n=1}^{\infty}\left\{\omega \,\middle|\, \sigma_\omega(n)<\frac{\beta\sigma(n)}{2(\log(n+1))^{1/\alpha}}\right\}\right)\leqslant\sum_{n=2}^{\infty}n^{2-K/\beta^\alpha} \tag{2.11}$$

and it is clear that we can find a $\beta>0$, depending only on p, so that the right side of (2.11) is less than 1/2. This completes the proof of Lemma 2.1 since we can take

$$\Omega_0=\bigcap_{n=1}^{\infty}\left\{\omega \,\middle|\, \sigma_\omega(n)\geqslant\frac{\beta\sigma(n)}{2(\log(n+1))^{1/\alpha}}\right\}$$

and $\beta(p)=\beta/2$.

We will also use the following variant of Lemma 2.1:

LEMMA 2.4. *In the above notation let $t_1,\dots,t_n$ be arbitrary points in T. There exists a subset $\Omega_1\subset\Omega$ with $P(\Omega_1)>1/2$ such that for all $\omega\in\Omega_1$ and all $i, k\in\{1,\dots,n\}$,*

$$d_\omega(t_i,t_k)\geqslant\gamma(p)\frac{d(t_i,t_k)}{(\log n)^{1/\alpha}}, \tag{2.12}$$

where $\gamma(p)>0$ is a constant depending only on p. (If $n=1$ we will consider both sides of (2.12) *equal to zero.)*

Proof. By Lemma 2.2 we have that for all $\varepsilon>0$

$$P\left(\bigcup_{i,k\leqslant n}\{\omega|d_\omega(t_i,t_k)<\varepsilon d(t_i,t_k)\}\right)\leqslant n^2\exp-\frac{K}{\varepsilon^\alpha},$$

where K is a constant depending only on p. Therefore we can find a $\gamma>0$ sufficiently small so that

$$n^2\exp\left(-\frac{K}{\gamma^\alpha}(\log n)\right)<\frac{1}{2}$$

for all $n\geqslant2$ and we get

$$P\left(\bigcup_{1\leqslant i,k\leqslant n}\left\{\omega \,\middle|\, d_\omega(t_i,t_k)\leqslant\frac{\gamma d(t_i,t_k)}{(\log n)^{1/\alpha}}\right\}\right)<\frac{1}{2}.$$

This proves the lemma since we can choose such a γ depending only on p.

Our first application of the preceding lemmas is a generalization of Slepian's lemma to the p-stable case. A number of people have observed that Slepian's lemma does not go over directly to p-stable processes (although we remarked following

Theorem C that it does for strongly stationary p-stable processes). Nevertheless, we can obtain interesting generalizations by comparing general p-stable processes with Gaussian processes.

THEOREM 2.5. *Let T be a finite set of cardinality n, and let $(X(t))_{t\in T}$ and $(Y(t))_{t\in T}$ be two stochastic processes, real or complex, such that $(X(t))_{t\in T}$ is p-stable, $0<p\leq 2$ and $(Y(t))_{t\in T}$ is a Gaussian process (i.e. $p=2$). For each s, $t\in T$, we denote by $d_Y(s,t)$ and $d_X(s,t)$ the parameters of $Y(s)-Y(t)$ and $X(s)-X(t)$ respectively. Assume,*

$$d_Y(s,t)\leq d_X(s,t), \quad \forall s,\ t\in T. \tag{2.13}$$

Then, for each $r<p$, there exists a constant $B_{p,r}$ depending only on r and p such that

$$\left(E\sup_{s,t\in T}|Y(s)-Y(t)|^r\right)^{1/r}\leq B_{p,r}(\log n)^{1/p-1/2}\left(E\sup_{s,t\in T}|X(s)-X(t)|^r\right)^{1/r}. \tag{2.14}$$

In the real case, if $1<p\leq 2$, we also have

$$E\sup_{t\in T}Y(t)\leq B_{p,1}(\log n)^{1/p-1/2}E\sup_{t\in T}X(t). \tag{2.15}$$

Furthermore when $p=2$ (2.14) holds for $0\leq r\leq 2$.

Proof. Recall that the Sudakov version of Slepian's lemma (see [10], [1]) is simply (2.15) with $p=2$ and $B_{2,1}=1$. Our proof is based on this result. In the real case, by symmetry, we have

$$E\sup_{t\in T}Y(t)=\tfrac{1}{2}E\sup_{s,t\in T}|Y(s)-Y(t)| \tag{2.16}$$

and similarly for $(X(t))_{t\in T}$. By a well known result of Fernique and Landau and Shepp (see e.g. [10]) we know that for any Gaussian process $(Y(t))_{t\in T}$ and any $0<r<\infty$,

$$\begin{aligned}\frac{1}{\Delta_r}E\sup_{s,t\in T}|Y(t)-Y(s)| &\leq \left(E\sup_{s,t\in T}|Y(t)-Y(s)|^r\right)^{1/r}\\ &\leq \Delta_r E\sup_{s,t\in T}|Y(t)-Y(s)|\end{aligned} \tag{2.17}$$

where $\Delta_r>0$ is a constant depending only on r. Therefore, if $(Y(t))_{t\in T}$ and $(X(t))_{t\in T}$ are two Gaussian processes satisfying (2.13) we can use (2.15), (2.16) and (2.17) to obtain

$$\left(E\sup_{s,t\in T}|Y(t)-Y(s)|^r\right)^{1/r}\leq \Delta_r^2\left(E\sup_{s,t\in T}|X(t)-X(s)|^r\right)^{1/r}. \tag{2.18}$$

We will now prove Theorem 2.5 in the real case, the complex case follows by looking at the real and imaginary parts separately. We use Lemma 1.6 (a) and assume that $(X(t))_{t\in T}$ is defined on $(\Omega, \mathscr{A}, P)\times(\Omega', \mathscr{A}', P')$ and that for $\omega\in\Omega$ fixed, the process $(X(t;\omega,\cdot))_{t\in T}$ is a Gaussian process on $(\Omega', \mathscr{A}', P')$. Denote by $d_\omega(s,t)$ the parameter of the Gaussian variable $X(t;\omega,\cdot)-X(s;\omega,\cdot)$. Then by Lemma 2.4 we can find a set $\Omega_1\subset\Omega$ with $P(\Omega_1)>1/2$ such that for all ω in Ω_1 we have

$$d_\omega(s,t)\geq\gamma(p)(\log n)^{-1/\alpha}d_X(s,t),\quad \forall s,\ t\in T.$$

By (2.13) we have that for all ω in Ω_1

$$d_\omega(s,t)\geq\gamma(p)(\log n)^{-1/\alpha}d_Y(s,t),\quad \forall s,\ t\in T.$$

Now, since both processes $(Y(t))_{t\in T}$ and $(X(t;\omega,\cdot))_{t\in T}$ are Gaussian processes we apply (2.18) to obtain

$$\left(E\sup_{s,t\in T}|Y(t)-Y(s)|^r\right)^{1/r}\leq(\log n)^{1/\alpha}\gamma(p)^{-1}\Delta_r^2(E_{\omega'}|X(t;\omega,\cdot)-X(s;\omega,\cdot)|^r)^{1/r},\quad (2.19)$$

for each $\omega\in\Omega_1$. We raise both sides of (2.19) to the rth power, take the expectation with respect to Ω_1 and then take the rth root to obtain

$$\left(\frac{1}{2}E\sup_{s,t\in T}|Y(t)-Y(s)|^r\right)^{1/r}\leq(\log n)^{1/\alpha}\gamma(p)^{-1}\Delta_r^2\left(E\sup_{s,t\in T}|X(t)-X(s)|^r\right)^{1/r}.$$

This is exactly (2.14) with $B_{p,r}=2^{1/r}\gamma(p)^{-1}\Delta_r^2$. (In the complex case use $2B_{p,r}$.) Inequality (2.15) is obtained by using (2.16) in (2.14) with $r=1$.

The preceding result was used in the Gaussian case by Sudakov to prove the next statement in the particular case $p=2$. For $p<2$, Theorem 2.6 is the first non-trivial necessary condition for the a.s. continuity or boundedness of a general p-stable process of which we are aware.

THEOREM 2.6. *Let T be a compact metric space and let $(X(t))_{t\in T}$ be a p-stable stochastic process, real or complex, with $0<p\leq 2$. Let d_X be as above and let q be the conjugate of p, i.e. $1/q+1/p=1$.*

(i) *If $(X(t))_{t\in T}$ has a.s. continuous sample paths then, necessarily*

$$\lim_{\varepsilon\downarrow 0}\varepsilon(\log N(T,d_X;\varepsilon))^{1/q}=0.\quad (2.20)$$

(ii) *If $(X(t))_{t\in T}$ has a.s. bounded sample paths then necessarily*

$$\sup_{\varepsilon>0} \varepsilon(\log N(T, d_X; \varepsilon))^{1/q} < \infty; \tag{2.21}$$

moreover, for each $r<p$

$$\sup_{\varepsilon>0} \varepsilon(\log N(T, d_X; \varepsilon))^{1/q} \leq c_{p,r} \cdot \left(E \sup_{s,t\in T} |X(t)-X(s)|^r\right)^{1/r} \tag{2.22}$$

where $c_{p,r}$ is a constant depending only on p and r.

Proof. As in the proof of Theorem 2.5 we use Lemma 1.6(a) and assume that $(X(t))_{t\in T}$ is defined on $(\Omega, \mathscr{A}, P)\times(\Omega', \mathscr{A}', P')$ and that for each $\omega \in \Omega$, $(X(t;\omega, \cdot))_{t\in T}$ is a Gaussian process. By Fubini's theorem if $(X(t))_{t\in T}$ has a.s. continuous sample paths then there exists a set $\Omega_1 \subset \Omega$ with $P(\Omega_1)=1$ such that for $\omega \in \Omega_1$ the process $(X(t;\omega, \cdot))_{t\in T}$ is an a.s. continuous Gaussian process with respect to $(\Omega', \mathscr{A}', P')$. We apply Sudakov's theorem, i.e. (2.20) and (2.21) with $p=q=2$ (cf. [10]), to this process and obtain that for $\omega \in \Omega_1$,

$$\lim_{\varepsilon \downarrow 0} \varepsilon(\log N(T, d_\omega; \varepsilon))^{1/2} = 0$$

or, equivalently

$$\lim_{n\to\infty} \sigma_\omega(n)(\log n)^{1/2} = 0.$$

Using Lemma 2.1 we get

$$\lim_{n\to\infty} \sigma(n)(\log n)^{1/q} = 0$$

or, equivalently (2.20).

The argument for the first part of (ii) is similar to the above argument. It remains to prove (2.22). This is known in the Gaussian case, ([10], p. 83). For any Gaussian process $(Y(t))_{t\in T}$ we have

$$\sup_{\varepsilon>0} \varepsilon(\log N(T, d_Y; \varepsilon))^{1/2} \leq CE \sup_{s,t\in T} |Y(s)-Y(t)|$$

where C is an absolute constant. We apply this inequality, for each $\omega \in \Omega$, to the Gaussian process $(X(t;\omega, \cdot))_{t\in T}$ and obtain

$$\begin{aligned}\sup_{\varepsilon>0} \varepsilon(\log N(T, \sigma_\omega; \varepsilon))^{1/2} &\leq CE_{\omega'} \sup_{s,t\in T} |X(t;\omega,\omega')-X(s;\omega,\omega')| \\ &\leq C\Delta_r\left(E_{\omega'} \sup_{s,t\in T} |X(t;\omega,\omega')-X(s;\omega,\omega')|^r\right)^{1/r}\end{aligned}$$

where Δ_r is given in (2.17). Equivalently, this is

$$\sup_{n>0} \sigma_\omega(n)(\log n)^{1/2} \leq C\Delta_r \Big(E_{\omega'} \sup_{s,t\in T} |X(t;\omega,\omega')-X(s;\omega,\omega')|^r \Big)^{1/r}.$$

Finally, we have by Lemma 2.1

$$E_\omega E_{\omega'} \sup_{s,t\in T} |X(t;\omega,\omega')-X(s;\omega,\omega')|^r \geq \Big(\frac{1}{C\Delta_r}\Big)^r \int_{\Omega_1} dP(\omega) \Big(\sup_{n>0} \sigma_\omega(n)(\log n)^{1/2}\Big)^r$$
$$\geq \frac{1}{2}\Big(\frac{\beta(p)}{C\Delta_r} \sup_{n>0} \sigma(n)(\log n)^{1/q}\Big)^r,$$

and this immediately implies (2.22).

COROLLARY 2.7. *Let $(\theta_1, \ldots, \theta_n)$ be a p-stable sequence in $\mathbf{R}^n$ (i.e. $(\theta(t))_{t\in T}$ is a p-stable stochastic process with $T=\{1,2,\ldots,n\}$). Then for $r<p$*

$$E \sup_j |\theta_j| \geq D_{r,p} \inf_{1\leq i \neq k\leq n} (E|\theta_i-\theta_k|^r)^{1/r} (\log n)^{1/q} \tag{2.23}$$

where $D_{r,p}$ is a constant depending only on p and r.

Proof. We use (2.22). By (1.15′) the pseudo-metric for this process, can be expressed as

$$d(i,k) = \delta(r,p)^{-1} (E|\theta_i-\theta_k|^r)^{1/r}.$$

Now, if $d(i,k)\geq\delta$ then $N(T,d;\delta/2)>n$. Therefore by (2.22) and (2.16)

$$E\sup_j |\theta_j| \geq (c_{p,r})^{-1} (\log n)^{1/q} \delta.$$

The result follows since we can take $\delta = \inf_{1\leq i\neq k\leq n} d(i,k)$.

Remark 2.8. The lower bound in (2.23) is best possible to a constant multiple. Let $1<p<2$ and let $(\bar\varepsilon_1, \ldots, \bar\varepsilon_n)$ be i.i.d. symmetric Bernoulli random variables each one taking on the values $\pm 2^{-1/q}$, $(1/p+1/q=1)$. Let $\{(\bar\varepsilon_{j1}, \ldots, \bar\varepsilon_{jn})\}_{j=1}^\infty$ be i.i.d. copies of $(\bar\varepsilon_1, \ldots, \bar\varepsilon_n)$. Consider

$$(\theta_1, \ldots, \theta_n) = \delta(r,p)^{-1} \sum_{j=1}^\infty (\Gamma_j)^{-1/p} (\bar\varepsilon_{j1}, \ldots, \bar\varepsilon_{jn}).$$

It follows by Proposition 1.5 that $(\theta_1, \ldots, \theta_n)$ is a p-stable sequence in $\mathbf{R}^n$. By (1.15'), $\forall i \neq j = 1, \ldots, n$

$$(E|\theta_i - \theta_r|^r)^{1/r} = (E|\varepsilon_i - \varepsilon_j|^p)^{1/p} = 1$$

and by Lemma 1.2 and (1.24)

$$E \sup_{1 \leq k \leq n} |\theta_k| \sim E \sup_{1 \leq k \leq n} \left| \sum_{j=1}^{\infty} \frac{r_{jk}}{j^{1/p}} \right| \sim (\log n)^{1/q}$$

where $\{r_{jk}\}_{j,k=1}^{\infty}$ are independent Rademacher random variables and the final estimate follows from the fact that

$$P\left[\sum_{j=1}^{\infty} \frac{r_{jk}}{j^{1/p}} > \lambda \right] \sim e^{-c\lambda^q}$$

for λ sufficiently large and c a constant. (See Lemma 3.1 below.)

On the other hand if $(\theta_1, \ldots, \theta_n)$ are i.i.d. p-stable random variables, normalized so that $(E|\theta_i - \theta_j|^r)^{1/r} = 1$, it is elementary to check that

$$E \sup_{1 \leq k \leq n} |\theta_k| \sim n^{1/p}.$$

In this case (2.23) is very weak.

The next theorem is an extension of Fernique's lower bound for a.s. continuous stationary Gaussian processes to strongly stationary p-stable processes, $1 \leq p < 2$.

THEOREM 2.9. *Let G be a locally compact Abelian group with dual group Γ and let $K \subset G$ be a fixed compact neighborhood of 0 in G. Let $(X(t))_{t \in G}$ be a strongly stationary p-stable random process with associated pseudo-metric d_X. We assume that d_X is continuous on $G \times G$.*

Let $1 < p \leq 2$. If $(X(t))_{t \in G}$ has a.s. locally bounded paths, or equivalently, if $(X(t))_{t \in K}$ has a.s. bounded paths, then necessarily

$$J_q(d_X) = \int_0^{\infty} (\log N(K, d_X; \varepsilon))^{1/q} \, d\varepsilon < \infty, \tag{2.24}$$

where $1/p + 1/q = 1$. Moreover, for each $r < p$, we have

$$J_q(d_X) \leq F_{p,r}(K) \left(E \sup_{s,t \in K} |X(t) - X(s)|^r \right)^{1/r} \tag{2.25}$$

where $F_{p,r}(K)$ is a constant depending only on p, r and K.

In the case $p=1$, if $(X(t))_{t\in K}$ has a.s. bounded paths then necessarily

$$J_\infty(d_X)=\int_0^\infty \log^+\log N(K,d_X;\varepsilon)\,d\varepsilon<\infty. \tag{2.26}$$

Moreover, for each $r<1$ we have

$$J_\infty(d_X)\leq F_{1,r}(K)\Big(E\sup_{s,t\in K}|X(t)-X(s)|^r\Big)^{1/r} \tag{2.27}$$

where $F_{1,r}(K)$ is a constant depending only on r and K.

Proof. In the case $q=p=2$, (2.25) is due to Fernique. Assume now that $1\leq p<2$ and consider $(X(t))_{t\in G}$ as given in the hypothesis of this theorem. By Lemma 1.6(b) we can consider that $(X(t))_{t\in G}$ is defined on a product space $(\Omega,\mathscr{A},P)\times(\Omega',\mathscr{A}',P')$ and that for each fixed $\omega\in\Omega$ the process $(X(t;\omega,\cdot))_{t\in G}$ is a stationary Gaussian process. Continuing the above notation, we have by Fernique's lower bound for stationary Gaussian processes (cf. [10] p. 83, [9] p. 48), there is a constant β_K such that

$$\begin{aligned} a_2\sum_{n=1}^\infty \frac{\sigma_\omega(n)}{n(\log(n+1))^{1/2}} &\leq \beta_K E_{\omega'}\sup_{s,t\in K}|X(t;\omega,\omega')-X(s;\omega,\omega')| \\ &\leq \beta_K\,\Delta_r\Big(E_{\omega'}\sup_{s,t\in K}|X(t;\omega,\omega')-X(s;\omega,\omega')|^r\Big)^{1/r} \end{aligned}$$

where we also use (1.46) and (2.17). We now apply Lemma 2.1 to the process $(X(t;\omega,\omega'))_{t\in K}$. We can find a set $\Omega_1\subset\Omega$ with $P(\Omega_1)>1/2$ so that for $\omega\in\Omega_1$

$$a_2\beta(p)\sum_{n=1}^\infty \frac{\sigma(n)}{n(\log(n+1))^{1/p}}\leq \beta_K\,\Delta_r\Big(E_{\omega'}\sup_{s,t\in K}|X(t;\omega,\omega')-X(s;\omega,\omega')|^r\Big)^{1/r}. \tag{2.28}$$

If we raise (2.28) to the rth power and integrate over Ω_1 we obtain

$$a_2\beta(p)\sum_{n=1}^\infty \frac{\sigma(n)}{n(\log(n+1))^{1/p}}\leq \beta_K\,\Delta_r\,2^{1/r}\Big(E\sup_{t,s\in K}|X(t)-X(s)|^r\Big)^{1/r}.$$

Using (1.46) and the comment immediately following (1.47) we get (2.25) and (2.27). The fact that (2.24) and (2.26) hold is implicit in the above proof. Fernique's result implies that

$$\sum_{n=1}^\infty \frac{\sigma_\omega(n)}{n(\log(n+1))^{1/2}}<\infty \quad \text{a.s. } \omega;$$

hence we can find an $\omega \in \Omega_1$ for which this holds. Then Lemma 2.1 implies that

$$\sum_{n=1}^{\infty} \frac{\sigma(n)}{n(\log(n+1))^{1/p}} < \infty$$

and this is equivalent to (2.24) (or (2.6) when $p=1$) by (1.45) and (1.47).

Remark 2.10. We recall that the Fernique–Landau–Shepp result on the integrability of semi-norms of Gaussian random variables has an extension to the p-stable case. In particular, let $(X(t))_{t\in T}$ (T is any set) be an a.s. bounded p-stable process. Then

$$\sup_{c>0} c^p P\left(\sup_{s,t\in T} |X(t)-X(s)| > c\right) < \infty$$

and, a fortiori

$$E \sup_{s,t\in T} |X(t)-X(s)|^r < \infty, \quad \forall r<p.$$

Moreover, there is a constant $\Delta(r,p)$ depending only on r and p such that if M is either $\sup_{s,t\in T}|X(t)-X(s)|$ or $\sup_{t\in T}X(t)$ we have for each $r<p$,

$$\left(\sup_{c>0} c^p P\{M>c\}\right)^{1/p} \leqslant \Delta(r,p)(EM^r)^{1/r}. \tag{2.29}$$

We refer the reader to [2] for details.

Finally we note that when G is compact we can improve Theorem 2.9.

COROLLARY 2.11. *Let G be a compact group with the Haar measure of G equal to 1. Then under the hypotheses of Theorem* 2.7

$$\int_0^\infty (\log N(G, d_X; \varepsilon))^{1/q} d\varepsilon \leqslant F_{p,r} E\left(\sup_{s,t\in G} |X(t)-X(s)|^r\right)^{1/r}$$

and

$$\int_0^\infty \log^+ \log N(G, d_X; \varepsilon)\, d\varepsilon \leqslant F_{1,r} E\left(\sup_{s,t\in G} |X(t)-X(s)|^r\right)^{1/r}$$

where $F_{p,r}$ and $F_{1,r}$ are constants depending only on p and r and not on G.

Proof. This is an immediate consequence of the proof of Theorem 2.9 and the corresponding result in the Gaussian case (cf. [21], p. 11).

The last result of this section, an application of these methods to functional analysis, concerns extensions of nonlinear contractions on finite subsets of L^p.

THEOREM 2.12. *Assume* $1<p\leq 2$. *Let* (S,Σ,μ) *be a measure space and* T *a finite subset of* $L^p(S,\Sigma,\mu)$ *of cardinality* n. *Let* H *be a Hilbert space and let* $\Phi\colon T\to H$ *be a contraction, i.e.*

$$\|\Phi(s)-\Phi(t)\|_H \leq \|s-t\|_{L^p(\mu)}, \quad \forall s,\ t\in T. \tag{2.30}$$

Then there exists an extension $\tilde{\Phi}\colon L^p(\mu)\to H$ *of* Φ *such that*

$$\|\tilde{\Phi}(s)-\tilde{\Phi}(t)\|_H \leq c_p\,(\log n)^{1/\alpha}\|s-t\|_{L^p(\mu)}, \quad \forall s,\ t\in L^p(\mu), \tag{2.31}$$

where $1/\alpha=1/p-1/2$ *and* c_p *is a constant depending only on* p.

Proof. By classical results (cf. e.g. [5]) there exists a p-stable process $(X(t))_{t\in L^p(\mu)}$ such that the associated metric $d_X(s,t)$ satisfies

$$d_X(s,t)=\|s-t\|_{L^p(\mu)}, \quad \forall s,\ t\in L^p(\mu).$$

Using Proposition 1.5 it is easy to show this. Without loss of generality we may assume that μ is a probability measure on S. We define for $t\in L^p(\mu)$,

$$X(t)=c(p)\,\sigma\sum_{j=1}^{\infty} g_j(\Gamma_j)^{-1/p}u_j(t)$$

where $u_j(t)=t(u_j)$ and $\{u_j\}$ are i.i.d. S valued random variables with $P(u_j\in A)=\mu(A)$, $\forall A\in\Sigma$. The remaining terms are the same as in (1.34). It follows from (1.15) that, with $u(s)=s(u)$, $\forall s\in L^p(\mu)$,

$$d_X(s,t)=\left(\int |u(s)-u(t)|^p\,d\mu(u)\right)^{1/p}=\|s-t\|_{L^p(\mu)}.$$

By Lemma 1.6 we can assume that $(X(t))_{t\in L^p(\mu)}$ is defined on $(\Omega,\mathcal{A},P)\times(\Omega',\mathcal{A}',P')$ and that for each $\omega\in\Omega$, $(X(t;\omega,\cdot))_{t\in L^p(\mu)}$ is a Gaussian process on $(\Omega',\mathcal{A}',P')$. Let $d_\omega(s,t)$ be the parameter of $X(s;\omega,\cdot)-X(t,\omega,\cdot)$, i.e.,

$$d_\omega(s,t)=(aE_{\omega'}|X(s;\omega,\cdot)-X(t;\omega,\cdot)|^2)^{1/2}, \tag{2.32}$$

where $a=1/2$ in the real case and $a=1/4$ in the complex case.

By Lemma 2.4 there is a set $\Omega_1 \subset \Omega$ with $P(\Omega_1) > 1/2$ such that, for each $\omega \in \Omega_1$, we have

$$\|s-t\|_{L^p(\mu)} \leq \gamma(p)^{-1} \log n^{1/\alpha}\, d_\omega(s,t), \quad \forall s,\ t \in T,$$

and, consequently, by (2.30)

$$\|\Phi(s)-\Phi(t)\|_H \leq \gamma(p)^{-1} (\log n)^{1/\alpha}\, d_\omega(s,t), \quad \forall s,\ t \in T. \tag{2.33}$$

For each $\omega \in \Omega_1$ and $t \in T$ we define the function

$$\Phi_\omega(X(t;\omega,\cdot)) = \Phi(t) \tag{2.34}$$

and observe that by (2.33) and (2.32)

$$\begin{aligned}\|\Phi_\omega(X(s;\omega,\cdot)) - \Phi_\omega(X(t;\omega,\cdot))\|_H &= \|\Phi(s)-\Phi(t)\|_H \\ &\leq \gamma(p)^{-1}(\log n)^{1/\alpha} \|X(s;\omega,\cdot) - X(t;\omega,\cdot)\|_{L^2(dP')}.\end{aligned}$$

Thus Φ_ω is a contraction on a subset of the Hilbert space $L^2(\Omega', P', \mathscr{A}')$ with values in a Hilbert space H. By a well known result (cf. [30], Theorem 11.3) Φ_ω can be extended to a contraction on the whole space, i.e. $\Phi_\omega: L^2(\Omega', \mathscr{A}', P') \to H$ such that

$$\|\Phi_\omega(u) - \Phi_\omega(v)\|_H \leq \gamma(p)^{-1} (\log n)^{1/\alpha} \|u-v\|_{L^2(dP)}, \quad \forall u,\ v \in L^2(\Omega', \mathscr{A}', P').$$

Therefore, we have by (2.32)

$$\|\Phi_\omega(X(s;\omega,\cdot)) - \Phi_\omega(X(t;\omega,\cdot))\|_H \leq \gamma(p)^{-1} (\log n)^{1/\alpha}\, d_\omega(s,t), \quad \forall s,\ t \in L^p(\mu).$$

We define

$$\tilde{\Phi}(t) = P(\Omega_1)^{-1} \int_{\Omega_1} \Phi_\omega(X(t;\omega,\cdot))\, dP(\omega),$$

so that we have, by (2.34) that $\tilde{\Phi}(t) = \Phi(t)$, $\forall t \in T$ and

$$\|\tilde{\Phi}(t) - \tilde{\Phi}(s)\|_H \leq \gamma(p)^{-1} (\log n)^{1/\alpha} \tfrac{1}{2} E_\omega\, d_\omega(s,t)$$

for all $s,\ t \in L^p(\mu)$. Furthermore by (1.15′)

$$\begin{aligned} d_X(s,t) &= \delta(1,p)^{-1} E_\omega\, E_{\omega'} |X(s) - X(t)| \\ &= \delta(1,p)^{-1} E_\omega [\delta(1,2)^{-1} d_\omega(s,t)] = c_p\, E_\omega\, d_\omega(s,t), \end{aligned}$$

where $c_p > 0$ is a constant depending only on p.

Therefore, we finally have

$$\|\tilde{\Phi}(s)-\tilde{\Phi}(t)\|_H \leq \gamma(p)^{-1}(\log n)^{1/\alpha}(2c_p)^{-1}d_X(s,t), \quad \forall s,\ t\in L^p(\mu),$$

which gives the desired result.

3. A sufficient condition for a.s. continuity

In this section we show that the "entropy condition" (i.e. $J_q(d_X)<\infty$) is sufficient for the a.s. continuity of the sample paths of a strongly stationary p-stable process, $1<p<2$. The idea of the proof is different from that of Section 2. In Section 2 we considered p-stable processes as mixtures of Gaussian processes. Here we will consider them as mixtures of certain Rademacher series which have tail behavior on the order of magnitude $\exp-\delta c^q$ as $c\to\infty$, ($\delta>0$, $1/p+1/q=1$). Since $q>2$ such series are "better behaved" than Gaussian variables which, of course, have tail behavior on the order of $\exp-\delta c^2$ as $c\to\infty$. It seems to us that it is necessary to use Rademacher rather than Gaussian series in what follows.

We will need several preliminary results which are of independent interest. We begin by recalling some standard notation. Let $0<p<\infty$ and let I be an index set. We denote by $l_{p,\infty}(I)$ the space of all families $(\alpha_i)_{i\in I}$ of complex numbers such that

$$\sup_{t>0} t^p \operatorname{card}\{i\in I: |\alpha_i|>t\}<\infty$$

and we define

$$\|(\alpha_i)_{i\in I}\|_{p,\infty} = \left(\sup_{t>0} t^p \operatorname{card}\{i\in I: |\alpha_i|>t\}\right)^{1/p}.$$

It is well known that for $p>1$, the functional $\|\cdot\|_{p,\infty}$ is equivalent to a norm on $l_{p,\infty}(I)$ with which $l_{p,\infty}(I)$ is a Banach space. The space $l_{p,\infty}(\mathbf{N})$ is often referred to as "weak l_p", ($\mathbf{N}$ denotes the integers).

For any family $(\alpha_i)_{i\in I}$ of complex numbers tending to zero at infinity, we can define a sequence $(\alpha_n^*)_{n\in\mathbf{N}}$ which is the non-increasing rearrangement of $(|\alpha_i|)_{i\in I}$. It is well known and easy to check that

$$\|(\alpha_i)_{i\in I}\|_{p,\infty} = \sup_{n\geq 1} n^{1/p}\alpha_n^*. \tag{3.1}$$

We note for further use that, obviously,

$$|\alpha_i|\leq|\beta_i|, \quad \forall i\in I, \quad \text{implies} \quad \|(\alpha_i)\|_{p,\infty}\leq\|(\beta_i)\|_{p,\infty}. \tag{3.2}$$

Let $\varphi: \mathbf{R}_+ \to \mathbf{R}_+$ be an increasing convex function with $\varphi(0)=0$. For any probability space $(\Omega, \mathscr{A}, P)$ we denote by $L^{\varphi}(dP)$ the so called "Orlicz space" formed by all the measurable functions $f: \Omega \to \mathbf{C}$ for which there is a $c>0$ such that

$$\int \varphi\left(\frac{|f|}{c}\right) dP < \infty.$$

We equip this space with the norm

$$\|f\|_{\varphi} = \inf\left\{c > 0 : E\varphi\left(\frac{|f|}{c}\right) \leq 1\right\}.$$

Throughout this paper we will denote by ψ_q the function

$$\psi_q(x) = \exp|x|^q - 1.$$

The Orlicz space $L^{\psi_q}(dP)$ will be used repeatedly.

Let $\{\varepsilon_i\}_{i\in I}$ be a Rademacher sequence on some probability space. It is easy to check that for $p<2$, $l_{p,\infty}(I) \subset l_2(I)$. Therefore, if $(a_i)_{i\in I}$ is in $l_{p,\infty}(I)$, the series $S=\Sigma_{i\in I} a_i \varepsilon_i$ converges a.s.

We can now state the first lemma.

LEMMA 3.1. *If $(a_i)_{i\in I}$ belongs to $l_{p,\infty}(I)$, $1<p<2$, then S belongs to $L^{\psi_q}(dP)$ and we have*

$$k_p^{-1}\|(a_i)_{i\in I}\|_{p,\infty} \leq \|S\|_{\psi_q} \leq k_p\|(a_i)_{i\in I}\|_{p,\infty} \tag{3.3}$$

where k_p is a constant depending only on p.

This lemma is rather elementary. A proof is given in [25] and the result is mentioned in [28]. Quite possibly it was recorded earlier but we have not found such a reference. Since this result is important in what follows we will include the proof given in [25].

Proof. Clearly we may assume that $I=\mathbf{N}$ and $(|a_k|)_{k\in\mathbf{N}}$ is non-increasing. We first show the right side of (3.3). Assume that $\sup_n n^{1/p}|a_n| \leq 1$ and let $S_n=\Sigma_{k=1}^n a_k \varepsilon_k$. Then for all $c>0$

$$P(|S|>2c) \leq P(|S_n|>c)+P(|S-S_n|>c).$$

We pick $c=qn^{1/q} \geq \Sigma_{k=1}^n k^{-1/p}$, $1/p+1/q=1$, so that $P(|S_n|>c)=0$. By a well known estimate for subgaussian series (cf. [16], p. 43) we get

$$\begin{aligned}P(|S-S_n|>c) &\leq 2\exp\left\{-\frac{c^2}{2}\left(\sum_{k>n}|\alpha_k|^2\right)^{-1}\right\}\\ &\leq 2\exp\left\{-\frac{c^2}{2}\left(\sum_{k>n}k^{-2/p}\right)^{-1}\right\}\\ &\leq 2\exp\left\{-\frac{q(q-2)}{2}n\right\}\end{aligned}$$

for all integers n. From this it is easy to see that

$$\|S\|_{\psi_q}\leq k_p$$

for some constant k_p depending only on p. The right side of (3.3) follows by homogeneity.

To prove the left side of (3.3) we first note that $\Sigma_{k=1}^n \varepsilon_k=n$ with probability 2^{-n}. Therefore

$$E\exp\left\{\left|\frac{\beta\sum_{k=1}^n \varepsilon_k}{n^{1/p}}\right|^q\right\}\geq e^n$$

for $\beta=(1+\log 2)^{1/q}$. It follows that

$$\left\|\sum_{k=1}^n \varepsilon_k\right\|_{\psi_q}\geq \beta^{-1}n^{1/p} \tag{3.4}$$

where β depends only on q (or p). We also have that

$$\begin{aligned}\left\|\sum_{k=1}^\infty \alpha_k\varepsilon_k\right\|_{\psi_q} &\geq \left\|\sum_{k=1}^n \alpha_k\varepsilon_k\right\|_{\psi_q}\\ &\geq \frac{1}{2}\left\|\sum_{k=1}^n \varepsilon_k\right\|_{\psi_q}\inf_{k\leq n}|\alpha_k| \\ &\geq \beta^{-1}n^{1/p}\alpha_n^*\end{aligned} \tag{3.5}$$

and this completes the proof of the lemma. Note that the first inequality in (3.5) follows by convexity. The second inequality follows from the fact that

$$E\exp\left|\sum_{k=1}^n \alpha_k\varepsilon_k\right|^q\geq E\exp\left|2^{-1}\left(\inf_k|\alpha_k|\right)\sum_{k=1}^n \varepsilon_k\right|^q$$

which is a consequence of the contraction principle in the complex case (cf. [21], p. 45) and the last inequality follows from (3.4).

The second lemma is a well known variant of Dudley's theorem (cf. [22]).

LEMMA 3.2. *Let (T, d) be a compact metric space and assume that for* $0<q<\infty$,

$$J_q(d) = \int_0^\infty (\log N(T, d; \varepsilon))^{1/q} d\varepsilon < \infty.$$

Then any random process $(X(t))_{t\in T}$ in $L^{\psi_q}(dP)$ satisfying

$$\|X(t)-X(s)\|_{\psi_q} \leq d(s,t), \quad \forall s, t \in T, \tag{3.6}$$

has a version with continuous sample paths and

$$E \sup_{s,t\in T} |X(s)-X(t)| \leq D_q(J_q(d)+\hat{d})$$

where $\hat{d}=\sup_{s,t\in T} d(s,t)$ and D_q is a constant depending only on q.

We refer the reader to [21], p. 25 where a proof of this result is given for $q=2$. The case $q>0$ is entirely similar since (3.6) implies

$$P\left[\frac{|X(s)-X(t)|}{d(s,t)} > u\right] \leq 2e^{-|u|^q}, \quad \forall u > 0.$$

For a more general discussion see [26], [12]. The following is the major new step in proving sufficient conditions for continuity of strongly stationary p-stable processes. It should also be of independent interest.

THEOREM 3.3. *Let $\{Z_n\}$ be a sequence of independent positive random variables. Then for any $0<p<\infty$ and all $c>0$*

$$c^p P\{\|\{Z_n\}\|_{p,\infty} > c\} \leq \lambda \sup_{t>0} t^p \sum_n P(Z_n > t) \tag{3.7}$$

where $\lambda=262$.

Proof. It is enough to show (3.7) with $p=1$. The general case follows by applying (3.7) with $p=1$ to the sequence $\{Z_n^p\}$. Let

$$A = \sup_{t>0} t \sum_{n=1}^\infty P(Z_n > t)$$

and assume that $A \leq 1$. For a fixed $c>0$ we will show

$$P\{\|\{Z_n\}\|_{1,\infty} > c\} \leq \lambda c^{-1} \tag{3.8}$$

and, by homogeneity, this will conclude the proof of the theorem. Thus it remains to show (3.8). Let

$$U_n = Z_n 1_{\{Z_n \leq c\}}$$

$$V_n = Z_n 1_{\{Z_n > c\}}$$

so that $Z_n = \max(U_n, V_n)$. Using the representation

$$\|\{Z_n\}\|_{1,\infty} = \sup_{t>0} t \sum_{n=1}^{\infty} 1_{\{Z_n > t\}}$$

it is easy to see that

$$\|\{Z_n\}\|_{1,\infty} \leq \|\{U_n\}\|_{1,\infty} + \|\{V_n\}\|_{1,\infty}.$$

Therefore

$$\begin{aligned} P\{\|\{Z_n\}\|_{1,\infty} > 2c\} &\leq P\{\|\{U_n\}\|_{1,\infty} > c\} + P\{\|\{V_n\}\|_{1,\infty} > c\} \\ &\leq c^{-2} E\|\{U_n\}\|_{1,\infty}^2 + P\left\{\sup_{n\geq 1} V_n \neq 0\right\} \\ &\leq c^{-2} E\|\{U_n\}\|_{1,\infty}^2 + \sum_{n=1}^{\infty} P\{Z_n > c\}. \end{aligned} \tag{3.9}$$

We now estimate the first term in the last line of (3.9). Let $\{U_n'\}$ be an independent copy of $\{U_n\}$ defined on $(\Omega', \mathscr{F}', P')$ and let E' denote expectation with respect to this space. Note that since

$$t \sum_{n=1}^{\infty} P'(U_n' > t) \leq t \sum_{n=1}^{\infty} P(Z_n > t)$$

we have

$$\begin{aligned} \sup_{t>0} t \sum_{n=1}^{\infty} 1_{\{U_n > t\}} &\leq \sup_{t>0} \left[t \sum_{n=1}^{\infty} (1_{\{U_n > t\}} - P'(U_n' > t)) + t \sum_{n=1}^{\infty} P(Z_n > t) \right] \\ &\leq \sup_{t>0} \left[t \sum_{n=1}^{\infty} (1_{\{U_n > t\}} - P'(U_n' > t)) \right] + A. \end{aligned}$$

Using this we get

$$E\|\{U_n\}\|_{1,\infty}^2 = E\left(\sup_{t>0} t\sum_{n=1}^{\infty} 1_{\{U_n>t\}}\right)^2$$

$$\leq E\left(\sup_{t>0}\left[t\sum_{n=1}^{\infty}(1_{\{U_n>t\}}-P'(U'_n>t))\right]+A\right)^2 \tag{3.10}$$

$$\leq 2E\left(\sup_{t>0}\left[t\sum_{n=1}^{\infty}(1_{\{U_n>t\}}-E'1_{\{U'_n>t\}})\right]\right)^2+2$$

since $A\leq 1$. By convexity we can majorize the last term by

$$2EE'\left(\sup_{t>0}\left[t\sum_{n=1}^{\infty}(1_{\{U_n>t\}}-1_{\{U'_n>t\}})\right]\right)^2+2. \tag{3.10'}$$

Now let $\{\varepsilon_n\}$ be a Rademacher sequence independent of $\{U_n\}$ and $\{U'_n\}$. By symmetry (3.10') is equal to

$$2EE'\left(\sup_{t>0}\left[t\sum_{n=1}^{\infty}\varepsilon_n(1_{\{U_n>t\}}-1_{\{U'_n>t\}})\right]\right)^2+2$$

$$\leq 4E\left[\sup_{t>0} t\sum_{n=1}^{\infty}\varepsilon_n 1_{\{U_n>t\}}\right]^2+4E'\left[\sup_{t>0} t\sum_{n=1}^{\infty}\varepsilon_n 1_{\{U'_n>t\}}\right]^2+2 \tag{3.11}$$

$$\leq 8E\left[\sup_{t>0} t\sum_{n=1}^{\infty}\varepsilon_n 1_{\{U_n>t\}}\right]^2+2.$$

We use the following elementary lemma to estimate the last expression.

LEMMA 3.4. *For any sequence $\{u_n\}$ of positive numbers we have*

$$E\left[\sup_{t>0} t\sum_{n=1}^{\infty}\varepsilon_n 1_{\{u_n>t\}}\right]^2 \leq 8\sum_{n=1}^{\infty} u_n^2.$$

Proof of Lemma 3.4. There is nothing to prove unless $u_n\to 0$ so assume that this is the case. Let $\{u_n^*\}$ be a non-increasing rearrangement of $\{u_n\}$. Let $\pi: N\to N$ be a permutation such that $u_{\pi(n)}=u_n^*$. We have

$$\sup_{t>0} t\sum_{n=1}^{\infty}\varepsilon_n 1_{\{u_n>t\}} = \sup_{k\geq 1} u_k^*\sum_{n=1}^{k}\varepsilon_{\pi(n)}. \tag{3.12}$$

Let

$$S_k = u_k^*\sum_{n=1}^{k}\varepsilon_{\pi(n)}$$

and

$$T_k = \sum_{n=1}^{k} u_n^* \varepsilon_{\pi(n)}.$$

Following the argument used in the proof of Kronecker's lemma we have

$$\begin{aligned} S_k &= u_k^* \sum_{n=1}^{k} (T_n - T_{n-1}) u_n^{*-1} \\ &= T_k - u_k^* \sum_{n=1}^{k-1} T_n (u_{n+1}^{*-1} - u_n^{*-1}) \\ &\leqslant T_k + u_k^* \sum_{n=1}^{k-1} \sup_{1 \leqslant n \leqslant k} |T_n| (u_{n+1}^{*-1} - u_n^{*-1}). \end{aligned}$$

Therefore

$$\sup_{k \geqslant 1} |S_k| \leqslant 2 \sup_{k \geqslant 1} |T_k|$$

so that

$$E\left(\sup_{k \geqslant 1} S_k\right)^2 \leqslant 4E\left(\sup_{k \geqslant 1} |T_k|\right)^2. \tag{3.13}$$

It is obvious that T_k is equal to $\Sigma_{n=1}^k u_n^* \varepsilon_n$ in distribution. Hence by Levy's inequality (cf. [16], p. 12) we have

$$E \sup_{k \geqslant 1} |T_k|^2 \leqslant 2 \sum_{n=1}^{\infty} u_n^{*2} = 2 \sum_{n=1}^{\infty} u_n^2. \tag{3.14}$$

Finally, (3.12), (3.13) and (3.14) give Lemma 3.4.

Completion of proof of Theorem 3.3. Let us denote by E_ε expectation with respect to $\{\varepsilon_n\}$. Let $(\Omega, \mathscr{F}, P)$ denote the probability space of $\{Z_n\}$ and let E denote the corresponding expectation operator. For a fixed $\omega \in \Omega$ we have by Lemma 3.4

$$E_\varepsilon \left(\sup_{t>0} t \sum_{n=1}^{\infty} \varepsilon_n 1_{\{U_n(\omega)>t\}} \right)^2 \leqslant 8 \sum_{n=1}^{\infty} U_n^2(\omega).$$

Therefore,

$$EE_\varepsilon \left(\sup_{t>0} t \sum_{n=1}^{\infty} \varepsilon_n 1_{\{U_n(\omega)>t\}} \right)^2 \leqslant 8E\left(\sum_{n=1}^{\infty} U_n^2(\omega) \right)$$

$$
\begin{aligned}
&= 8\sum_{n=1}^{\infty} EU_n^2 \\
&= 8\sum_{n=1}^{\infty}\int_0^c 2tP(U_n>t)\,dt \qquad (3.15)\\
&\leqslant 16cA \leqslant 16c.
\end{aligned}
$$

Combining (3.9) and (3.10), and evaluating (3.10) by (3.11), Lemma 3.4 and (3.15) we obtain

$$P\{\|\{Z_n\}\|_{1,\infty} > 2c\} \leqslant \frac{128}{c} + \frac{2}{c^2} + \frac{1}{c} \leqslant 131/c$$

for $c \geqslant 1$ and of course it is also true for $c<1$. This gives (3.8) with $\lambda=262$, concluding the proof of the theorem.

The next corollary strengthens Theorem 3.3. It also generalizes a well known result on empirical distribution functions which is sometimes referred to as Daniels' theorem (cf. [27]).

COROLLARY 3.5. *Let $\{Z_n\}$ be a sequence of independent positive random variables. Assume that $\Sigma_{n=1}^{\infty} P(Z_n \geqslant t) < \infty$ for all $t>0$. Let*

$$Z^* = \sup_{t>0} \frac{\sum_{n=1}^{\infty} 1_{\{Z_n \geqslant t\}}}{\sum_{n=1}^{\infty} P(Z_n \geqslant t)}.$$

Then for all $c>0$

$$cP(Z^* > c) \leqslant \lambda \qquad (3.16)$$

where $\lambda=262$.

Proof. Note that

$$\sup_{t>0} t\sum_{k=1}^{N} I_{[Z_k>t]} = \sup_{t>0} t\sum_{k=1}^{N} I_{[Z_k \geqslant t]}$$

and, similarly

$$\sup_{t>0} t\sum_{k=1}^{N} P(Z_k > t) = \sup_{t>0} t\sum_{k=1}^{N} P(Z_k \geqslant t).$$

Using these two equalities in (3.7), in the case $p=1$, we get

$$cP\left\{\sup_{t>0} t\sum_{k=1}^{N} I_{[Z_k\geqslant t]}>c\right\}\leqslant\lambda\sup_{t>0} t\sum_{k=1}^{N} P(Z_k\geqslant t) \tag{3.17}$$

where $\lambda=262$. Let

$$\varphi(t)=\left(\sum_{k=1}^{N} P(Z_k\geqslant t)\right)^{-1}.$$

This function, $\varphi(t)$, is non-decreasing, left continuous and has right hand limits. Therefore we can find a sequence of functions $\{\varphi_n(t)\}$ such that $\varphi_n(t)$ is continuous, strictly increasing in t and is increasing in n, $\varphi_n(0)=0$ and $\lim_{n\to\infty}\varphi_n(t)=\varphi(t)$, $\forall t>0$. By (3.17) with $\varphi_n(Z_k)$ replacing Z_k

$$cP\left\{\sup_{t>0} t\sum_{k=1}^{N} I_{[\varphi_n(Z_k)\geqslant t]}>c\right\}\leqslant\lambda\sup_{t>0} t\sum_{k=1}^{N} P[\varphi_n(Z_k)\geqslant t]. \tag{3.18}$$

We have

$$\begin{aligned}\sup_{t>0} t\sum_{k=1}^{N} P[\varphi_n(Z_k)\geqslant t] &= \sup_{s>0}\varphi_n(s)\sum_{k=1}^{N} P[\varphi_n(Z_k)\geqslant\varphi_n(s)]\\ &= \sup_{s>0}\varphi_n(s)\sum_{k=1}^{N} P\{Z_k\geqslant s\}\leqslant 1,\end{aligned}$$

and similarly

$$\sup_{t>0} t\sum_{k=1}^{N} I_{[\varphi_n(Z_k)\geqslant t]}=\sup_{s>0}\varphi_n(s)\sum_{k=1}^{N} I_{[Z_k\geqslant s]}.$$

Using these two relations in (3.18) we get

$$cP\left\{\sup_{t>0} t\sum_{k=1}^{N} I_{[\varphi_n(Z_k)\geqslant t]}>c\right\}\leqslant\lambda. \tag{3.19}$$

Now, since $\varphi_n(s)$ increases monotonically to $\varphi(s)$,

$$\lim_{n\to\infty} cP\left[\sup_{s>0}\varphi_n(s)\sum_{k=1}^{N} I_{[Z_k\geqslant s]}>c\right]=cP\left[\sup_{s>0}\varphi(s)\sum_{k=1}^{N} I_{[Z_k\geqslant s]}>c\right].$$

Using this in (3.19) we have

$$cP\left[\sup_{s>0}\frac{\sum_{k=1}^{N} I_{[Z_k\geq s]}}{\sum_{k=1}^{N} P[Z_k\geq s]}>c\right]\leq\lambda$$

and since this holds for all N we get (3.16).

Remark 3.6. Daniels' theorem states that for $\{Z_n\}_{n=1,\dots,N}$ i.i.d. with continuous distribution function,

$$P\left[\sup_{t}\frac{\frac{1}{N}\sum_{n=1}^{N} 1_{\{Z_n\leq t\}}}{P\{Z_1\leq t\}}>c\right]=\frac{1}{c},\quad \forall c\geq 1. \tag{3.20}$$

Assume that $Z_n\geq 0$, $n=1,\dots,N$. If we replace Z_n by Z_n^{-1} in Corollary 3.5 and take the sup over all $t>0$, we get

$$P\left[\sup_{t>0}\frac{\frac{1}{N}\sum_{n=1}^{N} 1_{[Z_n\leq t]}}{\frac{1}{N}\sum_{n=1}^{N} P(Z_n\leq t)}>c\right]\leq\frac{\lambda}{c} \tag{3.21}$$

where $\{Z_n\}_{n\leq N}$ are not necessarily identically distributed. Therefore, in some sense, (3.21) is a robust form of the upper bound in Daniels' theorem.

(While making final revisions of this paper prior to publication we discovered that (3.21) was proved by van Zuijlen [31], using completely different methods from ours. Furthermore, it is easy to see that (3.21) implies our Theorem 3.3. Nevertheless our proof seems to be more elementary than the one in [31]. Recently, Joel Zinn has found a new and simple proof of Theorem 3.3.)

COROLLARY 3.7. *Let $0<p<\infty$ and let $\{X_j\}$ be an i.i.d. sequence of random variables. Then if $E|X_1|^p<\infty$ the sequence $\{j^{-1/p}X_j\}$ is a.s. in $l_{p,\infty}(\mathbf{N})$ and we have*

$$c^pP\{\|\{j^{-1/p}X_j\}\|_{p,\infty}>c\}\leq\lambda E|X_1|^p. \tag{3.22}$$

A fortiori, for $0<r<p$

$$(E\|\{j^{-1/p}X_j\}\|_{p,\infty}^r)^{1/r}\leq h_{p,r}(E|X_1|^p)^{1/p} \tag{3.23}$$

where $h_{p,r}$ is a constant depending only on p and r.

Proof. The first inequality follows immediately from Theorem 3.3 and the well known observation that

$$\sum_{j=1}^{\infty} P(j^{-1/p}|X_j|>c) \le \sum_{j=1}^{\infty} P(|X_1|>j^{1/p}c) \le E\left|\frac{X_1}{c}\right|^p, \quad \forall c>0.$$

The second inequality is a consequence of the classical identity

$$EM^r = \int_0^\infty P(M^r>c)\,dc. \tag{3.24}$$

The next corollary is an immediate consequence of Theorem 3.3.

COROLLARY 3.8. *Let $0<p<\infty$ and let Γ be an arbitrary set. Let $\{\xi_\gamma\}_{\gamma\in\Gamma}$ be a family of independent random variables such that*

$$P(|\xi_\gamma|>c) \le c^{-p}, \quad \forall\gamma\in\Gamma, \forall c>0. \tag{3.25}$$

Let $\{a_\gamma\}_{\gamma\in\Gamma}$ be complex numbers such that $\Sigma_{\gamma\in\Gamma}|a_\gamma|^p<\infty$. Then $\{a_\gamma\xi_\gamma\}_{\gamma\in\Gamma}$ is a.s. in $l_{p,\infty}(\Gamma)$ and we have

$$c^pP(\|\{a_\gamma\,\xi_\gamma\}_{\gamma\in\Gamma}\|_{p,\infty}>c) \le \lambda\sum_{\gamma\in\Gamma}|a_\gamma|^p, \quad \forall c>0.$$

Remark 3.9. (i) It is an elementary consequence of the Three Series Theorem that $\{a_\gamma\xi_\gamma\}_{\gamma\in\Gamma}$ is a.s. in $l_{p+\varepsilon}(\Gamma)$ for each $\varepsilon>0$ but that $\{a_\gamma\xi_\gamma\}_{\gamma\in\Gamma}$ is not necessarily in $l_p(\Gamma)$. The preceding corollary is a refinement of these observations i.e. $\{a_\gamma\xi_\gamma\}_{\gamma\in\Gamma}$ is in "weak l_p".

(ii) Theorem 3.3 and its corollaries are not valid if we drop the assumption of independence of the random variables. This is rather obvious except in the context of the last corollary. This point puzzled us for some time until W. Beckner kindly showed us a counter example.

(iii) Corollary 3.8 can be obtained as a direct consequence of Doob's inequality. It is easy to see that

$$M_t = \sum_{\gamma\in\Gamma}|a_\gamma|^p\frac{1_{\{|a_\gamma\,\xi_\gamma|>t\}}}{P(|a_\gamma\,\xi_\gamma|>t)}, \quad t>0$$

is a martingale. By Doob's maximal inequality we have

$$c^pP(\sup_{t>0}M_t>c^p) \le \sup_{t>0}EM_t = \sum_{\gamma\in\Gamma}|a_\gamma|^p. \tag{3.26}$$

Also, by (3.25)

$$t^p \sum_{\gamma\in\Gamma} 1_{\{|a_\gamma \xi_\gamma|>t\}} \leq M_t.$$

Using this in (3.26) we immediately obtain

$$c^p P(\|\{a_\gamma \xi_\gamma\}_{\gamma\in\Gamma}\|_{p,\infty} > c) \leq \sum_{\gamma\in\Gamma} |a_\gamma|^p.$$

This alternate proof has the additional advantage that it yields Corollary 3.7 with a constant equal to 1 instead of λ. We suspect that there is also a simpler proof of Theorem 3.3 based on martingale methods but we have not been able to find one.

We can now prove a sufficient condition for a.s. continuity of strongly stationary p-stable random processes.

THEOREM 3.10. *Let* $1<p<2<q<\infty$ *where* $1/p+1/q=1$. *Let* G, Γ, K, $(X(t))_{t\in G}$ *and* m *be as in Theorem A and let* d_X *and* $J_q(d_X)$ *be as given in* (0.1) *and* (0.2). *If* $J_q(d_X)<\infty$ *then* $(X(t))_{t\in K}$ *has a version with continuous sample paths. Moreover, we have*

$$E \sup_{s,t\in K} |X(s)-X(t)| \leq c_p(K)\,(J_q(d_X)+m(\Gamma)^{1/p}) \tag{3.27}$$

where $c_p(K)$ *is a constant depending only on* p *and* K.

Proof. We may assume (multiplying $(X(t))_{t\in G}$, if necessary, by $m(\Gamma)^{-1/p}$) that m is a probability measure on Γ. Let $\{Y_j\}_{j\geq 1}$ be an i.i.d. sequence of random variables with values in Γ such that the probability distribution of Y_1 is equal to m. Let $\{\Gamma_j\}_{j\geq 1}$ and $\{w_j\}_{j\geq 1}$ be as in Proposition 1.5. We take $\{\Gamma_j\}_{j\geq 1}$, $\{Y_j\}_{j\geq 1}$ and $\{w_j\}_{j\geq 1}$ to be independent of each other and define them all on the probability space $(\Omega, \mathcal{A}, P)$. Let $\{\varepsilon_j\}_{j\geq 1}$ be a Rademacher sequence defined on a different probability space $(\Omega', \mathcal{A}', P')$. The random process

$$W(t;\omega,\omega') = c(p)\sum_{j=1}^{\infty} \varepsilon_j(\omega')\,\Gamma_j^{-1/p}(\omega)\,w_j(\omega)\,Y_j(t,\omega), \quad t\in G \tag{3.28}$$

has the same distribution as $(X(t))_{t\in G}$. This follows from Proposition 1.5 since (3.28) and (1.34) are clearly equal in distribution.

We introduce the random metric δ_ω, defined for each fixed $\omega\in\Omega$ by

$$\delta_\omega(s,t) = \|\{\Gamma_j(\omega)^{-1/p} w_j(\omega)\,(Y_j(s;\omega)-Y_j(t;\omega))\}_{j\geq 1}\|_{p,\infty}, \quad \forall s,\, t\in G.$$

Since $|w_j Y_j| \leq 1$ we have, by (3.2), that

$$\sup_{s,t \in G} \delta_\omega(s,t) \leq 2\|\{\Gamma_j^{-1/p}|w_j|\}\|_{p,\infty} \leq \pi \|\{\Gamma_j^{-1/p}\}\|_{p,\infty} \tag{3.29}$$

and by (1.25) and (3.23)

$$E \sup_{s,t \in G} \delta_\omega(s,t) \leq \pi \alpha_p \equiv B_p. \tag{3.30}$$

Since Y_j takes values in Γ, it is clear that $\delta_\omega(s,t) = \delta_\omega(0,t-s)$ and, similarly, $d_X(s,t) = d_X(0,t-s)$. To simplify the notation we set $\sigma_\omega(t-s) = \delta_\omega(s,t)$ and $\sigma_X(t-s) = d_X(s,t)$. Now, for fixed $\omega \in \Omega$, consider $W(s;\omega,\omega') - W(t,\omega,\omega')$ as a Rademacher series. By Lemma 3.1, we have

$$\|W(s;\omega,\cdot) - W(t;\omega,\cdot)\|_{\psi_q} \leq k_p\, \delta_\omega(s,t), \quad \forall s,\ t \in G. \tag{3.31}$$

Therefore, by Lemma 3.2 and (3.29)

$$E_{\omega'} \sup_{s,t \in K} |W(t;\omega,\cdot) - W(s;\omega,\cdot)| \leq k_p\, D_q[J_q(\delta_\omega) + \pi \|\{\Gamma_j^{-1/p}\}\|_{p,\infty}] \tag{3.32}$$

where, as in (0.2), we have defined

$$J_q(\delta_\omega) = \int_0^\infty (\log N(K, \delta_\omega; \varepsilon))^{1/q}\, d\varepsilon.$$

Taking the expectation of (3.32) with respect to $(\Omega, \mathcal{A}, P)$ and using (3.30) we get

$$E \sup_{s,t \in K} |W(t) - W(s)| \leq k_p\, D_q[E_\omega\, J_q(\delta_\omega) + B_p]. \tag{3.33}$$

To complete the proof of this theorem we will need the following lemma which will be proved after Remark 3.12.

LEMMA 3.11. *Let δ_ω be a random, translation invariant, pseudo-metric on G. Let $\delta(s,t)$ be the pseudo-metric defined by*

$$\delta(s,t) = E_\omega\, \delta_\omega(s,t), \quad \forall s,\ t \in G.$$

Then

$$E_\omega\, J_q(\delta_\omega) \leq \Lambda_q(K) \left[J_q(\delta) + E_\omega \sup_{s,t \in K} \delta_\omega(s,t) \right], \tag{3.34}$$

where $\Lambda_q(K)$ is a constant depending only on q and K.

Completion of proof of Theorem 3.10. Note that by Lemma 1.2, (3.2) and (3.23)

$$\begin{aligned}\delta(s,t) &= E_\omega\, \delta_\omega(s,t)\\ &\leq E\left\{\sup_{j\geq 1}(j\Gamma_j^{-1})^{1/p}\cdot \|\{j^{-1/p}w_j(Y_j(s)-Y_j(t))\}_{j\geq 1}\|_{p,\infty}\right\} \qquad (3.35)\\ &\leq \alpha_p(\pi/2)\,h_{p,1}\,d_X(s,t)\equiv \chi_p\, d_X(s,t).\end{aligned}$$

Therefore, by (3.30) and (3.35) we get

$$J_q(\delta)+E_\omega \sup_{s,t\in K} \delta_\omega(s,t)\leq \chi_p\, J_q(d_X)+B_p. \qquad (3.36)$$

Finally, note that if $J_q(d_X)$ is finite then by Lemma 3.11 and (3.36), $J_q(d_\omega)$ is finite a.s. with respect to $(\Omega, \mathscr{A}, P)$. It follows from (3.31) and Lemma 3.2 that there exists a set $\bar{\Omega}\subset\Omega$, $P(\bar{\Omega})=1$, such that for each $\omega\in\bar{\Omega}$, $W(t;\omega,\cdot)$ has a version with continuous sample paths with respect to $(\Omega', \mathscr{A}', P')$. Therefore, by Fubini's theorem $(V(t))_{t\in K}$ has a version with continuous sample paths. Furthermore, by (3.33), (3.34) and (3.36) we have

$$\begin{aligned}E\sup_{s,t\in K}|W(t)-W(s)| &\leq k_p\, D_q[\Lambda_p(K)\chi_p\, J_q(d_X)+(\Lambda_p(K)+1)B_p]\\ &\leq C_p(K)\,[J_q(d_X)+1].\end{aligned}$$

Since $(W(t))_{t\in K}$ and $(X(t))_{t\in K}$ have the same distribution this completes the proof of Theorem 3.10. (The reader will recall that we have normalized so that $m(\Gamma)=1$.)

Remark 3.12. By (3.27) and (2.29)

$$\left(\sup_{c>0} c^p P\left\{\sup_{s,t\in K}|X(s)-X(t)|>c\right\}\right)^{1/p}\leq \Delta(1,p)\,C_p(K)\,(J_q(d_X)+m(\Gamma)^{1/p}). \qquad (3.37)$$

This can also be proved by considering $(W(t))_{t\in K}$ instead of $(X(t))_{t\in K}$ and using an inequality of Hoffmann-Jørgensen (cf. [13]). In fact, it is also possible to slightly modify the proof of Theorem 3.10 and obtain the improved result (3.37) directly.

Proof of Lemma 3.11. We use an idea which first appeared in [20] in a similar context. Since our argument is essentially the same as that of [21], Chapter II, Lemmas 2.3 and 3.6, we will not give too many details (see also [11]). Note that the fact that the pseudo-metrics $\delta_\omega(s,t)$ are translation invariant is essential in this proof.

First we will obtain (0.12). Consider $\sigma_X(u)=d_X(0,u)$ as defined prior to (0.10) and

$\overline{\sigma_X(u)}$ as defined in (0.11). Clearly, we can find a number $b(p)>0$ large enough so that $u^{-1}(\log b(p)/u)^{-1/p}$ is non-increasing on $[0,1]$. Now, exactly as in the proof of (3.30), Chapter II, [21], we can find a constant $D_p(K)>0$ depending only on b and K such that

$$\begin{aligned} D_p(K)^{-1}\left[\sup_{s,t\in K} d_X(s,t)+J_q(d_X)\right] &\leq I_p(\sigma_X)+\sup_{s,t\in K} d_X(s,t) \\ &\leq D_p(K)\left[\sup_{s,t\in K} d_X(s,t)+J_q(d_X)\right] \end{aligned} \tag{3.38}$$

where $I_p(\sigma_X)$ is defined in (0.13). This immediately gives (0.12) since $\sup_{s,t\in K} d_X(s,t)\leq 2m(\Gamma)^{1/p}$.

Recall that following (3.30) we defined $\sigma_\omega(u)=\delta_\omega(0,u)$. As we just did for $\sigma_X(u)$, we define $\overline{\sigma_\omega(u)}$ by (0.11). We now use (3.38) with σ_ω and δ_ω instead of σ_X and d_X and obtain

$$\begin{aligned} D_p(K)^{-1}\left[\sup_{s,t\in K} \delta_\omega(s,t)+J_q(\delta_\omega)\right] &\leq I_p(\sigma_\omega)+\sup_{s,t\in K} \delta_\omega(s,t) \\ &\leq D_p(K)\left[\sup_{s,t\in K} \delta_\omega(s,t)+J_q(\delta_\omega)\right]. \end{aligned} \tag{3.39}$$

We also define $\sigma(u)=\delta(0,u)$ so that, by the hypothesis of this lemma

$$\sigma(u)=E\sigma_\omega(u).$$

We have, by Lemma (2.3), Chapter II, [21], that

$$E_\omega I_p(\sigma_\omega)\leq I_p(\sigma). \tag{3.40}$$

We can now obtain (3.34) since by (3.39), (3.40) and (3.38) with σ and δ instead of σ_X and d_X, we get

$$\begin{aligned} E_\omega J_q(d_\omega) &\leq D_p(K)\left[E_\omega I_p(\sigma_\omega)+E_\omega \sup_{s,t\in K} \delta_\omega(s,t)\right] \\ &\leq D_p(K)\left[I_p(\sigma)+E_\omega \sup_{s,t\in K} \delta_\omega(s,t)\right] \\ &\leq D_p^2(K)\left[J_q(\delta)+\sup_{s,t\in K} \delta(s,t)+E_\omega \sup_{s,t\in K} \delta_\omega(s,t)\right] \\ &\leq 2D_p^2(K)\left[J_q(\delta)+E_\omega \sup_{s,t\in K} \delta_\omega(s,t)\right], \end{aligned}$$

where at the last step we use the obvious inequality, $\sup_{s,t\in K}\delta(s,t)\leq E_\omega \sup_{s,t\in K}\delta_\omega(s,t)$.

Aside from the representation of $X(t)$ by $W(t)$, none of the steps in the proof of Theorem 3.10 require that the process is exactly p-stable. To demonstrate this we will prove Theorem B, (i) in which the independent random variables $\{\xi_\gamma\}$ need not be stable.

Proof of Theorem B, (i): This proof is similar to the proof of Theorem 3.10 but somewhat simpler since in this case, we can obtain an easier proof of Corollary 3.8 using Remark 3.9, (iii).

Let $\{\varepsilon_\gamma\}_{\gamma\in\Gamma}$ be a Rademacher sequence defined on a probability space $(\Omega', \mathscr{A}', P')$. Without loss of generality, we can replace $Y(t)$ given in (0.7) by

$$Y(t;\omega,\omega') = \sum_{\gamma\in\Gamma} a_\gamma\, \varepsilon_\gamma(\omega')\, \xi_\gamma(\omega)\, \gamma(t), \quad t\in K,$$

defined on $(\Omega\times\Omega', \mathscr{A}\times\mathscr{A}', P\times P')$. By Lemmas 3.1 and 3.2 we have

$$E_{\omega'} \sup_{s,t\in K} |Y(t;\omega,\cdot)-Y(s;\omega,\cdot)| \leq k_p\, D_q \left[J_q(\delta_\omega) + \sup_{s,t\in K} \delta_\omega(s,t)\right]$$

where

$$\delta_\omega(s,t) = \|\{a_\gamma \xi_\gamma(\omega)(\gamma(s)-\gamma(t))\}_{\gamma\in\Gamma}\|_{p,\infty}.$$

By Corollary 3.8 and (3.24) we have

$$E_\omega\, \delta_\omega(s,t) \leq b_p\, d_Y(s,t)$$

where b_p is a constant depending only on p. Also by (3.2)

$$\sup_{s,t} \delta_\omega(s,t) \leq 2\|\{a_\gamma\, \xi_\gamma\}_{\gamma\in\Gamma}\|_{p,\infty}$$

and by Corollary 3.8

$$E \sup_{s,t} \delta_\omega(s,t) \leq C \left(\sum |a_\gamma|^p\right)^{1/p}$$

where C is constant. Finally, arguing as in Lemma 3.11 we get

$$\begin{aligned} E_\omega\, J_q(\delta_\omega) &\leq \Lambda'_p(K) \left\{ J_q(E_\omega\, \delta_\omega) + \left(\sum_{\gamma\in\Gamma} |a_\gamma|^p\right)^{1/p}\right\} \\ &\leq b_p\, \Lambda'_p(K) \left\{ J_q(d_Y) + \left(\sum_{\gamma\in\Gamma} |a_\gamma|^p\right)^{1/p}\right\}. \end{aligned}$$

The proof of Theorem B, (i) can now be completed exactly as in the proof of Theorem 3.10 taking into account (2.29) (see also Remark 3.12).

Remark 3.14. The only place that the nature of K came into the proof of Theorem 3.10 or the proof of Theorem B, (i) was in (3.38). This is because the inequalities in (3.38) involve $\mu(K)$, $\mu(K+K)$ and $\mu(K+K+K+K)$, (where $K+K+K+K=\{s+t+u+v: s\in K,\ t\in K,\ u\in K,\ v\in K\}$), and in general these three sets have different μ measure. If G is a compact group we take $K=G$ and of course $K+K=G$ and $K+K+K+K=G$ so this problem does not arise. Therefore, for example, we can replace $\Lambda_p(K)$ in (0.8) by Λ_p, a constant depending only on p and independent of G.

Remark 3.15. Let $\{x_j\}$ be a sequence in an arbitrary Banach space. By an argument similar to that of Lemma 1.2 one can show that $\Sigma_{j=1}^{\infty}\varepsilon_j\Gamma_j^{-1/p}x_j$ converges a.s. iff $\Sigma_{j=1}^{\infty}\varepsilon_j j^{-1/p}x_j$ converges a.s. This reduces the study of p-stable processes to series of the latter form. As an example of this we can obtain an interesting relationship. To be specific, let $v=\{v(t),\ t\in[0,1]\}$ be a real valued stochastic process with continuous sample paths satisfying $\sup_{t\in[0,1]}E|v(t)|^p<\infty$. Let m denote the measure induced by v on $C[0,1]$ and let $\{v_j\}$ be i.i.d. copies of v. Then $\Sigma_{j=1}^{\infty}\varepsilon_j j^{-1/p}v_j(t)$, $t\in[0,1]$, converges uniformly a.s. iff the p-stable process with m as its spectral measure has a version with continuous sample paths.

4. Proofs of Theorems A, B and C

In this section we briefly mention how the results of Sections 2 and 3 are put together to prove Theorems A, B and C. We also mention some other results which follow easily from these theorems and some results in [21].

Proof of Theorem A. The fact that $J_q(d_X)<\infty$ is a necessary and sufficient condition and that $J_\infty(d_X)<\infty$ is a necessary condition for X to have a version with continuous sample paths follows immediately from Theorem 2.9 and Theorem 3.10. To obtain the lower inequality in (0.3) we have by (3.27) of Theorem 3.10 that

$$E\sup_{t\in K}|X(t)|\leq E|X(0)|+C_p(K)\,(J_q(d_X)+m(\Gamma)^{1/p}). \tag{4.1}$$

Furthermore by (1.11)

$$E|X_0|=\delta'(1,p)\,m(\Gamma)^{1/p}. \tag{4.2}$$

Using (4.1) and taking Remark 2.10 into account we get the lower inequality in (0.3). The upper inequality in (0.3) follows from (2.25) of Theorem 2.9, (4.2) and Remark 2.10.

Remark 4.1. The reader probably has noticed that the choice of the compact set K does not affect the qualitative results in this paper. Indeed, let $(X(t))_{t\in G}$ be any stationary process (i.e. a process such that $(X(t))_{t\in G}$ has the same distribution as $(X(t+s))_{t\in G}$ for each $s\in G$). If K_1 and K_2 are two compact subsets of G with non-empty interiors, then $(X(t))_{t\in K_1}$ has a version with continuos sample paths iff $(X(t))_{t\in K_2}$ does. This is obvious since each of the sets K_1 and K_2 can be covered by finitely many translates of the other. Consequently, if G is the union of a countable sequence of compact sets, then $(X(t))_{t\in K}$ has a version with continuous sample paths iff the entire process $(X(t))_{t\in G}$ does also. Of course, this applies in the most important cases such as $G=\mathbf{R}^n$.

Proof of Theorem B. Part (i) was proved at the end of Section 3. Part (ii) follows from Theorem 2.9, applied in the case when the spectral measure m is discrete, and a comparison theorem from [14]. To be more specific suppose that $(Y(t))_{t\in K}$ is a.s. continuous, then by a result of Ito and Nisio (cf. Theorem 3.4, p. 95, [15]), $\Sigma_{\gamma\in\Gamma} a_\gamma \xi_\gamma \gamma(t)$, $t\in K$, must be a.s. convergent in $C(K)$ in any chosen ordering. Let $\{\tilde{\theta}_\gamma\}_{\gamma\in\Gamma}$ be i.i.d. complex valued p-stable random variables. Since we assume in (ii) that

$$\inf_{\gamma\in\Gamma} P\{|\xi_\gamma|>c\}\,(P\{|\theta_\gamma|>c\})^{-1}$$

is bounded below for c sufficiently large, it follows from Theorem 5.1 [14] that the series $\Sigma_{\gamma\in\Gamma} a_\gamma \theta_\gamma \gamma(t)$, $t\in K$ converges a.s. in $C(K)$ and this implies, by Theorem 2.9 that $J_q(d_X)<\infty$, and when $p=1$, $J_\infty(d_X)<\infty$. (Even though Theorem 5.1 [14] is written for real valued random variables it is also valid for complex valued random variables. Also, clearly, we could have used it to prove Theorem B, (i) as well.)

Remark 4.2. In [21] we consider series of the form

$$Y(t)=\sum a_\gamma\,\varepsilon_\gamma\,\xi_\gamma\,\gamma(t),\quad t\in K \tag{4.3}$$

where $\{\varepsilon_\gamma\}$ is a Rademacher sequence and $\{\xi_\gamma\}$ are complex valued random variables not necessarily independent but with $\{\varepsilon_\gamma\}$ and $\{\xi_\gamma\}$ independent of each other. In this case if we add the hypothesis $\sup_\gamma E|\xi_\gamma|^p<\infty$ our proof shows that the series in (4.3) converges uniformly a.s. and we get an expression like (0.8) for the same d_Y as in Theorem B but with the left side replaced by $(E \sup_{s,t\in K} |Y(t)-Y(s)|^p)^{1/p}$. This follows because $\|\{a_\gamma\, \xi_\gamma |\gamma(u)-1|\}\|_{p,\infty} \leq (\Sigma |a_\gamma|^p |\xi_\gamma|^p |\gamma(u)-1|^p)^{1/p}$.

Remark 4.3. Exactly as in Chapter 7, § 2, [21] we can prove the following: Let $(X(t))_{t\in T}$ be a strongly stationary, real or complex valued p-stable process, $1<p\leq 2$. Then $(X(t))_{t\in G}$ admits a version with paths bounded on the whole of G if and only if

$$\int_0^\infty (\log N(G,\, d_X;\varepsilon))^{1/q}\, d\varepsilon < \infty\ , \tag{4.4}$$

$1/p+1/q=1$. Moreover if (4.4) holds then the paths of $(X(t))_{t\in G}$ must be a.s. almost periodic functions on G.

Remark 4.4. Let $G=\Gamma=\mathbf{R}^n$ and consider the random Fourier series

$$\sum_{k=1}^\infty a_k\, \theta_k\, e^{i\langle\lambda_k, t\rangle}, \quad t\in [-1,1]^n, \tag{4.5}$$

where $\{\theta_k\}$ are i.i.d. real or complex valued p-stable random variables. Then for $1<p\leq 2$

$$\sum_{n=1}^\infty \frac{\left(\sum_{|\lambda_k|\geq n} |a_k|^p\right)^{1/p}}{n(\log(n+1))^{1/p}} < \infty \tag{4.6}$$

is sufficient for the uniform convergence a.s. of the series (4.5), where $|\cdot|$ denotes the Euclidean norm on $\mathbf{R}^n$. This result follows because, exactly as in Chapter 7, § 1, [21], we can show that (4.6) implies $J_q(d_X)<\infty$, $1/p+1/q=1$.

Now let G be the circle group and consider

$$\sum_{k=1}^\infty a_k\, \theta_k\, e^{ikt}, \quad t\in [0,2\pi]. \tag{4.7}$$

The expression given in (4.6) but with $\lambda_k=k$ is a sufficient condition for the series in (4.7) to converge uniformly a.s. for $1\leq p\leq 2$. The new element here is that the result holds for $p=1$. The proof is rather technical and we will not give it at this time. Now let $|a_k|$ in (4.7) be non-increasing. In this case the series in (4.7) converges uniformly a.s. iff (4.6) holds (with $\lambda_k=k$), $1\leq p\leq 2$. This is proved by the same method used in [19] in the case $p=2$ except that we must use the version of Slepian's lemma for strongly stationary p-stable processes mentioned right after the statement of Theorem C.

It appears that Theorems 1.2 and 1.4, Chapter 7, [21] should also generalize for $1\leq p\leq 2$. However, at this writing, we have not yet done this.

Proof of Theorem C. Let us denote by μ normalized Haar measure on G and let γ_0 denote the identity element in Γ. We first observe that

$$\left(\sum_{\gamma\neq\gamma_0}|a_\gamma|^p\right)^{1/p}\leq\int_G\left(\sum_{\gamma\neq\gamma_0}|a_\gamma|^p|\gamma(t)-1|^p\right)^{1/p}d\mu(t)$$
$$\leq\sup\{d_X(s,t)\colon s,\ t\in G\}.$$

Therefore, if $\varepsilon<(\Sigma_{\gamma\neq\gamma_0}|a_\gamma|^p)^{1/p}$, we must have $N(G,\,d_X;\varepsilon)>1$ and consequently

$$(\log 2)^{1/q}\left(\sum_{\gamma\neq\gamma_0}|a_\gamma|^p\right)^{1/p}\leq J_q(d_X). \tag{4.8}$$

We can now obtain the left side of (0.9) by using (4.8), (0.8), Remark 3.14 and the following elementary inequality:

$$E\sup_{s,t\in G}|X(t)-X(s)|\geq E\sup_{t\in G}\left|X(t)-\int_G X(s)\,d\mu(s)\right|$$
$$\geq E\left\|\sum_{\gamma\neq\gamma_0}a_\gamma\,\theta_\gamma\,\gamma\right\|.$$

On the other hand it is obvious that

$$E\sup_{s,t\in G}|X(s)-X(t)|\leq 2E\|X\|$$

so that the right side of (0.9) follows immediately from Corollary 2.11.

5. Applications to harmonic analysis

Using some of the results of the preceding sections we will show that for $1<p<2$ the space of all p-stable a.s. continuous random Fourier series can be identified with the predual of a certain space of multipliers. Since our methods are similar to those used (in the case $p=2$) in [24], in Chapter VI of [21], in [25] and in [26] we will not give too many details.

Let G be a compact Abelian group with discrete dual group Γ. We denote by μ the normalized Haar measure on G. Recall that, by definition, a "pseudo-measure" f is a formal Fourier series $f=\Sigma_{\gamma\in\Gamma}\hat{f}(\gamma)\gamma$ such that $\sup_{\gamma\in\Gamma}|\hat{f}(\gamma)|<\infty$. Let $1\leq p\leq\infty$. We denote by F_p the space of all pseudo-measures f such that $\Sigma_{\gamma\in\Gamma}|\hat{f}(\gamma)|^p<\infty$ and equip this space with the norm

$$\|f\|_{F_p}=\left(\sum_{\gamma\in\Gamma}|\hat{f}(\gamma)|^p\right)^{1/p}.$$

Let φ_q denote the Orlicz function

$$\varphi_q(x)=x(1+\log(1+x))^{1/q}.$$

The functions φ_q and ψ_q (which was defined prior to Lemma 3.1), are in duality, in the sense that L^{ψ_q} is the dual of L^{φ_q}, and the corresponding norms are equivalent. For $1<p<2$ and $1/p+1/q=1$, we will denote by $A(p,\varphi_q)$ the space of all functions f in F_p which can be written as

$$f=\sum_{n=1}^{\infty}h_n * k_n$$

with

$$\sum_{n=1}^{\infty}\|h_n\|_{F_p}\|k_n\|_{\varphi_q}<\infty.$$

We define

$$\|f\|_{A(p,\varphi_q)}=\inf\left\{\sum_{n=1}^{\infty}\|h_n\|_{F_p}\|k_n\|_{\varphi_q}\right\}$$

where the infimum runs over all such representations.

Let $(\theta_\gamma)_{\gamma\in\Gamma}$ be an i.i.d. sequence of p-stable random variables with parameter 1, defined on $(\Omega,\mathscr{A},P)$. We denote by $C^p_{\text{a.s.}}$ the space of all f in F_p such that the series $\Sigma_{\gamma\in\Gamma}\hat{f}(\gamma)\,\theta_\gamma\gamma$ is a.s. continuous, $(0<p\leq 2)$. If $p>1$ we equip this space with the norm

$$[\![f]\!]_p=E\left\|\sum_{\gamma\in\Gamma}\hat{f}(\gamma)\,\theta_\gamma\gamma\right\|_{C(G)}$$

where $\|\ \|_{C(G)}$ is the standard sup-norm on the space of continuous functions on G. It is not hard to see that $(C^p_{\text{a.s.}},[\![\ ]\!]_p)$ is a Banach space. If $p=1$, we define

$$[\![f]\!]_1=\sup_{c>0}c\mathbf{P}\left\{\left\|\sum_{\gamma\in\Gamma}\hat{f}(\gamma)\,\theta_\gamma\gamma\right\|_{C(G)}>c\right\}.$$

The space $C^1_{\text{a.s.}}$ equipped with this "norm" is a quasi Banach space. If $p<1$, it is easy to see that f belongs to $C^p_{\text{a.s.}}$ iff $\Sigma_{\gamma\in\Gamma}|\hat{f}(\gamma)|^p<\infty$, so this case is trivial.

Let f be in F_p. For $t\in G$, let f_t be the translated function, i.e. $f_t(x)=f(t+x)$. We introduce the pseudo-metric d^f_p defined by

$$d^f_p(s,t)=\|f_s-f_t\|_{F_p}=\left(\sum_{\gamma\in\Gamma}|\hat{f}(\gamma)|^p|\gamma(s)-\gamma(t)|^p\right)^{1/p}.$$

To simplify the notation we will write

$$E_{p,r}(f)=\int_0^\infty(\log N(G,d^f_p;\varepsilon))^{1/r}d\varepsilon+|\hat{f}(0)|.$$

We can now state the main result of this section.

THEOREM 5.1. *Let f be in F_p, $1<p<2<q<\infty$, $1/p+1/q=1$. The following properties are equivalent:*

(i) *f belongs to $C^p_{\text{a.s.}}$,*
(ii) *$E_{p,q}(f)<\infty$,*
(iii) *f belongs to $A(p,\varphi_q)$.*

Moreover, $[\![f]\!]_p$, $\|f\|_{A(p,\varphi_q)}$ and $E_{p,q}(f)$ are all equivalent quasi-norms.

Proof. We only sketch the proof of the equivalence of the three functionals under consideration. By Theorem 2.1 in [26], we can immediately deduce that

$$\|f\|_{A(p,\varphi_q)}\leq d_p\,E_{p,q}(f)$$

for some constant d_p. From Theorem C, we have

$$\frac{1}{C'_p}E_{p,q}(f)\leq[\![f]\!]_p\leq C'_p\,E_{p,q}(f)$$

for some constant C'_p. Therefore, it remains only to show that

$$[\![f]\!]_p\leq C''_p\|f\|_{A(p,\varphi_q)}$$

for some constant C''_p. This is an immediate consequence of the following lemma.

LEMMA 5.2. *If $h\in F_p$ and $k\in L^{\varphi_q}$ then $h*k\in C^p_{\text{a.s.}}$ and moreover*

$$[\![h * k]\!]_p \leqslant C_p'' \|h\|_{F_p} \|k\|_{\varphi_q}$$

for some constant C_p'' depending only on p.

To prove Lemma 5.2 we will need the following lemma.

LEMMA 5.3. *Let $(\varepsilon_\gamma)_{\gamma \in \Gamma}$ be an i.i.d. sequence of Rademacher random variables defined on some probability space $(\Omega', \mathscr{A}', P')$. Let $(a_\gamma)_{\gamma \in \Gamma}$ belong to $l_{p,\infty}(\Gamma)$. Then for almost all $\omega' \in \Omega'$ the function $\Sigma_{\gamma \in \Gamma} a_\gamma \varepsilon_\gamma(\omega')\gamma$ is in $L^{\psi_q}(d\mu)$. Moreover,*

$$E_{\omega'} \left\| \sum_{\gamma \in \Gamma} a_\gamma \varepsilon_\gamma(\omega') \gamma \right\|_{\psi_q} \leqslant r_q \|(a_\gamma)_{\gamma \in \Gamma}\|_{p,\infty}$$

for some constant r_q.

The proof of Lemma 5.3 is quite similar to that of Lemma 1.3, Chapter VI, [21] but using Lemma 3.1 instead of the corresponding result for $p=q=2$.

Proof of Lemma 5.2. We begin by recalling that $(\theta_\gamma)_{\gamma \in \Gamma}$ has the same distribution on Ω as $(\varepsilon_\gamma(\omega')\theta_\gamma(\omega))_{\gamma \in \Gamma}$ has on $\Omega \times \Omega'$, $(\omega \in \Omega,\ \omega' \in \Omega')$. Let $a_\gamma(\omega) = \theta_\gamma(\omega)\hat{h}(\gamma)$. Since $h \in F_p$, by Corollary 3.8, $(a_\gamma(\omega))_{\gamma \in \Gamma} \in l_{p,\infty}(\Gamma)$ ω a.s. Therefore, by Lemma 5.3 the function

$$H_{\omega,\omega'}(t) = \sum_{\gamma \in \Gamma} a_\gamma(\omega)\, \varepsilon_\gamma(\omega')\, \gamma(t)$$

belongs $\Omega \times \Omega'$ a.s. to the space $L^{\psi_q}(d\mu)$. Moreover, we have

$$\|E_{\omega,\omega'} H_{\omega,\omega'}\|_{\psi_q} \leqslant C_1(q)\|h\|_{F_p} \tag{5.1}$$

for some constant $C_1(q)$. By a well known duality argument, since k belongs to L^{φ_q} and $H_{\omega,\omega'}$ belongs to L^{ψ_q} a.s., the function $H_{\omega,\omega'} * k$ belongs to $C(G)$ almost surely and furthermore

$$\|H_{\omega,\omega'} * k\|_{C(G)} \leqslant C_2(q)\|H_{\omega,\omega'}\|_{\psi_q} \|k\|_{\varphi_q} \tag{5.2}$$

for some constant $C_2(q)$. Therefore, by (5.1) and (5.2) we have

$$E\|H_{\omega,\omega'} * k\|_{C(G)} \leqslant C_1(q)\, C_2(q) \|h\|_{F_p} \|k\|_{\varphi_q}.$$

This completes the proof of Lemma 5.2 and consequently of Theorem 5.1.

When p and q are no longer assumed to be conjugates we can extend Theorem 5.1 as follows:

THEOREM 5.4. *Assume* $1<p<2$, $1/p+1/q=1$ *and* $q<r<\infty$. *Let* f *be in* F_p.

(i) *Then* f *belongs to* $A(p,\varphi_r)$ *iff* $E_{p,r}(f)<\infty$ *and moreover, there is a constant* $C_{p,r}$ *depending only on* p *and* r *such that*

$$C_{p,r}^{-1}\, E_{p,r}(f)\leqslant \|f\|_{A(p,\varphi_r)}\leqslant C_{p,r}\, E_{p,r}(f), \quad \forall f\in A(p,\varphi_r).$$

(ii) *The space* $A(p,\varphi_r)$ *coincides with the interpolation space* $[A(p,\varphi_q),F_p]_{\theta,1}$ *obtained by the Lions-Peetre interpolation method* (*cf.* [3]) *where* θ *is defined by the relation* $1/r=(1-\theta)/q+\theta/\infty$. *Moreover, the corresponding norms are equivalent on* $A(p,\varphi_r)$.

In the particular case $p=q=2$ Theorem 5.4 is proved in detail in [26]. Now that we have established Theorem 5.1 the general case $1<p<2<q<r$ can be proved by a trivial modification of the arguments of [26].

Remark 5.5. Using the ideas of [25] and [26] relating interpolation spaces and the functionals $E_{p,r}(\cdot)$, along with the preceding results, it is not hard to prove the inclusions

$$[C^1_{\text{a.s.}}, C^2_{\text{a.s.}}]_{\theta,1}\subset C^p_{\text{a.s.}}\subset [F_1, C^2_{\text{a.s.}}]_\theta$$

where $[\ ,\ ]_\theta$ denotes the complex interpolation functor and $1/p=(1-\theta)/1+\theta/2$, $1<p<2$. We do not know if $C^p_{\text{a.s.}}$ (or equivalently $A(p,\varphi_q)$) coincides with a suitable interpolation space either between F_1 and $C^2_{\text{a.s.}}$ or between $C^1_{\text{a.s.}}$ and $C^2_{\text{a.s.}}$.

Remark 5.6. (i) It would be quite interesting to find a direct proof of the fact that the functional $E_{p,r}$ is equivalent to a norm if $1/p+1/r\leqslant 1$ and $1<p\leqslant 2$. It is not hard to see that this is no longer the case if $1/p+1/r>1$. However, we conjecture that $E_{p,r}$ is equivalent to a norm when $1/p+1/r\leqslant 1$ and $p>2$. (Unfortunately we can not find a substitute for Theorems 5.1 and 5.4 in this case.) Note that for $p>2$, $1/p+1/q=1$, it is rather clear that $\|\cdot\|_{A(p,\varphi_q)}$ and $E_{p,q}(\cdot)$ are no longer equivalent functionals. This can be seen by considering lacunary series.

(ii) By a well known comparison principle (cf. [14]) we know that if $1<p_1<p_2\leqslant 2$

$$[\![f]\!]_{p_2}\leqslant C(p_1,p_2)[\![f]\!]_{p_1}, \quad \forall f\in C^{p_1}_{\text{a.s.}},$$

where $C(p_1,p_2)$ is a constant depending only on p_1 and p_2. By Theorem 5.1 this implies

$$E_{p_1,q_2}(f)\leqslant C'(p_1,p_2)\,E_{p_1,q_1}(f) \tag{5.3}$$

where $1/p_1+1/q_1=1$, $1/p_2+1/q_2=1$ and $C'(p_1,p_2)$ is a constant depending only on p_1 and p_2. However, we know of no direct proof of (5.3).

(iii) It is natural to raise the following question: *Problem:* Is (5.3) true if $2\leqslant p_1<p_2\leqslant\infty$? In particular, is $E_{\infty,1}(f)$ dominated by a multiple of $E_{2,2}(f)$?

Remark 5.7. Let $(\theta_\gamma)_{\gamma\in\Gamma}$ be as above. By exactly the same proof as the proof of Corollary 1.10, Chapter VI, [21], we can show that for each p, $1<p<2$, there is a number $\delta_p>0$ such that: For any finite subset $A\subset\Gamma$ of cardinality n, we have

$$E\left\|\sum_{\gamma\in A} a_\gamma\,\theta_\gamma\gamma\right\|_{C(G)}\geqslant\delta_p\,n^{1/p}(\log n)^{1/q}\frac{1}{n}\sum_{\gamma\in A}|a_\gamma|. \tag{5.4}$$

On the other hand it follows easily from the results of Section 3 that

$$E\left\|\sum_{k=1}^{n}\theta_k\,e^{ikt}\right\|_{C(G)}\leqslant C_p\,n^{1/p}(\log(n+1))^{1/q}$$

for some constant C_p independent of n. Therefore the left side of (5.4) is essentially minimal when $(\gamma)_{\gamma\in A}$ are consecutive integers.

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