The index theorem for topological manifolds

by

NICOLAE TELEMAN [1]

State University of New York,
Stony Brook, N.Y., U.S.A.

Dedicated to M. F. Atiyah and I. M. Singer

§ 0. Introduction

The famous Atiyah–Singer index theorem expresses the Fredholm index of any elliptic pseudo-differential operator on an arbitrary closed smooth manifold as the period of a universal algebraic combination of the Pontrjagin classes of the manifold, and of the Chern character of the symbol of the operator.

In 1969 M. F. Atiyah [A_1]—see also [A_2]—by taking as axioms some of the basic properties of the elliptic pseudo-differential operators of order zero on closed smooth manifolds, introduces the notion of abstract elliptic operator, for any compact topological space.

In 1970 I. M. Singer [Si] exposes a comprehensive program aimed to extend the theory of ellipic operators and their index to more general situations.

D. Sullivan's theorem [Su_1] about existence of an orientation class in the K-theory, in the world of odd primes, of PL-bundles over PL-manifolds, gives evidence that at least the *symbol* of *signature operators* on PL-manifolds should exist, see I. M. Singer [Si] § 3. One of the problems I. M. Singer formulated in [Si] was that of realizing, geometrically, signature operators on PL-manifolds, and at the same time, pointed out that such a realization could lead to an analytical proof of Novikov's theorem about the topological invariance of the rational Pontrjagin classes.

In 1975, L. G. Brown, R. G. Douglas, P. A. Fillmore [B.D.F.], and G. G. Kasparov [K] show, independently, in different contexts, that the stably-homotopic classes of abstract elliptic operators on the compact metric space X form an abelian group, which is naturally isomorphic to $K_0(X)$, the Spanier–Whitehead dual of $K^0(X)$. The same year P. Baum, W. Fulton, R. MacPherson [B.F.M.] prove the Riemann–Roch theorem for singular varieties.

In 1976, D. Sullivan discovers that any topological manifold of dimension $\neq 4$ admits a unique—up to homeomorphisms close to the identity—Lipschitz structure.

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In 1977, N. Teleman $[T_1]$ obtains preliminary results for a Hodge theory on pseudo-manifolds. In 1978–79, N. Teleman $[T_2]$ constructs a Hodge theory on *PL*-manifolds, and J. Cheeger [C] produces Hodge theory on a very general class of pseudo-manifolds.

In 1980, D. Sullivan and N. Teleman [S.T.] obtain an analytical proof of Novikov's theorem by showing that the index of signature operators on Lipschitz manifolds, introduced in $[T_3]$, are topological invariants. The proof realizes at the same time a constructive procedure for the definition of the rational Pontrjagin classes of topological manifolds.

In the first section of this paper we extend the Hirzebruch–Atiyah–Singer signature theorem on topological manifolds; we show, Theorem 1.1, that the index of the signature operators with values in continuous vector bundles over topological manifolds can be expressed in topological terms by the same formula as in the smooth case.

The proof of the topological signature Theorem 1.1 is based on the proof of the cobordism invariance of the index in the topological context. This last proof uses the excision Theorem 12.1 from $[T_3]$ in contrast with the method used by M. F. Atiyah, I. M. Singer in the smooth case, which is based on the study of the Cauchy problem.

The other sections of the paper are dedicated to the study of the index of the abstract elliptic operators on topological manifolds. We show that for any abstract elliptic operator on an arbitrary closed, oriented topological manifold, its topological index can be defined, in a natural way, and that it equals the Fredholm index of the operator, Theorem 6.3.

The Index Theorem 6.3. extends the Atiyah–Singer non-equivariant index theorem [P]. Our proof follows the main steps of the original Atiyah–Singer's proof.

The original Atiyah–Singer's proof consists of checking first the index theorem on the signature operators with values in vector bundles, and to show in a second moment, that these signature operators generate, by homotopying and stabilizing their symbols, all other elliptic operators, modulo torsion.

The first question toward an index theorem for abstract elliptic operators is how to define the topological index of such operators, given that, first of all, these operators do not have a symbol.

The symbol of an abstract elliptic operator on M might be defined as the class of the operator itself modulo compact operators. The equivalence relation used in the definition of the K-homology group $K_0(M)$ is stronger than the factorization by compact operators—see G. Kasparov [K]; therefore $K_0(M)$ may be thought of as the K-theory of stably-homotopic classes of symbols of abstract elliptic operators on M.

Having now a good definition for the symbol, a new problem arises: to make the signature operators $D_\xi^+ : W_1^+(M,\xi) \to W_0^-(M,\xi)$ from [S.T.] represent elements in $K_0(M)$.

This is a serious difficulty because the signature operators D_ξ^+ are operators of *order* 1, while the abstract elliptic operators are operators of *order* 0; that is, the last operators act between Hilbert spaces which are modules over the algebra of continuous functions over M, while the signature operators D_ξ^+ are defined on the spaces $W_1^+(M,\xi)$, which are analogues of the Sobolev spaces of order 1, and they are not modules over the algebra of continuous functions, in the obvious way. In the smooth case, the calculus of pseudo-differential operators permits to convert easily on operator of order one into an operator of order zero so that its symbol restricted at the cotangent sphere bundle remained unchanged.

We succeed to convert *abstractly*, instead, an operator of order one into an operator of order zero by preserving the K-homology class of its *symbol*. To do this, we create first a new K_*-homology, $K_*^{(1)}(-)$, for operators of order 1 (where the signature operators live) and, afterwards we show that the natural transformation of generalized homology functions $K_*(-) \to K_*^{(1)}(-)$ is an equivalence.

$K_*^{(1)}(-)$ is constructed along the lines in [K] by taking Hilbert space representations of the Banach algebra $C^1(-)$ of functions of class C^1 instead of representations of the algebra of continuous functions. Unfortunately, $C^1(-)$ is not a C^*-algebra, and this forces us to find a new proof for the suspension axiom.

The reason why we start the construction of our $K_*^{(1)}(-)$ from functions of class C^1 instead of Lipschitz functions—which appear naturally in the entire theory—is that the Banach algebra of Lipschitz functions is not finitely generated (as most of the Kasparov's considerations require), while $C^1(-)$ is finitely generated. Notice that we may not, of course, speak of C^1-functions on a topological manifold with Lipschitz structure; we may bypass this difficulty by taking C^1-functions on a tubular neighborhood of a Lipschitz embedding of the manifold in $\mathbf{R}^N$. (In other words, we use an embedding of the topological manifold M with Lipschitz structure, into $\mathbf{R}^N$, in order to select—in a fairly natural way—a finitely generated Banach sub-algebra of the algebra of Lipschitz functions on M.)

Existence of an index theorem for abstract elliptic operators which generalizes the non-equivariant Atiyah–Singer index theorem shows, in particular, that the axioms for the abstract elliptic operators are the only properties of the elliptic pseudo-differential operators on smooth manifolds which are essentially involved in the index theorem.

This paper was essentially written at the California Institute of Technology, Pasadena, California, in 1980. Its results were announced in [T_2]. The publication of this paper was delayed in connection with the previous papers [T_3] and [S.T.] upon which it depends.

It is a pleasure to express our warmest thanks to many mathematicians who, in different ways and, at different times, contributed to the appearance of this paper: J. Eells, E. Martinelli, C. Schochet, I. M. Singer, D. Sullivan and C. Teleman.

§ 1. The index theorem for signature operators on topological manifolds

It was shown in [T_3] that for any Lipschitz vector bundle ξ over an arbitrary closed, oriented, even-dimensional Lipschitz manifold M, signature operators D_ξ^+ can be constructed. These operators are natural, that is, they coincide with the Atiyah–Singer's signature operators when all objects involved in their construction are smooth. The important fact about these operators is that they are Fredholm operators and that the Index D_ξ^+ is a Lipschitz invariant of the pair (M, ξ).

A fundamental result by D. Sullivan [S_2] asserts that any topological manifold of dimension $\neq 4$ admits a Lipschitz structure, and that this structure is essentially unique. Sullivan's theorem makes then possible to construct signature operators not only on Lipschitz manifolds, but on an arbitrary closed, topological manifold of even dimension $\neq 4$. It was shown by Sullivan–Teleman [S.T.] the important result that Index D_ξ^+ is not merely a Lipschitz invariant of the pair (M, ξ)—for an arbitrary Lipschitz structure on M—but, it is a topological invariant of it.

An immediate consequence of this result is the famous theorem of Novikov on the topological invariance of the rational Pontrjagin classes [N]. This corollary provides a constructive way for the definition of the rational Pontrjagin classes on topological manifolds.

This section of the paper is intended to prove first an index theorem for signature operators on topological manifolds.

THEOREM 1.1 (the index theorem for topological signature operators). *For any continuous vector bundle ξ over the closed, oriented topological manifold M,* $\dim M \geq 6$,

$$\operatorname{Index} D_\xi^+ = \operatorname{ch}\xi \cup L(M)\,[M]; \tag{1.1}$$

here $\operatorname{ch}\xi$ *denotes the Chern character of* ξ, $L(M)$ *denotes the Hirzebruch polynomials on the rational Pontrjagin classes of* M, *and* $[M]$ *is the fundamental class of* m.

Proof of Theorem 1.1. The idea of the proof of Theorem 1.1 consists of reducing the verification of the formula (1.1) from the case of topological signature operators, to the case of smooth signature operators. This reduction is made possibly by the use of topological cobordisms. The use of cobordisms in the study of the signature was previously done by F. Hirzebruch [H_3] and M. F. Atiyah and I. M. Singer, see [P].

More precisely, the proof proceeds in this way. (1) The index of topological signature operators is invariant at topological cobordisms. (2) Any continuous vector bundle over an arbitrary topological manifold is **Q**-topologically cobordant to a smooth vector bundle over some smooth manifold. (3) For smooth signature operators, the formula (1.1) is a particular case of the Atiyah–Singer index theorem, see [A.S.].

From these three statements, only the first one is new for topological manifolds. The remainder of this section is devoted to its proof. At the end of it, we will give the precise formulation for the statement (2).

Let $\xi \to W$ be a continuous vector bundle over the oriented, compact topological manifold W of dimension $m+1$. Suppose that the boundary of W consists of two disjoint, oriented manifolds M_1 and $-M_0$. We set $\xi_j = \xi | M_j$, $j=0, 1$. Then (ξ, W) is called an *(oriented) topological cobordism* between the bundles (ξ_0, M_0) and (ξ_1, M_1); the bundles themselves will be called *topologically cobordant.*

THEOREM 1.2 (topological cobordism invariance of the index). *If* (ξ_0, M_0), (ξ_1, M_1) *are topologically cobordant vector bundles,* $\dim M_0 = \dim M_1 \geq 5$, *then*:

$$\operatorname{Index} D^+_{\xi_0} = \operatorname{Index} D^+_{\xi_1}. \tag{1.2}$$

The proof of Theorem 1.2 depends upon rather delicate known topological techniques, along with the excision Theorem 12.1. [T_3].

One of these techniques is contained in the following:

THEOREM 1.3 (see R. Kirby and L. C. Siebenmann [K.S.], Essay III, Theorem 2.1). *For any compact topological manifold W of dimension* $m+1 \geq 6$, *and for any clean* $(m+1)$*-submanifold* $W_0 \subset W$, *there exists a* handlebody decomposition, *i.e. a finite filtration*:

$$W_0 \subset W_1 \subset \ldots \subset W_r = W \tag{1.3}$$

by (closed) clean $(m+1)$*-submanifolds such that, if* H_{k+1} *denotes Closure* $(W_{k+1} \setminus W_k)$, *then*

$$(H_{k+1}, H_{k+1} \cap W_k) \tag{1.3'}$$

is homeomorphic to $(B^p, \partial B^p)\times B^q$, $p+q=m+1$, *where* B^p *is the closed p-dimensional unit ball.* H_{k+1} *is called a* handle.

If (ξ, W) is a cobordism between the bundles (ξ_0, M_0), (ξ_1, M_1), we take $W_0\subset W$ to be a collar on M_0, and we consider a filtration (1.3) as given by Theorem 1.3.

Notice that the boundary of W_i is a disjoint union of two closed, oriented m-manifolds; one of them is $-M_0$, and the complementary of it, which we call N_i. Notice also that $N_0\cong M_0$, and $N_r=M_1$.

Therefore, if (ξ, W) is a cobordism as above, we may extract from it a chain of vector bundles over closed, oriented topological m-dimensional submanifolds in W:

$$(\xi_0, M_0)\cong(\eta_0, N_0),\quad (\eta_1, N_1), \dots, (\eta_r, N_r)=(\xi_1, M_1). \tag{1.4}$$

Any two consecutive bundles (η_k, N_k), (η_{k+1}, N_{k+1}) in this chain have the following properties:

$$N_{k+1} \text{ results from } N_k \text{ by a surgery } \chi_k \text{ on } N_k, \tag{1.5 i}$$

$$\text{the bundles } \eta_k, \eta_{k+1} \text{ coincide over the common part } N_k\cap N_{k+1}, \tag{1.5 ii}$$

$$\begin{gathered} N_k\cap N_{k+1} \text{ is precisely that region of } N_k \text{ (and } N_{k+1})\\ \text{which is not affected by the surgery } \chi_k. \end{gathered} \tag{1.5 iii}$$

The regions of N_k and N_{k+1} which are affected by the surgery χ_k are (see (1.3′)):

$$\begin{aligned} N_k\cap H_{k+1} &\simeq \partial B^p\times B^q, \quad (\text{in } N_{k+1})\\ N_{k+1}\cap H_{k+1} &\simeq B^p\times \partial B^q, \quad (\text{in } N_k). \end{aligned} \tag{1.6}$$

We are going to prove now that the excision Theorem 12.1 [T3] implies:

$$\operatorname{Index} D^+_{\eta_k} = \operatorname{Index} D^+_{\eta_{k+1}}, \quad 0\leqslant k\leqslant r-1, \tag{1.7}$$

which will complete the proof of Theorem 1.2.

THEOREM 1.4 (N. Teleman [T3], Theorem 12.1). *Let* M_α, $\alpha=1, 2$, *be closed, oriented Lipschitz Riemannian manifolds, and let* ξ_α, θ_α *be two Lipschitz vector bundles over* M_α.

Suppose $U\neq\emptyset$ is an open common Lipschitz submanifold of M_1 and M_2, and let $V\subset U$ be an open subset in U such that $\bar{V}\subset U$.

Suppose that these vector bundles can be identified as in the diagram:

$$\begin{tikzcd} \xi_1 \arrow[rr, leftrightarrow, "\text{over } M_1\setminus\bar{V}"] \arrow[d, leftrightarrow, "\text{over } U"'] & & \theta_1 \arrow[d, leftrightarrow, "\text{over } U"] \\ \xi_2 \arrow[rr, leftrightarrow, "\text{over } M_2\setminus\bar{V}"'] & & \theta_2 \end{tikzcd} \tag{1.8}$$

in a compatible way over $U\setminus\bar{V}$.

Then, (i):

$$\operatorname{Index} D^+_{\xi_1} - \operatorname{Index} D^+_{\xi_2} = \operatorname{Index} D^+_{\theta_1} - \operatorname{Index} D^+_{\theta_2}; \tag{1.9}$$

(ii) if, in addition to the hypotheses, the bundles θ_1 and θ_2 are trivial bundles, and $\operatorname{Sig} M_1 = \operatorname{Sig} M_2$, then

$$\operatorname{Index} D^+_{\xi_1} - \operatorname{Index} D^+_{\xi_2} = 0. \tag{1.10}$$

The desired equality (1.7) follows from (1.10) for the following choices for the manifolds and bundles from Theorem 1.4:

$$\begin{aligned} M_1 &\equiv N_k, \quad M_2 \equiv N_{k+1} \\ \xi_1 &\equiv \eta_k, \quad \xi_2 \equiv \eta_{k+1}, \end{aligned} \tag{1.10}$$

and θ_1, θ_2 are taken to be product bundles over N_k, N_{k+1}; (let $T_1 \subset T_{1+\varepsilon} \subset N_k \cap N_{k+1}$ be two small closed collar neighborhoods of ∂H_{k+1} in $N_k \cap N_{k+1}$), we finally take:

$$\begin{aligned} U &= (N_k \cap N_{k+1}) \setminus T_1, \\ V &= (N_k \cap N_{k+1}) \setminus \mathring{T}_{1+\varepsilon}. \end{aligned} \tag{1.10$'$}$$

For which regards the diagram (1.8), we identify ξ_1 and ξ_2 over U in the natural way given that η_k, η_{k+1} coincide over $N_k \cap N_{k+1}$; we do the same with the bundles θ_1, θ_2.

Notice also that the bundle ξ is trivial over $H_{k+1} \cup T_{1+\varepsilon}$ because this space is contractible. A trivialization of ξ over $H_{k+1} \cup T_{1+\varepsilon}$ will provide, by restriction at $N_k \setminus V$ and $N_{k+1} \setminus V$ the required compatible horizontal identifications from the diagram (1.8).

Clearly, the manifolds N_k, N_{k+1} have the same signature because one is obtained from the other by surgery.

Finally, in order to be entitled to apply the Theorem 1.4 we have to make sure that the manifolds N_k, N_{k+1} have Lipschitz structures with the property that U from (1.10′) is an open Lipschitz submanifold in both N_k and N_{k+1}.

To show this, we take first a Lipschitz structure $\mathscr{L}_k$ on N_k. We take on $N_k \cap N_{k+1} \subset N_{k+1}$ the Lipschitz structure $\mathscr{L}_k | N_k \cap N_{k+1}$. We extend this last Lipschitz structure on the whole N_{k+1}, possibly by modifying it on a small neighbourhood of $\partial(N_k \cap N_{k+1})$, see D. Sullivan [S2] and P. Tukia and J. Väisälä [T.V.] §4.

This completes the proof of Theorem 1.2.

Remark 1.5. The proof of the topological cobordism invariance Theorem 1.2, given here, provides also an alternative proof for the smooth cobordism invariance theorem by Atiyah–Singer [P].

In order to complete the proof of the Theorem 1.1, we need to compare the bordism groups $\Omega_*^{STOP}(-)$, respectively $\Omega_*^{SO}(-)$, of oriented topological, respectively, smooth manifolds.

THEOREM 1.6 (Proposition 10.3, Annex C by L. C. Siebenmann, in [L.C.]). *The natural homomorphism:*

$$i_m \otimes \mathbf{1}_{\mathbf{Q}} : \Omega_m^{SO}(point) \otimes \mathbf{Q} \to \Omega_m^{STOP}(point) \otimes \mathbf{Q}, \quad m \in \mathbf{N}, \tag{1.11}$$

is an isomorphism.

This theorem implies that the homology functors:

$$\Omega_*^{SO}(-) \otimes \mathbf{Q}, \quad \Omega_*^{STOP}(-) \otimes \mathbf{Q}$$

are naturally equivalent over the category of CW-complexes. In particular, if BU denotes the classifying space for the infinite unitary group, the homomorphism:

$$i_* \otimes \mathbf{1}_{\mathbf{Q}} : \Omega_*^{SO}(BU) \otimes \mathbf{Q} \to \Omega_*^{STOP}(BU) \otimes \mathbf{Q}, \quad * \in \mathbf{N}, \tag{1.12}$$

is an isomorphism. The groups involved in the isomorphism (1.12) are precisely the groups of smooth, respectively topological, cobordisms of complex vector bundles; compare [P] Chapter XVIII.

§2. Preliminaries for K-homology of operators of order $\geqslant 0$

[*Note* added September 8, 1983. In the following sections 2 and 3 we produce a K-homology theory for operators of order $\geqslant 0$. In the section 4 we show that the groups

$K_*(-)$ which we construct are modules over $K^0(-)$. We need this theory in order to make the (first order) signature operators, introduced in [T_3], represent K_0-homology classes, as it will be shown in the section 5.

A few months ago, important and elegant contributions in this theory were made by S. Baaj, P. Julg [B.J.], and M. Hilsum [H_1]. Specifically, S. Baaj and P. Julg constructed a theory in which the K-homology classes are represented by self-adjoint unbounded operators with additional properties; their theory is very natural for first order operators. Moreover, M. Hilsum shows that the signature operators introduced in [T_3] fit naturally into the Baaj–Julg construction, and gives alternative, elegant proofs for two basic results from [T_3].]

We are especially interested in the complex K-homology. For this reason we don't refer to the real or *real* K-homologies, even though their constructions would require only minor formal modifications, see [K].

In which follows, H denotes a separable complex Hilbert space, while $L(H)$ the C^*-algebra of continuous linear operators on H. Let $K(H)\subset L(H)$ denote the ideal of compact operators on H, $A(H)=L(H)/K(H)$ the Calkin algebra, and $\Pi: L(H)\to A(H)$ the canonical projection.

If $S, T\in L(H)$, we write $S\sim T$ iff $S-T\in K(H)$.

We denote by $L^U(H)\subset L(H)$ the group of unitary operators on H.

$C_{p,q}$ denotes the Clifford algebra of the quadratic form on $\mathbf{R}^{p+q}$:

$$-x_1^2-\ldots-x_p^2+x_{p+1}^2+\ldots+x_{p+q}^2;$$

$e_1, \ldots, e_p$, resp. $\varepsilon_1, \ldots, \varepsilon_q$, are the first p, resp. last q, unit vectors in $\mathbf{R}^{p+q}$.

An involution $\ast$ will be defined on $C_{p,q}$ acting on the generators in this way

$$\ast e_i=-e_i, \quad \ast \varepsilon_j=\varepsilon_j.$$

Let k be a fixed non-negative integer.

We will construct a K-homology functor on the category $\mathscr{C}^k$ described as follows.

The objects $\mathscr{C}^k$ are compact subspaces $X\subset\mathbf{R}^N$, N arbitrary, having the property that the interior of X is sense in X.

If $X\subset\mathbf{R}^{N_1}$ and $Y\subset\mathbf{R}^{N_2}$ are two objects in $\mathscr{C}^k$, then $\mathrm{Hom}_{\mathscr{C}^k}(X, Y)$ iff $f: X\to Y$ is a mapping of class C^k. We say that f is of class C^k iff f has an extension $\tilde{f}: \tilde{X}\to\mathbf{R}^{N_2}$ of class C^k defined on a neighborhood $\tilde{X}\supset X$.

For any $X\in \mathcal{O}b\mathscr{C}$, we define:

$$C^k(X) = \{f|f: X \to \mathbf{C}, f \text{ of class } C^k\}.$$

We introduce on $C^k(X)$ the norm $\| \|_k^{\hat{}}$

$$\|f\|_k^{\hat{}} = \sum_{|\alpha| \leq k} \sup_{x \in X} \left| \frac{\partial^\alpha f(x)}{\partial x^\alpha} \right|$$

$\{C^k(X), \| \|_k^{\hat{}}\}$ is a unital, commutative, Banach algebra with involution, the involution being the complex conjugation: $f \mapsto \bar{f}$.

$C^k(X)$ is finitely generated because the polynomials are dense in $C^k(X)$. (This is the main reason why we selected this subalgebras of the algebra of Lipschitz functions, in order to construct an appropriate K-homology functor along the lines in [K].)

Unfortunately, $C^k(X)$ is not a C^*-algebra if $k \geq 1$ and X is not discrete.

For any $X \in \mathcal{O}b\mathcal{C}^k$, the cone $C(X)$, the cone $C(X)$, the suspension $\Sigma(X)$ and the cone Ci of any inclusion $i: A \hookrightarrow X$ are objects in $\mathcal{C}^k$. Specifically, these objects are realized as follows. We define, for $X \subset \mathbf{R}^N$,

$$C^{\pm}(X) = \{tx \pm (1-t)\, e_{N+1} | x \in X,\ 0 \leq t \leq 1\} \subset \mathbf{R}^{N+1},$$

with $e_{N+1} = (0, \ldots, 0, 1) \in \mathbf{R}^{N+1}$.

Then $C(X) = C^+(X)$, and $\Sigma(X) = C^-(X) \cup C^+(X)$. We define $Ci \subset \mathbf{R}^{N+1}$ so:

$$Ci = \{(x,t) \in \mathbf{R}^N \times \mathbf{R},\ x \in X,\ t \in [-1, 0]\} \cup C^+A.$$

We introduce also, for any $0 \leq a \leq b \leq 1$,

$$C^+_{(a,b]}(X) = \{tx + (1-t)\, e_{N+1} | x \in X,\ a < t \leq b\} \subset C^+(X).$$

If a group G acts on X, and $A \subset X$ is kept fixed, we extend naturally this action onto $C^{\pm}(X)$, $\Sigma(x)$, and Ci in the obvious way.

In which follows, G denotes an arbitrary compact topological group. We suppose that any action of G on an object X in C^k is admissible, i.e. any transformation in X is of class C^k.

For the following definition and other definitions given here very briefly, we refer to [K].

Definition 1.1. For any $X \in \mathcal{O}b\mathcal{C}^k$, with G acting on X, we define $\mathscr{E}^{(k),G}_{p,q}(X)$ to be the class of all quadruples (χ, φ, ψ, F), where:

(i) $\chi: G \to L^U(H)$ is a continuous homomorphism.

(ii) $\varphi: C^k(X) \to L(H)$ is a homomorphism of Banach algebras which is: unital, equivariant, and quasi-involutive (not necessarily involutive as Kasparov [K] requires).

We say that φ is equivariant iff for any $g \in G$ and $f \in C^k(X)$

$$\varphi(gf) = \chi(g) \circ \varphi(f) \circ \varkappa(g^{-1}),$$

where $gf \in C^k(X)$ is the function

$$gf(x) = f(gx), \quad \text{for any } x \in X.$$

We say that φ is quasi-involutive iff for any $f \in C^k(X)$

$$\varphi(\bar{f}) \sim (\varphi(f))^*.$$

Notice, however, that $\Pi \circ \varphi: C^k(X) \to A(H)$ is involutive. This will suffice in order to construct products # in K-homology.

(iii) $\psi: C_{p,q+1} \to L(H)$ is an involutive homomorphism which commutes with φ (i.e. ψ is "local") and with χ,

(iv) $F \in L(H)$ is a Fredholm operator which satisfies the properties:

(1) $F^* \sim -F$

(2) $F^2 \sim -1$ (1)

(3) F commutes with χ (F is equivariant)

(4) F anticommutes with $\psi(e_i)$, $1 \leq i \leq p$, and with $\psi(\varepsilon_j)$, $1 \leq j \leq q+1$, and

(5) for any $f \in C^k(X)$,

$$\varphi(f) F \sim F \varphi(f),$$

i.e. F is *semi-local*.

Any element $(\chi, \varphi, \psi, F) \in \mathscr{E}_{p,q}^{(k),G}(X)$ will be called an *abstract equivariant elliptic (pseudo-differential) operator of order* $\leq k$ on X.

The notions of equivalence, homotopy, and degenerated elements in $\mathscr{E}_{p,q}^{(k),G}(X)$ are defined as in [K], § 1.

Let $\bar{\mathscr{E}}_{p,q}^{(k),G}(X)$, resp. $\bar{\mathscr{D}}_{p,q}^{(k),G}(X)$, denote the equivalent classes of elements in $\mathscr{E}_{p,q}^{(k),G}(X)$, resp. $\mathscr{D}_{p,q}^{(k),G}(X)$.

$\bar{\mathscr{E}}_{p,q}^{(k),G}(X)$ and $\bar{\mathscr{D}}_{p,q}^{(k),G}(X)$, are abelian semigroups under the sum operation defined by the direct sum of quadruples.

We define:

(1) This requirement may be omitted, see [K] Proposition 2, § 1, and we shall ignore it in the sequel.

$$K_{p,q}^{(k),G}(X)=\frac{\bar{\mathscr{C}}_{p,q}^{(k),G}(X)}{\bar{\mathscr{D}}_{p,q}^{(k),G}(X)}.$$

If (X,A) is a pair in $\mathscr{C}^k$, and G transform A in itself, then the restriction homomorphism $C^k(X)\to C^k(A)$ is an epimorphism and hence we may define $K_{p,q}^{(k),G}(X,A)$ as in [K], § 1, Definition 6. The Definition 7 [K] will be used in order to define $K_{p,q}^{(k),G}(X,\varnothing)$.

It turns out that $K_{p,q}^{(k),G}(X,A)$ is an abelian group.

If $f\colon(X,A)\to(Y,B)$ is an equivariant morphism of pairs in the category $\mathscr{C}^k$, then f induces a homomorphism of abelian groups:

$$f_*\colon K_{p,q}^{(k),G}(X,A)\to K_{p,q}^{(k),G}(Y,B).$$

§ 3. *K*-homology of the algebra of functions of class C^k

The following theorem states that $K_n^{(k),G}(-,-)$ is a generalized homology functor.

THEOREM 3.1. *Let k,p,q be arbitrary non-negative integers. Then:*

(1) $K_{p,q}^{(k),G}(-,-)$ *is a covariant functor from the category of pairs in $\mathscr{C}^k$, with G-action to the category of abelian groups.*

(2) *For any pair* (X,A) in $\mathscr{C}^k$ *there exists a natural isomorphism:*

$$K_{p,q}^{(k),G}(X,A)\xrightarrow{\cong}K_{p+1,q+1}^{(k),G}(X,A).$$

(3) *For any $X\in\mathscr{O}\mathscr{C}^k$, there exists a natural isomorphism*:

$$i_*\colon K_{p,q}^{(k),G}(X)\to K_{p,q}^{(k),G}(X,\varnothing).$$

(4) *There exists a natural transformation of functors*

$$\partial\colon K_{p,q}^{(k),G}(X,A)\to K_{p-1,q}^{(k),G}(A).$$

(5) (Exactness axiom.) *For any pair (X,A) in $\mathscr{C}^k$, the two-sided long exact sequence*

$$\dots\to K_n^{(k),G}(A)\xrightarrow{i_*}K_n^{(k),G}(X)\xrightarrow{j_*}K_n^{(k),G}(X,A)\xrightarrow{\partial}K_{n-1}^{(k),G}(A)\to\dots$$

is exact, where

$$K_n^{(k),G}(X,A)=\begin{cases} K_{n,0}^{(k),G}(X,A), & \text{for } n\geq 0\\ K_{0,-n}^{(k),G}(X,A), & \text{for } n\leq 0;\end{cases}$$

and $i: A\hookrightarrow X$ *is the inclusion.*

(6) (Homotopy axiom.) *If* $H:(X,A)\times[0,1]\to(Y,B)$ *is a homotopy in* $\mathscr{C}^k$, *then for any* $n\in\mathbf{Z}$,

$$(H_0)_*=(H_1)_*: K_n^{(k),G}(X,A)\to K_n^{(k),G}(Y,B),$$

where $H_t(-)=H(-,t):(X,A)\to(Y,B)$, $0\leq t\leq 1$.

(7) (Suspension axiom.) *For any* $X\in\mathscr{O}\mathscr{C}^k$ *there exists a natural isomorphism*

$$\Sigma_n: \tilde{K}_n^{(k),G}(\Sigma X)\to\tilde{K}_{n-1}^{(k),G}(X),$$

where $\tilde{K}_{n-1}^{(k),G}(X)$ *denote the reduced groups.*

Proof. For the proof to the statements (1)–(6) we refer to [K]. However, some comments are in order.

(a) Notice that the relative homotopy axiom can be deduced from the absolute homotopy axiom, exactness axiom and the five-lemma.

(b) Among the statements (1)–(7), only (3) and (6) require quite delicate analysis techniques. Among them, integrals of the form

$$\int_{[0,2\pi]} F(t)\,d(\mathbf{1}_H\otimes\mu)$$

are involved. Here, F denotes a continuous function:

$$F:[0,2\pi]\to L(H\otimes L_2([0,2\pi])),$$

$\mu: C^0([0,2\pi])\to L(L_2([0,2\pi]))$ is the natural C^*-representation given by multiplication of L_2-functions on $[0,2\pi]$ by continuous functions, while $d(\mathbf{1}_H\otimes\mu)$ denotes the spectral measure given by the *involutive*, unital homomorphism:

$$\mathbf{1}_H\otimes\mu: C^0([0,2\pi])\to L(H\otimes L_2([0,2\pi]));$$

this is the reason why we may apply the same proof as in [K], even if φ from (ii) Definition 1.1 is quasi-involutive instead of being involutive.

(c) And now, a short complement to the proof of Theorem 1, §5 [K]. The operators M, N used there may be taken to be of the form:

$$M=\begin{pmatrix} M' & 0 \\ 0 & M'' \end{pmatrix}, \quad N=\begin{pmatrix} N' & 0 \\ 0 & N'' \end{pmatrix},$$

where (M',N'), resp. (M'',N'') satisfy the requirements for (M,N) in the spaces $H\otimes L_2([0,2\pi/3])$, resp. $H\otimes L_2([2\pi/3,2\pi])$, instead of $H\otimes L_2([0,2\pi])$. This is possible because $T_1(b)$ is a pseudo-differential operator whose symbol is identically -1 on $[\pi/3,2\pi]$; therefore $T_1(b)$ may be modified, slightly, so that $T_1(b)$ preserves its symbol and becomes local (multiplication by -1) on $[\pi/2,2\pi]$.

Then M, N will, clearly, commute with projections on $H\otimes L_2([0,2\pi/3])$ and $H\otimes L_2([2\pi/3,2\pi])$.

(d) The proof of (7) will follow from the Lemma 3.1 and 3.2.

LEMMA 3.1 (weak excision axiom). *Let* $Y\in\mathcal{O}\ell\mathscr{C}^k$ *and* $X=\Sigma(Y)$, $A=C^+(Y)$, $U=C^+_{(1/2,1]}(Y)$.

Then, the homomorphism

$$i_*: K^{(k),G}_{p,q}(X\setminus U, A\setminus U)\to K^{(k),G}_{p,q}(X,A)$$

induced by the inclusion

$$i: (X\setminus U, A\setminus U)\hookrightarrow (X,A),$$

is an isomorphism.

Proof. We shall adopt the following convention in order to avoid introducing too many symbols. If $(\chi,\varphi,\psi,F)\in\mathscr{E}^{(k),G}_{p,q}(X,A)$, and $f:(X,A)\to(Y,B)$ is a morphism in $\mathscr{C}^k$, then

$$f_*(\chi,\varphi,\psi,F)=(\chi,f_*\varphi,\psi,F)\in\mathscr{E}^{(k),G}_{p,q}(Y,B),$$

where $f_*\varphi=\varphi\circ f^*$, $f^*: C^k(Y)\to C^k(X)$, and

$$f_*[(\chi,\varphi,\psi,F)]=[(\chi,f_*\varphi,\psi,F)]\in K^{(k),G}_{p,q}(Y,B).$$

We shall construct a homomorphism

$$j: K^{(k),G}_{p,q}(X,A)\to K^{(k),G}_{p,q}(\Sigma(Y)\setminus U, C^+(Y)\setminus U)$$

which will be the inverse for i_*.

Let $\lambda: [0,1]\times[0,1]\to[0,1]$ be a function of class C^{k+1} with the following properties:

(a) $\lambda(s,t)=s$ for $\forall(s,t)\in[0,\frac{1}{4}]\times[0,1]$

(b) $\lambda(s, 0)=s$ for $\forall s \in [0, 1]$

(c) $\lambda(s, t)=1$ for $\forall s \geq 1-\frac{2}{3}t$, and $t>\frac{1}{2}$

(d) for any fixed $t \in [0, 1]$, $\lambda(s, t)$ is a non-decreasing monotone function in s.

Then $\lambda_t(-)=\lambda(-, t)$ is a C^{k+1}-homotopy of the identity mapping of $[0, 1]$.

We use the homotopy λ_t in order to construct a homotopy h_t of class C^k on $\Sigma(Y)$ which *smashes* neighborhoods of the vertex of $C^+(Y)$ progressively. This homotopy is:

$$\begin{cases} h_t|C^-(Y) = \mathrm{Id}_{C^-(Y)} \\ h_t(se_{N+1}+(1-s)x) = \lambda_t(s)\cdot e_{N+1}+(1-\lambda_t(s))x \end{cases}$$

for any $x \in Y$.

Let $\alpha\colon \Sigma(Y) \to [0, 1]$ be a C^k-function:

$$\alpha(x) = \begin{cases} 1 & \text{for } x \in C^-(Y) \cup C^+_{[0, 1/3]}(Y) \\ 0 & \text{for } x \in U = C^+_{[1/2, 1]}(Y). \end{cases}$$

If

$$(\chi, \varphi, \psi, F_s) \in \mathscr{E}^{(k), G}_{p,q}(\Sigma(Y), C^+(Y)),$$

we define

$$j(\chi, \varphi, \psi, F_s) \in \mathscr{E}^{(k), G}_{p,q}(\Sigma(Y) \setminus U, C^+(Y) \setminus U)$$

by the formula

$$j(\chi, \varphi, \psi, F_s) = (\chi, j(\varphi), \psi, F_s),$$

where the representation $j(\varphi)\colon C^k(\Sigma(Y) \setminus U) \to L(H)$ is

$$(j(\varphi))(f) = \varphi(h_1^*(\alpha\cdot f))$$

for any $f \in C^k(\Sigma(Y) \setminus U)$.

Here, $\alpha\cdot f$ is thought of as a function on $\Sigma(Y)$. Of course $j(\varphi)$ is a homomorphism of Banach algebras because $h_1^*(\alpha\cdot f)$ depends only on

$$(\alpha f)\big|\big(C^-(Y) \cup C^+_{[0, 1/3]}(Y)\big) = f\big|\big(C^-(Y) \cup C^+_{[0, 1/3]}(Y)\big).$$

For any $(\chi, \varphi, \psi, F_s) \in \mathscr{E}^{(k), G}_{p,q}(\Sigma(Y) \setminus U, C^+(Y) \setminus U)$, we have

$$j \circ i_*(\chi, \varphi, \psi, F_s) = (\chi, j(i_*(\varphi)), \psi, F_s).$$

For any $f \in C^k(\Sigma(Y) \setminus U)$, we get

$$\begin{aligned}(j(i(\varphi)))(f) &= (i_*(\varphi))(h_1^*(\alpha\cdot f)) \\ &= \varphi((h_1^*(\alpha\cdot f))|(\Sigma(Y) \setminus U)).\end{aligned}$$

The inclusion:

$$l\colon (\Sigma(Y)\setminus C^+_{(1/3,1]}(Y), C^+(Y)\setminus C^+_{(1/3,1]}(Y)) \hookrightarrow (\Sigma(Y)\setminus C^+_{(1/2,1]}(Y), C^+(Y)\setminus C^+_{(1/2,1]}(Y))$$

is a C^k-homotopy equivalence, and therefore l induces an isomorphism l_* between the corresponding groups $K^{(k),G}_{p,q}(-,-)$.

In order to show that $j\circ i_*=1$, it would be enough to show that

$$(j\circ i_*)\circ l_* = l_*.$$

Because $\alpha|(\Sigma(Y)\setminus C^+_{(1/3,1]})\equiv 1$, and h_1 is the identity mapping on $\Sigma(Y)\setminus C^+_{(1/3,1]}(Y)$, it follows that for any quadruple

$$(\chi',\varphi',\psi',F'_s)\in \mathscr{E}^{(k),G}_{p,q}(\Sigma(Y)\setminus C^+_{(1/3,1]}(Y), C^+(Y)\setminus C^+_{(1/3,1]}(Y)),$$

we have

$$j\circ i_*\circ l_*(\chi',\varphi',\psi',F'_s) = l_*(\chi',\varphi',\psi',F'_s);$$

therefore $j\circ i_*=1$.

Now, let us take $(\chi,\varphi,\psi,F_x)\in \mathscr{E}^{(k),G}_{p,q}(\Sigma(Y),C^+(Y))$. We get

$$i_*\circ j(\chi,\varphi,\psi,F_s) = (\chi, i_*(j(\varphi)),\psi,F_s).$$

For any function $f\in C^k(\Sigma(Y))$, we have:

$$\begin{aligned}(i_*(j(\varphi)))(f) &= (j(\varphi))[f|(\Sigma(Y)\setminus C^+_{(1/2,1]}(Y))]\\ &= \varphi\{h_1^*[\alpha\cdot(f|(\Sigma(Y)\setminus C^+_{(1/2,1]}(Y)))]\}\\ &= \varphi(h_1^*(f));\end{aligned}$$

therefore,

$$i_*\circ j = (h_1)_* = 1$$

because h_1 is C^k-homotopic to the identity. This completes the proof of Lemma 3.1.

LEMMA 3.2 (weak Mayer–Vietoris exact sequence). *For any $Y\in \mathscr{OBC}^k$, let be*

$$X=\Sigma(Y)$$
$$B = C^-(Y)\cup C^+_{[0,2/3]}(Y)\subset \Sigma(Y)$$
$$C = C^+_{[1/3,1]}(Y)\subset\Sigma(Y)$$
$$A = B\cap C = C^+_{[1/3,2/3]}(Y).$$

Then the natural Mayer–Vietoris sequence

$$\dots \to K_n^{(k),G}(A) \xrightarrow{(k_*,-i_*)} K_n^{(k),G}(B) \oplus K_n^{(k),G}(C) \xrightarrow{j_*+l_*} K_n^{(k),G}(X) \xrightarrow{\partial'_n \varepsilon_n^{-1} q_*} K_n^{(k),G}(A) \to \dots$$

is exact. (The homomorphisms i_*, j_*, k_*, l_*, ∂'_n, ε_n, q_* will be explained in the proof.)

Proof. (Compare with Proof of Theorem 6.7 from [B.D.F.].) The homomorphisms i_*, k_*, j_*, l_*, ∂', q_*, are the natural homomorphisms from the following commutative diagram:

$$\begin{tikzcd}
\text{(I)} & \dots \arrow[r, "\partial'_{n+1}"] & K_n^{(k),G}(A) \arrow[r, "i_*"] \arrow[d, "k_*"] & K_n^{(k),G}(C) \arrow[r, "r"] \arrow[d, "l_*"] & K_n^{(k),G}(C,A) \arrow[r, "\partial'_n"] \arrow[d, "\varepsilon_n", "\simeq"'] & K_{n-1}^{(k),G}(A) \arrow[r, "i_*"] \arrow[d, "k_*"] & \dots \\
\text{(II)} & \dots \arrow[r, "\partial''_{n+1}"] & K_n^{(k),G}(B) \arrow[r, "j_*"] & K_n^{(k),G}(X) \arrow[r, "q_*"] & K_n^{(k),G}(X,B) \arrow[r, "\partial''_n"] & K_{n-1}^{(k),G}(B) \arrow[r, "j_*"] & \dots
\end{tikzcd}$$

with the lines (I) and (II) exact; ε_n is the excision isomorphism from Lemma 3.1.

It is easy to check that the composition of any two consecutive homomorphisms in the Mayer–Vietoris sequence is the zero homomorphism.

(1) Exactness at $K_n^{(k),G}(X)$. Let be $a \in K_n^{(k),G}(X)$, $\partial'_n \varepsilon_n^{-1} q_*(a)=0$. From the exactness of (I), we get there exists $c \in K_n^{(k),G}(C)$ such that $r(c)=\varepsilon_n^{-1} q_*(a)$. Now, $q_* l_*(c)=\varepsilon_n r(c)=\varepsilon_n \varepsilon_n^{-1} q_*(a)=q_*(a)$. Therefore $q_*(l_*(c)-a)=0$.

From exactness of (II), we deduce there exists $b \in K_n^{(k),G}(B)$ such that $j_*(b)=l_*(c)-a$, or $a=j_*(b)+l_*(c)$.

(2) Exactness at $K_{n-1}^{(k),G}(A)$. Let $a \in K_{n-1}^{(k),G}(A)$ such that $(k_*,-i_*)(a)=0$, or $k_*(a)=0$, $i_*(a)=0$. From exactness of (I), we get there exists $x \in K_n^{(k),G}(C,A)$ such that $\partial'_n(x)=a$. Then $0=k_*(a)=k_* \partial'_n(x)=\partial''_n \varepsilon_n(x)$.

From exactness of (II), we know there exists $y \in K_n^{(k),G}(X)$ such that $q_*(y)=\varepsilon_n(x)$. Then $\partial'_n \varepsilon_n^{-1} q_*(y)=\partial'_n \varepsilon_n^{-1} \varepsilon_n(x)=\partial'_n(x)=a$.

(3) Exactness at $K_n^{(k),G}(B) \oplus K_n^{(k),G}(C)$. Suppose that $b \in K_n^{(k),G}(B)$, $c \in K_n^{(k),G}(C)$, and $j_*(b)+l_*(c)=0$. Then $0=q_*(j_*(b)+l_*(c))=q_* l_*(c)=\varepsilon_n r(c)$; as ε_n is an isomorphism, we deduce $r(c)=0$. From exactness of (I) we get there exists $a_1 \in K_n^{(k),G}(A)$ such that $i_*(a_1)=c$. Then

$$0=j_*(b)+l_*(i_*(a_1))=j_*(b)+j_* k_*(a_1)=j_*(b+k_*(a_1)).$$

From exactness of (II), we get that there exists $x \in K_{n+1}^{(k),G}(X,B)$ such that $\partial''_{n+1}(x)=b+k_* a_1$. As ε_{n+1} is an isomorphism, we know there exists $y \in K_{n+1}^{(k),G}(C,A)$

such that $\varepsilon_{n+1}(y)=x$. We take $a=\partial'_{n+1}(y)-a_1$. An easy computation enables us to conclude that $k_*(a)=b$, and $-i_*(a)=c$.

Now we are in a position to prove the suspension axiom (7).

We rewrite the weak Mayer–Vietoris sequence proved before substituting the absolute $K_n^{(k),G}(-)$ groups by the reduced groups:

$$\tilde{K}_n^{(k),G}(X) = K_n^{(k),G}(X)/K_n^{(k),G}(\text{point}).$$

Of course, $\tilde{K}_n^{(k),G}(B)=\tilde{K}_n^{(k),G}(C)=0$, $\forall n \in \mathbf{Z}$, because B and C have the C^k-homotopy type of a point. Therefore the suspension homomorphism

$$\partial'_n\ \varepsilon_n^{-1} q_*: \tilde{K}_n^{(k),G}(\Sigma Y) \to \tilde{K}_{n-1}^{(k),G}(A) \simeq \tilde{K}_{n-1}^{(k),G}(Y)$$

is an isomorphism because A has the C^k-homotopic type of Y. Theorem 3.1 is completely proven.

There exists a natural transformation of functors ($K_{p,q}^G(-,-)$ denoting the Kasparov functor):

$$\Phi_{p,q}^G(-,-): K_{p,q}^G(-,-) \to K_{p,q}^{(k),G}(-,-)$$

defined as follows. For any quadruple $(\chi,\varphi,\psi,F_s) \in \mathscr{E}_{p,q}^G(X,A)$, we define

$$(\chi,\varphi',\psi,F_s) = \Phi_{p,q}^G(X,A)(\chi,\varphi,\psi,F_s) \in (\mathscr{E}_{p,q}^{(k),G}(X,A)$$

where $\varphi'=\varphi \circ i_k: C^k(X) \to L(H)$, and $i_k: C^k(X) \hookrightarrow C^0(X)$ is the inclusion of the algebra of functions of class C^k into the algebra of continuous functions.

Of course, $\Phi_{p,q}(X,A)$ defined at the quadruple-level, factorizes to $K_{p,q}^G(-,-)$ and is an homomorphism.

Now we show that $\Phi_{p,q}^G(-,-)$ is an isomorphism on a large enough class of spaces, when the group G is trivial. Even if the following proof requires G to be trivial, we expect that the following theorem be true for any compact group G.

When $G=\{1\}$ is the trivial group, we will omit to indicate the group.

THEOREM 3.2. *Suppose that $X \in \mathscr{OC}^k$, $0 \leq k \leq \infty$, is triangulable, and $G=\{1\}$ is the trivial group. Then, for any $p,q=0,1,\ldots,$*

$$\Phi_{p,q}(X): K_{p,q}(X) \to K_{p,q}^{(k)}(X)$$

is an isomorphism.

Remark. The proof of this theorem is not immediate, as the reader could expect, because the category $\mathscr{C}^k$ does not have enough quotient spaces X/A. Fortunately, if $i: A \hookrightarrow X$ is inclusion, $Ci \in \mathcal{O}b\mathscr{C}^k$ and Ci has the homotopic type of X/A.

Proof. We need three lemmas.

LEMMA 3.3. *$\Phi_{p,q}$ (point) is an isomorphism.*

Proof of Lemma 3.3. Of course C^0 (point)$=C^k$ (point)$=\mathbf{C}$. On the other hand, the only difference between our quadruples, and Kasparov's quadruples is that the representation $\varphi: C^k \to L(H)$ has to be quasi-involutive, while in the Kasparov's case it has to be involutive. But, in both cases it has to be unital, and it is immediate to see that any quasi-involutive homomorphism $\varphi: \mathbf{C} \to L(H)$ is automatically involutive.

LEMMA 3.4. *For any pair* (X, A) in $\mathscr{C}^k$, $A \neq \varnothing$, *there is an isomorphism*

$$K_{p,q}^{(k),G}(X, A) \simeq \tilde{K}_{p,q}^{(k),G}(Ci),$$

where $i: A \hookrightarrow X$ is the inclusion.

Proof of Lemma 3.4. We denote

$$\begin{aligned} &B = C^+_{[1/3,\,1]}(A) \subset Ci,\\ &C = X \times [-1, 0] \cup C^+_{[0,\,2/3]}(A) \subset Ci,\\ &\tilde{A} = B \cap C = C^+_{[1/3,\,2/3]}(A),\\ &U = C^+_{(2/3,\,1]}(A). \end{aligned}$$

It may be proven exactly as in the proof of the weak excision axiom (Lemma 3.1) that the excisive-inclusion:

$$(C, \tilde{A}) = (Ci \setminus U, B \setminus U) \hookrightarrow (Ci, B)$$

induces an isomorphism

$$\varepsilon_n: K_n^{(k),G}(C, \tilde{A}) \to K_n^{(k),G}(Ci, B),$$

for any $n \in \mathbf{Z}$.

We will write again the commutative diagram used in the proof of the weak Mayer–Vietoris exact sequence, with A, B, C, X replaced by $\tilde{A}, B, C, Ci$:

$$\begin{array}{lccccccccc}
\text{(I)} & \cdots \xrightarrow{\partial'_{n+1}} & \tilde K_n^{(k),G}(\tilde A) & \xrightarrow{i_*} & \tilde K_n^{(k),G}(C) & \xrightarrow{r} & K_n^{(k),G}(C,\tilde A) & \xrightarrow{\partial'_n} & \tilde K_{n-1}^{(k),G}(\tilde A) & \to \cdots \\
& & \downarrow{\scriptstyle k_*} & & \downarrow{\scriptstyle l_*} & & \simeq\downarrow{\scriptstyle \varepsilon_n} & & \downarrow{\scriptstyle k_*} & \\
\text{(II)} & \cdots \longrightarrow & \tilde K_n^{(k),G}(B) & \xrightarrow{j_*} & K_n^{(k),G}(Ci) & \underset{\simeq}{\xrightarrow{q_*}} & K_n^{(k),G}(Ci,B) & \xrightarrow{\partial''_n} & \tilde K_{n-1}^{(k),G}(B) & \longrightarrow \cdots
\end{array}$$

Because B has the C^k-homotopy type of point $\tilde K_n^{(k),G}(B)=0$, and therefore q_* is an isomorphism. Notice also that $(C,\tilde A)$ is C^k-homotopy equivalent to (X,A). The succession of isomorphisms

$$K_n^{(k),G}(X,A) \simeq K_n^{(k),G}(C,\tilde A) \xrightarrow{\varepsilon_n} K_n^{(k),G}(Ci,B) \underset{\simeq}{\xrightarrow{q_*^{-1}}} \tilde K_n^{(k),G}(Ci)$$

will bring us to the proof of Lemma 3.4.

LEMMA 3.5. *Let* $\{X_i\}$ $1\leq i\leq k$ *be a finite family of objects in* $\mathscr{C}^k$ *with base point* $x_0 \in X_i$, $1\leq i\leq k$, *and with no other intersections.*

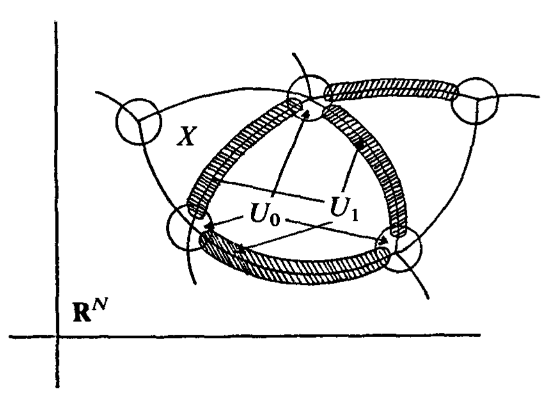
Then:

$$\tilde K_{p,q}^{(k),G}(X_1 \vee \ldots \vee X_k) = \bigoplus_{i=1}^{k} \tilde K_{p,q}^{(k),G}(X_i)$$

$$(X_1 \vee \ldots \vee X_k) = \bigcup_{i=1}^{k} X_i \text{ is the wedge).}$$

Proof. See proof of Theorem 1.7 from P. Hilton [H_1]. We pass to the proof of Theorem 3.2.

We take a regular covering $U_0, U_1, \ldots, U_N$ of $X \subset \mathbf{R}^N$, where $U_0 \cup U_1 \cup \ldots \cup U_i$ is a neighborhood of the i-scheleton of the triangulation of X.



It is not difficult to show (by using, for example, the principle of the proof of Lemma 3.5) that for a finite disjoint union $\cup$ of spaces X_i,

$$K_n^{(k)}\left(\bigcup_i X_i\right) = \bigoplus_i K_n^{(k)}(x_i) \quad \text{and}$$
$$K_n\left(\bigcup_i X_i\right) = \bigoplus_i K_n(X_i), \quad \forall n \in \mathbf{Z}.$$

This property, along with Lemma 3.3, proves that $\Phi_{p,q}(U_0)$ is an isomorphism.

Let i_k: $U_0 \cup U_1 \cup \ldots \cup U_{k-1} \hookrightarrow U_0 \cup U_1 \cup \ldots \cup U_k$ denote the inclusion.

If we are able to prove that all $\Phi_{p,q}(Ci_k)$, $0 \leq k \leq N$, are isomorphisms, then an induction argument on k—(by using the exact sequences of the pair

$$(U_0 \cup \ldots \cup U_k, U_0 \cup \ldots \cup U_{k-1}),$$

for both functors $K_{p,q}(-)$ and $K_{p,q}^{(k)}(-)$, along with the five lemma)—would conduct us to the conclusion that $\Phi_{p,q}(X)$ is an isomorphism.

In order to prove that $\Phi_{p,q}(Ci_k)$ is an isomorphism, we construct a space $N^k \subset \mathbf{R}^N$, in $\mathscr{C}^k$, having the properties:

(1) $N^k = \mathbf{V}_\alpha N_\alpha^k$, whereas α is any k-simplex of the triangulation of X,

(2) each N_α^k is C^k-homotopy equivalent to the sphere $\mathbf{S}^k \subset \mathbf{R}^{k+1}$.

We shall prove that Ci_k is C^k-homotopy equivalent to N^k.

Indeed, Ci_k is C^0-homotopy equivalent to $\mathbf{V}_\alpha\, \mathbf{S}_\alpha^k$, and therefore C^0-homotopy equivalent to N^k. That is, there exist two continuous functions

$$f\colon Ci_k \to N^k, \quad g\colon N^k \to Ci_k$$

with the property that $f \circ g \overset{H''}{\approx} \mathbf{1}_N k$, $g \circ f \overset{H'}{\approx} \mathbf{1}_{Ci_k}$, where the homotopies H', H'' are continuous.

The Whitney approximation theorem permits us to suppose that f and g are of class C^k (we may replace f and g by two other functions of class C^k which are C^0-homotopic to them). At this point, we may approximate the homotopies H', H'' by C^k-homotopies so as at the beginning and the end of these homotopies, they remain, respectively, equal to $f \circ g$ and $\mathbf{1}_{N^k}$, resp. $g \circ f$ and $\mathbf{1}_{Ci_k}$. Therefore, Ci_k has the C^k-homotopy type of N^k.

Therefore, $\Phi_{p,q}(N^k)$ is an isomorphism in view of the Lemma 3.5, Lemma 3.3, and the suspension axiom.

§ 4. Products

From now on we restrict ourselves to the non-equivariant case, $G=\{1\}$, and G will be omitted from the notation.

4.1. In this section we intend to construct a natural bilinear homomorphism:

$$\cap: K_{p,q}^{(k)}(X)\times K^0(X)\to K_{p,q}^{(k)}(X). \tag{4.1}$$

Pairings of this form are, of course, known in the Brown–Douglas–Fillmore theory and in the Kasparov theory, but, due to the specific way in which our signature operators (with values in vector bundles) occur, it is necessary to describe them in a different manner.

Let X be an object in $\mathscr{C}^k$. Let $(1,\varphi,\psi,F)\in\mathscr{E}_{p,q}^{(k),G}(X)$, $F: H\to H$, and let ξ be a complex vector bundle of class C^k over X. We will define $(1,\varphi,\psi,F)\#\xi=(1,\varphi_\xi,\psi_\xi,F_\xi)\in\mathscr{E}_{p,q}^{(k),G}(X)$.

Let $i: \xi\hookrightarrow\mathbf{N}$ be a C^k-embedding of ξ into the product bundle $\mathbf{N}: X\times\mathbf{C}^N\to X$, $\mathbf{N}$ being endowed with the obvious Hermitian structure; let $\xi^\perp$ be the orthogonal complement to ξ in $\mathbf{N}$.

In general, if $\eta\to X$ is a C^k-vector bundle, we denote by $C^k(\eta)$ the space of C^k-sections in η, and by $C^k(X)$ the complex valued functions of class C^k on X. Notice that $C^k(\eta)$ is a $C^k(X)$-module.

We have

$$C^k(\mathbf{N}) = C^k(\xi)\oplus C^k(\xi^\perp); \tag{4.2}$$

let $p_\xi, p_{\xi^\perp}$, denote the projections:

$$p_\xi: C^k(\mathbf{N})\to C^k(\xi),\quad p_{\xi^\perp}: C^k(\mathbf{N})\to C^k(\xi^\perp).$$

Let s_i, $1\leq i\leq N$, denote the constant section in

$$\mathbf{N}: s_i(x) = (x,(0,\dots,\underset{(i)}{1},\dots,0)),\quad x\in X.$$

Notice that the quadruple $(1,\varphi,\psi,F)$ being given $\varphi: C^k(X)\to L(H)$ defines a $C^k(X)$-module structure on H. We define then:

$$H_\xi = H\otimes_{C^k(X)} C^k(\xi);$$

here $\otimes_{C^k(X)}$ denotes the algebraic tensor product over $C^k(X)$. If no confusion could occur, the basic ring $C^k(X)$ will be omitted from the tensor product symbol.

From (4.2) we get

$$H_{\mathbf{N}} = H_\xi \oplus H_{\xi^\perp}. \tag{4.4}$$

But

$$H_{\mathbf{N}} = H \otimes_{C^k(X)} C^k(\mathbf{N}) = \underbrace{H \oplus \ldots \oplus H}_{N},$$

which allows us to introduce a Hilbert structure on $H_{\mathbf{N}}$ (direct sum). We endow $H_\xi, H_{\xi^\perp}$, which are subspaces of $H_{\mathbf{N}}$, with the induced norms. Of course, $\mathbf{1}_H \otimes p_\xi, \mathbf{1}_H \otimes p_{\xi^\perp}$ are complementary projections, with ranges $H_\xi, H_{\xi^\perp}$.

PROPOSITION 4.1. *H_ξ and $H_{\xi^\perp}$ are closed subspaces $H_{\mathbf{N}}$; hence they are Hilbert spaces.*

Proof. Let $x_1, \ldots, x_n, \ldots$ be any sequence in H_ξ which converges in $H_{\mathbf{N}}$. Any element x_n may be represented uniquely:

$$x_n = \sum_{\alpha=1}^{N} h_n^\alpha \otimes s_\alpha, \quad h_n^\alpha \in H.$$

Because $\{x_n\}$ converges in $H_{\mathbf{N}}$, the limits

$$h^\alpha = \lim_{n \to \infty} h_n^\alpha$$

exist in H for any $1 \leq \alpha \leq N$.

Let be

$$p_\xi(s_\alpha) = \sum_{\alpha,\beta} A_\alpha^\beta s_\beta, \quad A_\alpha^\beta \in C^k(X).$$

Because $x_n \in H_\xi$, we have:

$$x_n = (1 \otimes p_\xi)(x_n) = \sum_\alpha h_n^\alpha \otimes p_\xi(s_\alpha) = \sum_{\alpha,\beta} \varphi(A_\alpha^\beta) h_n^\alpha \otimes s_\beta.$$

As $\varphi(A_\alpha^\beta)$ is a bounded operator on H, we have

$$\lim_{n\to\infty} x_n = \sum_{\alpha,\beta} \varphi(A_\alpha^\beta)(\lim_{n\to\infty} h_n^\alpha)\otimes s_\beta = \sum_{\alpha,\beta} h^\alpha \otimes A_\alpha^\beta s_\beta = (1\otimes p_\xi)\sum_\alpha h^\alpha\otimes s_\alpha,$$

which belongs to H_ξ.

Remark. If the representation φ were involutive, then $H_\xi, H_{\xi^\perp}$ would be orthogonal one to another.

Now, we pass to define $(1, \varphi_\xi, \psi_\xi, F_\xi)$.

H_{N} is a $C^k(X)$-module by the homomorphism:

$$\varphi_{\mathrm{N}}: C^k(X) \to L(H_{\mathrm{N}}) \tag{4.7}$$

$$\varphi_{\mathrm{N}}(f): \sum_{\alpha=1}^N h^\alpha\otimes s_\alpha \mapsto \sum_{\alpha=1}^N \varphi(f)(h^\alpha)\otimes s_\alpha,$$

for any $f\in C^k(X)$.

The subspace $H_\xi \subset H_{\mathrm{N}}$ is stable at φ_N, and we denote by φ_ξ the induced representation by φ_{N} on H_ξ. φ_ξ is quasi-involutive.

We define

$$F_\xi = (1\otimes p_\xi)(F\otimes 1_\xi): H_\xi \to H_\xi;$$

because F is not, in general, a C^k-homomorphism, F_ξ has to be precised better. If $x\in H_\xi$, we write x, uniquely,

$$x = \sum_\alpha h^\alpha\otimes s_\alpha, \quad h^\alpha \in H, \tag{4.8}$$

and we define:

$$F_\xi(x) = (1\otimes p_\xi)\sum_\alpha (Fh^\alpha)\otimes s_\alpha = \sum_{\alpha,\beta} F(h^\alpha)\otimes A_\alpha^\beta s_\beta. \tag{4.9}$$

We check that the operator F_ξ satisfies the requirements (IV) from Definition 2.1.

Let $x=\Sigma_\alpha h^\alpha\otimes s_\alpha$, $y=\Sigma_\beta k^\beta\otimes s_\beta$ be any two elements in H_ξ. We have:

$$\begin{aligned}\langle F_\xi x, y\rangle &= \left\langle \sum_{\alpha,\beta} F(h^\alpha)\otimes A_\alpha^\beta s_\beta, \sum_\beta k^\beta\otimes s_\beta\right\rangle \\ &= \sum_{\alpha,\beta} (\varphi(A_\alpha^\beta)F(h^\alpha), k^\beta)_H \\ &= \sum_{\alpha,\beta} (F(\varphi(A_\alpha^\beta)h^\alpha) + K_\alpha^\beta(h^\alpha), k^\beta)_H\end{aligned}$$

(where $K_\alpha^\beta=\varphi(A_\alpha^\beta)\circ F-F\circ\varphi(A_\alpha^\beta)$ are compact operators)

$$=\sum_{\alpha,\beta}(\varphi(A_\alpha^\beta)h^\alpha+K_\alpha^\beta(h^\alpha),\ (-F+K)(k^\beta))_H$$

(where $K=F^*+F$ is a compact operator)

$$\simeq\sum_{\alpha,\beta}(\varphi(A_\alpha^\beta)h^\alpha,\ -F(k^\beta))_H$$

(where $\simeq$ means that we ignore terms which stay in the range of compact operators)

$$\simeq\sum_{\alpha,\beta}(h^\alpha,-\varphi(\bar{A}_\alpha^\beta)F(k^\beta))_H$$

$$=\left\langle\sum_\alpha h^\alpha\otimes s_\alpha,-\sum_{\alpha,\beta}\varphi(\bar{A}_\alpha^\beta)F(k^\beta)\otimes s_\alpha\right\rangle$$

(because p_ξ is a Hermitian projection, there is $\bar{A}_\alpha^\beta=A_\beta^\alpha$, and we have further:)

$$=\left\langle x,-\sum_{\alpha,\beta}\varphi(A_\beta^\alpha F(k^\beta)\otimes s_\alpha\right\rangle$$

$$=\left\langle x,-\sum_\beta F(k^\beta)\otimes p_\xi(s_\beta)\right\rangle=\langle x,-F_\xi(y)\rangle,$$

or

$$F_\xi^*\sim-F_\xi.$$

In order to prove that the operator $F_{\xi^\perp}$ is a Fredholm operator, we consider also the operator $F_{\xi^\perp}$, analogously defined:

$$F_{\xi^\perp}\left(\sum_\alpha h^\alpha\otimes s_\alpha\right)=(1\otimes p_{\xi^\perp})\sum_{\alpha,\beta}(Fh^\alpha)\otimes B_\alpha^\beta s_\beta,$$

where $\|B_\alpha^\beta\|=p_{\xi^\perp}$, and $\Sigma_\alpha\, h^\alpha\otimes s_\alpha\in H_{\xi^\perp}$.

We have, clearly:

$$F_\xi\otimes F_{\xi^\perp}+(1\otimes p_\xi)(F\otimes 1_N)(1\otimes p_{\xi^\perp})+(1\otimes p_{\xi^\perp})(F\otimes 1_N)(1\otimes p_\xi)=F\otimes 1_N=F\oplus F\oplus\ldots\oplus F,$$

which is a Fredholm operator; notice that the last two operators of the first line in this formula are compact operators and, then both F_ξ and $F_{\xi^\perp}$ are Fredholm operators. Indeed, we have for any $x=\Sigma_\alpha h^\alpha \otimes s_\alpha$ in $H_{\mathbf{N}}$:

$$\begin{aligned}(1\otimes p_{\xi^\perp})(F\otimes 1_{\mathbf{N}})(1\otimes p_\xi)\left(\sum_\alpha h^\alpha\otimes s_\alpha\right) &= (1\otimes p_{\xi^\perp})(F\otimes 1_{\mathbf{N}})\left(\sum_{\alpha,\beta} A_\alpha^\beta h^\alpha\otimes s_\beta\right)\\ &= (1\otimes p_{\xi^\perp})\left[\sum_{\alpha,\beta} F(A_\alpha^\beta h^\alpha)\otimes s_\beta\right]\\ &= \sum_{\alpha,\beta,\gamma} B_\beta^\gamma F(A_\alpha^\beta h^\alpha)\otimes s_\gamma\\ &\cong \sum_{\alpha,\beta,\gamma} F(B_\beta^\gamma A_\alpha^\beta h^\alpha)\otimes s_\gamma = 0.\end{aligned}$$

In a similar way one could prove that for any $f\in C^k(X)$,

$$\varphi_\xi(f)\cdot F_\xi \sim F_\xi\cdot\varphi_\xi(f).$$

We define

$$\begin{gathered}\psi_\xi: C_{p,q+1}\to L(H_\xi)\\ \psi_\xi(e)(h\otimes s)=\psi(e)(h)\otimes s,\end{gathered} \tag{4.10}$$

for any $e\in C_{p,q+1}$, $h\in H$, and $s\in C^k(\xi)$.

Then $(\mathbf{1},\varphi_\xi,\psi_\xi,F_\xi)\in\mathscr{E}^{(k)}_{p,q}(X)$.

Remark. If $\theta:\xi\to\xi'$ is a C^k-isomorphism of vector bundles, then $1\otimes\theta: H_\xi\to H_{\xi'}$ is an algebraic isomorphism. One may show that $1\otimes\theta$ is a homeomorphism.

We intend to show now that the operator F_ξ, modulo compact operators, does not depend on the embedding $\xi\hookrightarrow\mathbf{N}$. To this aim, let us choose a finite covering $\mathscr{U}=\{U_a\}_a$ of X by contractible open subsets U_a, and let $\{\varrho_a\}_a$ be a subordinated partition of unity. We denote by $s_1^a,\ldots,s_n^a$, $n=\operatorname{Rank}\xi$, a C^k-frame in ξ over $\bar U_a$.

We decompose F_ξ in a sum of operators $F_{\xi,a}$ with *support* in U_a:

$$F_{\xi,a}=F_\xi\cdot\varphi_\xi(\varrho_a);$$

because φ_ξ is unital, we have:

$$F_\xi=\sum_a F_{\xi,a}.$$

If $s\colon U_a \to \xi$ is any C^k-section, and $h \in H$, then we define:

$$F_{\xi,a}(h \otimes s) = F_\xi(h \otimes (\varrho_a s)).$$

$i\colon \xi \hookrightarrow \mathbf{N}$ denoting the inclusion, we have

$$i \circ s_k^a = \sum_{\alpha=1}^{N} S_k^\alpha s_\alpha, \quad 1 \leqslant k \leqslant n,$$

where $S_k^\alpha \in C^k(U_a)$.

We have then:

$$\begin{aligned} F_{\xi,a}\left(\sum_{k=1}^{n} h^k \otimes s_k^a\right) &= F_\xi\left(\sum_k h^k \otimes \varrho_a(i \circ s_k^a)\right) \\ &= F_\xi\left(\sum_{k,\alpha} h^k \otimes \varrho_a S_k^\alpha s_\alpha\right) = \sum_{k,\alpha,\beta} F(\varphi(\varrho_a S_k^\alpha) h^k) \otimes A_\alpha^\beta s_\beta \\ &\simeq \sum_{k,\alpha,\beta} F(h^k) \otimes \varrho_a S_k^\alpha A_\alpha^\beta s_\beta \\ &= \sum_{k,\beta} F(h^k) \otimes \varrho_a\left(\sum_\alpha S_k^\alpha p_\xi(s_\alpha)\right) \\ &= \sum_k F(h^k) \otimes \left(\varrho_a p_\xi\left(\sum_\alpha S_k^\alpha s_\alpha\right)\right) \\ &= \sum_k F(h^k) \otimes \varrho_a(i \circ s_k^a), \end{aligned}$$

which shows that $(\mathbf{1}_H \otimes i)^{-1} F_{\xi,a}$, modulo compact operators, does not depend on the inclusion i.

These considerations enable us to conclude that the correspondence

$$(\mathbf{1}, \varphi, \psi, F) \times \xi \mapsto (\mathbf{1}, \varphi_\xi, \psi_\xi, F_\xi)$$

gives a well defined pairing

$$\cap\colon K_{p,q}^{(k)}(X) \times K^0(X) \to K_{p,q}^{(k)}(X) \tag{4.11}$$

The following propositions will be needed in section §5.

PROPOSITION 4.2. *If $\Phi_{p,q}(-)$ denotes the isomorphism from Theorem 2.2, then the following diagram:*

$$\begin{array}{ccc} K^{(0)}_{p,q}(X)\times K^0(X) & \xrightarrow{\cap} & K^{(0)}_{p,q}(X) \\ \downarrow{\scriptstyle \Phi_{p,q}(X)\times 1} & & \downarrow{\scriptstyle \Phi_{p,q}(X)} \\ K^{(k)}_{p,q}(X)\times K^0(X) & \xrightarrow{\cap} & K^{(k)}_{p,q}(X) \end{array} \tag{4.12}$$

is commutative.

Proof. If ξ is a vector bundle of class C^k, then $H\otimes_{C^k(X)}C^k(\xi)$ is canonically isomorphic to $H\otimes_{C^0(X)}C^0(\xi)$.

PROPOSITION 4.3. *Let $A\overset{j}{\hookrightarrow}X$ be an inclusion in the category $\mathscr{C}^k$. Then, for any $\xi\in K^0(X)$, and $D\in K^{(k)}_{p,q}(A)$, we have*

$$j_*(D\cap j^*\xi)=(j_*D)\cap\xi \quad \text{in } K^{(k)}_{p,q}(X). \tag{4.13}$$

Proof. Let $(1,\varphi,\psi,F)\in\mathscr{C}^{(k)}_{p,q}$ represent the element $D, F\colon H\to H$, and let ξ be a vector bundle. We may think of ξ as a sub-bundle of the product bundle $\mathbf{N}$ over X.

Let

$$T(\xi)\colon H\otimes_{C^k(X)}C^k(\xi)\to H\otimes_{C^k(A)}C^k(\xi|A)$$

be the homomorphism given by the restriction homomorphism of the basic ring: $C^k(X)\to C^k(A)$, and by the restriction homomorphism $C^k(\xi)\to C^k(\xi|A)$. We will show that $T(\xi)$ is a isomorphism, and afterwards, (4.13) will follow easily.

In order to check that $T(\xi)$ is an isomorphism, it is enough to show that $T(\xi)\oplus T(\xi^{\perp})$ is an isomorphism. But both the domain of definition, and the range of this last homomorphism, are canonically isomorphic to $H\oplus\ldots\oplus H$ (N times); we use these canonical isomorphisms as identifying isomorphisms. With these identifications, $T(\xi)\oplus T(\xi^{\perp})$ is the identity mapping.

PROPOSITION 4.4. *For any $D\in K^{(k)}_{p,q}(X)$, and $\xi,\zeta\in K^0(X)$, we have:*

$$(D\cap\xi)\cap\zeta=D\cap(\xi\cdot\zeta), \tag{4.14}$$

where $\cdot$ denotes the multiplication in $K^0(X)$.

Proof. Immediate from the definitions.

Since now on, we will be interested only in $K^0(X)$ and $K^{(0)}_0(X)\simeq K^{(1)}_0(X)$. If X has a finite regular covering, as in the proof of Lemma 2.5, then all these three groups are finitely generated. We introduce:

$$K^0_{\mathbf{Q}}(-)=K^0(-)\otimes_{\mathbf{Z}}\mathbf{Q}$$

$$K^{(0)}_{0,\mathbf{Q}}(-)=K^{(0)}_0(-)\otimes_{\mathbf{Z}}\mathbf{Q},\ K^{(1)}_{0,\mathbf{Q}}(-)=K^{(1)}_0\otimes_{\mathbf{Z}}\mathbf{Q}.$$

PROPOSITION 4.5.

$$\dim_{\mathbf{Q}}K^0_{\mathbf{Q}}(-)=\dim_{\mathbf{Q}}K^{(0)}_{0,\mathbf{Q}}(-)=\dim_{\mathbf{Q}}K^{(1)}_{0,\mathbf{Q}}(-). \tag{4.15}$$

Proof. This result is implicitly contained in [B.D.F.] and [K].

§ 5. Signature operators versus K-homology

In this section we show that the signature operators D^+_ξ constructed in [T₃]:

$$D^+_\xi: W^+_1(M,\xi)\to W^-_0(M,\xi) \tag{5.1}$$

represent well defined elements in $K_0(M)$; however, some technical difficulties arise. The aim of the sections §§1–4 was exactly that of preparing the instruments needed in surpassing these incoming difficulties.

The Hilbert spaces $W^{\pm}_1(M,\xi)$ are not $C(M)$-modules under the usual multiplication operation. Instead, $W^{\pm}_1(M,\xi)$, and of course $W^{\pm}_0(M,\xi)$, are modules over the algebra $\mathscr{L}(M)$ of Lipschitz functions on M. Unfortunately, $\mathscr{L}(M)$ is not a finitely generated Banach algebra, as G. Kasparov's theory [K] requires. We can select though, in a fairly natural way, a finitely generated Banach subalgebra of $\mathscr{L}(M)$ as follows.

We embed M^m as a Lipschitz sub-manifold in $\mathbf{R}^N$. Then $\{f, f=F|M, F\in C^1(\mathbf{R}^N)\}$ is a finitely generated Banach subalgebra of $\mathscr{L}(M)$. However, because it is not easy to keep control on the first order partial derivatives of such restrictioned functions to a rather irregular sub-manifold as M is, it is more practical, instead, to consider the Banach algebra of all complex valued functions of class C^1 defined on a tubular neighborhood U of M, provided that we would be able to relate $K_*(U)$ to $K_*(M)$.

A first step toward this objective is the following:

PROPOSITION 5.1. (i) (J. Luukkainen, P. Tukia [L.T.], Corollary 4.12.) *For any Lipschitz manifold M^m, there exists a closed locally Lipschitz flat embedding $j: M^m \hookrightarrow \mathbf{R}^{3m}$, and*

(ii) *a tubular neighborhood U of $j(M^m)$ in $\mathbf{R}^{3m}$ such that*

$$j: M^m \hookrightarrow U$$

is a homotopy equivalence;

(iii) *U is triangulable.*

Proof. Let j be a flat Lipschitz embedding as in (i). Then $j(M)$ has a normal disc bundle $\pi\colon U\to j(M)$ in $\mathbf{R}^{3m}$, see C. P. Rourke–B. J. Sanderson [R.S.] Corollary 5.5; this proves (ii).

From the local triviality of this normal bundle, and from the fact that M^m is a manifold, we conclude that U is a topological sub-manifold of dimension N in $\mathbf{R}^N$. From Proposition 2.1 § 2 and the Classification theorem § 0, both in Essay IV, [K.S.] by R. Kirby, L. Siebenmann, we deduce that U is triangulable.

By virtue of the Theorem 3.1 (6) and Theorem 3.2, we have the following sequence of isomorphisms:

$$K_n'(M) \xrightarrow[\simeq]{j_{*,n}} K_n(U) \xrightarrow[\simeq]{\Phi_n(U)} K_n^{(1)}(U), \quad n\in\mathbf{Z}; \tag{5.2}$$

we will make the signature operators represent elements in $K_0^{(1)}(U)$.

With U chosen as above, $W_1^{\pm}(M,\xi)$ and $W_0^{\pm}(M,\xi)$ are $C^1(U)$-modules in a natural way. Let φ_1, resp. φ_0, denote the homomorphisms:

$$\varphi_i\colon C^1(U)\to L(W_i^{\pm}(M,\xi)), \quad i=0,1,$$

which define these $C^1(U)$-module structures.

We associate with D_ξ^+ the operator:

$$D_\xi\colon H_\xi\to H_\xi, \quad \text{where } H_\xi=W_1^+(M,\xi)\oplus W_0^-(M,\xi),$$
$$D=\begin{pmatrix}0 & -(D_\xi^+)^*\\ D_\xi^+ & 0\end{pmatrix},$$

see N. Teleman [T$_3$], § 11. The generator ε_1 of the Clifford algebra $C_{0,1}$ is acting on H_ξ in this way:

$$\psi_\xi(\varepsilon_1)=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix};$$

we define also $\varphi_\xi\colon C^1(U)\to L(H_\xi)$ to be $\varphi_\xi=\varphi_1\oplus\varphi_0$. Then $\tilde{D}_\xi=(1,\varphi_\xi,\psi_\xi,D_\xi)\in\mathscr{E}_{0,1}^{(1)}(U)$. This statement is proven in [T$_3$].

It remains, though, to prove the following:

LEMMA 5.2. *The homomorphism*

$$\varphi_\xi\colon C^1(U)\to L(H_\xi)$$

is quasi-involutive.

Proof. The action of $C^1(U)$ on $W_0^{\mp}(M,\xi)$ is, clearly, involutive; we need, then, to show that for any Lipschitz function f on M, the multiplication by f:

$$\mu_f\colon W_1^{\mp}(M,\xi)\to W_1^{\mp}(M,\xi) \tag{5.3}$$

has the property:

$$\mu_f^*=\mu_f+K_f, \tag{5.4}$$

where K_f is a compact operator.

In order to prove (5.4), we need to report to the definition of H_ξ in § 4. Supposing that the bundle ξ is realized as a Lipschitz sub-bundle of the product $\mathbf{N}$, and that the projection of the constant section $s_\alpha\colon M\to\mathbf{N}$, $1\leqslant\alpha\leqslant N$, on ξ is:

$$p_\xi\circ s_\alpha=\sum_\beta A_\alpha^\beta s_\beta,$$

where A_α^β are Lipschitz functions, then any element x in $W_1^{\mp}(M,\xi)$ may be written (not uniquely):

$$x=\sum_{\alpha,\gamma}A_\alpha^\gamma h^\alpha\otimes s_\gamma,\quad\text{with } h^\alpha\in W_1^{\pm}(M,\mathbf{1}).$$

For any other element $y\in W_1^{\mp}(M,\mathbf{1})$,

$$y=\sum_{\beta,\gamma}A_\beta^\gamma k^\beta\otimes s_\gamma,$$

and f a Lipschitz function on M, we have

$$\begin{aligned}\langle\mu_f(x),y\rangle_1&=\sum_{\alpha,\beta,\gamma}(fA_\alpha^\gamma h^\alpha,A_\beta^\gamma k^\beta)_1\\&=\sum_{\alpha,\beta,\gamma}[(fA_\alpha^\gamma h^\alpha,A_\beta^\gamma k^\beta)_0+(d(fA_\alpha^\gamma h^\alpha),d(A_\beta^\gamma k^\beta))_0+\\&\qquad+(\delta(fA_\alpha^\gamma h^\alpha),\delta(A_\beta^\gamma k^\beta))_0]\end{aligned}$$

$$= \langle x, \mu_f(y)\rangle_1 + \sum_{\alpha,\beta,\gamma} [(df \wedge A^\gamma_\alpha h^\alpha, d(A^\gamma_\beta k^\beta))_0 \tag{5.5}$$
$$-(A^\gamma_\alpha h^\alpha, df \wedge A^\gamma_\beta k^\beta)_0 + (df \lrcorner A^\gamma_\alpha h^\alpha, \delta(A^\gamma_\beta k^\beta))_0$$
$$-(\delta(A^\gamma_\alpha h^\alpha), df \lrcorner A^\gamma_\beta k^\beta)_0].$$

If we take in (5.5) $x=(\mu_f^*-\mu_f)(y)$ and, if we use the Cauchy–Schwartz inequality, we get, for $C=\operatorname{ess\,sup}_M|df|$, and $C_1=\|\mu_f^*-\mu_f\|$

$$\|(\mu_f^*-\mu_f)y\|_1^2 \leq 2CC_1 \cdot \|y\|_0 \cdot \|y\|_1. \tag{5.6}$$

Let $\{y_i\}_{i\in N}$, be any bounded sequence in $W_1^{\pm}(M,\xi)$. Because $\mu_f^*-\mu_f$ is a bounded operator in $W_1^{\mp}(M,\xi)$, and as the inclusion $W_1^{\mp}(M,\xi)\hookrightarrow W_0^{\mp}(M,\xi)$ is compact, see N. Teleman [T$_3$], it follows that there exists a subsequence of $\{y_i\}_i$, which we may suppose to be the sequence itself, with the property that the sequence $\{(\mu_f^*-\mu_f)y_i\}_{i\in N}$ converges in $W_0^{\mp}(M,\xi)$. From (5.6), we get then:

$$0 \leq \lim_{i,j\to\infty} \|(\mu_f^*-\mu_f)(y_i-y_j)\|_1^2$$
$$\leq 4C \lim_{i,j\to\infty} \|(\mu_f^*-\mu_f)(y_i-y_j)\|_0 \sup_{i\in N} \|y_i\|_1 = 0,$$

which shows that $\{(\mu_f^*-\mu_f)(y_i)\}_{i\in \mathbf{N}}$ converges in $W_1^{\pm}(M,\xi)$; this proves (5.4).

To summarize, $\tilde{D}_\xi$ represents a well defined class in $K_0^{(1)}(U)$. Taking into account (5.2), we define:

$$D_\xi = j_{*,0}^{-1} \circ \Phi_0(U)^{-1} [\tilde{D}_\xi] \in K_0(M).$$

The element D_ξ might depend on the embedding j, but the image of D_ξ in $K_0(M)\otimes \mathbf{Q}$ is independent of J, as follows from the discussion in section § 6.

We denote

$$D_M = D_1, \quad D_M \in K_0(M),$$

and by virtue of Propositions 4.2, 4.3 and 4.4, we have:

$$D_\xi = D_M \cap \xi, \quad \text{for any } \xi \in K^0(M)$$
$$D_\xi \cap \zeta = D_{\xi\cdot\zeta}, \quad \text{for any } \xi, \zeta \in K^0(M).$$

§ 6. The index theorem for topological manifolds

PROPOSITION 6.1 (rational Thom isomorphism). *For any compact, oriented topological manifold M^m, $m\geqslant 6$, $m\equiv 0 \pmod 2$, $\partial M=\varnothing$, the Thom homomorphism:*

$$T: K^0_{\mathbf{Q}}(M)\to K_{0,\mathbf{Q}}(M)$$
$$T: \xi\mapsto D_M\cap\xi, \tag{6.1}$$

where D_M is the signature operator defined in § 5, is an isomorphism.

Proof. By virtue of Proposition 4.5, it is enough to show that the Thom homomorphism (6.1) is a monomorphism. Then, let ξ be any element in $K^0_{\mathbf{Q}}(M)$, and suppose that $T(\xi)=0$. From (5.9) and (5.10), we have, a fortiori, for any $\zeta\in K^0_{\mathbf{Q}}(M)$:

$$0=T(\xi)\cap\zeta=(D_M\cap\xi)\cap\zeta=D_M\cap(\xi\cdot\zeta)=D_{\xi\cdot\zeta}. \tag{6.2}$$

From the Theorem 1.1 we get then:

$$0=\operatorname{Index} D_{\xi\cdot\zeta}=\operatorname{ch}(\xi\cdot\zeta)\cup L(TM)[M]$$
$$=\operatorname{ch}\xi\cup\operatorname{ch}\zeta\cup L(TM)[M]. \tag{6.3}$$

From (6.3) along with the Poincaré duality, and the fact that ch is epimorphic, we deduce the $\operatorname{ch}\xi\cup L(TM)=0$. As $L(TM)=\pm 1+$ higher terms, it is invertible in $H^*(M,\mathbf{Q})$, (see e.g. R. Palais [P], Chapter XV, § 4) we get that $\operatorname{ch}\xi=0$, and finally, that $\xi=0$, which proves that T is a monomorphism.

COROLLARY 6.2. *There exists one and only one homomorphism:*

$$\mathrm{Ch}: K_0(M)\to H^{ev}(M,\mathbf{Q}) \tag{6.4}$$

such that, for any $\xi\in K^0(M)$

$$\mathrm{Ch}(D_M\cap\xi)=\operatorname{ch}\xi.$$

Proof. $K_0(M)$ is a finitely generated group. From the rational Thom isomorphism (6.1), we get that $D_M\cap K^0(M)$ is a sub-group of finite index in $K_0(M)$, and that Ch is well defined on this sub-group.

Let S be any pseudo-differential operator of order zero on the circle $\mathbf{S}^1$, whose index is $+1$. Then S represents an element in $K_0(\mathbf{S}^1)$.

For any $D\in\mathscr{E}_0(M)$, the external product

$$D\#S \in K_0(M\times \mathbf{S}^1)$$

is defined, see [K], § 4, Theorem 2.

Now, we are in a position to define the homomorphism Ch on odd-dimensional manifolds by the formula:

$$\mathrm{Ch}\,(D) = \mathrm{Ch}\,(D\#S) \cap [\mathbf{S}^1]. \tag{6.5}$$

If $\dim M \leq 5$, and D is an abstract elliptic operator over M, we define:

$$\mathrm{Ch}\,(D) = \mathrm{Ch}\,(D\#\underbrace{S\#\ldots\#S}_{5 \text{ factors}}) \cap \underbrace{[\mathbf{S}^1\times\ldots\times\mathbf{S}^1]}_{5 \text{ factors}} \tag{6.6}$$

Definition. For any abstract elliptic operator $D=(1,\varphi,\psi,F)\in \mathscr{E}_{0,1}(M)$ over the topological manifold M, we define its analytical index:

$$a\text{-Ind}\,(D) = \dim_{\mathbf{C}} \mathrm{Ker}^+ F - \dim_{\mathbf{C}} \mathrm{Ker}^- F \tag{6.7}$$

where $\mathrm{Ker}^{\pm} F$ denote the ± 1-eigenspaces of the involution $\varphi(\varepsilon_1)$ on $\mathrm{Ker}\,F$.

The topological index of D is, by definition:

$$t\text{-Ind}\,(D) = \mathrm{Ch}\,(D \cup L(TM)\,[M]. \tag{6.8}$$

THEOREM 6.3. (Index theorem for topological manifolds.) *For any compact, oriented boundary free topological manifold M, of any dimension, and for any abstract elliptic operator D on M,*

$$a\text{-Ind}\,(D) = t\text{-Ind}\,(D). \tag{6.9}$$

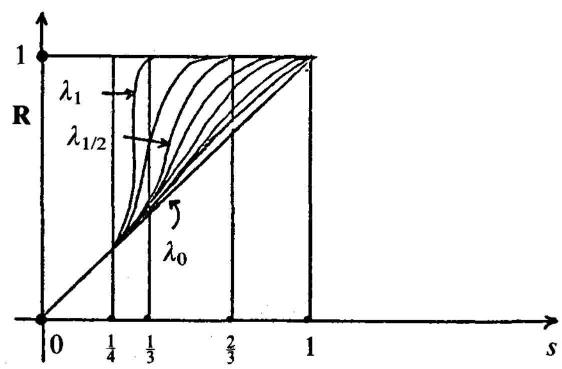
(*Notice that D is an arbitrary elliptic operator over the C*-algebra of* continuous *functions on M.*)

Proof. We consider first the case when M is even-dimensional. Then, the topological signature Theorem 1.1 asserts that the homomorphisms a-Ind and t-Ind coincide on the finite index sub-group $D_M \cap K^0(M)$ of $K_0(M)$; therefore, they coincide on $K_0(M)$.

The proof of the index theorem for manifolds of odd-dimension and of dimension lower than 6, follows now easily by multiplication with circles.

§ 7. Appendix

The following graph explains how the one-parameter family of functions $\lambda_t(-)$ would look like:



We intend also to explain, here, why the homotopy h_t:

$$h_t((1-s)x+s\cdot e_{N+1})=(1-\lambda_t(s))x+\lambda_t(s)\cdot e_{N+1}$$

is of class C^k, if $\lambda(-)$ is of class C^{k+1}, and its partial derivatives are continuous functions on t. The function h_t, expressed in cartesian coordinates, is:

$$h_t(y,s)=\begin{cases}\left(\dfrac{1-\lambda_t(s)}{1-s}\cdot y,\lambda_t(s)\right), & 0\leqslant s<1\\ \left(\dfrac{d\lambda_t(s)}{ds}\Big|_{s=0}\cdot y,1\right), & s=1,\end{cases}$$

for any $y\in\mathbf{R}^N$.

The only trouble occurs with the function $(1-\lambda_t(s))/(1-s)$ in $s=1$.

In order to show that this function is of class C^k in $s=1$, use the Taylor expansion of this function around $s=1$ and compute, by increasing induction, its derivatives in $s=1$.

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