The Heights theorem for quadratic differentials on Riemann surfaces

by

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Soon after Thurston announced his theory of measured foliations on surfaces, Hubbard and Masur recognized that this concept fits in perfectly with quadratic differentials. Their main theorem makes this precise: On a fixed compact Riemann surface (without boundary) there is a one-to-one correspondence between equivalence classes of measured foliations and holomorphic quadratic differentials. The correspondence is in terms of the horizontal trajectory structure and vertical measure determined by a quadratic differential.

From the point of view of complex analysis, the achievement of Hubbard and Masur completed a line of investigation initiated by Teichmüller in the late 1930's. It was further developed in the 1950's primarily by Jenkins and in the 1960's and 70's by Strebel. Quadratic differentials appear in association with solutions of extremal mapping problems involving variation of conformal structure. Understanding the geometry of these differentials bears substantially on understanding the geometry of the extremal mappings themselves.

In their paper, Hubbard and Masur developed the subject independently of the Thurston theory. But their analysis was quite complicated, involving the local variation over Teichmüller space of differentials with multiple zeros. Later in his thesis, Kerckhoff showed how to derive their main result by using the techniques of the Thurston theory.

Our purpose here is to develop the subject within the context of the "classical" theory of quadratic differentials. We believe that this allows for a simpler approach and one that clearly exhibits the geometry of differentials. At the same time, it allows us to develop the theory in a more general context (for example, for parabolic surfaces of infinite topological type). What makes such an approach possible is the discovery by Strebel of a direct proof of the Heights theorem. This is the fundamental uniqueness theorem of the subject and our paper is built around it. Using it, we explore questions of approximation by and convergence of sequences of the geometrically simplest differentials (Chapters 5 and 6). The information so obtained is then applied toward the understanding of how Teichmüller mappings are geometrically determined (Chapter 7). In a later paper we plan to develop the Thurston theory of pseudo-Anosov diffeomorphisms in the context of quadratic differentials as well.

Technically the basic problem is how to deal with recurrent trajectories. We use the method of strips that was introduced by Strebel in 1970 (and foreshadowed by Jenkins in 1960) and that has been successful in a number of different connections in getting at their properties. To this we add the technique of Thurston that shows how to form a single simple loop in a controlled manner from any number of mutually disjoint ones. Thurston's concept of a convergent sequence of simple loops forms one pillar of the bridge between the geometric and the analytic. The other is formed by the simple Jenkins-Strebel differentials. The bridge itself carries the happy traveller between the flexible geometric world of measured foliations and the rigid analytic world of quadratic differentials.

An exposition of some of the results here is contained in [11].

This paper was written while the first named author was a guest of the Forschungsinstitut für Mathematik, E.T.H., Zürich. Part of the theory was developed for lectures presented at the University of California at San Diego by him in 1980. It has been a great privilege indeed to have been a member of those institutes. In addition, the work was supported in part by the National Science Foundation (U.S.A.).

1. Basic properties of quadratic differentials

1.1. We will work with Riemann surfaces R which do not necessarily have finite topological type and which may have a *border* ∂R. By definition of ∂R, each point

$p \in \partial R$ has a neighborhood conformally equivalent to $\{z \in \mathbf{C}: |z|<1, \operatorname{Im} z \geq 0\}$. The components of ∂R consist of open intervals and closed curves.

An ideal boundary component of R is called a *puncture* if it has a neighborhood conformally equivalent to the once punctured disk. Points removed from a surface become punctures and conversely punctures put back in a surface become points.

Correspondingly, one can also speak of punctures on the border of a surface.

In the best cases R will be compact and the border if nonempty will be a finite union of curves. We will speak of a compact surface R possibly with boundary ∂R. In addition we want to allow the possibility that a surface R comes from such a compact surface by the removal of a finite number of points, some of which may be on the boundary. Thus the terminology, "R is a compact surface, possibly with boundary ∂R and possibly with a finite number of punctures".

1.2. Our investigation concerns *quadratic differentials* $\varphi\, dz^2$ on a Riemann surface R. These are invariant forms with *holomorphic* $\varphi(z)$ in the local coordinate neighborhood governing z, and holomorphic on the border ∂R too, if $\partial R \neq \emptyset$.

We will always assume that at the punctures of R, $\varphi\, dz^2$ has *at most* simple poles. The *norm* of $\varphi\, dz^2$ is defined as

$$\|\varphi\| = \iint_R |\varphi|\, dx\, dy.$$

On a compact surface, possibly with border, possibly with a finite number of punctures, the norm is automatically finite.

A *normalized differential* is one which has finite norm and for which

$$\|\varphi\| = 1.$$

We will describe a number of known results. For a systematic development of the subject see [7] or [20].

The differential $\varphi\, dz^2$ is called *real* if $\operatorname{Im} \varphi\, dz^2 = 0$ along ∂R. If in addition $\varphi\, dz^2 \geq 0$ (resp., ≤ 0) on ∂R, it is called *positive* (resp., *negative*). Let $\hat{R}$ denote the double of R across ∂R and $J: \hat{R} \to \hat{R}$ the anti-conformal involution fixing ∂R. A differential $\psi\, dz^2$ on $\hat{R}$ is called even if $\psi(Jz) = \overline{\psi(z)}$ and odd if $\psi(Jz) = -\overline{\psi(z)}$, in terms of the local coordinate z about some $p \in \hat{R}$ and $\overline{J(z)}$ about $J(p)$. The negative differentials on R are the restrictions of the odd ones on $\hat{R}$, and the positive differentials the restrictions of the even ones.

When R has genus g, $p \geq 0$ punctures all in the interior, and $b \geq 0$ boundary components, then the real differentials form a real vector space of dimension

$6g+3b+2p-6$ (assuming this is positive). On a torus $g=1$, $b=p=0$, they form a real vector space of dimension two. If the tori are realized as lattices in $\mathbf{C}$, the differentials are of the form

$$c\,dz^2, \quad c\in\mathbf{C}.$$

1.3. In working with a quadratic differential $\varphi\,dz^2$ on R, we will use the term *critical point* to refer to either (i) a zero of φ, or (ii) a puncture of R (where φ may or may not have a zero or simple pole). On a compact surface R of genus $g\geq 1$ without boundary, a differential $\varphi\,dz^2$ has exactly $4g-4$ zeros, counted according to multiplicity.

Away from the critical points the expression

$$\Phi(z)=\int^z \sqrt{\varphi}\,dz$$

determines a local homeomorphism into the complex plane. The preimages in R of the horizontal lines in $\mathbf{C}$ (resp. vertical lines), extended as far as possible by analytic continuation are called *horizontal* (resp., vertical) trajectories.

That is, the horizontal (resp., vertical) trajectories are the integral curves of the line field determined by the expression $\varphi\,dz^2>0$ (resp., $\varphi\,dz^2<0$). Through each non-critical point ζ runs exactly one horizontal and one vertical trajectory. Given one of the two horizontal or vertical directions at ζ, we speak of the horizontal or vertical *trajectory ray* starting from ζ in that direction.

We distinguish five possibilities for a trajectory ray:

(i) It closes up forming a simple loop.

(ii) It runs into a critical point. It is then called a *critical ray*.

(iii) It is *recurrent*: it continues indefinitely without ever crossing itself but comes arbitrarily close to its initial point infinitely often.

(iv) It approaches the ideal boundary, i.e. it leaves every compact set. It is then called a boundary ray.

(v) It has limit points on the ideal boundary: it contains a sequence of points which approach the ideal boundary.

From a zero of $\varphi\,dz^2$ of order $n\geq 1$, $(n+2)$ horizontal and $(n+2)$ vertical rays emanate. From a simple pole, only one horizontal and one vertical ray appears.

1.4. Each differential $\varphi\,dz^2$ gives rise to a singular flat metric on R,

$$ds = |\varphi(z)|^{1/2}\,|dz|,$$

with area

$$\|\varphi\| = \int\int |\varphi(z)|\,dx\,dy \leqslant \infty.$$

To gain some insight into this metric assume that R is a compact surface, possibly with boundary, possibly with a finite number of punctures. Making allowance for certain limiting cases the following statement is true:

Given points ζ_1, ζ_2 on R and a homotopy class of paths between them, there exists a unique φ-geodesic in this class. In particular, we must allow for the possibility that the geodesic runs through some of the punctures. If the homotopy class contains a simple arc, the φ-geodesic will also be simple, or a limiting case of simple arcs.

The φ-geodesic is a finite union of φ-straight segments (each the preimage under Φ of a Euclidean line segment in the plane). Each end point of each segment is either one of the points ζ_1, ζ_2, or is a critical point.

Suppose instead that the free homotopy class of a simple loop is prescribed on R, and assume that it is not retractable to a puncture (or a point in R). Allowing again for limiting cases, there is a φ-geodesic in the class which is also a simple loop (it may run through punctures). Either it is unique, or for some ϑ, $e^{i\vartheta}\varphi\,dz^2$ has a closed horizontal trajectory in that class. The φ-geodesic, when unique, is a union of φ-straight segments whose end points are critical points.

1.5. We continue working with a compact surface R, possibly with boundary, possibly with a finite number of punctures. Suppose that $\varphi\,dz^2$ is a negative differential on R. Then a non-closed vertical trajectory of finite φ-length has both end points at critical points (recall that, by definition, punctures are critical points).

Let R_φ denote the open set resulting from the removal of all non-closed vertical trajectories of finite length (the "critical graph"). The components A of R_φ are of two kinds:

(i) A is an *annular domain* swept out by closed vertical trajectories not retractable to a point or puncture of R.

(ii) A is a *spiral domain* in which each vertical ray of infinite φ-length from a point of A is dense in A (A may be punctured). A is not doubly or triply connected.

Suppose now that $\varphi\,dz^2$ more generally is real on ∂R. Then in addition to the possibilities listed above, a vertical trajectory of finite length may be a *cross-cut*. That is, both its ends lie on components of ∂R.

If $\varphi\,dz^2$ is not negative, then in addition to the two possibilities listed above for components A of R_φ is the certainty of a third:

(iii) A is a *cross-cut* domain swept out by parallel vertical trajectories whose end points lie on ∂R. These trajectories are not retractable into $\partial R \setminus \{\text{critical points}\}$.

A (non-critical) vertical ray starting from a point of ∂R can only end at another point of ∂R and therefore determines a cross-cut domain.

1.6. Decomposition of a spiral domain into strips. This important technique was introduced by Strebel in [15]. We continue to work with a finite surface R as in §§ 1.4, 1.5. Taking in hand a spiral domain A, fix a short horizontal φ-segment α. Regard this as two sided α^+ and α^-. On each of these sides mark the finite set of points $\{x_i^+\}$, $\{x_i^-\}$ determined as follows: (i) the vertical trajectory ray leaving from x_i^+ or x_i^- hits a critical point (necessarily in A^-) before returning to α, or (ii) x_i^+ or x_i^- is one of the two end points of α_+ or α_-, or (iii) the vertical trajectory ray from x_i^+ or x_i^- hits an end point of α before otherwise meeting α.

The points $\{x_i^+\}$ and $\{x_i^-\}$ divide α^+ and α^- respectively into a number of intervals $\{I_j^+\}$ and $\{I_j^-\}$. The total number of intervals is even and they are arranged in pairs as follows.

To each interval I there corresponds I' such that the pair (I, I') have the following property: A vertical trajectory departing α from an interior point of I first returns to α (without leaving A) by hitting an interior point of I'. The intervals I, I' are either on the same side of α, or they are on opposite sides. The two intervals (I, I') of all pairs are on opposite sides if and only if a branch of $\sqrt{\varphi}\,dz$ can be globally defined in A.

Moreover corresponding to each pair (I, I') is a "strip" or φ-rectangle S whose interior lies in A. The two segments I, I' from the two horizontal sides of S. Each vertical side goes through a critical point of order $n \geqslant -1$ subtending in S the angle $2\pi/(n+2)$ at that point ($n=0$ means a puncture with no singularity), and/or has as one of its end points an end point of α.

The interior of S is mapped by any choice of $\Phi(z) = \int^z \sqrt{\varphi}\,dz$ onto a proper rectangle in the plane whose sides are parallel to the coordinate axes. Its height b and width a are independent of the choice of branch Φ. Its area is

$$\iint_S |\Phi'(z)|^2\,dx\,dy = \iint_S |\varphi|\,dx\,dy = ab.$$

The union of all the (closed) rectangles S fills the closure A^-.

For later purposes it is very convenient to introduce a technical refinement in this strip decomposition. A strip is of the *first kind* if its horizontal sides lie on opposite sides of α. We leave these alone but observe that the totality of their horizontal sides covers the same length of α^+ as of α^-. Each of the remaining strips, those of the *second kind*, has both its horizontal sides on the same side of α. The totality of these covers the same length of α^+ as of α^-. Hence by further subdividing these strips, they can be arranged (not uniquely) in distinct pairs (S, S'). The φ-widths of S and S' are the same, but one has its horizontal sides on α^+, the other has its on α^-.

2. Heights

2.1. We return to the general situation of §§ 1.1 and 1.2 of a Riemann surface R, possibly with border ∂R, and a quadratic differential $\varphi\, dz^2$ on R with at most simple poles at the punctures. For each simple loop γ on R define its *φ-height* to be the number,

$$h_\varphi(\gamma) = \inf_{\tilde\gamma \sim \gamma} \int_{\tilde\gamma} |\mathrm{Im}\sqrt{\varphi}\, dz| = \inf_{\tilde\gamma \sim \gamma} \int |dv|.$$

Here the infimum is taken over all simple loops in the free homotopy class of γ and the integral is the total variation of the imaginary part dv,

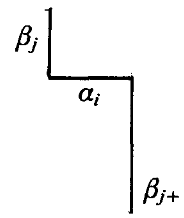
$$dw = du + i\,dv = \Phi'(z)\,dz = \sqrt{\varphi}\, dz,$$

(locally away from the critical points). We will automatically assume the class is non-trivial, that is, not retractable to a point or puncture of R. In any case for those cases, the height is always zero.

Heights can just as easily be defined for *cross-cuts*. By a cross-cut on R we mean a simple closed arc γ whose end points lie on the border ∂R of R (so if $\partial R = \varnothing$ there are no cross-cuts in this sense). Two cross-cuts γ_1, γ_2 are called freely homotopic (homotopic modulo ∂R) if there is a continuous map of a closed rectangle into R taking one vertical side to γ_1, the other to γ_2, and the horizontal sides into ∂R. When γ is a cross-cut, $h_\varphi(\gamma)$ is defined exactly as was done when γ is a simple loop. Again we exclude the trivial cases that γ is retractable to a point or puncture in ∂R.

Remark. In the competition for the infimum one can include not only simple loops and arcs but also those with self-intersections without lowering the infimum. This fact follows from Lemma 2.9 and the remark following it.

2.2. The height of a simple loop or cross-cut γ composed of horizontal and vertical φ-segments such that the two vertical arcs β_j, β_{j+1} meeting a horizontal arc α_i end on different sides of α_i (forming a step as shown)



is computed by the simple formula

$$h_\varphi(\gamma) = \Sigma_j b_j.$$

Here b_j is the φ-length of β_j. In particular, the height of a closed vertical trajectory is simply its φ-length while the height of a closed horizontal one is zero.

When the surface is compact, possibly with boundary, possibly with a finite number of punctures, and the differential is real, more can be said. There are always simple loops or arcs in the free homotopy class over which the integral achieves its infimum (allowing, as usual, for limiting cases). One such curve or arc is the φ-geodesic. The height is zero if and only if the free homotopy class contains (possibly as a limiting case) an element made up entirely of φ-horizontal segments.

2.3. On a finitely punctured compact surface, suppose real differentials $\varphi_n \to \varphi$ converge locally uniformly (uniformly on compact subsets), hence uniformly. It is clear from what was said above in § 2.2 that for each simple loop or cross-cut γ, $\lim h_{\varphi_n}(\gamma) = h_\varphi(\gamma)$. On the other hand for a general surface, matters are not so simple.

The rest of the chapter will be occupied by the proof of the following important general fact.

PROPOSITION 2.3. *Suppose R is an arbitrary Riemann surface possibly with border ∂R and $\{\varphi_n\}$ is a sequence of real quadratic differentials that converges locally uniformly to φ. Then for every simple loop or cross-cut γ,*

$$\lim h_{\varphi_n}(\gamma) = h_\varphi(\gamma).$$

2.4. *Proof.* Assume first that γ is a non-trivial simple loop. We will make use of the annular covering surface $A(\gamma)$ of R corresponding to γ (see [20]) which is conformally equivalent to a standard annulus. It is characterized by the following property; $\pi: A(\gamma) \to R$ denotes the projection. Namely for each simple loop α in the free homotopy class $[\gamma]$, exactly one component α^* of $\{\pi^{-1}(\alpha)\}$ in $A(\gamma)$ is a loop and for that $\pi^{-1}: \alpha \to \alpha^*$ is a homeomorphism. The other components are open simple arcs, each covering α infinitey often. Conversely, if α_1^* is a simple loop separating the contours of $\partial A(\gamma)$, then $\pi(\alpha_1^*)\in[\gamma]$ but it is not necessarily a simple loop.

A differential $\varphi\, dz^2$ on R can also be lifted to a differential $\varphi^*\, dz^2$ on $A(\gamma)$. In particular it will satisfy

$$\int_{\alpha^*} |\mathrm{Im}\sqrt{\varphi^*}\, dz| = \int_{\alpha} |\mathrm{Im}\sqrt{\varphi}\, dz|.$$

This holds even if $\alpha=\pi(\alpha^*)$ is not simple. Thus for the heights,

$$h_{\varphi^*}(\gamma) = h_{\varphi}(\gamma)$$

where for the left, the infimum is taken over all simple loops in $A(\gamma)$ separating the boundary contours.

2.5. LEMMA 2.5. *Suppose $\sigma\subset A(\gamma)$ is a simple loop separating ∂A and is the union of φ^*-straight segments. There is an integer M with the following property. Suppose $A_0\subset A(\gamma)$ is a simply connected region whose boundary ∂A_0 is the union of an arc of σ and a φ^*-horizontal segment α. Then A_0 contains at most M critical points of φ^*.*

Proof. ∂A_0 is a φ^*-polygon with certain vertices $\{\zeta_j\}$ (including critical points on ∂A_0) and interior angles $\{\vartheta_j\}$ at those. According to Teichmüller's lemma (see [20])

$$\Sigma\left(1-\vartheta_j\frac{n_j+2}{2\pi}\right) = 2+\Sigma n_i,$$

where $n_j\geqslant 0$ is the order of ζ_j and $\{n_i\}$ with $n_i\geqslant 1$ denotes the orders of the critical points of φ^* lying in A_0. All the quantities on the left are determined by σ except for the two angles at the end points of α. There is an upper bound for all possible values the left side can have. Therefore Σn_i has an upper bound $M>0$ as well.

2.6. COROLLARY 2.6. *With σ as in Lemma 2.5 there is a number M_1 with the following property. There exist at most M_1 mutually disjoint arcs $\{\sigma_i\}$ of σ such that to each σ_i corresponds a φ^*-horizontal segment α_i, with the same end points as σ_i but otherwise disjoint from σ, such that $\sigma_i\cup\alpha_i$ bounds a simply connected region in $A(\gamma)$.*

Proof. Suppose A_0 is a simply connected region such that ∂A_0 is the union of a φ-straight segment σ_0 and a horizontal segment α. If ∂A_0 has no critical points then there are exactly two vertices and the interior angles are ϑ and $\pi-\vartheta$. By Teichmüller's lemma,

$$2+\Sigma n_i=(1-\vartheta/\pi)+(1-(\pi-\vartheta)/\pi)=1$$

which is impossible (that is, α is the unique geodesic in its homotopy class). Such regions A_0 do not exist. The proof of Corollary 2.6 follows from this fact.

2.7. LEMMA 2.7. *With the hypothesis on σ and the horizontal cross-cut α as in Lemma 2.5, let d denote the φ^*-length of σ. Then the φ^*-length of α does not exceed d.*

Proof. The horizontal φ^*-segment α is the unique φ^*-geodesic between its end points (in its homotopy class).

2.8. Suppose σ is as in Lemma 2.5, and α is a φ^*-horizontal segment meeting σ only at its end points. The end points of α divide σ into two (connected) arcs exactly one of which, say σ_0, has the property that $\sigma_0 \cup \alpha$ bounds a simply connected region in $A(\gamma)$. It will be convenient to say that σ_0 is the arc of σ *determined by the horizontal cut α.*

We will also use the terminology *band of horizontal cuts of height b* to refer to the following situation. That (i) there is an open vertical segment β of height b, and a family α_t of horizontal segments (without critical points) such that (ii) each α_t intersects β once and $\{\alpha_t\}$ is indexed by the height parameter t on β, $0<t<b$, and (iii) the end points of α_t lie on σ but otherwise α_t does not meet σ.

LEMMA 2.8. *With σ as in Lemma 2.5 suppose that σ_0 is the arc of σ determined by a horizontal cut α_0. Set $h=\int_{\sigma_0}|\mathrm{Im}\sqrt{\varphi^*}\,dz|$. Then in the simply connected region $A_0\subset A(\gamma)$ bounded by $\sigma_0\cup\alpha_0$ there exists a band of horizontal cuts $\{\alpha_t\}$ of height $h/2(3M)^2S$. The arc σ_t of σ determined by α_t satisfies $\int_{\sigma_t}|\mathrm{Im}\sqrt{\varphi^*}\,dz|>h/2(3M)^2S$ for each t. Here S is the number of straight segments of σ.*

Proof. From the at most M critical points (according to multiplicity) in A_0 draw the critical horizontal rays until they meet σ. These critical rays divide A_0 into at most $3M$ simply connected regions $\{B_i\}$. For some i, say $i=1$,

$$\int_{\partial B_1}|\mathrm{Im}\sqrt{\varphi^*}\,dz|\geq h/3M.$$

There are at most $3M$ arcs of σ contained in ∂B_1. Therefore at least one of them, say σ_1, has φ^*-height not less than $h/(3M)^2S$. There is a band of horizontal cuts $\{\alpha_t\}$ of height $h/(3M)^2S$ leaving σ_1 and lying in B_1. (Actually the vertical interval β used in the definition may have to be replaced by some steps composed of horizontal and vertical segments.) But instead of taking the whole band take only half of it, cut off at the 1/4 and 3/4 level. Then we can be sure that the arc σ_t of σ determined by α_t has φ^*-height greater than $h/2(3M)^2 S$.

2.9. *The ε-height condition.* A simple loop $\sigma\subset A(\gamma)$ separating the contours will be said to satisfy the ε-height condition if it has the following properties: (i) σ is a union of a finite number S of φ^*-horizontal and vertical segments without critical points, and (ii) for any finite collection $\{\sigma_i\}$ of mutually disjoint subarcs of σ which are determined respectively by horizontal cuts $\{\alpha_i\}$, it is true that

$$\sum_i \int_{\sigma_i} |\mathrm{Im}\sqrt{\varphi^*}\,dz| < \varepsilon.$$

LEMMA 2.9. *Suppose σ satisfies the ε-height condition for some $\varepsilon>0$. Let τ be any simple loop in $A(\gamma)$, separating its boundary contours. Then*

$$\int_\tau |\mathrm{Im}\sqrt{\varphi^*}\,dz| > \int_\sigma |\mathrm{Im}\sqrt{\varphi^*}\,dz| - 2\varepsilon.$$

Proof. For simplicity assume first τ is disjoint from σ, say it lies to the left of σ. Consider the horizontal rays $\{\alpha_p\}$ leaving from the left side of interior points $\{p\}$ of vertical segments of σ. If α_p returns to σ before meeting τ it determines an arc σ_p of σ such that σ_p and a segment of α_p bound a simply connected region A_p in $A(\gamma)$. Take the union $\bigcup A_p$ of all such regions. In view of Corollary 2.6 this union has a finite number of components. Each is simply connected and bounded by an arc σ_i of σ and a horizontal segment α_i which may touch τ without crossing it and/or pass through a critical point. Let $\sigma'=\sigma\setminus\bigcup\sigma_i$.

Recall the following elementary fact. If K is a simply connected relatively compact region and ∂K is the union of (i) two horizontal segments which may pass through critical points, (ii) a vertical segment β, and (iii) some other arc γ then

$$\int_\gamma |\mathrm{Im}\sqrt{\varphi^*}\,dz| \geqslant \int_\beta |\mathrm{Im}\sqrt{\varphi^*}\,dz|.$$

It follows easily from this fact that

$$\int_\tau |\mathrm{Im}\sqrt{\varphi^*}\,dz| \geqslant \int_{\sigma'} |\mathrm{Im}\sqrt{\varphi^*}\,dz| \geqslant \int_\sigma |\mathrm{Im}\sqrt{\varphi}\,dz| - \varepsilon.$$

In the more general case we can proceed as follows. Among the components of

$A(\gamma)\setminus(\tau\cup\sigma)$ there are exactly two, B and B', that are not relatively compact in $A(\gamma)$; say B is adjacent to the contour of $\partial A(\gamma)$ on the left of σ and B' to the contour on the right. The relative boundary components b of B and b' of B' can be decomposed as

$$b=\bigcup\sigma_i\cup\tau_j,\quad b'=\bigcup\sigma_i'\cup\tau_j'$$

where $\{\sigma_i,\sigma_i'\}$ are mutually disjoint arcs of σ and $\{\tau_j,\tau_j'\}$ of τ (ignoring end points).

Since σ and b, as well as τ and b', are "essentially" mutually disjoint (can be pulled apart by an arbitrarily small deformation) the argument above proves,

$$\int_b |\operatorname{Im}\sqrt{\varphi^*}\,dz|>\int_\sigma |\operatorname{Im}\sqrt{\varphi^*}\,dz|-\varepsilon$$

$$\int_{b'} |\operatorname{Im}\sqrt{\varphi^*}\,dz|>\int_\sigma |\operatorname{Im}\sqrt{\varphi^*}\,dz|-\varepsilon.$$

Adding these two inequalities, using the decomposition of b and b' as well as the fact that $\{\sigma_i,\sigma_i'\}$ are mutually disjoint on σ and $\{\tau_j,\tau_j'\}$ on τ, we end up with

$$\int_\tau |\operatorname{Im}\sqrt{\varphi^*}\,dz|\geq\int_{\bigcup\tau_j\cup\tau_j'} |\operatorname{Im}\sqrt{\varphi^*}\,dz|>\int_\sigma |\operatorname{Im}\sqrt{\varphi^*}\,dz|-2\varepsilon.$$

Remark. Lemma 2.9 holds even when τ is not a simple loop by essentially the same proof.

2.10. LEMMA 2.10. *Given $\varepsilon>0$ there exists a simple loop $\sigma\subset A(\gamma)$ satisfying the ε-height condition and for which*

$$\int_\sigma |\operatorname{Im}\sqrt{\varphi^*}\,dz|\leq h_\varphi(\gamma)+\varepsilon.$$

Furthermore σ may be chosen so that $\pi(\sigma)\subset R$ is also a simple loop.

Proof. The loop σ is selected from a minimizing sequence for $h_\varphi(\gamma)$. Critical points can be avoided by arbitrarily short detours.

2.11. *Completion of the proof of the proposition.* We may assume that $\varphi\not\equiv0$ for otherwise Proposition 2.3 is obvious. Given $\varepsilon>0$ choose σ as in Lemma 2.10 but with ε there replaced by $\varepsilon/2(3M)^2M_1S$ where M is given by Lemma 2.5 and M_1 by Corollary

2.6. Lift the φ_n to φ_n^* in $A(\gamma)$. Then $\varphi_n^* \to \varphi^*$, uniformly on compact subsets of $A(\gamma)$.

Let U be a relatively compact annular neighborhood of σ whose closure contains no critical points of φ^*. For some N, φ_n^* for $n \geq N$ has no critical points there either. When N is sufficiently large, we can construct for each $n \geq N$ a simple loop $\sigma_n \subset U$ with the following properties: (i) σ_n is the union of S φ_n^*-horizontal and vertical segments, and (ii) these segments can be indexed with those of σ so that as $n \to \infty$, each segment converges to the corresponding segment of σ. (In fact $\pi(\sigma_n) \subset R$ will be a simple loop too.)

It is important to observe that the constants M, M_1 hold uniformly for all σ_n, $n \geq N$, as well as for σ. We may also assume that σ_n for $n \geq N$ satisfies Lemma 2.7 with d there replaced by $(d+\varepsilon)$.

We claim that for all large n, σ_n has the ε-height property with respect to φ_n^*. For suppose to the contrary that an infinite subsequence $\{\sigma_k\}$ does not. By Corollary 2.10 there is a φ_k^*-horizontal cut of σ_k which determines an arc τ_k of σ_k whose φ_k^*-height is $\geq \varepsilon/M_1$.

According to Lemma 2.8, there is a φ_k^*-band of horizontal σ_k cuts of φ_k^*-height $\varepsilon/2(3M)^2 M_1 S$. Further, each cut α_{kt} in this band bounds a simply connected region in $A(\gamma)$ with some arc σ_{kt} of σ_k whose φ_k^*-height exceeds $\varepsilon/2(3M)^2 M_1 S$.

We may orient the band and so locate the two arcs β_k, β_k' of σ_k such that the band consists of the trajectory segments leaving the points of β_k from the right and arriving from the left at the points of β_k'. Passing to a subsequence if necessary we can assume that the arcs $\{\beta_k\}$, $\{\beta_k'\}$ converge to arcs β, β' of σ. Since β_k, β_k' have equal φ_k^*-height of $\varepsilon/2(3M)^2 M_1 S$ so β, β' have φ^*-height also of $\varepsilon/2(3M)^2 M_1 S$.

Suppose by taking a suitable subsequence we can get one cut α_{kt} of the k-band to converge to a φ^*-horizontal cut α_t of σ. Then α_t determines an arc σ_t of σ whose φ^*-height is not less than $\varepsilon/2(3M)^2 M_1 S$. But this is in contradiction to the choice of σ to satisfy the $\varepsilon/2(3M)^2 M_1 S$-height condition.

The k-band has an upper (horizontal) edge and a lower edge. We designate the lower edge to be the one that, with an arc of σ_k, bounds a simply connected region containing the rest of the band. In fact the k-band can be regarded as a φ_k^*-rectangle. The modulus of this, namely the ratio of its φ_k^*-height to its φ_k^*-width, is uniformly bounded above zero and less than infinity as $k \to \infty$ by Lemma 2.7. Therefore if the lower edge has limit points on the inner contour, say, of $A(\gamma)$, the upper edge cannot. Actually (after passing to a subsequence if necessary) all the interior cuts α_{kt} of the band converge to φ^*-horizontal cuts α_t of σ. But we had just concluded that this is impossible.

Consequently the situation does not arise in the first place: for all large n, σ_n satisfies the ε-height condition. Now apply Lemma 2.9 to φ_n^*. It implies

$$h_{\varphi_n}(\gamma) \geq \int_{\sigma_n} |\text{Im}\sqrt{\varphi_n^*}\, dz| - 2\varepsilon.$$

Therefore in view of our choice of σ_n,

$$\liminf h_{\varphi_n}(\gamma) \geq h_\varphi(\gamma) - 2\varepsilon.$$

The opposite inequality,

$$\limsup h_{\varphi_n}(\gamma) \leq h_\varphi(\gamma),$$

is clear. Since ε is arbitrary the proof of the proposition is complete when γ is a simple loop.

2.12. The case that γ is a cross-cut (end points on ∂R) is handled in a very similar manner. We first find a replacement for the annular covering surface. The double $\hat{\gamma}$ of γ in the double $\hat{R}$ of R is a simple loop invariant under the anticonformal involution J of $\hat{R}$ that fixes ∂R. The lift $\hat{J}$ of J to the annular covering surface $A(\hat{\gamma})$ is an anticonformal involution that fixes exactly two open arcs. These arcs separate $A(\hat{\gamma})$ into two "rectangles" exactly one of which $A(\gamma)'$ contains a cross-cut which projects one-to-one onto γ. We specify the two fixed arcs under $\hat{J}$ to be the two vertical sides of $A(\gamma)'$. Let $A(\gamma)$ denote the result of removing from $A(\gamma)'$ all those points not lying over R. The "rectangle" $A(\gamma)$ has the same vertical sides as $A(\gamma)'$ and there is exactly one lift of γ that runs between them, in fact that even touches them. If τ is a cross-cut of $A(\gamma)$ between the vertical sides, the projection of τ to R is freely homotopic to γ. Conversely, each arc freely homotopic to γ is the projection of exactly one such τ.

The lifted differential φ^* to $A(\gamma)$ is real on the vertical sides. That is, the vertical sides are the union of φ^*-horizontal and vertical segments.

Using $A(\gamma)$ with its vertical sides distinguished we can follow the proof above step-by-step. In particular, the corresponding ε-condition is easily formulated. The desired conclusion follows.

2.13. *Remark.* The analogous statement holds for lengths $L_\varphi(\gamma)$ (see § 4.3) as well: if $\{\varphi_n\}$ converges locally uniformly to φ then $\lim L_{\varphi_n}(\gamma) = L_\varphi(\gamma)$ for all simple loops and cross-cuts γ.

3. A minimum norm property

3.1. Heights can be defined for more general objects yet which are still modeled on quadratic differentials. Let R be a general Riemann surface, possibly with border ∂R. Denote by $\tilde{v}$ a system of real C^1 function elements defined locally on R, outside a discrete set of points E (without accumulation points in R). Assume that on overlapping domains the elements are related as

$$\tilde{v}_2 = \pm\tilde{v}_1 + \text{constant}.$$

For example, $\tilde{v}=\operatorname{Im}(\int\sqrt{\varphi}\,dz)$ for a quadratic differential φ. Outside E, $|d\tilde{v}|=|\tilde{v}_x\,dx+\tilde{v}_y\,dy|$ is invariantly defined. We assume that it has finite Dirichlet integral,

$$\|dv\|^2 = \int_R (\tilde{v}_x^2+\tilde{v}_y^2)\,dx\,dy < \infty.$$

For such a system $\tilde{v}$ and any simple loop or cross-cut γ define the $\tilde{v}$-height as

$$h_{\tilde{v}}(\gamma) = \inf_{\tilde{\gamma}\sim\gamma} \int_{\tilde{\gamma}} |d\tilde{v}|.$$

As before (§ 2.1), the infimum is taken over the free homotopy class.

3.2. A surface R is called *parabolic* if it has no Green's function. This means that the ideal boundary is small: if $R\subset\mathbf{C}$ the complement is a set of capacity zero. For our purposes this means that for a differential φ of finite norm, the set of its horizontal trajectories that tend to the ideal boundary in both directions covers a set of φ-area zero.

The remainder of this chapter will be occupied by the proof of the following basic result.

THEOREM 3.2. *Assume that R is a parabolic Riemann surface or surface with border ∂R whose double across ∂R is parabolic. Let $\varphi\,dz^2$ be a real quadratic differential of finite norm on R and $\tilde{v}$ a system of function elements as specified in* § 3.1. *Suppose that for all simple loops and cross-cuts γ on R the heights satisfy*

$$h_{\tilde{v}}(\gamma) \geqslant h_{\varphi}(\gamma).$$

Then

$$\|d\tilde{v}\|^2 \geqslant \|\varphi\|$$

with equality if and only if $(d\bar{v})^2=(\operatorname{Im}\sqrt{\varphi}\,dz)^2$. *If* $\varphi\,dz^2$ *is negative, then only simple loops* γ *need be considered for the conclusion to hold.*

3.3. We will first prove the theorem under the assumption that R is a compact surface, possibly with boundary ∂R, possibly with a finite number of punctures. As in § 1.5 let R_φ denote the open set resulting from the removal of all critical vertical trajectories of finite length. The proof proceeds by analyzing each component A of R_φ separately.

Case 1. A is an annular domain. Fix a horizontal cross-cut α of A joining its two boundary components. Cut A along α and map the result onto a rectangle in the $w=u+iv$ plane, say $\{0<u<a,\ 0<v<b\}$, by a choice of $w=\Phi(z)=\int^z\sqrt{\varphi}\,dz$.

For the closed vertical trajectories β in A, $h_\varphi(\beta)=b$. In the rectangle they become the vertical cross-cuts β_u, $0<u<a$. By hypothesis,

$$b=h_\varphi(\beta_u)\leq h_{\bar{v}}(\beta_u)\leq\int_{\beta_u}|d\bar{v}|=\int_0^b\left|\frac{\partial\bar{v}(u,v)}{\partial v}\right|dv,$$

where we have represented $d\bar{v}$ in the w-coordinate system. Integration in du yields,

$$ab\leq\int_0^a\int_0^b\left|\frac{\partial\bar{v}(u,v)}{\partial v}\right|du\,dv.$$

Then we apply the Schwarz inequality to obtain,

$$(ab)^2\leq ab\int_0^a\int_0^b\left(\frac{\partial\bar{v}}{\partial v}\right)^2du\,dv\leq ab\int_0^a\int_0^b\left[\left(\frac{\partial\bar{v}}{\partial u}\right)^2+\left(\frac{\partial\bar{v}}{\partial v}\right)^2\right]du\,dv,$$

or more simply,

$$\iint_A|\varphi|\,dx\,dy=ab\leq\iint_A\left[\left(\frac{\partial\bar{v}}{\partial x}\right)^2+\left(\frac{\partial\bar{v}}{\partial y}\right)^2\right]dx\,dy. \tag{1}$$

The at most finite number of exceptional points in A cause no trouble.

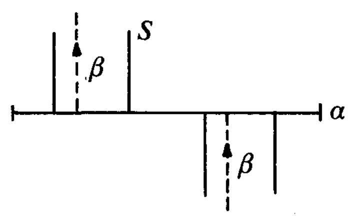
Case 2. A is a cross-cut domain. This is exactly the same as Case 1 except there is no need for a preliminary horizontal cut in order to obtain a rectangle.

Case 3. A is a spiral domain. Given a small $\varepsilon>0$ fix a non-critical horizontal segment α of φ-length $<\varepsilon$ in A which does not meet the exceptional set E. Construct a strip decomposition of A based on α as described in § 1.6. Consider each strip S

separately if its horizontal sides lie on opposite sides of α (first kind); otherwise consider S together with its partner S' (second kind).

As above, a finite number of points of E may lie in A but because of our hypothesis of finite Dirichlet integral these do not affect the area integrals.

Case 3 (a). S is of the first kind. The vertical segments between the horizontal sides of S all have the same φ-length, say b. For each such vertical segment there is one way to add a segment of α between its end points there so as to form a simple loop β^*.



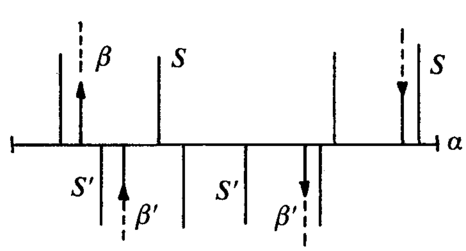
The added segment of α of course has φ-length $<\varepsilon$ and $h_\varphi(\beta^*)=b$. Using the $w=u+iv=\Phi(z)$ coordinates in S where Φ is chosen to map S onto $\{0<u<a, 0<v<b\}$, in view of our hypothesis on heights,

$$b\leq h_{\tilde v}(\beta^*)\leq \int_{\beta^*}|d\tilde v|\leq \int_0^b\left|\frac{\partial \tilde v}{\partial v}\right|dv+M\varepsilon.$$

Here M is an upper bound for $|\partial\tilde v/\partial u|$ on α. Integrating in the u-direction,

$$ab\leq \int\int_S\left|\frac{\partial \tilde v}{\partial v}\right|du\,dv+M\varepsilon a.$$

Case 3 (b). S and S' are a pair of strips of the same width of the second kind. Let a denote the common width. Parametrize the vertical lines β_u in S and β'_u in S' so that they sweep out $S\cup S'$ as u increases from 0 to a.



Let b denote the common φ-length of the β_u and b' of the β'_u. To $\beta_u \cup \beta'_u$ two (not necessarily disjoint) segments of α can be added so as to obtain essentially a simple loop β^*_u (it will be a simple loop if one of the horizontal segments is pushed a little off α). Again in $w=u+iv=\Phi(z)$ coordinates,

$$b+b' \leq h_{\tilde{v}}(\beta^*) \leq \int\int_{\beta_u+\beta'_u} \left|\frac{\partial \tilde{v}}{\partial v}\right| dv+2M\varepsilon.$$

Integrating in the u-direction we obtain

$$(b+b')a \leq \int\int_{S\cup S'} \left|\frac{\partial \tilde{v}}{\partial v}\right| du\,dv+2M\varepsilon a.$$

Sum the expressions from Case 3 (a) and 3 (b) over all strips using the notation $\|\varphi\|_A=\int\int_A du\,dv$,

$$\|\varphi\|_A \leq \int\int_A \left|\frac{\partial \tilde{v}}{\partial v}\right| du\,dv+M\varepsilon^2.$$

The integral is not independent of the choice of local coordinates so it must be interpreted in terms of fixed local patches of the Φ-coordinates. Let $\varepsilon \to 0$ and then apply the Schwarz inequality to the result. We obtain

$$\|\varphi\|_A^2 \leq \|\varphi\|_A \int\int_A \left|\frac{\partial \tilde{v}}{\partial v}\right|^2 du\,dv \leq \|\varphi\|_A \int\int_A \left[\left(\frac{\partial \tilde{v}}{\partial u}\right)^2+\left(\frac{\partial \tilde{v}}{\partial v}\right)^2\right] du\,dv,$$

and finally,

$$\|\varphi\|_A \leq \int\int_A \left[\left(\frac{\partial \tilde{v}}{\partial u}\right)^2+\left(\frac{\partial \tilde{v}}{\partial v}\right)^2\right] du\,dv. \tag{2}$$

Therefore the same inequality (1), (2) holds irrespective of whether A is a ring or cross-cut domain or a spiral domain. Suppose for some A equality holds. Then $\partial\tilde{v}(u,v)/\partial u \equiv 0$ in $A \setminus A \cap E$ so that $\tilde{v}=\tilde{v}(v)$ in terms of the natural parameter v in A. But there is still equality in the application of Schwarz inequality and this forces $|d\tilde{v}/dv|=$ constant on $A \setminus A \cap E$. The equality $b=\int_{\beta^*}|d\tilde{v}|$ in Case 1 or 2, or the analogous thing in Case 3, implies that $|d\tilde{v}/dv| \equiv 1$ in $A \setminus A \cap E$. Therefore $\tilde{v}=\pm v+$constant in $A \setminus A \cap E$ and $d\tilde{v}=\pm dv=\pm \operatorname{Im}\sqrt{\varphi}\,dz$ with either one sign or the other holding locally.

Finally to obtain Theorem 3.2, sum up the inequalities (1) and (2) over all components A of the decomposition R_φ of R.

3.4. It remains to deal with the general case. Start by finding the annular domains and cross-cut domains, if any, in R. In particular each non-critical point on the border ∂R about which $\varphi\, dz^2>0$ lies in (or rather, on the edge of) a cross-cut domain. The boundary of the annular domains, and the relative boundary in $\operatorname{Int} R$ of the cross-cut domains, is a countable union of possibly critical vertical trajectories of finite length. For each of these domains Case 1 or 2 from above applies and we arrive at (1).

Let then R_0 denote the complement of the closure of the union of the annular and cross-cut domains. The relative boundary ∂R_0 of R_0 in R, which may include all or part of ∂R, is a union of possibly critical vertical trajectories: $\varphi\, dz^2\leq 0$ on ∂R_0. The double of R_0 across ∂R_0 is parabolic, or more precisely, each component of the double is parabolic.

We will apply the technique introduced in [19] to obtain a strip decomposition of R_0.

Let α be a φ-horizontal segment and α^+, α^- its two sides. On $\alpha^+\cup\alpha^-$ mark those points $\{x\}$ from which the vertical ray (i) hits a critical point of φ, (ii) has limit points on the ideal boundary of R_0, or (iii) hits an end point of α before otherwise meeting α. The set $\{x\}$ is closed on $\alpha^+\cup\alpha^-$ because the vertical ray from any point ζ not in it again meets α and there is a maximal neighborhood of ζ from which the ray-segments sweep out a rectangle whose horizontal sides are components of $\alpha^+\cup\alpha^-\setminus\{x\}$. Because the double of R_0 is parabolic, $\{x\}$ has linear measure zero (cf. [19, § 1]). It need not be countable.

The complement $F(\alpha)$ of $\{x\}$ on $\alpha^+\cup\alpha^-$ is a finite or countably infinite union of intervals. These are arranged in pairs, each pair comprising the two horizontal sides of an open φ-rectangle. Its vertical sides tend to critical points, end points of α, and/or ideal boundary components of R_0.

Each rectangle in the strip decomposition can be classified of the first kind or second kind according to whether its horizontal sides are on opposite sides of α or on the same side. The lengths of α^+ and α^- occupied by sides of the first kind are the same. Therefore, since $\{x\}$ has measure zero, the lengths of α^+ and α^- occupied by sides of the second kind are the same too. As was done for compact surfaces in § 1.6, here too, after subdividing if necessary, the rectangles of the second kind can be arranged in pairs (S, S'), where S with sides on α^+ corresponds to S^- with horizontal sides of the same length but on α^-.

Suppose α_1 is another horizontal segment, disjoint from α. If α_1 does not intersect the closure of the union of the vertical strips for α, then the strips for α_1 do not intersect those for α. If on the other hand α_1 lies in one of the strips for α, then the strips for α_1 are contained in the closure of the union of those for α.

Now R_0 can be covered up to a set of φ-area zero by a countable number of φ-rectangles whose horizontal sides are of φ-length less than some prescribed ε but we do this in the following way. Given $\delta>0$ choose compact sets C and C' with $C \subset \operatorname{Int} C'$ such that (i) C' contains no points of E or critical points of φ, and (ii) $\|\varphi\|_C \geq \|\varphi\| - \delta$. Given $\varepsilon>0$ we can find a finite collection of φ-rectangles contained in C' yet which cover C and each has φ-width less than ε. From within these choose horizontal segments $\{\alpha_i\}$, each of length $<\varepsilon$, for which the associated strip decompositions cover C without overlap up to a set of φ-area zero. Let $\{A_{ij}\}$ denote the strips (φ-rectangles) based on α_i.

Let a_i denote the length of α_i. Then $L = \Sigma a_i < \infty$. Also since all $\alpha_i \subset C'$, there is a constant $M < \infty$ such that

$$\left| \frac{\partial \bar{v}}{\partial u} \right| < M$$

on each α_i, in terms of the natural φ-coordinate $w = u + iv$ on it.

For each α_i, exactly as in Case 3 of § 3.3 even though there may now be infinitely many rectangles $\{A_{ij}\}$ based on α_i,

$$\|\varphi\|_{A_i} \leq \iint_{A_i} \left| \frac{\partial \bar{v}}{\partial v} \right| du\, dv + a_i M \varepsilon.$$

Here $A_i = \cup A_{ij}$. Summing over i, since $\cup A_i$ covers C up to a set of φ-area zero,

$$\|\varphi\|_C \leq \iint_R \left| \frac{\partial \bar{v}}{\partial v} \right| du\, dv + LM\varepsilon.$$

Again the integral must be interpreted in terms of a fixed covering by φ-rectangles.

First let $\varepsilon \to 0$. Then let $\delta \to 0$ so that $C \to R_0$. We get

$$\|\varphi\|_{R_0} \leq \iint_{R_0} \left| \frac{\partial \bar{v}}{\partial v} \right| du\, dv.$$

The proof is completed with Schwarz's inequality as in § 3.3.

4. The Heights theorem and other corollaries

4.1. The most important consequence of Theorem 3.2 is the Heights theorem. For compact surfaces it was first proved by Hubbard and Masur [4] using quite different techniques.

HEIGHTS THEOREM. *Assume R is a parabolic Riemann surface or a surface with border ∂R whose double across ∂R is parabolic. Suppose φ and ψ are real quadratic differentials of finite norm satisfying $h_\varphi(\gamma)=h_\psi(\gamma)$ for all simple loops and cross-cuts γ. Then $\varphi\equiv\psi$. If both φ and ψ are negative, then only simple loops γ need be considered.*

Proof. Recall that φ is real if $\operatorname{Im}(\varphi\, dz^2)=0$ along ∂R and negative if more restrictively $\varphi\, dz^2\leq 0$ along ∂R. If $\partial R=\emptyset$, by definition there are no cross-cuts so only the statement involving simple loops is operative. Let $\tilde{v}_1$ denote the function elements determined by $\operatorname{Im}(\int\sqrt{\varphi}\, dz)$ and $\tilde{v}_2$ by $\operatorname{Im}(\int\sqrt{\psi}\, dz)$. The singular set E is the union of the critical points of φ and ψ. Theorem 3.2 applied to $\tilde{v}_1$ and ψ shows that $\|\varphi\|=\|d\tilde{v}_1\|^2\geq\|\psi\|$ and applied to $\tilde{v}_2$ and φ shows that $\|\psi\|=\|d\tilde{v}_2\|^2\geq\|\varphi\|$. Hence $\|d\tilde{v}_1\|=\|\psi\|$ and $\operatorname{Im}\Phi=\pm\operatorname{Im}\Psi+\text{constant}$, locally for the integrated functions. Hence locally, $\Phi=\pm\Psi+\text{constant}$, and then $\varphi\, dz^2=(\Phi'(z)\, dz)^2=(\Psi'(z)\, dz)^2=\psi\, dz^2$.

4.2. COROLLARY 4.2. *Suppose R is a parabolic surface or surface with border whose double across ∂R is parabolic. If $\varphi\, dz^2\not\equiv 0$ is a differential of finite norm then there is a simple loop or cross cut γ on R with $h_\varphi(\gamma)\neq 0$.*

4.3. There is a theorem analogous to the Heights theorem with lengths in place of heights. Given $\varphi\, dz^2$ define the φ-length of the free homotopy class of the simple loop or cross-cut γ as

$$L_\varphi(\gamma)=\inf_{\tilde{\gamma}\sim\gamma}\int_{\tilde{\gamma}}|\varphi|^{1/2}|dz|$$

when $\tilde{\gamma}$ runs over all (locally rectifiable) simple loops or cross-cuts in the free homotopy class of γ.

THEOREM 4.3. *Assume R is a parabolic Riemann surface or a surface with border ∂R whose double across ∂R is parabolic. Suppose φ and ψ are real quadratic differentials of finite norm satisfying $L_\varphi(\gamma)=L_\psi(\gamma)$ for all simple loops and cross-cuts γ. Then $\psi\equiv e^{i\vartheta}\varphi$ for some constant angle ϑ with $\vartheta=0$ or π if $\partial R\neq\emptyset$. If both φ and ψ are negative only simple loops need be considered and $\vartheta=0$ (if $\partial R\neq\emptyset$).*

Proof. As $|\varphi|^{1/2}|dz|=|dw|\geq|dv|$, where $w=u+iv$ and $w=\Phi(z)=\int^z\sqrt{\varphi}\, dz$,

$$\int_{\tilde{\gamma}}|dv|\leq\int_{\tilde{\gamma}}|\varphi(z)|^{1/2}|dz|.$$

Therefore $h_\varphi(\gamma)\leq L_\varphi(\gamma)$. Our hypothesis thus implies that $h_\varphi(\gamma)\leq L_\psi(\gamma)$ for all γ. Now refer back to the proof of Theorem 3.2 and carry out the same calculations but with $|\psi|^{1/2}|dz|$ in place of $|d\bar{v}|$. We obtain

$$\|\varphi\| \leq \int\int_R |\psi(w)|^{1/2} du\, dv.$$

By Schwarz's inequality

$$\|\varphi\|^2 \leq \int\int_R du\, dv \int\int_R |\psi(w)|\, du\, dv = \|\varphi\|\,\|\psi\|.$$

Consequently $\|\varphi\| \leq \|\psi\|$ and by symmetry there is equality. But equality forces $|\psi(w)| \equiv$constant and this constant must be one. Therefore $\psi(w) \equiv e^{i\vartheta}$ for some ϑ. In the original z-coordinates this equation becomes $\psi(z) = e^{i\vartheta}\varphi(z)$.

4.4. *Example*. Let R be a domain in the plane bounded by $b \geq 2$ smooth curves. A harmonic measure in R is a real harmonic function which is constant on each boundary contour. The harmonic measures form a real vector space of dimension $(b-1)$.

To each harmonic measure u associate the quadratic differential

$$\varphi\, dz^2 = (du + i\, dv)^2$$

where dv is the conjugate differential to du; $\varphi\, dz^2$ is negative. For these differentials our definitions become,

$$h_\varphi(\gamma) = \inf_{\tilde{\gamma} \sim \gamma} \int_{\tilde{\gamma}} |dv|,$$

$$L_\varphi(\gamma) = \inf_{\tilde{\gamma} \sim \gamma} \int_{\tilde{\gamma}} |\mathrm{grad}\, u|\,|dz|,$$

where $|\mathrm{grad}\, u| = (u_x^2 + u_y^2)^{1/2}$. The Heights theorem, Theorem 4.3, and homology can be compared as follows:

COROLLARY 4.4. *For two harmonic measures u_1, u_2 in the plane domain R, the following are equivalent*

(i) $u_2 = \pm u_1 + \mathit{constant}$

(ii) $h_{\varphi_2}(\gamma) = h_{\varphi_1}(\gamma)$, *all simple loops* γ

(iii) $L_{\varphi_2}(\gamma) = L_{\varphi_1}(\gamma)$, *all simple loops* γ

(iv) $\int_\gamma dv_2 = \int_\gamma dv_1$, *or* $= -\int_\gamma dv_1$, *all simple loops* γ.

4.5. We record one more application of the proof of Theorem 3.2. This is to a useful uniqueness theorem; for example it implies the uniqueness of Teichmüller maps between compact surfaces. It shows that subregions bounded by trajectories of quadratic differentials are essentially rigid. For the case of compact surfaces, see Ioffe [5]. We will not explore its consequences here.

THEOREM 4.5. *Assume that R is a parabolic Riemann surface or a surface with border ∂R whose double across ∂R is parabolic. Let $\varphi\, dz^2$ be a real quadratic differential of finite norm on R. Suppose $R_0 \subset R$ is an open set, not necessarily connected, resulting from the removal of at most countably many vertical trajectories or trajectory segments from R. If $f: R_0 \to R$ is a C^1 homeomorphism homotopic in R to the identity then in terms of the φ-coordinates $w = u + iv = \int^z \sqrt{\varphi}\, dz$,*

$$\|\varphi\| \leq \iint_{R_0} |\mathrm{Re}\,(f_w - f_{\bar{w}})|^2\, du\, dv \leq \iint_{R_0} |f_w - f_{\bar{w}}|^2\, du\, dv.$$

Equality occurs between the ends if and only if $\partial f/\partial v \equiv \pm 1$.

Proof. A simple loop or cross-cut γ is mapped by f onto a freely homotopic one (f is not required to keep ∂R pointwise fixed).

We work with the individual components of R_0 and sum the results in the end. In each component S express f in the local φ-coordinates: $\tilde{w} = f(w)$ and

$$d\tilde{w} = d\tilde{u} + i\, d\tilde{v} = f_w\, dw + f_{\bar{w}}\, \overline{dw}.$$

Refer back to the proof of Theorem 3.2. In place of $d\tilde{v}$ there use the present expression

$$d\tilde{v} = \mathrm{Im}\,(f_w\, dw + f_{\bar{w}}\, \overline{dw}).$$

Along a φ-vertical segment this becomes,

$$|d\tilde{v}| = |\mathrm{Im}\,(f_w\, i\, dv - f_{\bar{w}}\, i\, dv)| = |\mathrm{Re}\,(f_w - f_{\bar{w}})|\, dv.$$

As in the proof of Theorem 3.2 we obtain,

$$\|\varphi\| \leq \iint_{R_0} |\mathrm{Re}\,(f_w - f_{\bar{w}})|\, du\, dv.$$

From this Schwarz's inequality gives

$$\|\varphi\|^2 \leq \|\varphi\| \iint_{R_0} |\mathrm{Re}\,(f_w - f_{\bar{w}})|^2\, du\, dv.$$

Furthermore if equality holds, then

$$|\mathrm{Re}\,(f_w - f_{\bar{w}})| = |\mathrm{Im}\,\partial f/\partial v| = 1.$$

If there is equality in addition on the right terms in Theorem 4.5, then $|f_w - f_{\bar{w}}| = 1$. The two together imply that $\partial f/\partial v \equiv \pm 1$.

Remark. If instead horizontal slits are used,

$$\|\varphi\| \leq \int\int_{R_0} |\mathrm{Re}\,(f_w + f_{\bar{w}})|^2\,du\,dv \leq \int\int_{R_0} |f_w + f_{\bar{w}}|^2\,du\,dv$$

with equality at the ends if and only if $\partial f/\partial u \equiv \pm 1$.

5. Convergence of simple differentials

5.1. Let R be an arbitrary Riemann surface, with or without border ∂R, and $\gamma \subset R$ a simple loop not rectractible to a point or puncture (we call these non-trivial loops). There exists a quadratic differential $\varphi[\gamma]\,dz^2$ of finite norm, uniquely determined up to a positive scalar multiple by the following properties:

(i) the non-closed trajectories cover a set of area zero,
(ii) each closed trajectory is freely homotopic to γ.

That is $\varphi[\gamma]\,dz^2$ has an annular domain A which is dense in R. The boundary of A is a union of horizontal trajectories of finite total length. In particular ∂A contains ∂R and $\varphi[\gamma]\,dz^2$ is a positive differential.

Quadratic differentials of finite norm which satisfy property (i) are called *Jenkins-Strebel* differentials. The simplest of these are the differentials $\{\varphi[\gamma]\}$ and for that reason are called *simple Jenkins-Strebel differentials*, or *simple differentials* for short. These were discovered as the solution of extremal problems by Jenkins [6], Strebel [14], Jenkins-Suita [9] and Strebel [16]. We will always *normalize* them so that $\|\varphi[\gamma]\| = 1$.

When it is preferable to work with vertical trajectories, the corresponding simple differentials are $\{-\varphi[\gamma]\,dz^2\}$.

5.2. In this section we record two properties of the normalized simple differential $\varphi[\gamma]$. The first compares it to other normalized differentials ψ on R. The second concerns its dependence upon R. We stick to the notation $h_\varphi(\gamma)$ for height introduced in § 2.1 and $L_\varphi(\gamma)$ for length in § 4.3. Because we have not introduced a "horizontal" analogue of height, we will work with $-\varphi[\gamma]$ instead.

LEMMA 5.2.1. (i) $h_\psi(\gamma)<h_{-\varphi[\gamma]}(\gamma)$ *for all* $\psi \neq -\varphi[\gamma]$.

(ii) $L_\psi(\gamma)<L_{\varphi[\gamma]}(\gamma)$ *for all* $\psi \neq e^{i\vartheta}\varphi[\gamma]$.

LEMMA 5.2.2. *If R is properly contained in a larger surface* R^* *and the simple loop* $\gamma \subset R \subset R^*$ *is associated with the normalized simple differentials* $\varphi[\gamma]$ *on R and* $\varphi[\gamma]^*$ *on* R^*, *then*

$$L_{\varphi[\gamma]^*}(\gamma) = h_{-\varphi[\gamma]^*}(\gamma) < h_{-\varphi[\gamma]}(\gamma) = L_{\varphi[\gamma]}(\gamma).$$

Proof of Lemma 5.2.1. Let A denote the maximal annular domain swept out by the closed horizontal trajectories of $\varphi[\gamma]$ and fix a vertical cross-cut β of A. When A is cut by β it becomes a φ-rectangle A_0: $w=\Phi(z)=\int^z\sqrt{\varphi}\,dz=u+iv$ maps A_0 onto a rectangle in the w-plane. Denote its width by a ($=\varphi$-length of closed trajectories$=(-\varphi)$-height of closed trajectories) and denote its height by b ($=\varphi$-length of β). Then $ab=\|\varphi[\gamma]\|=1$.

Index the horizontal cross-cuts of A_0 by α_v, $0\leq v\leq b$. Each of these has φ-length equal to a and closes up in R. Using the w-coordinates in A_0, since $|\mathrm{Im}\sqrt{\psi}\,dw|=|\mathrm{Im}\sqrt{\psi}\,|du$ along α_v,

$$h_\psi(\gamma)\leq\int_{\alpha_v}|\mathrm{Im}\sqrt{\psi}\,|du \quad (\text{resp., } L_\psi(\gamma)\leq\int_{\alpha_v}|\psi|^{1/2}du),$$

$$bh_\psi(\gamma)\leq\int\int_{A_0}|\mathrm{Im}\sqrt{\psi}\,|du\,dv \quad (\text{resp., } bL_\psi(\gamma)\leq\int\int_{A_0}|\psi|^{1/2}du\,dv).$$

Apply Schwarz's inequality, using $\int\int_{A_0}du\,dv=1$, to get,

$$b^2h_\psi(\gamma)^2\leq\int\int_{A_0}|\mathrm{Im}\sqrt{\psi}\,|^2du\,dv\leq\int\int_{A_0}|\psi|\,du\,dv=1$$

(resp., $b^2L_\psi(\gamma)^2\leq\int\int_{A_0}|\psi|\,du\,dv=1$).

Inequality (i) and (ii) result from the relations

$$1/b=a=h_{-\varphi[\gamma]}(\gamma)=L_{\varphi[\gamma]}(\gamma).$$

If there is equality in (i) then $|\mathrm{Im}\sqrt{\psi}\,|=1$ and also $|\psi|^{1/2}=1$. Therefore Re $\sqrt{\psi}=0$, $\sqrt{\psi}=\pm i$ and $\psi=-1$. Equality in (ii) is analyzed similarly.

Proof of Lemma 5.2.2. Recall that the modulus of the annulus $\{1<|z|<k\}$ is $(\log k)/2\pi$ and this can also be interpreted as the reciprocal of the extremal length of the class of curves separating the contours. Similarly on the surface R, the extremal length of the class of curves in the free homotopy class of γ is a/b where a is the width and b the height of the $\varphi[\gamma]$-rectangle A_0 described above.

Move on to R^*. We assume that $R^*\setminus R$ contains an open set. On R^* the free homotopy class is larger so its extremal length is smaller. In fact it is strictly smaller (because otherwise the extremal metrics would coincide). That is,

$$\frac{a}{b} > \frac{a^*}{b^*}$$

where a^*, b^* denote the corresponding quantities for $\varphi[\gamma]^*$ on R^*. Since however the differentials are normalized on their respective domains, $ab=1=a^*b^*$. Consequently,

$$h_{-\varphi[\gamma]}(\gamma) = a > a^* = h_{-\varphi[\gamma]^*}(\gamma).$$

We remark that the assertions made regarding extremal length are established by the same type of length/area arguments already used.

5.3. The *geometric intersection number* $i(\sigma,\gamma)$ between two simple loops on a Riemann surface R is defined as follows:

$$i(\sigma,\gamma) = \inf \operatorname{card}(\tilde{\sigma} \cap \tilde{\gamma} : \tilde{\sigma} \sim \sigma, \tilde{\gamma} \sim \gamma)$$

where $\tilde{\sigma}$ runs through all simple loops freely homotopic to σ and $\tilde{\gamma}$ through those freely homotopic to γ. In computing the infimum we need only consider loops $\tilde{\sigma}$, $\tilde{\gamma}$ that intersect a finite number of times, and cross when they meet.

In particular $i(\sigma,\gamma)=0$ if there are representatives of the two classes which are disjoint.

Suppose now σ is a cross-cut of R. By our earlier definition (§ 2.1) this means that R has a border ∂R and σ is a simple closed arc whose ends lie on ∂R. The above definition holds just as well if σ is a cross-cut and γ a simple loop (see § 2.1 for the definition of free homotopy class of a cross-cut).

A useful way to view the intersection number is in terms of the Poincaré metric on R (analogous assertions can be made for those few surfaces which don't carry one). There is a unique Poincaré geodesic in each free homotopy class of a simple loop. If σ and γ are simple loops, and σ_0, γ_0 denote the corresponding Poincaré geodesics, then exactly

$$i(\sigma,\gamma)=\operatorname{card}\sigma_0\cap\gamma_0.$$

When σ is instead a cross-cut there are also analogues to the geodesic σ_0 so that the above formula continues to hold. We will indicate how to find one such σ_0. Let $\pi\colon \mathbf{H}\to R$ denote the natural projection from the universal covering surface, say the unit disk. Consider a component σ^* of $\{\pi^{-1}(\operatorname{Int}\sigma)\}$. It has well defined end points on the unit circle $\partial\mathbf{H}$. Let σ_0^* be the non-Euclidean line with the same end points. The projection $\sigma_0=\pi(\sigma_0^*)$ has the same end points as σ on ∂R and is a simple arc homotopic to it. For the simple Poincaré geodesic γ_0, $\operatorname{card}(\sigma_0\cap\gamma_0)$ does not depend on the choice of σ, hence σ_0, in its free homotopy class.

5.4. Suppose R is a subsurface of R^* so that the relative boundary ∂R of R in $\operatorname{Int}R^*$ is a finite union of simple loops. Suppose further that each of these bounds a component of $R^*\setminus R$ which is not simply or doubly connected. The following two results are required to go back and forth from R to R^*.

LEMMA 5.4.1. *Assume γ is a simple loop in R and σ is a simple loop or cross-cut in R^* which is not freely homotopic in R^* to one in R. Position σ in its free homotopy class so that it crosses each component C of ∂R exactly $i(\sigma,C)$ times. With this done denote the components of $\sigma\cap R$ by $\{\sigma_i\}$; these are all cross-cuts of R. Then*

$$i(\sigma,\gamma)=\Sigma i(\sigma_i,\gamma)$$

where each intersection number $i(\sigma_i,\gamma)$ is computed in R.

Proof. In the Poincaré metric on R^* we may assume that γ and each component of ∂R is a geodesic, and that σ is as well, using the modification above if it is a cross-cut. Then $i(\sigma,\gamma)=\Sigma\operatorname{card}(\sigma_i\cap\gamma)$. By looking in the universal covering surface of R^*, we see that $\operatorname{card}(\sigma_i\cap\gamma)=i(\sigma_i,\gamma)$ follows directly from the convexity of R in R^*.

LEMMA 5.4.2. *Assume σ is a cross-cut of R.*

(i) *Suppose both end points of σ lie on the same component C of ∂R. Let σ_1 be a cross-cut in $R^*\setminus R$ joining the end points of σ and which is not retractable in $R^*\setminus R$ into C. Let σ' denote the simple loop resulting from joining σ and σ_1. Then for any simple loop γ in R, $i(\sigma,\gamma)=i(\sigma',\gamma)$ (with $i(\sigma',\gamma)$ computed in R^*).*

(ii) *Suppose the end points of σ lie on different components C_1, C_2 of ∂R. Choose a simple loop σ_i in the component of $R^*\setminus R$ bounded by C_i, with origin at the end point of σ, and not retractable in $R^*\setminus R$ into C_i. Then form an (essentially) simple loop σ' by going along σ toward C_1, around σ_1 in a suitable direction, back along σ to C_2,* *and then around σ_2 (in symbols, $\sigma\sigma_1\sigma^{-1}\sigma_2$). Then for any simple loop γ in R, $i(\sigma', \gamma)=2i(\sigma, \gamma)$.*

Proof. Again this is most easily analyzed in the universal covering surface by using the hyperbolic metric. We omit the details.

Remark. Without changing the conclusions above, we may allow more general types of embeddings $R \subset R^*$. For example, (i) a component of ∂R may be a (non-trivial) cross-cut of R^*, and/or (ii) the relative boundary of a component of $R^* \setminus R$ may consist of more than one component of ∂R. In the example of (i), if a cross-cut σ of R has both end points on such components, σ can simply be extended in $R^* \setminus R$ to a cross-cut of R^*.

5.5. The following fact ties the geometric intersection number to quadratic differentials.

LEMMA 5.5. *For the annular domain A associated with the normalized $\varphi[\gamma]\,dz^2$, let $a=L_{\varphi[\gamma]}(\gamma)$ denote the $\varphi[\gamma]$-length of the closed horizontal trajectories in A and b denote the $\varphi[\gamma]$-length of the vertical cross-cuts of A. Then for any simple loop or cross-cut σ of R,*

$$h_{\varphi[\gamma]}(\sigma) = bi(\sigma, \gamma) = \frac{i(\sigma, \gamma)}{a}.$$

Proof. Let α be a closed horizontal trajectory of $\varphi[\gamma]$. Then deform σ in its free homotopy class so that $\operatorname{card} \sigma \cap \alpha = i(\sigma, \gamma)$. Next look at the components of $\sigma \cap A$. Those which do not cross α, i.e. whose end points both lie on the same component of ∂A, can be pushed into ∂A. The other components can be deformed to vertical cross-cuts. We end up with a loop σ' freely homotopic to σ which is a union of $i(\alpha, \gamma)$ vertical cross-cuts of A, and horizontal segments lying in ∂A. For this σ',

$$\int_{\sigma'} |\operatorname{Im} \sqrt{\varphi[\gamma]}\, dz| = bi(\sigma, \gamma).$$

Although σ' may not be a simple loop it can be made into one by arbitrarily small deformations. The integral expression on the left is indeed $h_{\varphi[\gamma]}(\sigma)$ because for any simple loop σ_0 freely homotopic to σ, $\sigma_0 \cap A$ must contain at least $i(\sigma, \gamma)$ components whose end points lie on the opposite components of A. Finally, recall that normalization implies $ab=\|\varphi[\gamma]\|=1$.

5.6. The key to a geometric understanding of general quadratic differentials in terms of simple ones is the following concept. It is due to Thurston and forms the basis of his theory. Actually here we are modifying it slightly to include cross-cuts (closed simple arcs whose end points lie on the border of a surface). See [3].

Definition 5.6. A sequence of simple loops $\{\gamma_n\}$ on a surface R, possibly with border ∂R, is said to *converge* if there is a sequence $\{t_n\}$ of positive numbers such that

$$\lim \frac{i(\sigma, \gamma_n)}{t_n}$$

exists for all simple loops and cross-cuts σ *and* if for at least one such σ, the limit is not zero.

Note that the sequence $\{t_n\}$ is uniquely determined, up to a positive factor, asymptotically by $\{\gamma_n\}$. Namely if $\{s_n\}$ is another sequence that can be used in the definition then for some $C \neq 0$,

$$\lim s_n/t_n = C.$$

For set $C(\sigma) = \lim i(\sigma, \gamma_n)/t_n$ and $C'(\sigma) = \lim i(\sigma, \gamma_n)/s_n$. Then $C(\sigma_1) \neq 0$ for some σ_1 and $C'(\sigma_2) \neq 0$ for some σ_2. Therefore $\lim s_n/t_n = C(\sigma_2)/C'(\sigma_2) = C(\sigma_1)/C'(\sigma_1)$.

5.7. At this point it is well to make some remarks concerning the convergence of a sequence of normalized real differentials $\{\varphi_n\}$ to a differential φ on a Riemann surface R, possibly with border ∂R.

Suppose first that R is a compact surface, possibly with boundary ∂R, possibly with a finite number of punctures. Then the following are equivalent:

(i) $\{\varphi_n\}$ converges locally uniformly (i.e. uniformly on compact subsets) to φ,
(ii) $\{\varphi_n\}$ converges uniformly on R to φ (at puncture $z=0$ parameterize by $\sqrt{z}$),
(iii) $\{\varphi_n\}$ converges in norm to φ, i.e. $\lim \|\varphi_n - \varphi\| = 0$.

In addition, because all $\|\varphi_n\| = 1$ so also $\|\varphi\| = 1$.

On a general surface R matters are much more complicated. We will usually assume that φ_n converges locally uniformly to φ. But in general we cannot claim in addition that (a) $\|\varphi\| = 1$, or equivalently, (b) $\{\varphi_n\}$ converges to φ in norm, or even that (c) $\varphi \neq 0$. In fact if $\|\varphi\| < 1$ the normalized sequence $\{(\varphi_n - \varphi)/\|\varphi_n - \varphi\|\}$ converges locally uniformly to the zero differential.

However in all cases it is true that a sequence of normalized differentials has a subsequence that converges locally uniformly to a differential on the surface with norm ≤ 1.

5.8. LEMMA 5.8. *Suppose R is a parabolic surface or a surface with border ∂R whose double across ∂R is parabolic. If for a sequence $\{\gamma_n\}$ of simple loops on R the normalized simple differentials $\{\varphi[\gamma_n]\}$ converge locally uniformly to a differential ψ, then for every simple loop or cross-cut σ,*

$$\lim \frac{i(\sigma, \gamma_n)}{a_n} = h_{\psi}(\sigma).$$

Here a_n denotes the $\varphi[\gamma_n]$-length of its closed trajectories. In particular, if $\psi \not\equiv 0$, the sequence $\{\gamma_n\}$ also converges.

Proof. The principal statement is the union of Proposition 2.3 and Lemma 5.5. Then see Corollary 4.2. Note that ψ is also a positive differential.

5.9. The main result of the chapter is this.

THEOREM 5.9. *Suppose R is a parabolic Riemann surface or surface with border ∂R whose double across ∂R is parabolic. Assume that $\{\gamma_n\}$ is a sequence of simple loops all contained in a relatively compact surface R_0 of R and let $\{\varphi[\gamma_n]\}$ be the corresponding normalized simple differentials on R. Then $\{\gamma_n\}$ is a convergent sequence on R if and only if $\{\varphi[\gamma_n]/\|\varphi[\gamma_n]\|_{R_0}\}$ converges locally uniformly to a differential on R. In the latter case the limit differential has finite norm and is not identically zero.*

Proof. First assume that $\{\gamma_n\}$ converges with associated sequence $\{t_n\}$. Consider the situation in R_0. There let $\{\varphi[\gamma_n]_0\}$ denote the normalized simple differentials. In view of Lemma 5.4.2, $\{\gamma_n\}$ also converges with respect to R_0 with the same $\{t_n\}$ (we may assume R_0 satisfies the hypothesis). Take any subsequence $\{\gamma_m\}$ of $\{\gamma_n\}$ for which $\{\varphi[\gamma_m]_0\}$ converges on R_0. Let $(a_m)_0$ denote the $\varphi[\gamma_m]_0$-length of its closed horizontal trajectories. By Lemma 5.8 and § 5.6, there exists

$$C_0 = \lim t_m/(a_m)_0 \neq 0.$$

Now move on to R itself. By Lemma 5.2.2, $(a_m)_0 > a_m$. Hence from Lemma 5.5,

$$h_{\varphi[\gamma_m]}(\sigma) = \frac{i(\sigma, \gamma_m)}{a_m} > \frac{i(\sigma, \gamma_m)}{(a_m)_0}.$$

Therefore if σ is suitably chosen,

$$\liminf h_{\varphi[\gamma_m]}(\sigma) \geq C_0 \lim \frac{i(\sigma, \gamma_m)}{t_m} \neq 0.$$

From this we conclude that no convergent subsequence of $\{\varphi[\gamma_m]\}$ converges (locally uniformly) to the zero differential. Going back a step to the choice of $\{\gamma_m\}$ we can also conclude that no subsequence of the original $\{\varphi[\gamma_n]\}$ converges to the zero differential.

Consider two convergent subsequences of $\{\varphi[\gamma_n]\}$: $\{\varphi[\gamma_p]\}$ that converges to ψ_1 and $\{\varphi[\gamma_q]\}$ that converges to ψ_2. Set

$$C_1 = \lim t_p/a_p, \quad C_2 = \lim t_q/a_q$$

as we may by Lemma 5.8 and § 5.6. For any σ in R,

$$h_{\psi_1}(\sigma) = C_1 \lim \frac{i(\sigma, \gamma_p)}{t_p}, \quad h_{\psi_2}(\sigma) = C_2 \lim \frac{i(\sigma, \gamma_q)}{t_q}.$$

Consequently for any σ,

$$h_{\psi_1}(\sigma)/C_1 = h_{\psi_2}(\sigma)/C_2.$$

This calls for the Heights theorem (§ 4.1)! Because ψ_1 and ψ_2 are real, in fact positive, differentials

$$\psi_1/C_1 \equiv \psi_2/C_2.$$

If ψ_1 and ψ_2 were known to have unit norm, as would be the case if R were a finitely punctured compact bordered surface, then we could conclude that $C_1 = C_2$, $\psi_1 = \psi_2$, and $\{\varphi[\gamma_n]\}$ converges. But in general we do not know this and to deal with it we in effect change the normalization.

Set

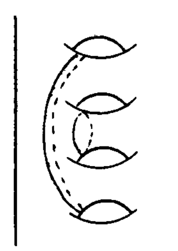
$$\varphi_0[\gamma_n] = \varphi[\gamma_n]/\|\varphi[\gamma_n]\|_{R_0} \quad \text{and} \quad \psi_i' = \psi_i/\|\psi_i\|_{R_0}.$$

Then locally uniformly, $\lim \varphi_0[\gamma_p] = \psi_1'$ and $\lim \varphi_0[\gamma_q] = \psi_2'$. But $\|\psi_1\|_{R_0}/C_1 = \|\psi_2\|_{R_0}/C_2$ so that now for all σ, $h_{\psi_1'}(\sigma) = h_{\psi_2'}(\sigma)$. Thus $\psi_1' \equiv \psi_2'$ and we can conclude that $\{\varphi_0[\gamma_n]\}$ itself converges.

Conversely suppose $\{\varphi_0[\gamma_n]\}$ converges locally uniformly on R to a differential ψ. Because $\{\gamma_n\}$ lies in R_0 it has a convergent subsequence $\{\gamma_m\}$ on R_0 by Lemma 5.8 and then on R by Lemma 5.4.1. The argument just completed shows that $\{\varphi_0[\gamma_m]\}$ converges on R to a non-zero differential of finite norm. That can only be ψ. Now we can apply Lemma 5.8 to the full sequence $\{\gamma_n\}$.

We remark that in place of R_0 we could have just as well used any relatively compact subset of R to "renormalize".

5.10. *Examples.* Without the assumption that the free homotopy classes of $\{\gamma_n\}$ contain representatives in a fixed compact part of R, the corresponding sequence of normalized differentials $\{\varphi[\gamma_n]\}$ may indeed tend to the zero differential, uniformly on compact subsets, while $\{\gamma_n\}$ converges. For an example take the doubly infinite ladder so constructed that it is a parabolic surface R. Take the loops $\gamma_1, \gamma_2, \ldots$ as indicated.



Then $\{\gamma_n\}$ converges with associated sequence $\{t_n\}$ where all $t_n=1$. The sequence $\{\varphi\{[\gamma_n]\}$ tends to zero. For suppose there were a subsequence $\{\varphi[\gamma_m]\}$ with non-zero limit ψ. Let α be a horizontal trajectory of ψ through a point $p \in R$. The trajectory α_m of $\varphi[\gamma_m]$ through p—we may assume it is a closed trajectory—tends to α uniformly on compact subsets. Given any compact set K, for large m, α_m meets the complement of K. Therefore α does as well. So given a vertical segment β for ψ, the horizontal rays from β all meet the complement of any prescribed compact set in R. But such rays must have linear measure zero on β. This is impossible.

Another example of this phenomenon is the following. Remove from the real axis an infinite sequence $\{x_k\}$, $-\infty<k<\infty$, with $x_k<x_{k+1}$ for all k. Let γ_n be a simple loop, symmetric in $\mathbf{R}$ that separates $\{x_k\}$, $0\leq k\leq n$, from all other points. The modulus of the free homotopy class of γ_n is given by means of the conformal mapping onto an annulus of the sphere with two slits, one along $\mathbf{R}$ from x_0 to x_n, the other from x_{-1} to x_{n+1} along $\mathbf{R}$ and over ∞ to be disjoint from the first. By suitably spacing the $\{x_k\}$ we can get the modulus to approach zero. The corresponding normalized simple differentials can likewise only converge to zero, while for each simple loop σ, $\lim i(\sigma, \gamma_n)$ exists.

6. Approximation by simple differentials

6.1. Let $\{\gamma_j\}$, $1 \leq j \leq r$, be a system of simple loops on a surface R, none retractable to an ideal boundary component (or a point) of R. In this section we will describe a method, due to Thurston, of joining them together in a controlled manner to form a single simple loop.

In addition to the loops $\{\gamma_j\}$ we start with the following:

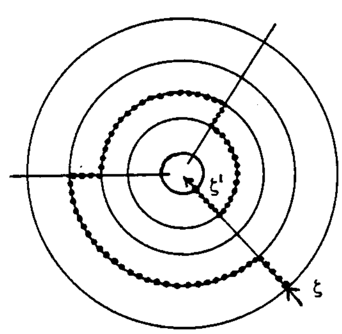
(i) a simple loop τ with $i(\tau, \gamma_j) \neq 0$ and equal to the actual number of crossings of τ and γ_j, $1 \leq j \leq r$,

(ii) the least common multiple M of the numbers $\{i(\tau, \gamma_j)\}$,

(iii) mutually disjoint thin annular neighborhoods $\{S_j\}$ about the $\{\gamma_j\}$.

In each S_j replace the single loop γ_j by M mutually disjoint strands, each parallel to γ_j.

We will describe the procedure by focusing on one of the $\{S_j\}$, say on S_p. Represent S_p as a proper annulus with the strands as M concentric circles and the components of $\tau \cap S_p$ as $i(\tau, \gamma_p) \neq 0$ radial cross-cuts. Start with a point $\zeta \in \tau \cap \partial S_p$, say ζ lies on the outer contour. Move along $\tau \cap S_p$ from ζ until hitting the first strand. Turn *left* along that strand and continue along that until again hitting $\tau \cap S_p$. At that point turn *right* along $\tau \cap S_p$ until the next point of intersection with a strand. Then turn left along that strand. And so on.



$M = i(\tau, \gamma_p) = 3$

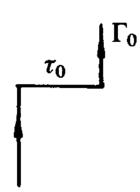
After making M right turns from a strand onto $\tau \cap S_p$, we end up at a point ζ' on the inner contour. That final point ζ' and the initial ζ are the end points of the same radial cross-cut—a component τ_{1p} of $\tau \cap S_p$.

In following this process we have traced out a simple arc. This arc is homotopic to what results to τ_{1p} if the outer contour is held fixed while the inner is rotated clockwise exactly $M/i(\tau, \gamma_p)$ times. Or equivalently if the inner is fixed and the outer rotated counterclockwise $M/i(\tau, \gamma_p)$. This is called a *Dehn twist* of order $M/i(\tau, \gamma_p)$. In particular it is in the *positive direction* and this direction is determined solely by the orientation of R (and then S_p), not by choice of inner and outer contour of S_p.

To continue with the construction, we have entered S_p at ζ and left it at ζ', the other end of the component τ_{1p} of $\tau \cap S_p$. Following along τ we next enter some S_q. Possibly $S_q = S_p$ but unless $\tau \cap S_p$ is connected we will then enter S_p along a different component of $\tau \cap S_p$. Repeat the process in S_q. And so continue.

In the end we come back to the initial point ζ. In the process, every segment of τ and of each of the strands is covered exactly once. Adjusting slightly at the corners we end up with a simple loop Γ_0. $\Gamma_0 \cap S_p$ consists of $i(\tau, \gamma_p)$ parallel spiral cross-cuts, each twisting $M/i(\tau, \gamma_p)$ times around in the positive direction.

If τ_0 is a segment of τ in Γ_0, then following along Γ_0 near τ_0, we turn right on τ_0 from some strand, and then turn left along another. Deform Γ_0 slightly to Γ which is now transverse to (crosses) τ. Because of the observation just made, the number of crossings of Γ and τ is exactly $i(\Gamma, \tau)$. This is seen in the universal covering surface of R, by following the construction there.



6.2. The following fact is important.

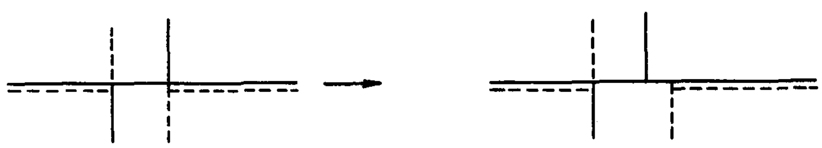
LEMMA 6.2. *Let σ be a simple loop or cross-cut on R and Γ as constructed from $\{\gamma_j\}$ and τ as above. Then*

$$M \Sigma\, i(\sigma, \gamma_j) - i(\sigma, \tau) \leq i(\sigma, \Gamma) \leq M \Sigma\, i(\sigma, \gamma_j) + i(\sigma, \tau).$$

Proof. (After A. Fathi [3, p. 68].) Position σ in its free homotopy class so that it crosses each γ_j exactly $i(\sigma, \gamma_j)$ times and τ exactly $i(\sigma, \tau)$ times. Then the right inequality is clear.

Now position σ so that it crosses Γ exactly $i(\sigma, \Gamma)$ times and τ exactly $i(\sigma, \tau)$ times. We know that σ must cross at least $M\Sigma i(\sigma, \gamma_j)$ times the Mr distinct loops of the system

$\mathcal{S}$ obtained by replacing each γ_j by M strands. Think of forming the point set $\Gamma \cup \tau$ from $\mathcal{S}$ as follows. Let little arcs grow out from $\mathcal{S}$ coalescing to form $\mathcal{S} \cup \tau$. Then along τ pull apart the



corners slightly to form $\Gamma_0 \cup \tau$ (this is the critical part: because τ is included there is no essential change in the local structure). Finally deform Γ_0 to Γ to form $\Gamma \cup \tau$. We conclude that σ cannot cross $\Gamma \cup \tau$ fewer than $M \Sigma i(\sigma, \gamma_j)$ times. That is,

$$i(\sigma, \Gamma) + i(\sigma, \tau) \geqslant M \Sigma\, i(\sigma, \gamma_j).$$

Note that we have allowed the possibility that a number of the $\{\gamma_j\}$ are parallel to each other.

6.3. LEMMA 6.3. *Let $\{\gamma_j\}$ be a finite collection of mutually disjoint non-trivial simple loops on a surface R, none retractable to an ideal boundary component of R. There exists a simple loop τ with $i(\tau, \gamma_j) \neq 0$ for all j.*

Proof (cf. Lemma 5.4.2). We will prove by induction a more general statement. Namely if R has a border ∂R and one or two of the components of ∂R are prescribed, then τ can either be chosen as a simple loop as stated, or as a cross-cut with end points on the designated component(s) of ∂R.

The assertion is clear if there is only one loop γ. Assume it is true for $(k-1)$ loops on any R. Suppose then we have $\{\gamma_j\}$, $1 \leqslant j \leqslant k$ on R with γ_k not parallel to any γ_j with $j<k$. Denote the two sides of γ_k by γ_k^+, γ_k^- and cut R along γ_k.

Case 1. The resulting surface R' is connected. Then γ_k^+, γ_k^- are two components of $\partial R'$. There are two cases. (i) Apply the induction hypothesis to find a cross-cut τ_0 of R' whose ends are on the opposite sides of γ_k and such that $i(\tau_0, \gamma_j) \neq 0$, $1 \leqslant j \leqslant k-1$. On R, τ_0 closes up to form a simple loop τ with $i(\tau, \gamma_k) = 1$ while $i(\tau, \gamma_j)$ is the same as $i(\tau_0, \gamma_j)$ computed in R', $j<k$. Or (ii) find a cross-cut τ_0 of R' from a point of γ_k^+ to one of the designated components of ∂R with $i(\tau_0, \gamma_j) \neq 0$, $j<k$. Because τ_0 does not divide R' there is a disjoint cross-cut τ_0' from γ_k^- to the other designated component of ∂R. On R, $\tau = \tau_0 \cup \tau_0'$ comes together to form a cross-cut of R with $i(\tau, \gamma_k) = 1$ and $i(\tau, \gamma_j) \geqslant i(\tau_0, \gamma_j)$ for $j<k$.

Case 2. R' has two components R'_1 with γ_k^+ in $\partial R'_1$ and R'_2 with γ_k^- in $\partial R'_2$. To (i) get a simple loop τ on R join a suitable cross-cut of R'_1 with both ends on γ_k^+ with one of R'_2 with both ends on γ_k^-. If (ii) the designated components of ∂R lie in different components R'_1, R'_2 a suitable cross-cut τ of R is found similarly. If (iii) the designated components of ∂R lie in say R'_1, then two disjoint cross-cuts of R'_1 must be joined with one of R'_2.

As with § 5.2, these constructions are best carried out in the universal covering surface with non-Euclidean geometry.

6.4. We start with the simplest situation that illustrates the method.

LEMMA 6.4. *Suppose R is a compact Riemann surface, possibly with boundary ∂R, possibly with a finite number of punctures. Assume that $\varphi\, dz^2$ has a spiral domain which is dense in R. Then $\varphi\, dz^2$ can be approximated by simple differentials.*

Proof. Fix a short vertical segment β of φ in its spiral domain (which may be all R). Let $\{S_k\}$ be the horizontal strip decomposition based on β (see § 1.6 where instead vertical strips are described). Denote the corresponding intervals on $\beta^+ \cup \beta^-$ as $\{\beta_j\}$. These are arranged in pairs, the two intervals of a pair forming the vertical sides of some φ-rectangle S_k.

Since by assumption φ has no closed trajectories the φ-lengths of the $\{\beta_j\}$ are not all rational multiples of the φ-length $|\beta|$ of β. Given a large integer N we now deform R to a new surface R_N as follows. Take the vertical segment β but change the position of the end points of the $\{\beta_j\}$ slightly, each by an amount less than $1/N$, to get new intervals $\{\beta_{Nj}\}$ such that (a) all their φ-lengths are rational multiples of $|\beta|$, and (b) they remain paired as before, two intervals of a pair having the same length. Exactly as R is constructed from the φ-rectangles $\{S_k\}$ based on the $\{\beta_j\}$ so construct R_N from rectangles $\{S_{Nk}\}$ with the same widths but based on the $\{\beta_{Nj}\}$. Just as $\varphi\, dz^2$ on R results by pasting together the Euclidean elements dw^2 on the rectangles $\{S_k\}$ in the $w=\Phi(z)=\int^z\sqrt{\varphi}\, dz$ plane, so a differential $\varphi_N dz^2$ on R_N is formed by pasting together the elements dw^2 on the $\{S_{Nk}\}$.

By construction there is a piecewise affine map $f_N: R \to R_N$ in terms of the Φ-coordinates on R and Φ_N-coordinates on R_N. It sends each β_j to β_{Nj} and S_k to S_{Nk}. In general f_N has different values on the two borders β^+, β^- of β so that it is not actually a homeomorphism of R. However to each simple loop γ in R_N transverse to β corresponds a well-determined "preimage $f_N^{-1}(\gamma)$" in R which is (essentially) a simple loop obtained by inserting short segments of β to adjust for the discontinuities. As $N\to\infty$, f_N converges uniformly to the identity because each $S_{Nk}\to S_k$ in the Φ-plane.

Take Q_N to be any integer such that the length of each β_{Nj} is an integral multiple of $|\beta|/Q_N$. As $N\to\infty$, $Q_N\to\infty$.

Divide each horizontal strip S_{Nk} into $\mathrm{ht}(S_{Nk})\,Q_N/|\beta|$ congruent horizontal substrips S_{Nkd} (where $\mathrm{ht}(S_{Nk})$ denotes the φ_N-height, i.e., the φ-length of the modified β_{Nj} forming a vertical side). Let α_{Nkd} denote the middle horizontal cross-cut of S_{Nkd}. Altogether there are Q_N of the horizontal segments $\{\alpha_{Nkd}\}$. They join end-to-end to form a number (which depends on N) of mutually disjoint simple loops $\{\tilde{\alpha}_{Nm}\}$. Correspondingly the rectangles $\{S_{Nkd}\}$ join together end-to-end to form annuli $\{A_{Nm}\}$.

Next, take a simple loop σ on R and consider the corresponding loop $f_N(\sigma)=\sigma_N$ on R_N. It can be positioned in its free homotopy class so that it crosses each $\tilde{\alpha}_{Nm}$ exactly $i(\sigma_N,\tilde{\alpha}_{Nm})$ times. In fact we can assume that the intersection of σ_N with each A_{Nm} is a union of horizontal segments on ∂A_{Nm}, possibly running through punctures, and $i(\sigma_N,\tilde{\alpha}_{Nm})$ vertical segments between the components of ∂A_{Nm}. In this form the φ_N-height is easily computed to be

$$h_{\varphi_N}(\sigma_N)=(|\beta|/Q_N)\,\Sigma_m\, i(\sigma_N,\tilde{\alpha}_{Nm})$$

since the height of each A_{Nm} is $|\beta|/Q_N$.

On the other hand on R_N, $h_{\varphi_N}(\sigma_N)$ can also be computed by integration on σ_N realized as a finite union of φ_N-straight segments between critical points. Therefore $\lim h_{\varphi_N}(\sigma_N)=h_\varphi(\sigma)$.

We can find a simple loop τ on R such that for an infinite sequence of $\{N\}$, $i(\tau_N,\tilde{\alpha}_{Nm})\neq 0$ for all corresponding $\{m\}$, where $\tau_N=f_N(\tau)$. For example if β is chosen so that there is a horizontal segment α connecting one end point of β to the other without otherwise meeting β, then τ can be taken as $\alpha\cup\beta$.

Apply the twisting procedure of § 6.1 to obtain from τ_N and the $\{\tilde{\alpha}_{Nm}\}$ a single simple loop Γ_N on R_N. Do this for each N.

From Lemma 6.2, dropping the index from τ and σ,

$$M_N\,\Sigma_m\, i(\sigma,\tilde{\alpha}_{Nm})-i(\sigma,\tau)\leqslant i(\sigma,\Gamma_N)\leqslant M_N\,\Sigma_m\, i(\sigma,\tilde{\alpha}_{Nm})+i(\sigma,\tau).$$

Comparing this with the formula above for $h_N(\sigma_N)=h_{\varphi_N}(\sigma_N)$,

$$h_N(\sigma)-\frac{i(\sigma,\tau)\,|\beta|}{Q_N M_N}\leqslant\frac{i(\sigma,\Gamma_N)}{(Q_N M_N/|\beta|)}\leqslant h_N(\sigma)+\frac{i(\sigma,\tau)\,|\beta|}{Q_N M_N}.$$

In the limit when $N\to\infty$ this becomes, (reintroducing the index)

$$h_\varphi(\sigma)=\lim_{N\to\infty}\frac{i(\sigma_N,\Gamma_N)}{(Q_N M_N/|\beta|)}=\lim_{N\to\infty}\frac{i(\sigma,\Gamma'_N)}{(Q_N M_N/|\beta|)}$$

where $\Gamma'_N=f_N^{-1}(\Gamma_N)$ is a simple loop on R.

By Theorem 5.9, the normalized simple differentials $\{\varphi[\Gamma'_N]\}$ on R converge. They converge to a differential with the same heights as $c\varphi$ for some constant $c>0$. We may assume φ is normalized too. Then by the Heights theorem, in fact $\lim\varphi[\Gamma_N]=\varphi$, which we had set out to prove.

6.5. COROLLARY 6.5. *Suppose R is a compact surface without boundary, possibly with a finite number of punctures. Any quadratic differential $\varphi\,dz^2$ on R can be approximated by simple ones.*

Proof. We can choose ϑ so that there is no φ-straight segment of slope ϑ (i.e. $\arg\varphi\,dz^2=2\vartheta$) going from one critical point to another. For such a segment would be the unique geodesic between its end points, in its homotopy class. There are finitely many critical points, and between any two of them countably many homotopy classes. In each homotopy class there is a unique φ-geodesic composed of a finite number of φ-straight segments between critical points. The slopes $\{\vartheta\}$ of the countably many segments that appear form a countable set. As long as ϑ is not in it, no such segment of slope ϑ will exist.

Even more, we can find a sequence $\vartheta_n\to 0$ of such angles. Each differential $e^{-2i\vartheta_n}\varphi\,dz^2$ will then have a spiral domain dense in R. Moreover the sequence $\{e^{-2i\vartheta_n}\varphi\,dz^2\}$ converges to $\varphi\,dz^2$. By Lemma 6.4 each $e^{-2i\vartheta_n}\varphi\,dz^2$ can be approximated by simple differentials, and therefore $\varphi\,dz^2$ can as well.

6.6. LEMMA 6.6. *Suppose R is a parabolic surface or a surface with border ∂R whose double across ∂R is parabolic. Assume that $\varphi\,dz^2$ is a positive differential on R with closed horizontal trajectories which sweep out a finite number of annular domains. Assume none of these is retractable to an ideal boundary component of R. Then φ can be approximated by simple differentials.*

Proof. Denote the annular domains by $\{A_j\}$ and fix a vertical cross-cut β_j of φ-length $|\beta_j|$ in each of them. Also fix a closed horizontal trajectory γ_j in each. By Lemma 6.3 there is a simple loop τ with $i(\tau,\gamma_j)\neq 0$ for all j.

Let σ be a simple loop or cross-cut on R. If all $\{A_j\}$ are relatively compact we can position σ in its free homotopy class so that $\sigma \cap A_j$ consists of exactly $i(\sigma, \gamma_j)$ cross-cuts, for each j, and the remainder of σ consists of horizontal segments. Then

$$h_\varphi(\sigma) = \Sigma |\beta_j| \, i(\sigma, \gamma_j).$$

The same formula holds in the general case and can be proven, for example, by exhausting the $\{A_j\}$.

For each large integer N, determine the positive integer p_{Nj} and number q_{Nj}, $0 \leq q_{Nj} < 1$, such that

$$N|\beta_j| = p_{Nj} + q_{Nj}.$$

Replace each γ_j by p_{Nj} parallel simple loops $\{\gamma_{jk}\}$. With τ and the mutually disjoint system $\{\gamma_{jk}\}$ in j and k, construct a simple loop Γ_N as in § 6.1. Because

$$\Sigma_k \, i(\sigma, \gamma_{jk}) = p_{Nj} \, i(\sigma, \gamma_j),$$

Lemma 6.2 gives the inequality

$$\Sigma|\beta_j| i(\sigma, \gamma_j) - \frac{i(\sigma, \tau) + M \Sigma q_{Nj} i(\sigma, \gamma_j)}{MN} \leqslant \frac{i(\sigma, \Gamma_N)}{MN} \leqslant \Sigma |\beta_j| \, i(\sigma, \gamma_j) + \frac{i(\sigma, \tau) - M \Sigma q_{Nj} i(\sigma, \gamma_j)}{MN},$$

in which M also depends on N. When $N \to \infty$ this becomes,

$$h_\varphi(\sigma) = \lim \frac{i(\sigma, \Gamma_N)}{MN}.$$

It follows from Theorem 5.9 that for any relatively compact subregion K of R, the sequence $\{\varphi[\Gamma_N]/\|\varphi[\Gamma_N]\|_K\}$ converges locally uniformly to a differential of finite norm. By the Heights theorem this can only be $\varphi/\|\varphi\|_K$. Thus φ is the limit of a (suitably normalized) sequence of simple differentials as asserted.

Remarks. (1) If R is a compact surface, possibly with boundary ∂R, possibly with a finite number of punctures, then the sequence of normalized differentials $\{\varphi[\Gamma_N]\}$ itself converges to φ, if also $\|\varphi\| = 1$.

(2) The condition that no annular domain is parallel to the ideal boundary is also necessary. For suppose one component C of ∂R is a simple loop and $\varphi \, dz^2$ is a differential in R satisfying $\varphi \, dz^2 > 0$ along C. Assume that some sequence of simple differentials $\{\varphi[\Gamma_N]\}$ converges locally uniformly to φ, in particular uniformly on C. For each large N, $\varphi[\Gamma_N]$ must have at least one critical point on C (unless φ is already a simple differential). But then φ must as well, a contradiction.

6.7. We now put together the techniques of Lemmas 6.4 and 6.6 to the following end.

COROLLARY 6.7. *Suppose R is a compact surface, possibly with boundary ∂R, possibly with a finite number of punctures. Assume that $\varphi\, dz^2$ is a positive differential with no closed trajectories parallel to a component of ∂R. Then $\varphi\, dz^2$ can be approximated by simple differentials.*

Proof. The complement R_φ of the critical graph of φ (critical trajectories of finite length) is a finite union of annular domains $\{A_i\}$ and spiral domains $\{B_j\}$. For each large integer N, we can build a model domain B_{jN} for each B_j as in § 6.4. But then we attach them to the $\{A_i\}$ and to each other to form a surface R_N, exactly as the $\{A_i\}$ and $\{B_j\}$ are attached to each other to form R. Just as before there is a differential φ_N on R_N (arising from $dw^2 = \varphi\, dz^2$ on the flat pieces) and a piecewise affine map $f_N\colon R \to R_N$ in terms of the Φ-coordinates on R and Φ_N-coordinates on R_N. As $N \to \infty$, f_N converges uniformly to the identity.

On R_N we take the following collection of mutually disjoint simple loops: (i) the loops $\{\tilde{\alpha}_{Nm}\}$ in each B_{jN} as constructed in § 6.4, and (ii) a closed trajectory γ_j in each A_j. By Lemma 6.6 there exists a sequence of simple loops $\{\Gamma_{NQ}\}$ on R_N such that

$$h_N(\sigma_N) = \lim_{Q\to\infty} \frac{i(\sigma_N, \Gamma_{NQ})}{t_Q} = \lim_{Q\to\infty} \frac{i(\sigma, \Gamma'_{NQ})}{t_Q}$$

for some sequence $\{t_Q\}$ depending on N, and for any simple loop or cross-cut σ on R. Here $h_N(\sigma_N)$ denotes the φ_N-height of $\sigma_N = f_N(\sigma)$ and $\Gamma'_{NQ} = f_N^{-1}(\Gamma_{NQ})$.

Again as in § 6.4, $\lim h_N(\sigma_N) = h_\varphi(\sigma)$. Therefore we can choose a diagonal sequence $N \to \infty$, $Q = Q(N) \to \infty$, such that for the sequence of simple loops $\{\Gamma'_{NQ}\}$ on R and any σ,

$$h_\varphi(\sigma) = \lim_{N\to\infty} \frac{i(\sigma, \Gamma'_{NQ})}{t_Q}.$$

Consequently the sequence $\{\varphi[\Gamma'_{NQ}]\}$ of normalized simple differentials on R converges (Theorem 5.9), and if φ is normalized, it converges to φ.

Remark. A different proof of Corollary 6.7 is contained in Theorem 6.8 below. It was first proved by Masur [12] (for no boundary and no punctures).

6.8. We do not know whether every differential of finite norm on a parabolic surface of infinite topological type can be approximated by simple ones. However the following result is useful at least in some of those cases (§§ 6.9, 6.10). The proof uses an area method, rather than a height-intersection number method, to reduce it to a point where Lemma 6.6 can be brought in.

By a *spiral domain* on R for a differential φ we mean a domain (i) whose closure in R is a compact surface, possibly with boundary, possibly with a finite number of punctures, and (ii) which has the properties with respect to the horizontal trajectories of φ as described in § 1.5.

THEOREM 6.8. *Suppose R is a parabolic Riemann surface or a surface with border ∂R whose double across ∂R is parabolic. Let $\varphi\, dz^2$ be a positive differential of finite norm on R such that it has no closed horizontal trajectory parallel to a component of ∂R. Assume that up to a set of φ-area zero, R is the union of annular domains and spiral domains for the horizontal trajectories of $\varphi\, dz^2$. Then $\varphi\, dz^2$ is the locally uniform limit of simple differentials.*

Proof. We will show following [18] that φ is the limit (in fact even in norm) of Jenkins-Strebel differentials $\{\varphi_N\}$ each with a finite number (depending on N) of free homotopy classes of closed trajectories. For this part the assumption that no closed trajectory be parallel to a component of ∂R is not required. Theorem 6.8 follows at once from this fact coupled with Lemma 6.6.

Fix a vertical cross-cut in each annular domain. Given $\varepsilon>0$, choose a non-critical vertical segment of length less than ε in each spiral domain. Denote the possibly countably infinite set of vertical segments by $\{\beta_j\}$. Given an integer $r>0$, let R_r denote the union of the annular and spiral domains corresponding to β_j for $1\leqslant j\leqslant r$; R_r does not depend on ε.

We now focus our attention on one of the vertical segments β_j, $1\leqslant j\leqslant r$; denote that one simply by β and assume that it corresponds to a spiral domain A. Following the procedure and notation of § 6.4, take the horizontal strip decomposition $\{S_k\}$ based on β. Given a large integer N, build the model surface A_N formed by rectangles $\{S_{Nk}\}$. Measuring Euclidean area in the Φ-plane we can arrange matters so that

$$|(\text{area} \cup_k S_{Nk})-(\text{area} \cup_k S_k)| < C/Q_N \tag{1}$$

for some constant C independent of N.

Also consider the mutually disjoint simple loops $\{\tilde{\alpha}_{Nm}\}$ on A_N and their "pre-images" $\alpha_{Nm}=f^{-1}(\tilde{\alpha}_{Nm})$ in A. These are composed of horizontal and vertical (which lie in β) φ-segments in such a way that successive horizontal segments meet the intervening vertical segment on opposite sides. Therefore [15, p. 364] the φ-length $L_\varphi(\alpha_{Nm})$ of the free homotopy class satisfies,

$$\Sigma_p c_{Nmp} a_p \leq L_\varphi(\alpha_{Nm}) \leq \Sigma_p c_{Nmp} a_p + Q_{Nm}\varepsilon. \tag{2}$$

Here c_{Nmp} is the number of components of $\tilde{\alpha}_{Nm} \cap S_{Np}$, a_p is the length of S_{Np}=φ-length of S_p, and

$$Q_{Nm} = \Sigma_p c_{Nmp}; \quad \Sigma Q_{Nm} = Q_N$$

(we had divided the $\{S_{Nk}\}$ into a total of Q_N sub-rectangles each of height $|\beta|/Q_N$).

We have worked with only one of the original $\{\beta_j\}$, $1 \leq j \leq r$, and one which corresponds to a spiral domain. The same construction with the same integer Q_N (if that is suitably chosen) can be carried out for each of the spiral domains among the j, $1 \leq j \leq r$. We may even assume that for each of those, the heights $|\beta_j| < \varepsilon$ are the same.

Let now $\{\alpha_{Nm}\}$ denote the complete collection of (essentially) simple loops from all the spiral domains and also one closed horizontal trajectory from each annular domain in R_r. Consider the following extremal problem: among all differentials ψ of finite norm on R which satisfy

$$L_\psi(\alpha_{Nm}) \geq \Sigma_p c_{Nmp} a_p \tag{3}$$

for all simple loops in our collection $\{\alpha_{Nm}\}$ find the one of smallest norm $\|\psi\|$. (Here if α_{Nm} is just a closed trajectory, we interpret the right side as being a_m, its φ-length.) According to [9], there is a unique solution $\psi_N dz^2$. In particular, it satisfies

$$\|\psi_N\| \leq \|\varphi\|. \tag{4}$$

Restrict the indices $\{N\}$ to a subsequence for which $\{\psi_N\}$ converges locally uniformly to a differential ψ_∞. We need an inequality in the opposite direction.

Return again to a spiral domain A of R_r. In the rectangles $\{S_{Nkd}\}$ of height $s_0 = |\beta|/Q_N$ arising by subdivision of the $\{S_{Nk}\}$, parameterize the horizontal lines as $\tilde{\alpha}_{Nkd}(s)$, $0 < s < s_0$. Choose the orientation for s so that when the S_{Nkd} are joined end-to-end to form annuli, for each s the $\{\tilde{\alpha}_{Nkd}(s)\}$ simultaneously join end-to-end to form simple loops parallel to the corresponding $\{\tilde{\alpha}_{Nm}\}$. Project each S_{Nkd} isometrically into S_k (they have the same length), filling up S_k as much as possible but there may also have to be some overlap. That will be governed by (1).

Denote the projection of $\tilde{\alpha}_{Nkd}(s)$ into S_k by $\alpha_{Nkd}(s)$. Integrating in R in the $w = u + iv = \Phi(z)$ coordinates associated with φ,

$$\Sigma_k \Sigma_d \int_{\alpha_{Nkd}(s)} du \leq \Sigma_m L_\varphi(\alpha_{Nm}),$$

$$\Sigma_m L_{\psi_N}(\alpha_{Nm}) \leq \Sigma_k \Sigma_d \int_{\alpha_{Nkd}(s)} |\psi_N|^{1/2} du + M\varepsilon Q_N$$

The first inequality is just (2). The second follows from the definition of minimum length $L_{\psi_n}(\alpha_{Nm})$; here M is the maximum of $|\psi_N|^{1/2}$ on β with respect to the Φ-coordinates. Therefore from (2) and (3),

$$\Sigma_k \Sigma_d \int_{\alpha_{Nkd}(s)} du \leq \Sigma_k \Sigma_d \int_{\alpha_{Nkd}(s)} |\psi_N|^{1/2} du + (M+1)\varepsilon Q_N. \tag{5}$$

We claim that on integrating this in s, $0 \leq s \leq |\beta|/Q_N$, there results an inequality,

$$\|\varphi\|_A \leq \int\int_A |\psi_N|^{1/2} du\, dv + (M+1)\varepsilon|\beta| + C_1/Q_N \tag{6}$$

for some constant C_1. By (1) the integral of the left side of (5) differs from the φ-area $\|\varphi\|_A$ of A by less than C/Q_N. On the other hand by (4)

$$\left(\int\int |\psi_N|^{1/2} du\, dv\right)^2 \leq \|\varphi\| \int\int du\, dv,$$

for integration over any subregion of A. Therefore if there is overlap, the error caused by that in integrating the right side of (5) is bounded by $C\|\varphi\|/Q_N$. This establishes (6).

In (6) let $N \to \infty$. By assumption A is the interior of a finitely punctured bordered surface so that $\psi_N \to \psi_\infty$ uniformly on A. Therefore

$$\|\varphi\|_A \leq \int\int_A |\psi_\infty|^{1/2} du\, dv + \varepsilon(M+1)|\beta|.$$

Now $\psi_\infty = \psi_\infty[\varepsilon; r]$ depends on both ε and r. First take a sequence $\varepsilon \to 0$ so that $\{\psi_\infty\}$ converges locally uniformly on R to some $\psi_{\infty\infty}[r]$. Then take a sequence $r \to \infty$ so that $\{\psi_{\infty\infty}[r]\}$ converges locally uniformly too, to some ψ_* on R. It satisfies

$$\|\varphi\|_A \leq \int\int_A |\psi_*|^{1/2} du\, dv. \tag{7}$$

Inequality (7) holds for each spiral domain in R since $R_r \to R$ (to a set of φ-area zero) as $r \to \infty$. But it also holds for the annular domains. For if A is one with cross-cut

β, we can parameterize the closed trajectories in A as $\alpha(s)$, $0<s<|\beta|$. By (3),

$$a \leqslant \int_{\alpha(s)} |\psi_N|^{1/2} du,$$

where a is the φ-length of the $\{\alpha(s)\}$. Letting $N\to\infty$, then $\varepsilon\to 0$ and $r\to\infty$ this becomes for each s,

$$a \leqslant \int_{\alpha(s)} |\psi_*|^{1/2} du.$$

To get (7) integrate in s.

Summing (7) over all spiral and annular domains in R and then applying the Schwarz inequality, we find that $\|\varphi\| \leqslant \|\psi_*\|$. On the other hand as a consequence of (4), $\|\psi_*\| \leqslant \|\varphi\|$ and there is equality. In particular there is equality in the application of Schwarz's inequality to ψ_*. That forces $|\psi_*|=1$ where ψ_* is the representation of the differential in terms of the φ-coordinates. Hence for some constant ϑ, $0\leqslant\vartheta\leqslant\pi/2$, $\psi_* \equiv e^{2i\vartheta}\varphi$.

Before dealing with this, we point out that ψ_* is the limit *in norm* of (necessarily positive) Jenkins-Strebel differentials on R, namely a "diagonal" sequence taken from the collection $\{\psi_N[\varepsilon; r]\}$. Each ψ_N was obtained as the solution of an extremal problem for a certain system of loops $\{\alpha_{Nm}\}$. It has the additional property that an annular domain lies in the free homotopy class of each α_{Nm}, and these domains cover R to a set of area zero. Because

$$\|\psi_*\| \leqslant \liminf \|\psi_N\| \leqslant \limsup \|\psi_N\| \leqslant \|\varphi\| = \|\psi_*\|,$$

$\lim \|\psi_N\| = \|\psi_*\|$ which implies $\lim \|\psi_N - \psi_*\| = 0$.

If then $\partial R \neq \varnothing$, because both φ and ψ_* are positive, $\vartheta=0$. But $\vartheta=0$ in general. For otherwise for large N the ψ_N-length of its closed trajectories would be smaller than their φ-length, by a factor independent of N. This would imply that the ψ_N-area $\|\psi_N\|$ of R is uniformly bounded less than the φ-area $\|\varphi\|$, a contradiction. For the details we refer to [18]. This completes the argument for Theorem 6.8.

6.9. COROLLARY 6.9. *Let R be the complement with respect to $\mathbf{C}$ of a countably infinite set of points $\{z_k\}$ without limit points. Every quadratic differential of finite norm in R can be approximated by simple differentials uniformly on compact sets.*

Proof. The surface R is parabolic. If $\varphi\, dz^2$ has finite norm on R, it follows from a result of Bers [2] and Reich [13] that $\varphi\, dz^2$ can be approximated *in norm* by the restrictions to R of rational differentials $\varphi_n dz^2$ on $\mathbf{C} \cup \{\infty\}$. Furthermore the poles of $\varphi_n dz^2$ all lie in the set $\{z_n\}$ plus ∞ (and are necessarily simple ones). Let S_n denote the finite set of poles of φ_n. On the sphere punctured at the points of S_n, φ_n is the limit (in norm) of simple differentials by Theorem 6.8. Each simple differential used in the approximation of φ_n, when restricted to R, in general is no longer simple. Rather it has a finite or countably infinite number of annular domains which cover R to a set of area zero. But as a consequence of Theorem 6.8, such differentials can be locally uniformly on R approximated by simple ones. Therefore $\varphi\, dz^2$ itself can be so approximated.

6.10. In this section we will restrict our attention to those quadratic differentials which are squares of first order holomorphic ones. We will establish the following result.

COROLLARY 6.10. *Assume that R is a parabolic Riemann surface and $\varphi\, dz^2 = (\alpha\, dz)^2$ is a quadratic differential of finite norm which is the square of a first order differential. Then $\varphi\, dz^2$ can be approximated by simple differentials, uniformly on compact subsets.*

Proof. The first step is to recall from [1] the facts concerning reproducing differentials on the parabolic surface R. Given a 1-cycle c on R there is a uniquely determined real harmonic differential $\sigma(c) \in \Gamma_h$ (= the Hilbert space of real harmonic differentials with finite Dirichlet integral) with the properties,

$$\int_c \omega = (\omega, \sigma(c)^*) = -\int\int_R \omega \wedge \sigma(c), \tag{1}$$

$$\int_d \sigma(c) = c \times d \quad \text{(algebraic intersection number)}, \tag{2}$$

for any $\omega \in \Gamma_h$ and 1-cycle d.

Furthermore if c_1, c_2 are 1-cycles and m_1, m_2 are integers,

$$\sigma(m_1 c_1 + m_2 c_2) = m_1 \sigma(c_1) + m_2 \sigma(c_2). \tag{3}$$

In view of (1), the closure in Γ_h of the linear subspace spanned by the set of conjugates $\{\sigma(c)^*\}$ of the reproducing differentials is just Γ_h itself because no non-zero element is orthogonal to all of them. Since $\Gamma_h \equiv \Gamma_h^*$, Γ_h is the closure of the linear span of the reproducing differentials $\{\sigma(c)\}$ themselves. In fact, LEMMA 6.10.1. *Given $\omega \in \Gamma_h$ there is a sequence $\{c_n\}$ of* 1-*cycles and $\{q_n\}$ of positive integers such that*

$$\omega = \lim \frac{\sigma(c_n)}{q_n}, \quad \textit{convergence in norm.}$$

Proof. A differential $\Sigma_{i=1}^k x_i \sigma(c_i)$ is the limit in norm of a sequence $\{\Sigma_{i=1}^k x_{in} \sigma(c_i)\}$ where for each i, $\{x_{in}\}$ is a sequence of rationals converging to x_i. Find a common denominator and use (3).

To each $\sigma(c)$ we associate the analytic differential $\alpha(c)=\sigma(c)+i\sigma(c)^*$ and the quadratic differential

$$\psi(c)\, dz^2 = -\alpha(c)^2 = (i\alpha(c))^2.$$

The norm of ψ satisfies $\|\psi(c)\|=(\sigma(c), \sigma(c))$ where $(\sigma(c), \sigma(c))$ is the Dirichlet integral of $\sigma(c)$. The horizontal trajectories of $\psi(c)$ are the curves along which $\sigma(c)$ vanishes (level curves).

LEMMA 6.10.2. *Up to a set of $\psi(c)$-area zero, R is the union of annular domains of $\psi(c)$.*

Proof of Lemma 6.2. Let β be a vertical segment of $\psi(c)$ of $\psi(c)$-length <1 and denote the two sides of β as β^+ and β^- as usual. As in § 3.4 find the set of points $\{y\}$ on $\beta^+ \cup \beta^-$ with the property that the horizontal ray leaving y (i) hits a critical point of ψ(c), (ii) has limit points on the ideal boundary of R, or (iii) hits an end point of β before otherwise meeting β. The set $\{y\}$ is closed and because R is parabolic, the trajectories through $\{y\}$ cover a subset of R of $\psi(c)$-area zero [19].

Choose $\zeta \in \beta^+ \cup \beta^- \setminus \{y\}$ and consider the horizontal ray from ζ. This ray must eventually return to β. We claim that it returns exactly at ζ, on the opposite side of β from whence it started, and therefore the trajectory through ζ is closed. For suppose instead it returns first at $\zeta_1 \neq \zeta$. Consider the simple loop γ formed by the horizontal segment from ζ to ζ_1 and the vertical segment on β from ζ_1 back to ζ. For this,

$$0 \neq \left| \int_\gamma \operatorname{Im} \sqrt{\psi(c)}\, dz \right| = \left| \int_\gamma \sigma(c) \right| = |c \times \gamma|.$$

Thus the integral on the left is a non-zero integer by (2) while at the same time its value cannot exceed the length of β which is less than one. This contradiction establishes the fact that the trajectory through each point of $\beta^+ \cup \beta^- \setminus \{y\}$ is closed. The lemma follows at once from this property.

Completion of the proof of Corollary 6.10. Given $\varphi\, dz^2=(\alpha\, dz)^2$ we can approximate $i\alpha\, dz$ in the Dirichlet norm by some sequence $\{\alpha(c_n)/q_n\}$ by Lemma 6.10.1. Hence in norm, $\varphi\, dz^2$ is approximated by $\{-\alpha(c_n)^2 dz^2/q_n^2\}$. But $-\alpha(c_n)^2 dz^2=\psi(c_n)\, dz^2$ fulfills the conditions of Theorem 6.8 because of Lemma 6.10.2. Therefore it in turn can be approximated by simple differentials, uniformly on compact subsets. The job is completed by taking a diagonal sequence.

7. Geometric determination of Teichmüller mappings

7.1. A Teichmüller map $f\colon R\to S$ between two Riemann surfaces (possibly $R\equiv S$) is a quasiconformal homeomorphism that satisfies a Beltrami equation on R of the form,

$$f_{\bar z}=k\frac{\bar\varphi}{|\varphi|}f_z,\quad 0<k<1,$$

where $\varphi\, dz^2$ is a (holomorphic) quadratic differential on R. Automatically, there is also a quadratic differential $\psi\, dz^2$ on S, uniquely determined up to a positive constant factor, such that the inverse map $f^{-1}\colon S\to R$ satisfies,

$$(f^{-1})_{\bar z}=-k\frac{\bar\psi}{|\psi|}(f^{-1})_z.$$

The differentials φ, ψ are said to be *associated* with f. The maximal dilatation of f is the constant

$$K=\frac{1+k}{1-k},\quad 1<K<\infty.$$

Geometrically Teichmüller maps have the following structure. First, given f and its associated φ on R, then ψ (more precisely the positive constant factor implicit in ψ) is uniquely determined by the relation

$$\iint_{R_0}|\varphi|\, dx\, dy=\iint_{f(R_0)}|\psi|\, dx\, dy$$

for every relatively compact subsurface of R. Introduce the φ-local coordinates $w=\int^z\sqrt{\varphi}\, dz$ in R and the ψ-coordinates $\zeta=\int^z\sqrt{\psi}\, dz$ in S. Then f is area preserving in these coordinates, it maps the critical points of φ to those of ψ, and away from these it has the form,

$$\zeta=\frac{w+k\bar w}{\sqrt{1-k^2}}.$$

That is, in the φ- and ψ-coordinates, R and S become locally flat and f is an affine map between these flat pieces.

In particular on a torus represented as a lattice in the plane, every quadratic differential has the form

$$\varphi\, dz^2 = ae^{-2i\vartheta} dz^2, \quad a > 0,$$

and the Teichmüller maps are exactly the non-singular, orientation preserving, affine maps.

If R has a border ∂R, and $\varphi\, dz^2$ is real (or positive) on ∂R then $\psi\, dz^2$ is automatically real (or positive) on $f(\partial R) = \partial S$.

As a consequence of this geometry, if γ is a simple loop on R, or a cross-cut between ∂R where $\varphi\, dz^2 \geq 0$ on ∂R, then

$$h_\psi(f(\gamma)) = h_\varphi(\gamma)/\sqrt{K}. \tag{1}$$

If R is a compact surface, possibly with boundary ∂R, possibly with a finite number of punctures, and $f_0\colon R \to S$ is a homeomorphism, according to the celebrated theorem of Teichmüller there is a unique Teichmüller map f in the homotopy class $[f_0]$ of f_0. Its associated differential φ on R will have finite norm and be real on the boundary ∂R of R, if $\partial R \neq \emptyset$. For surfaces R of infinite topological type the situation is much less clear. However it is known that if $[f_0]$ contains a Teichmüller map of *finite norm* (that is, φ has finite norm on R and hence also ψ on S), it is the only such map there.

7.2. In the following two results, the simple differentials will as usual be normalized (unit norms). On the other hand the differentials associated with the Teichmüller maps will not even be assumed to have finite norms. For a discussion of convergence, we refer to § 5.7; the height $h_\varphi(\gamma)$ is defined in § 2.1 and length $L_\varphi(\gamma)$ in § 4.3 for a simple loop or (when there is a border) cross-cut γ.

LEMMA 7.2.1. *Assume $f\colon R \to S$ is a Teichmüller map of maximal dilatation K and associated differential $\varphi\, dz^2$ on R. Given a simple loop γ on R,*

$$\frac{1}{\sqrt{K}} L_{\varphi[\gamma]}(\gamma) \leq L_{\varphi[f(\gamma)]}(f(\gamma)) \leq \sqrt{K}\, L_{\varphi[\gamma]}(\gamma),$$

and for all simple loops and cross-cuts α,

$$\sqrt{K}\, h_{\varphi[\gamma]}(\alpha) \geq h_{\varphi[f(\gamma)]}(f(\alpha)) \geq \frac{1}{\sqrt{K}} h_{\varphi[\gamma]}(\alpha).$$

Equality occurs on the right (in the second inequality, for an α with $i(\gamma,\alpha)\neq 0$) if and only if $\varphi\equiv c\varphi[\gamma]$. Equality occurs on the left (in the second inequality, for an α with $i(\gamma,\alpha)\neq 0$) if and only if $\varphi\equiv -c\varphi[\gamma]$. Here c is a positive constant.

LEMMA 7.2.2. *Let $f: R\to S$ be a Teichmüller map of maximal dilatation K and associated differentials φ on R and ψ on S. Suppose that $\{\gamma_n\}$ is a sequence of simple loops on R with the three properties:*

(i) $$\operatorname{Lim}\frac{L_{\varphi[f(\gamma_n)]}(f(\gamma_n))}{L_{\varphi[\gamma_n]}(\gamma_n)}=\sqrt{K}\quad\left(\textit{resp.}, \frac{1}{\sqrt{K}}\right),$$

or that for some α with $i(\gamma_n,\alpha)\neq 0$ for all n,

$$\operatorname{Lim}\frac{h_{\varphi[f(\gamma_n)]}(f(\alpha))}{h_{\varphi[\gamma_n]}(\alpha)}=\frac{1}{\sqrt{K}}\quad(\textit{resp.}, \sqrt{K}).$$

(ii) *The sequence $\{\varphi[\gamma_n]\}$ on R (resp., $\{\varphi[f(\gamma_n)]\}$ on S) converges locally uniformly to a differential φ_∞ (resp., to ψ_∞), and*

(iii) *the sequence of image differentials $\{\varphi[f(\gamma_n)]\}$ on S (resp., $\{\varphi[\gamma_n]\}$ on R) converges in norm to a differential ψ_∞ (resp., in norm to φ_∞ on R).*

Then there is a constant c, $0<c\leq 1$, such that $\varphi_\infty\equiv c\varphi$ (resp., $\psi_\infty\equiv -c\psi$). In particular φ and ψ have finite norms.

COROLLARY 7.2.3. *In addition to the hypotheses of Lemma 7.2.2 assume*

(iv) *R is a parabolic surface or surface with border ∂R whose double across ∂R is parabolic, and*

(v) *$\varphi\,dz^2$ is real.*

Normalize φ and ψ to have unit norms. Then $\varphi_\infty\equiv\varphi$ and $\psi_\infty\equiv\psi$ (resp., $\varphi_\infty\equiv-\varphi$ and $\psi_\infty\equiv-\psi$).

In the statements of Lemma 7.2.2 and Corollary 7.2.3 the parenthetical statements are to be read as a unit and to substitute for the adjacent statement.

Proof of Lemma 7.2.1.(1) The $\varphi[\gamma]$-length of its closed trajectories is exactly $L_{\varphi[\gamma]}(\gamma)=a$. Similarly set $a'=L_{\varphi[f(\gamma)]}(f(\gamma))$. We will use $\varphi[\gamma]$-local coordinates $w=u+iv$ in R and $\varphi[f(\gamma)]$-local coordinates $\zeta=\xi+i\eta$ in S. From the relations,

$$\frac{\partial f}{\partial z}=\frac{\partial f}{\partial w}\frac{dw}{dz},\quad \frac{\partial f}{\partial \bar z}=\frac{\partial f}{\partial \bar w}\overline{\left(\frac{dw}{dz}\right)},$$

(1) The statement also follows from the moduli inequality of § 7.4, using (9) and (10).

it follows that,

$$(f_{\bar{w}}/f_w)_{w=w(z)} = k\frac{\overline{(\varphi/\varphi[\gamma])}}{|\varphi/\varphi[\gamma]|}.$$

If α denotes a closed trajectory of $\varphi[\gamma]$,

$$a' = L_{\varphi[f(\gamma)]}(f(\gamma)) \leq \int_{f(\alpha)} |d\zeta| = \int_\alpha |f_w + f_{\bar{w}}|\, du.$$

By integrating in dv, thinking of $\alpha \equiv \alpha_v$ sweeping out the annular domain of area one for $\varphi[\gamma]$, we obtain,

$$\frac{a'}{a} \leq \int_R |f_w + f_{\bar{w}}|\, du\, dv. \tag{1}$$

The expression

$$J_f[\gamma](w) = |f_w|^2 - |f_{\bar{w}}|^2$$

is the Jacobian of the map $\zeta = f(w)$ relative to the $\varphi[f(\gamma)]$-coordinates ζ in S and $\varphi[\gamma]$-coordinates w in R. The form $J_f[\gamma](w)\, du\, dv$ on R is however independent of the coordinates on R, but it still depends on γ. The following ratios are computed to be,

$$\frac{|f_w + f_{\bar{w}}|^2}{J_f[\gamma](w)} = \frac{\left|1 + k\dfrac{\overline{\varphi/\varphi[\gamma]}}{|\varphi/\varphi[\gamma]|}\right|^2}{1-k^2} \tag{2}$$

$$\frac{(|f_w| + |f_{\bar{w}}|)^2}{J_f[\gamma](w)} = K. \tag{3}$$

Applying the Schwarz inequality to (1) and making use of (3) we obtain,

$$\left(\frac{a'}{a}\right)^2 \leq \int\int_R |f_w + f_{\bar{w}}|^2\, du\, dv \leq \int\int_R (|f_w| + |f_{\bar{w}}|)^2\, du\, dv = K,$$

since $\int\int_R J_f[\gamma](w)\, du\, dv = \int\int_S d\xi\, d\eta = 1$. Note that if there is equality at the ends, then

$$|f_w + f_{\bar{w}}| = |f_w| + |f_{\bar{w}}|.$$

Inserting (2) we obtain

$$\left(\frac{a'}{a}\right)^2 \leq (1-k^2)^{-1} \int\int_R \left|1 + k\frac{\overline{\varphi/\varphi[\gamma]}}{|\varphi/\varphi[\gamma]|}\right|^2 J_f[\gamma](w)\, du\, dv \leq K. \tag{4}$$

In the case of equality when $a'/a=\sqrt{K}$,

$$\left|1+k\frac{\overline{\varphi/\varphi[\gamma]}}{|\varphi/\varphi[\gamma]|}\right|^2=(1+k)^2. \tag{5}$$

Squaring and comparing sides in (5) we find for the meromorphic function $\varphi/\varphi[\gamma]$ that

$$\operatorname{Re}\varphi/\varphi[\gamma]=|\varphi/\varphi[\gamma]|. \tag{6}$$

Therefore $\varphi/\varphi[\gamma]$ is a positive constant c. In particular φ has finite norm and $c=\|\varphi\|$.

When $\varphi\equiv\varphi[\gamma]$, f consists of a stretch of magnitude $\sqrt{K}$ along the trajectories of $\varphi[\gamma]$ and a compression of $1/\sqrt{K}$ along its vertical trajectories. The inverse map f^{-1} does the opposite and stretches by a factor $\sqrt{K}$ the vertical trajectories of $\varphi[f(\gamma)]$. That is, when $\varphi\equiv\varphi[\gamma]$, the associated differential $\psi\,dz^2$ of f on S is

$$\psi\equiv\varphi[f(\gamma)].$$

Consequently when $a'/a=1/\sqrt{K}$, by considering f^{-1} in place of f, we conclude that $\varphi\equiv-c\varphi[\gamma]$ where $c=\|\varphi\|$.

For the height inequality in Lemma 7.2.1 recall from § 5.5 that

$$h_{\varphi[\gamma]}(\alpha)=bi(\gamma,\alpha)$$

where b is the $\varphi[\gamma]$-height of its annular domain. Correspondingly in S,

$$h_{\varphi[f(\gamma)]}(f(\alpha))=b'i(f(\gamma),f(\alpha))=b'i(\gamma,\alpha)$$

where b' is the $\varphi[f(\gamma)]$-height. Because the differentials are normalized,

$$bL_{\varphi[\gamma]}(\gamma)=b'L_{\varphi[f(\gamma)]}(f(\gamma))=1.$$

Consequently,

$$L_{\varphi[f(\gamma)]}(f(\gamma))\,h_{\varphi[f(\gamma)]}(f(\alpha))=L_{\varphi[\gamma]}(\gamma)\,h_{\varphi[\gamma]}(\alpha), \tag{7}$$

and the remainder of the lemma follows from the first part.

Proof of Lemma 7.2.2. By (7) the two conditions in (i) are equivalent. Also by (7), φ_∞ is not the zero differential, since ψ_∞, having unit norm as the limit in norm of normalized differentials, cannot be zero.

Before going to the limit in (4), we must clarify the dependence of $J_f[\gamma](w)$ on γ. Instead of using $\varphi[f(\gamma)]$-coordinates on S use the $\varphi[f(\alpha)]$-coordinates for some fixed α.

Then

$$J_f[\gamma](w) = \left| \frac{\varphi[f(\gamma)](f(w))}{\varphi[f(\alpha)](f(w))} \right| J_f(w), \quad w \in R$$

where $J_f(w)$ is the Jacobian relative to the $\varphi[f(\alpha)]$-coordinates on S. Set

$$J_f^*(w) = \left| \frac{\psi_\infty(f(w))}{\varphi[f(\alpha)](f(w))} \right| J_f(w).$$

Note the behavior of the Jacobians of f with respect to the various coordinates on S: Let U be a small open set on R then

$J_f[\gamma]\, du\, dv$, $J_f^*\, du\, dv$ on U transforms to $|\varphi[f(\gamma)]|\, dx\, dy$, $|\psi_\infty|\, dx\, dy$ on $f(U)$.

Here $z=x+iy$ is the "generic" coordinate system on S.

To deal with (4) we rewrite it as,

$$(1-k^2)\left(\frac{a'}{a}\right)^2 \leqslant \iint_R \left| 1+k\frac{\overline{\varphi/\varphi[\gamma]}}{|\varphi/\varphi[\gamma]|} \right|^2 \left| \frac{\varphi[f(\gamma)]}{\varphi[f(\alpha)]}(w) \right| J_f(w)\, du\, dv \leqslant (1+k)^2.$$

We want to replace γ by γ_n and take the limit. If we can do this under the integral we will end up with

$$\iint_R \left| 1+k\frac{\overline{\varphi/\varphi_\infty}}{|\varphi/\varphi_\infty|} \right|^2 J_f^*(w)\, du\, dv = (1+k)^2 = \iint_R (1+k)^2 J_f^*(w)\, du\, dv. \tag{8}$$

This in turn will imply that

$$\operatorname{Re} \varphi/\varphi_\infty = |\varphi/\varphi_\infty|$$

and that $\varphi_\infty = c\varphi$. It will follow that φ has finite norm and $c=\|\varphi_\infty\|/\|\varphi\|$. Concerning $\|\varphi_\infty\|$ we only know that $\|\varphi_\infty\| \leqslant 1$.

To justify the limit set

$$g[\gamma](w) = \left| 1+k\frac{\overline{\varphi/\varphi[\gamma]}}{|\varphi/\varphi[\gamma]|} \right|^2$$

so that $|g[\gamma](w)| \leqslant (1+k)^2$. Also write $g^*(w)$ for the result of replacing $\varphi[\gamma]$ by φ_∞. Now

$$g[\gamma_n] J_f[\gamma_n] - g^* J_f^* = g[\gamma_n](J_f[\gamma_n] - J_f^*) + (g[\gamma_n] - g^*) J_f^*.$$

Working on each term on the right separately,

$$\left|\iint_R g[\gamma_n](J_f[\gamma_n]-J_f^*)\,du\,dv\right| = \left|\iint_S (g[\gamma_n]\circ f^{-1})(|\varphi[f(\gamma_n)]|-|\psi_\infty|)\,dx\,dy\right|$$
$$\leq (1+k)^2\,\|\varphi[f(\gamma_n)]-\psi_\infty\|,$$

and for the other, the dominated convergence theorem implies,

$$\lim\iint_R |g[\gamma_n]-g^*|J_f^*\,du\,dv=0.$$

Together, these facts justify (8).

Finally, to prove the parenthetical facts, work on S as the domain of f^{-1}.

Proof of Corollary 7.2.3. Because f is quasiconformal, S has the same type (parabolic, etc.) as R. The new ingredient that can be used here is the Heights theorem. From (7) and Proposition 2.3,

$$h_{\psi_\infty}(f(\alpha))=\frac{1}{\sqrt{K}}h_{\varphi_\infty}(\alpha)=\frac{c}{\sqrt{K}}h_\varphi(\alpha),$$

and this holds for any α, even cross-cuts. As noted in § 7.1 (1),

$$h_\varphi(\alpha)=\sqrt{K}\,h_\psi(f(\alpha)).$$

Consequently by the Heights theorem, $\psi_\infty=c\psi$. But both ψ_∞ and ψ have unit norm, so $c=1$.

The parenthetical statements are proved by working on S.

Remark. Note that the corollary implies that if $\partial R\neq\varnothing$, $\varphi\,dz^2$ is not only real on ∂R but in fact is positive ($\varphi\,dz^2\geq 0$).

7.3. It is worth restating Lemma 7.2.2 for the case when there is no trouble with convergence.

LEMMA 7.3. *Suppose R is a compact surface possibly with boundary ∂R and with at most a finite number of punctures. Assume $f\colon R\to S$ is a Teichmüller map of maximal dilatation K and associated quadratic differentials φ on R and ψ on S, where φ and ψ are real if $\partial R\neq 0$, and they have unit norms. Suppose $\{\gamma_n\}$ is a sequence of simple loops on R such that either*

$$\lim\frac{L_{\varphi[f(\gamma_n)]}(f(\gamma_n))}{L_{\varphi[\gamma_n]}(\gamma_n)}=\sqrt{K}\quad\left(\mathit{resp.},\ \frac{1}{\sqrt{K}}\right)$$

or that for some α *with* $i(\gamma_n, \alpha)\neq 0$ *for all* n,

$$\lim \frac{h_{\varphi[f(\gamma_n)]}(f(\alpha))}{h_{\varphi[\gamma_n]}(\alpha)} = \frac{1}{\sqrt{K}} \quad (\textit{resp.}, \sqrt{K}).$$

Then with convergence in norm,

$$\varphi = \lim \varphi[\gamma_n], \quad \psi = \lim \varphi[f(\gamma_n)] \quad (\textit{resp.}, -\varphi = \lim \varphi[\gamma_n], -\psi = \lim \varphi[f(\gamma_n)]).$$

7.4. The modulus of a domain conformally equivalent to an annulus $\{z \in \mathbf{C}: 1<|z|<r\}$ is defined to be $(\log r)/2\pi$. This quantity is the reciprocal of the extremal length of the family of curves separating the boundary contours.

The *modulus* $M(\gamma)$ of the free homotopy class determined by the simple loop γ is likewise defined to be the reciprocal of the extremal length of the family of curves forming the free homotopy class. It is a fact that this number is exactly the modulus of the ring domain determined by $\varphi[\gamma]\,dz^2$. With the usual normalization $\|\varphi[\gamma]\|=1$ then,

$$M(\gamma) = \frac{1}{(L_{\varphi[\gamma]})^2}$$

If $f: R \to S$ is a K-quasiconformal mapping then

$$K^{-1}M(\gamma)_R \leq M(f(\gamma))_S \leq KM(\gamma)_R$$

for all simple loops γ. It is the purpose of the following theorem to examine

$$\sup_{\gamma} \left(\text{resp.}, \inf_{\gamma}\right) \frac{M(f(\gamma))_S}{M(\gamma)_R}.$$

THEOREM 7.4. *Suppose R is a parabolic Riemann surface or a surface with border ∂R whose double across ∂R is parabolic. Let $f: R \to S$ be a Teichmüller map with maximal dilatation K and associated differentials $\varphi\,dz^2$ on R and $\psi\,dz^2$ on S, with φ real on ∂R.*

(a) *Assume that* (i) *φ and ψ have finite, unit, norms, and for a sequence $\{\gamma_n\}$ of simple loops,* (ii) *$\{\varphi[\gamma_n]\}$ converges to φ (resp., to $-\varphi$) uniformly on compact subsets,* (iii) *$\{\varphi[f(\gamma_n)]\}$ converges to a differential ψ_∞ on S, also uniformly on compact sets. Then $\psi \equiv \psi_\infty/\|\psi_\infty\|$ (resp., $-\psi \equiv \psi_\infty/\|\psi_\infty\|$) and*

$$\lim \frac{M(f(\gamma_n))_S}{M(\gamma_n)_R} = \frac{1}{K\|\psi_\infty\|^2} \quad \left(\textit{resp.}, \frac{K}{\|\psi_\infty\|^2}\right)$$

(b) *Assume for a sequence of simple loops* $\{\gamma_n\}$ *on* R *that*

(i) $$\lim \frac{M(f(\gamma_n))_S}{M(\gamma_n)_R} = \frac{1}{K} \quad (\textit{resp.}, K),$$

(ii) $\{\varphi[\gamma_n]\}$ *converges to a differential* φ_∞ *on* R (*resp.*, $\{\varphi[f(\gamma_n)]\}$ *to* ψ_∞ *on* S), *uniformly on compact subsets, and*

(iii) $\{\varphi[f(\gamma_n)]\}$ *converges in norm to a differential* ψ_∞ *on* S (*resp.*, $\{\varphi[\gamma_n]\}$ *in norm to* φ_∞ *on* R). *Then* φ *and* ψ *have finite norms and if normalized,*

$$\varphi_\infty \equiv \varphi \quad \textit{and} \quad \psi_\infty \equiv \psi \quad (\textit{resp.}, \varphi_\infty \equiv -\varphi \textit{ and } \psi_\infty \equiv -\psi).$$

Proof. Continuing the analysis of § 7.3 note the relations,

$$\frac{M(f(\gamma))_S}{M(\gamma)_R} = \left(\frac{L_{\varphi[\gamma]}(\gamma)}{L_{\varphi[f(\gamma)]}(f(\gamma))}\right)^2 \tag{9}$$

$$h_{\varphi[f(\gamma)]}(f(\alpha)) = \frac{L_{\varphi[\gamma]}(\gamma)}{L_{\varphi[f(\gamma)]}(f(\gamma))} h_{\varphi[\gamma]}(\alpha). \tag{10}$$

The latter is just (7) and α is any simple loop, or possibly cross-cut.

First we prove (a). Take a subsequence $\{\gamma_m\}$ so that

$$\lim \frac{M(f(\gamma_m))_S}{M(\gamma_m)_R} = \frac{1}{L}$$

for some $L \leq K$. Then by (10) and Proposition 2.3 for all α,

$$h_{\psi_\infty}(f(\alpha)) = \frac{1}{\sqrt{L}} h_\varphi(\alpha).$$

But from §7.1 (1),

$$h_\psi(f(\alpha)) = \frac{1}{\sqrt{K}} h_\varphi(\alpha).$$

Since ψ and ψ_∞ have finite norm, the Heights theorem implies that $\psi_\infty = \sqrt{L/K}\,\psi$. Therefore

$$L = K(\|\psi_\infty\|/\|\psi\|)^2 = K\|\psi_\infty\|^2$$

and is independent of the subsequence $\{\gamma_m\}$ of $\{\gamma_n\}$ chosen for the proof.

For the parenthetical case, proceeding as above we find

$$h_{\psi_\infty}(f(\alpha)) = \frac{1}{\sqrt{L}} h_{-\varphi}(\alpha).$$

From §7.1 (1) applied to f^{-1},

$$h_{-\psi}(f(\alpha)) = \sqrt{K}\, h_{-\varphi}(\alpha).$$

Consequently $\psi_\infty \equiv -\sqrt{LK}\,\psi$ where now $L = \|\psi_\infty\|^2 K^{-1}$.

Part (b) is just Corollary 7.2.3.

7.5. In the nicest case Theorem 7.4 reduces to the following assertion. Results in this direction were earlier found by Kerckhoff [10].

COROLLARY 7.5. *Suppose R is a compact surface possibly with boundary ∂R, possibly with a finite number of punctures. Assume $f: R \to S$ is a Teichmüller map with maximal dilatation K and associated positive, normalized quadratic differentials φ on R and ψ on S where no closed horizontal trajectory of φ is parallel to a component of ∂R. Then*:

(a)
$$\inf_{\gamma} \frac{M(f(\gamma))_S}{M(\gamma)_R} = \frac{1}{K}.$$

If $\{\gamma_n\}$ is any minimizing sequence then $\lim \varphi[\gamma_n] = \varphi$ *and* $\lim \varphi[f(\gamma_n)] = \psi$.

(b)
$$\sup_{\gamma} \frac{M(f(\gamma))_S}{M(\gamma)_R} = K.$$

If $\{\gamma_n\}$ is any maximizing sequence then $\lim \varphi[\gamma_n] = -\varphi$ *and* $\lim \varphi[f(\gamma_n)] = -\psi$.

Given a Teichmüller map $f: R \to S$, one commonly knows that the associated differentials φ on R and ψ on S are real. From Corollary 6.7 and §6.6, Remark (2), one knows that φ can be approximated by simple differentials if and only if (when $\partial R \neq \emptyset$) φ is positive and furthermore has no closed horizontal trajectories parallel to a component of ∂R. If, for example, $\varphi\, dz^2 > 0$ along a component of ∂R, then as a consequence of Corollary 7.5, both $\sup M(f(\gamma))/M(\gamma) < K$ and $\inf M(f(\gamma))/M(\gamma) > 1/K$.

7.6. On an arbitrary Riemann surface R denote the set of quadratic differentials with finite norm by $Q(R)$. There are two natural topologies that may be considered: The norm topology and the in general weaker topology of uniform convergence on compact subsets. Fix a relatively compact subsurface R_0 of R and denote by $\|\cdot\|_0$ the norm over that. We will focus on the following set: $\mathfrak{C}_0(R) = \{\varphi \in Q(R): \varphi$ is the locally uniform limit of the sequence $\{\varphi[\gamma_n]/\|\varphi[\gamma_n]\|_0\}$ for some sequence $\{\gamma_n\}$ of simple loops$\}$.

The differentials in $\mathfrak{C}_0(R)$ have finite norm on R and unit norm on R_0. The set

$\mathfrak{C}_0(R)$ is closed in the topology of uniform convergence on compact sets. It is also essentially independent of R_0. In the case of a compact surface R, possibly with boundary ∂R, possibly with a finite number of punctures, $\mathfrak{C}_0(R)$ is essentially the unit sphere in $Q(R)$.

THEOREM 7.6. *Suppose R is a parabolic Riemann surface or a surface with border ∂R whose double across ∂R is parabolic. Let $f: R \to S$ be a Teichmüller map with associated differentials φ on R, ψ on S with φ real on ∂R and φ, ψ so normalized that $\|\varphi\|_0 = \|\psi\|_0 = 1$. Then in the topology of uniform convergence on compact subsets, f determines a homeomorphism*

$$f_\#: \mathfrak{C}_0(R) \to \mathfrak{C}_0(S).$$

If $\varphi \in \mathfrak{C}_0(R)$ then also $\psi \in \mathfrak{C}_0(S)$ and $f_\#(\varphi) = \psi$. If $-\varphi \in \mathfrak{C}_0(R)$ then also $-\psi \in \mathfrak{C}_0(S)$ and $f_\#(-\varphi) = -\psi$.

Proof. First consider the situation that $\{\varphi[\gamma_n]\}$ converges to φ_∞ uniformly on compact subsets. By passing to a subsequence if necessary we can assume $\{\varphi[f(\gamma_n)]\}$ converges too, to a differential ψ_∞ on S, uniformly on compact subsets. Necessarily $\varphi_\infty \in Q(R)$, $\psi_\infty \in Q(S)$ since their norms do not exceed one.

For some simple loop or suitable cross-cut α on R consider the formulas

$$h_{\varphi[\gamma_n]}(\alpha) = i(\alpha, \gamma_n) M(\gamma_n)_R^{1/2}$$

$$h_{\varphi[(\gamma_n)]}(f(\alpha)) = i(\alpha, \gamma_n) M(f(\gamma_n))_S^{1/2}$$

where

$$M(\gamma_n)_R / K \leq M(f(\gamma_n))_S \leq K M(\gamma_n)_R,$$

with K the maximal dilatation of f. In view of Proposition 2.3, these show that

$$K^{-1} h_{\varphi_\infty}(\alpha) \leq h_{\psi_\infty}(f(\alpha)) \leq K h_{\varphi_\infty}(\alpha).$$

In particular $\varphi_\infty \not\equiv 0$ if and only if $\psi_\infty \not\equiv 0$.

Suppose $\varphi_\infty \not\equiv 0$. The formulas also show that

$$h_{\psi_\infty}(f(\alpha)) = c h_{\varphi_\infty}(\alpha), \quad c = \lim \left(\frac{M(f(\gamma_n))_S}{M(\gamma_n)_R} \right)^{1/2}, \tag{1}$$

where the positive number c exists. Suppose also $\lim \varphi[\delta_n] = \varphi_\infty$ but that

$\lim \varphi[f(\delta_n)]=\psi'_\infty$. In view of the Heights theorem, $\psi'_\infty=d\psi_\infty$ for a positive constant d. A similar conclusion holds if $\lim \varphi[f(\delta_n)]=\psi_\infty$ but $\lim \varphi[\delta'_n]=\varphi'_\infty$.

Pass now to the differently normalized differentials, $\varphi[\gamma]_0=\varphi[\gamma]/\|\varphi[\gamma]\|_0$, $\varphi[f(\gamma)]_0=\varphi[f(\gamma)]/\|\varphi[f(\gamma)]\|_0$. We have shown above that $\varphi[\gamma_n]_0$ converges in $\mathfrak{C}_0(R)$ if and only if $\varphi[f(\gamma_n)]_0$ converges in $\mathfrak{C}_0(S)$ (convergence uniform on compact subsets). Of course in this normalization, the limit differential is never the zero differential.

We are ready to define $f_\#$. Start with

$$f_\#: \varphi[\gamma]_0 \to \varphi[f(\gamma)]_0.$$

The analysis above shows that $f_\#$ not only has a well defined extension to $\mathfrak{C}_0(R)$ but in fact is a homeomorphism of $\mathfrak{C}_0(R)$ onto $\mathfrak{C}_0(S)$. This in the topology of uniform convergence on compact subsets.

Suppose for the differential φ associated with f, that $\varphi=\lim \varphi[\gamma_n]_0$. We know that $\{\varphi[f(\gamma_n)]_0\}$ also converges. Then according to Theorem 7.4, $\lim \varphi[f(\gamma_n)]_0=\psi$. The same theorem shows that if $-\varphi=\lim \varphi[\gamma_n]_0$ then $\lim \varphi[f(\gamma_n)]_0=-\psi$.

7.7. Again the classical case is nicest.

COROLLARY 7.7. *Suppose R is a compact Riemann surface with $b\geq 0$ boundary components and $p\geq 0$ punctures in the interior. Let g: $R\to S$ be a homeomorphism to another Riemann surface that preserves punctures. Denote by $Q_0(R)$ (resp., $Q_0(S)$) the unit sphere in the real vector space of real quadratic differentials $Q(R)$ (resp., $Q(S)$) of dimension $6g+2p+3b-6$. There exists a homeomorphism*

$$f_\#: Q_0(R) \to Q_0(S)$$

which depends only on the homotopy class modulo ∂R $[g]$ of g. It has the property that $f_\#(\varphi)=\psi$, $f_\#(-\varphi)=-\psi$ for the quadratic differentials φ on R, ψ on S associated with the Teichmüller map in $[g]$.

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