Actions of compact abelian groups on semifinite injective factors

by

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Introduction

The Clifford algebra construction allows one to associate with any unitary representation of a group an action of the group on an algebra in a functorial way. If the representation is infinite dimensional one must allow infinite dimensional algebras and one is led immediately to consider actions of groups on factors not of type I. Using this approach, Blattner showed in [1] that any separable locally compact group has a faithful action, by outer automorphisms and the identity, on the hyperfinite type II_1 factor $\mathcal{R}$. It was certainly not obvious at the time that one might hope to say much more about the actions even of finite cyclic groups on $\mathcal{R}$, but largely thanks to work of Connes, much progress has been made. This paper adds another step in a continuing project by giving a detailed description of all actions of a compact abelian group on semifinite injective factors.

The most general results on actions of abelian groups on von Neumann algebras appear in [10], where Connes and the second author established the relationship between their discrete and continuous decompositions of type III von Neumann algebras using the flow of weights, itself an action of $\mathbf{R}$ on an abelian von Neumann algebra.

The results of [10], powerful as they are, give precious little information on the

(1) This research was supported in part by NSF Grant no. MCS79-03041.

classification of actions of a given group on a von Neumann algebra, even in the case of actions of $\mathbf{Z}_2$ on $\mathcal{R}$. Indeed there would be no such classification available were it not for the idea of Connes in [7] to look at the action on central sequences and then use McDuff's result in [21] to split off explicitly controlled model actions in some tensor product factorization $\mathcal{R}\cong\mathcal{R}\otimes\mathcal{R}$. With this idea all actions of finite cyclic groups on $\mathcal{R}$ were classified up to conjugacy. The next step was taken by Connes himself in [4] where he classified actions of $\mathbf{Z}$ on $\mathcal{R}$ up to outer conjugacy and extended his central sequence technique to make it work for all separable factors $\mathcal{M}$ satisfying $\mathcal{M}\cong\mathcal{M}\otimes\mathcal{R}$.

An important contribution of [7] was the appearance of two algebraic invariants associated with actions. For a cyclic group they take a particularly simple form, one being a pair (p, γ) where $p\in\mathbf{N}$ and γ is a pth root of unity, the other being a probability measure on a finite cyclic group, defined up to translation. The first of these invariants was generalized to arbitrary (discrete) group actions by the first author in [14, 15]. It becomes the characteristic invariant, an element of a relative cohomology group. In [14] the first author extended Connes' classification to actions of an arbitrary finite group on $\mathcal{R}$. For this a generalization of the second invariant was required. The definition of this invariant in [14] was not optimal but it will be clear from this paper that it may be defined for any compact group action as a naturally occurring projection in the crossed product, modulo a certain equivalence relation.

The next step in the group action program was taken by Ocneanu in [23] where he shows that Connes' result on outer conjugacy of $\mathbf{Z}$ actions extends to arbitrary actions of amenable discrete groups, provided due consideration is given to the characteristic invariant. Since abelian groups are amenable, Ocneanu's result opens the way for a classification of actions of compact abelian groups on $\mathcal{R}$ via the duality result of [28]. A special case of this (when the crossed product is a factor) was done in [16]. The question of ergodic actions had already been solved by Olesen, Pedersen and the second author in [24]. This is somewhat simpler as the spectral subspace technique is all that is needed to obtain the result.

In this paper the authors consider the general question of actions of compact abelian groups on $\mathcal{R}$. In the course of the investigation it soon became apparent that the restriction to $\mathcal{R}$ was artificial and constraining so the setting was extended to actions on semifinite injective (separable) factors. We obtain a classification of such actions up to conjugacy though the sense in which it is a complete classification will be discussed in § 1.3.

The authors would like to thank O. Bratelli and N. Munch for pointing out a serious error in a previous version of this paper.

Chapter 1. Preliminaries

§ 1.1. Notation. Statement of the main theorem

Throughout this paper $\mathcal{M}$ and $\mathcal{N}$ will be separable von Neumann algebras with centers $\mathcal{Z}(\mathcal{M})$ and $\mathcal{Z}(\mathcal{N})$, and $\mathcal{A}$ will be an abelian von Neumann algebra. The unitary group of $\mathcal{M}$ will be written $\mathcal{U}(\mathcal{M})$. For each $u \in \mathcal{U}(\mathcal{M})$, $\mathrm{Ad}\, u$ will denote the inner automorphism: $x \mapsto uxu^*$. If $\mathcal{M}$ is a factor, $\mathcal{U}(\mathcal{Z}(\mathcal{M}))$ may be identified with $\{z \in \mathbf{C} \mid |z|=1\}$ and this group will be denoted $\mathbf{T}$. The automorphism group of $\mathcal{M}$ will be written $\mathrm{Aut}(\mathcal{M})$ and $\mathrm{Int}(\mathcal{M})$ will denote the normal subgroup of inner automorphisms. If $\mathcal{H}$ is a Hilbert space, $\mathcal{B}(\mathcal{H})$ will denote the algebra of all bounded operators on $\mathcal{H}$.

The symbols $\mathcal{R}$ and $\mathcal{R}_{0,1}$ will denote the injective factors of type II_1 and II_∞ respectively.

Compact abelian groups will be written A and B and the letters G, H and N will denote *discrete abelian groups*. The duality between A and $G=\hat{A}$ will be written $\langle g, a\rangle$. All groups will be written *multiplicatively* except when they appear as coefficients for some cohomology group. $L^2(A)$ will be the Hilbert space of square-integrable functions with respect to Haar measure da. (Similarly for G.) An *action* α of A (or G) on $\mathcal{M}$ will be a homomorphism α: $A \to \mathrm{Aut}\,\mathcal{M}$, $a \mapsto \alpha_a$, which is pointwise strongly continuous. The *crossed product* of $\mathcal{M}$ by an action α of A will be written $\mathcal{M} \rtimes_\alpha A$ (similarly for G). The dual action of $\hat{A}$ on $\mathcal{M} \rtimes_\alpha A$ will be written $\hat{\alpha}$.

If $\alpha: A \to \mathrm{Aut}\,\mathcal{M}$ is an action, a (unitary) cocycle for α will be a strongly continuous family $\{v_a\}$ of unitaries of $\mathcal{M}$ such that $v_a \alpha_a(v_b) = v_{ab}$. Two actions α and β of A on $\mathcal{M}$ and $\mathcal{N}$ will be called *conjugate* if there is an isomorphism $\theta: \mathcal{N} \to \mathcal{M}$ such that $\alpha_a = \theta\beta_a\theta^{-1}$ for all $a \in A$. They will be called *cocycle* conjugate if there is θ and a unitary cocycle $\{v_a\}$ for α such that $\mathrm{Ad}\, v_a \alpha_a = \theta\beta_a\theta^{-1}$ (similarly for G).

If α is an action of G and $N = \alpha^{-1}(\mathrm{Int}\,\mathcal{M})$, the *characteristic invariant* of α will be the pair (λ, μ), modulo coboundaries, defined by the following conditions

$$\alpha_h = \mathrm{Ad}\, w_h \quad \text{for } h \in N,$$

$$w_h w_k = \mu(h,k)\, w_{hk}, \quad h, k \in N,$$

$$\alpha_g(w_h) = \lambda(g,h)\, w_h, \quad g \in G,\ h \in N.$$

Thus $\lambda: G \times N \to \mathcal{U}(\mathcal{Z}(\mathcal{M}))$ and $\mu: N \times N \to \mathcal{U}(\mathcal{Z}(\mathcal{M}))$. We denote the characteristic invariant of α by χ_α. The characteristic invariant is a cocycle conjugacy invariant in the sense that if $\{\mathcal{M}, \alpha\}$ and $\{\mathcal{N}, \beta\}$ are cocycle conjugate covariant systems over G, and if θ is an isomorphism of $\mathcal{M}$ onto $\mathcal{N}$ and $\{v_g\}$ is an α-cocycle such that $\mathrm{Ad}\, v_g \circ \alpha_g = \theta\beta_g\theta^{-1}$, then θ maps naturally χ_β into χ_α. On the other hand, if we consider actions of G on a fixed $\mathcal{M}$, then the characteristic invariant χ_α is no longer invariant under conjugacy of α. Namely, if θ is an automorphism of $\mathcal{M}$ and if $\beta_g=\theta\alpha_g\theta^{-1}$, then we have $\chi_\beta=\theta(\chi_\alpha)$ as mentioned above. As seen later, $\theta(\chi_\alpha)$ need not be equal to χ_α. Therefore, in this case, the true cocycle conjugacy invariant is the orbit $[\chi_\alpha]$ of χ_α under the group $\mathrm{Aut}_G(\mathscr{Z}(\mathcal{M}))$ of automorphisms of $\mathscr{Z}(\mathcal{M})$ commuting with the restriction of α to $\mathscr{Z}(\mathcal{M})$. We call this orbit $[\chi_\alpha]$ the *characteristic orbit*.

The regular representation $a\to v_a$ on $L^2(A)$ gives an action $\varrho: A\mapsto \mathrm{Aut}(\mathscr{B}(L^2(A)))$, $\varrho_a=\mathrm{Ad}\, v_a$. Two actions α and β of A on $\mathcal{M}$ and $\mathcal{N}$ will be called *stably* conjugate if $\alpha\otimes\varrho$ and $\beta\otimes\varrho$ are conjugate. If the actions α and β are stably conjugate, $\mathcal{M}\otimes\mathscr{B}(L^2(A))$ and $\mathcal{N}\otimes\mathscr{B}(L^2(A))$ may be identified via an isomorphism θ conjugating $\alpha\otimes\varrho$ and $\beta\otimes\varrho$. The actions α and β will be said to have the same *inner invariant* if $\theta(1\otimes\int_A v_a\, da)=\sigma(1\otimes v_a\, da)$ for some σ commuting with $\alpha\otimes\beta$ (see § 3.2).

We now state the main theorem of this paper.

THEOREM 1. *Let α and β be two actions of A on the semifinite injective factors $\mathcal{M}$ and $\mathcal{N}$. Then*

(a) *α and β are stably conjugate iff the dual actions are cocycle conjugate.*

(b) *The dual actions $\hat{\alpha}$ and $\hat{\beta}$ are cocycle conjugate iff*

(i) *The crossed products are isomorphic*;

(ii) *$\hat{\alpha}$ and $\hat{\beta}$ have conjugate (ergodic) actions on $\mathscr{Z}(\mathcal{M}\rtimes_\alpha A)$ and $\mathscr{Z}(\mathcal{N}\rtimes_\beta A)$*;

(iii) *There exists an isomorphism θ of $\mathscr{Z}(\mathcal{M}\rtimes_\alpha A)$ onto $\mathscr{Z}(\mathcal{N}\rtimes_\beta A)$, which conjugates $\{\alpha, \mathscr{Z}(\mathcal{M}\rtimes_\alpha A)\}$ and $\{\beta, \mathscr{Z}(\mathcal{N}\rtimes_\beta A)\}$, such that $\theta(\chi_\alpha)=\chi_\beta$.*

(c) *Two stably conjugate actions are conjugate iff they have the same inner invariant.*

§ 1.2. Outline of the proof

The proof of Theorem 1 is somewhat involved and appeals to several results which are themselves rather difficult. We take this opportunity to outline the proof, stating clearly the main results which will be used.

The first assertion of Theorem 1 follows from the following theorem.

THEOREM 1.2.1 ([28]). *Let α be an action of the separable locally compact abelian group F on $\mathcal{M}$. Then $(\mathcal{M}\rtimes_\alpha F)\rtimes_{\hat{\alpha}} F$ is isomorphic to $\mathcal{M}\otimes\mathscr{B}(L^2(F))$ under an isomorphism which carries $\hat{\alpha}$ onto $\alpha\otimes\varrho$.*

The last assertion of Theorem 1 is almost obvious. Thus we are reduced to proving assertion (b).

The first thing to note is that by the following result $\mathcal{M}\rtimes_\alpha A$ is injective.

THEOREM 1.2.2 (Connes [5]). *Let $\mathcal{M}$ be an injective von Neumann algebra and G be a group with a countable amenable dense subgroup, acting on $\mathcal{M}$. Then the crossed product $\mathcal{M}\rtimes G$ is injective.*

Also by Theorem 1.2.1 we know that the action $\hat{\alpha}$ must leave invariant a fatihful normal semifinite trace on $\mathcal{M}\rtimes_\alpha A$ and that $\hat{\alpha}$ is ergodic on $\mathcal{Z}(\mathcal{M}\rtimes_\alpha A)$. Writing $\mathcal{M}\rtimes_\alpha A=\int^\oplus \mathcal{N}(x)\,d\mu(x)$ with $\mathcal{N}(x)$ factors, it follows that all the $\mathcal{N}(x)$ may be supposed semifinite. The next result shows that all the $\mathcal{N}(x)$ may be supposed injective.

THEOREM 1.2.3 (Connes [5]). *$\int^\oplus \mathcal{N}(x)\,d\mu(x)$ is injective iff μ-almost all algebras $\mathcal{N}(x)$ are injective. The only semifinite injective factors are $\mathcal{R}$, $\mathcal{R}_{0,1}$ and type I factors.*

The second half of Theorem 1.2.3 allows us to assert that $\mathcal{M}\rtimes_\alpha A=\mathcal{P}\otimes\mathcal{A}$ for an injective semifinite factor $\mathcal{P}$. By hypotheses (i) and (ii) of Theorem 1 we may, after a first conjugation of the form $\mathrm{id}\otimes\theta$, suppose that $\hat{\alpha}$ and $\hat{\beta}$ are equal on $\mathcal{Z}(\mathcal{M}\rtimes_\alpha A)$ and that $\mathcal{M}\rtimes_\alpha A=\mathcal{N}\rtimes_\beta A=\mathcal{P}\otimes\mathcal{A}$.

At this stage we may change our point of view and think of the actions $\hat{\alpha}$ and $\hat{\beta}$ as actions of the groupoids $G\times X$ (where $G=\hat{A}$, $\mathcal{A}=L^\infty(X,\mu)$) on the factor $\mathcal{P}$. If H is the kernel of the action of G on $\mathcal{Z}(\mathcal{M}\times_\alpha A)=\mathbf{C}\otimes\mathcal{A}$, the groupoid $(G/H)\times X$ is principal. In the measure-preserving case the next result was proved by Dye [11] and in the general case by Feldman and Lind, [18]:

THEOREM 1.2.4. *Let Q be a freely acting abelian countable discrete group of non-singular transformations of a standard measure space $\{X,\mu\}$. Then the principal measured groupoid $Q\times X$ is hyperfinite, or equivalently is generated by a single transformation.*

This result was further generalized recently to the case where Q is amenable by Connes, Feldman and Weiss [9].

From this result it also follows that the second cohomology of $(G/H)\times X$ vanishes (see [12]) so we may split the groupoid $G\times X$ as a product $H\times((G/H)\times X)$. This splits the problem as well. First one obtains a field of actions of H on the algebras $\mathcal{N}(x)$. By ergodicity they all have the same characteristic invariant, determined by those of $\hat{\alpha}$ and $\hat{\beta}$, so we may use the following result.

THEOREM 1.2.5 (Ocneanu [23]). *Let γ_1 and γ_2 be two approximately inner actions of an amenable discrete group H on $\mathcal{N}=\mathcal{R}$ or $\mathcal{R}_{0,1}$ then if $\gamma_1^{-1}(\mathrm{Int}\,\mathcal{N})=\gamma_2^{-1}(\mathrm{Int}\,\mathcal{N})$, γ_1 and γ_2 are cocycle conjugate iff they have the same characteristic invariant.*

Thus after a second conjugation we may suppose the fields of actions of H on $\mathcal{N}(x)$ to be all equal to the same action σ of H with given characteristic invariant.

The actions (coming from $\hat{\alpha}$ and $\hat{\beta}$) of the groupoid $H\times((G/H)\times X)$ now give rise to homomorphisms of the hyperfinite principal measured groupoid $(G/H)\times X$ into the group K of automorphisms of $\mathcal{N}$ commuting up to cocycles with σ. Given the appropriate topology this group is Polish and inner automorphisms are a normal dense Borel subgroup. To prove this density result one uses the following result.

THEOREM 1.2.6 (Ocneanu [23]). *If H is a countable amenable discrete group and $h\mapsto\gamma_h\in\mathrm{Aut}\,\mathcal{N}$ is a map which gives a free action on the algebra $\mathcal{N}_\omega$ of ω-centralizing sequences, then this action is stable, i.e. if $h\mapsto U_h\in\mathcal{U}(\mathcal{N}_\omega)$ is a cocycle then there is a $W\in\mathcal{U}(\mathcal{N}_\omega)$ such that $W^*\gamma_h(W)=U_h$.*

So we may apply the following theorem due to Bures, Connes, Krieger and Sutherland, a proof of which is given in the appendix.

THEOREM 1.2.7. *Let $\mathcal{G}$ be a hyperfinite measured groupoid, G a Polish group and ϱ_1, ϱ_2 be Borel homomorphisms of $\mathcal{G}$ into G such that*

$$\varrho_1\equiv\varrho_2 \mod \bar{H},$$

where H is a normal Borel subgroup of G and $\bar{H}$ means the closure of H. Then there exist Borel maps $h\colon\mathcal{G}\to H$ and $P\colon X=\mathcal{G}^{(0)}\to\bar{H}$ such that

$$\varrho_2(\gamma)=h(\gamma)P(r(\gamma))\varrho_1(\gamma)P(s(\gamma))^{-1}, \quad \gamma\in\mathcal{G},$$

where r and s denote the range and the source maps of $\mathcal{G}$ onto X, respectively.

After applying Theorem 1.2.7 to the two homomorphisms of $(G/H)\times X$ into K, and a further application of the vanishing 2-cohomology, we find that if π and ν are homomorphisms we may suppose $\pi(h,\gamma)=\mathrm{Ad}\,w_{h,\gamma}\nu(h,\gamma)$, where $w_{h,\gamma}$ are unitaries in $\mathcal{N}$ which are cocycles in each variable separately. We want $w_{h,\gamma}$ to be a cocycle for all of $H\times((G/H)\times X)$. There is one last obstruction in $H^1(H)\otimes H^1((G/H)\times X)$ which we remove by using a special model action. The cocycle property for $w_{h,\gamma}$ "integrates" to give the cocycle conjugacy of the actions $\hat{\alpha}$ and $\hat{\beta}$ and we are through.

§ 1.3. The meaning of the result

In what sense is Theorem 1 a classification of actions of compact abelian groups on injective semifinite factors? It is certainly not as complete a classification as that of [14] where, given a finite group, one can completely parametrize its actions on $\mathscr{R}$ by algorithmically calculable spaces. Indeed one of the invariants of our theorem is a non-singular ergodic action of a countable discrete group on a measurable space, defined up to conjugacy. All actions can occur, even the type III ones, but even in the case where the group is $\mathbf{Z}$ and the actions preserve a probability measure it is well known that there is no smooth parametrization of the actions. For more complicated abelian groups than $\mathbf{Z}$ the structure can only be worse.

If one accepts ergodic actions of $\hat{A}$ as parameters, then one has to work with the characteristic invariant first, which is at least as complicated as the first cohomology of an ergodic hyperfinite principal measured groupoid (see our Proposition 2.3.18)—known to be somewhat pathological (see [25]). Then one must find out the orbit structure of characteristic invariants under the action of the group S of automorphisms of the center $\mathscr{Z}=\mathscr{Z}(\mathcal{M}\rtimes_\alpha A)$ commuting with the restriction of $\hat{\alpha}$ to $\mathscr{Z}$. Very little is known about this space of characteristic orbits. Here a number of questions arise immediately. When does S act non-trivially on the space of characteristic invariants? Can the space of characteristic orbits be countable? For some examples see the end of § 2.1. We will pick up this topic elsewhere. Finally, the inner invariant is not particularly easy to tie down. For each stable conjugacy class one must determine the action of the group commuting with the second dual action on the space of projections equivalent (via automorphisms) to $1\otimes\int_A v_a\,da$ in $\mathcal{M}\otimes\mathscr{B}(L^2(A))$, which necessitates a different space for each conjugacy class. (Actually the situation here is not as hopeless as it might seem—the authors intend to say more about the inner invariant in a future publication.)

On the other hand, Theorem 1 does describe all actions of compact abelian groups on injective semifinite factors in terms of invariants which are drawn from ergodic theory and cohomology. So the situation is roughly the same as it is in the classification of injective type III_0 factors: the flow of weights is a complete but ill understood invariant from ergodic theory.

If other conditions (such as factoriality of the crossed product or ergodicity of the action) are imposed, the classifying spaces become much easier to handle than in the general case. We have described several such situations in § 3.2.

To conclude, Theorem 1 should be thought of as a structure theorem rather than as a classification in the sense of an enumeration of all possible actions.

Chapter 2. Discrete Abelian groups

§ 2.1. The characteristic invariant

Let G be a countable discrete abelian group, written multiplicatively, and N a subgroup of G. If A is a G-module on which N acts trivially, the group $\Lambda(G, N, A)$ is defined as follows, see [13, 14, 15]. Let $Z(G, N, A)$ be the abelian group of all pairs (λ, μ) where $\lambda: G \times N \mapsto A$ and $\mu: N \times N \mapsto A$ satisfying the conditions:

$$\mu(g, h) = 0 = \lambda(g, h) \quad \text{whenever } g \text{ or } h = 1 \text{ (normalization);} \tag{2.1.1}$$

$$\mu(f, h) + \mu(fh, k) = \mu(h, k) + \mu(f, hk); \tag{2.1.2}$$

$$\lambda(g, hk) - \lambda(g, h) - \lambda(g, k) = \mu(h, k) - g\mu(h, k); \tag{2.1.3}$$

$$\lambda(gg', h) = \lambda(g, h) + g\lambda(g', h); \tag{2.1.4}$$

$$\lambda(h, k) = \mu(h, k) - \mu(k, h), \tag{2.1.5}$$

for every $g, g' \in G$, $f, h, k \in N$. If $\sigma: N \mapsto A$ is any function with $\sigma(1)=0$, define the pair $\delta\sigma = (\delta_1 \sigma, \delta_2 \sigma)$ by

$$(\delta_1 \sigma)(g, h) = \sigma(h) - g\sigma(h), \quad g \in G,\ h \in N; \tag{2.1.6}$$

$$(\delta_2 \sigma)(h, k) = \sigma(hk) - \sigma(h) - \sigma(k), \quad h, k \in N. \tag{2.1.7}$$

It follows that $\delta\sigma \in Z(G, N, A)$. We define $B(G, N, A)$ to be the set of all such $\delta\sigma$'s, which is a subgroup of $Z(G, N, A)$. We set

$$\Lambda(G, N, A) = Z(G, N, A)/B(G, N, A).$$

Now suppose that $\mathcal{M}$ is a von Neumann algebra with center $\mathcal{A}$. If α is an action of G on $\mathcal{M}$, we set

$$N_\alpha = \{g \in G: \alpha_g \in \operatorname{Int}(\mathcal{M})\}. \tag{2.1.8}$$

Clearly, N_α acts trivially on the unitary group $\mathcal{U}(\mathcal{A})$ and we may consider the group $\Lambda(G, N_\alpha, \mathcal{U}(\mathcal{A}))$. The *characteristic invariant* χ_α is defined as the class in $\Lambda(G, N_\alpha, \mathcal{U}(\mathcal{A}))$ of the pair (λ, μ) defined by the following:

$$\alpha_h = \operatorname{Ad} u_h, \quad h \in N_\alpha,\ u_1 = 1; \tag{2.1.9}$$

$$\alpha_g(u_h) = \lambda(g, h)\, u_h, \quad g \in G,\ h \in N_\alpha; \tag{2.1.10}$$

$$u_h u_k = \mu(h, k)\, u_{hk}, \quad h, k \in N_\alpha. \tag{2.1.11}$$

(Note the changes from additive to multiplicative notations in the coefficient group.)

Remark 2.1.12. It is instructive to consider the case where $\mathcal{M}$ is a factor, i.e., $\mathcal{A}=\mathbf{C}$ and $\mathcal{U}(\mathcal{A})=\mathbf{T}$. The conditions (2.1.1)–(2.1.5) may be reduced to saying that $\lambda: G\times N\to\mathbf{T}$ is a bicharacter and μ is a normalized 2-cocycle with $\lambda(h,k)=\mu(h,k)\mu(k,h)^{-1}$, $h,k\in N_\alpha$. This relation in fact implies that λ determines the cohomology class of μ. For 2-cocycles μ, the antisymmetrization map determines an isomorphism between $H^2(N,\mathbf{T})$ and $\Lambda^2(N,\mathbf{T})$, the group of all antisymmetric bicharacters on $N\times N$, [14], [24]. In particular, if $\lambda|_{N\times N}\equiv 0$, then μ is a coboundary. The map: $[\lambda,\mu]\mapsto\lambda$ thus gives an isomorphism of $\Lambda(G,N,\mathbf{T})$ onto the group of all bicharacters on $G\times N$ whose restriction to $N\times N$ is antisymmetric. We shall think of elements of $\Lambda(G,N,\mathbf{T})$ as bicharacters via this isomorphism. An important conclusion of this observation is that $\Lambda(G,N,\mathbf{T})$ is a compact abelian group with respect to the pointwise convergence topology on $G\times N$.

A centrally ergodic action α of G on $\mathcal{M}$ means, by definition, that $\{\mathcal{A},G,\alpha\}$ is ergodic. In this case, the characteristic invariant χ_α determines the center of the crossed product $\mathcal{M}\rtimes_\alpha G$. To see this, introduce, for any $(\lambda,\mu)\in Z(G,N,\mathcal{U}(\mathcal{A}))$, the algebra $\mathcal{A}\rtimes_\mu N$, the twisted crossed product of $\mathcal{A}$ by N with respect to the trivial action of N and the 2 cocycle μ. A typical element may be written

$$x=\sum_{h\in N}x_h w_h,\quad x_h\in\mathcal{A},$$

where w_h and $\mathcal{A}$ commute and

$$w_h w_k=\mu(h,k)\,w_{hk},\quad h,k\in N.$$

Thus $\mathcal{A}\rtimes_\mu N$ is not abelian unless μ is a coboundary. Now, G acts on $\mathcal{A}\rtimes_\mu N$ by the following:

$$\sigma_g\left(\sum_{h\in N}a_h w_h\right)=\sum_{h\in N}\lambda(g,h)\,\alpha_g(a_h)\,w_h.$$

That the fixed point algebra $(\mathcal{A}\rtimes_\mu N)^\sigma$ for σ is isomorphic to the center of the crossed product $\mathcal{M}\rtimes_\alpha G$ follows from the next result:

PROPOSITION 2.1.13. *Let $\mathcal{P}=\mathcal{M}\rtimes_\alpha G$ and consider the dual action $\hat\alpha$ of $\hat G$.*

(i) *The dual action $\hat\alpha$ is centrally ergodic if and only if α is.*

Assume that α is centrally ergodic.

(ii) *The relative commutant* $\mathcal{M}' \cap \mathcal{P}$ *is anti-isomorphic to* $\mathcal{A} \rtimes_\mu N_\alpha$.

(iii) *The center of* $\mathcal{P}$ *is precisely the image of* $(\mathcal{A} \rtimes_\mu N_\alpha)^\sigma$ *under the anti-isomorphism in* (ii).

Thus, $\mathcal{P}$ *is a factor if and only if* α *is centrally ergodic and* σ *is ergodic.*

Proof. (i) It follows immediately from [28].

(ii) Let $y=\Sigma_{g\in G} a_g u_g \in \mathcal{R}$ be an element of the relative commutant $\mathcal{M}'$. Then $xa_g=\alpha_g(x)\,a_g$ for every $x\in\mathcal{M}$. But if $g\in G\setminus N_\alpha$, then α_g is free since G is abelian and the action is ergodic. Hence $y=\Sigma_{h\in N_\alpha} a_h v_h$. Now, choose $v_h\in\mathcal{U}(\mathcal{M})$ with $\alpha_h=\mathrm{Ad}(v_h)$ for $h\in N_\alpha$, and write

$$y=\sum_{h\in N_\alpha} b_h v_h^* u_h.$$

Since y commutes with $\mathcal{M}$, each b_h belongs to $\mathcal{A}$. But the algebra $\mathcal{B}$ of all such y's is anti-isomorphic to $\mathcal{A}\rtimes_\mu N_\alpha$. Conversely, every y of the above form commutes with $\mathcal{M}$.

(iii) In the previous arguments $\mathrm{Ad}\,u_g$ on $\mathcal{B}$ corresponds to the action σ_g on $\mathcal{A}\rtimes_\mu N_\alpha$. The center $\mathcal{C}$ of $\mathcal{P}$ is precisely $\{u_g\}'\cap\mathcal{B}$, which is the image of $(\mathcal{A}\rtimes_\mu N_\alpha)^\sigma$. Q.E.D.

Note that the conditions in Proposition 2.1.13 are determined entirely by $\{\mathcal{A}, G, \alpha\}$ and the corresponding characteristic invariant.

We can now state our main result and the rest of this chapter will be devoted to its proof.

THEOREM 2.1.14. *Let* α *and* β *be two centrally ergodic actions of a countable discrete abelian group* G, *preserving some faithful semifinite normal trace on a semifinite injective von Neumann algebra* $\mathcal{M}$. *Then* α *and* β *are cocycle conjugate if and only if*

(i) $\{\mathcal{A},\alpha\}$ *and* $\{\mathcal{A},\beta\}$ *are conjugate*;

(ii) $N_\alpha=N_\beta$ *and there exists an automorphism* θ *of* $\mathcal{A}$ *such that* $\theta\alpha_g\theta^{-1}=\beta_g$, $g\in G$, *on* $\mathcal{A}$ *and* $\theta(\chi_\alpha)=\chi_\beta$.

The necessity of these conditions is immediate. The sufficiency will be proved in §2.5.

Thus cocycle conjugacy classes of actions α of G with $N_\alpha=N$ and $\{\mathcal{A},\alpha\}$ fixed are parameterized by orbits of the action of $\mathrm{Aut}_G(\mathcal{A})$ on $\Lambda(G,N,\mathcal{U}(\mathcal{A}))$. But how does $\mathrm{Aut}_G(\mathcal{A})$ act? This is not an orbit equivalence invariant and must be determined for each conjugacy class of ergodic actions of G on $\mathscr{A}$. This is an invariant which has not been previously encountered, a fact which is explained by the following two results.

LEMMA 2.1.15. *Let A be a G-module on which N acts trivially. Since G is abelian, each element $g \in G$ belongs to $\mathrm{Aut}_G(A)$. Then the induced map $g^*: \Lambda(G, N, A) \mapsto \Lambda(G, N, A)$ is the identity.*

Proof. If $(\lambda, \mu) \in Z(G, N, A)$, we must show that $(\lambda - g\lambda, \mu - g\mu)$ is a coboundary. One easily checks that with $\sigma(g) = \lambda(g, h)$

$$(\delta_1 \sigma, \delta_2 \sigma) = (\lambda - g\lambda, \mu - g\mu). \qquad \text{Q.E.D.}$$

(This is a special case of a general result of the type "inner automorphisms do not act on cohomology" which could be established in the framework of [29].)

COROLLARY 2.1.16. *If $\mathscr{A} = L^\infty(X, \mu)$ and G acts transitively on X, then $\mathrm{Aut}_G(\mathscr{A})$ acts trivially on $\Lambda(G, N, \mathscr{U}(\mathscr{A}))$.*

Proof. This follows from the last lemma and the fact that $\mathrm{Aut}_G(\mathscr{A}) = \alpha(G)$. Q.E.D.

Thus, in the study of finite group actions on factors, the characteristic invariant of the dual action is a well-defined cocycle conjugacy invariant. This explains why the action of $\mathrm{Aut}_G(\mathscr{A})$ does not occur in [14].

But if the action of G on $\{X, \mu\}$ is properly ergodic, then the group $\mathrm{Aut}_G(\mathscr{A})$ may act non-trivially on $\Lambda(G, N, \mathscr{U}(\mathscr{A}))$ as we will show now. Ornstein's example in [30] shows that there are also cases where $\mathrm{Aut}_G(\mathscr{A})$ acts trivially on $\Lambda(G, N, \mathscr{U}(\mathscr{A}))$.

It will be established in the next section that an element of $\Lambda(G, N, \mathscr{U}(\mathscr{A}))$ has two parts, one of which is an element of $H^1(G/H, H^1(N, \mathscr{U}(\mathscr{A})))$, so it suffices to show that there is a non-trivial action on $H^1(G/H, H^1(N, \mathscr{U}(\mathscr{A})))$. For this, we choose $N = \mathbf{Z}$, so $H^1(N, \mathscr{U}(\mathscr{A})) = \mathscr{U}(\mathscr{A})$, and also choose $G/H = \mathbf{Z}$. Then we are in the simple situation of a single ergodic transformation T of a measure space $\{X, \mu\}$ and we want to find a transformation S on X commuting with T such that S acts on $H^1(\mathbf{Z}, \mathscr{U}(L^\infty(X, \mu)))$, where $\mathbf{Z}$ acts via T. Let $\{T, X, \mu\}$ be the irrational rotation transformation by angle θ on the unit circle X with the Lebesgue measure μ. Any transformation S commuting with T must be a rotation of angle φ. Let α (resp. β) be the automorphism of $\mathscr{A} = L^\infty(X, \mu)$ corresponding to T (resp. S). Let $\lambda \in \mathrm{Sp}(S)$. Then there exists $u \in \mathscr{U}(\mathscr{A})$ such that $\lambda = u^* \beta(u)$. Now, u and $\beta(u)$ are cohomologous in $Z(\mathbf{Z}, \mathscr{U}(\mathscr{A}))$ if and only if there exists

$v \in \mathcal{U}(\mathcal{A})$ such that $\beta(u)=v^*u\alpha(v)=uv^*\alpha(u)$, which is then equivalent to saying that $\lambda=u^*\beta(u)=v^*\alpha(u)$. This means that β acts trivially only if $\mathrm{Sp}(\beta)\subset\mathrm{Sp}(\alpha)$. But we have

$$\mathrm{Sp}(\alpha)=\{e^{2\pi i n\theta}: n\in\mathbf{Z}\} \quad \text{and} \quad \mathrm{Sp}(\beta)=\{e^{2\pi i n\varphi}: n\in\mathbf{Z}\}.$$

Thus, $\mathrm{Sp}(\beta)\subset\mathrm{Sp}(\alpha)$ implies that $\beta=\alpha^n$ for some n. Therefore, we conclude that for the irrational rotation system $\{T, X, \mu\}$, an element β of $\mathrm{Aut}_\alpha(\mathcal{A})$ acts on $\Lambda(G, N, \mathcal{U}(\mathcal{A}))$ trivially if and only if $\beta=\alpha^n$ for some n.

In general, we do not know the size of the space $\Lambda(G, N, \mathcal{U}(\mathcal{A}))/\mathrm{Aut}_G(\mathcal{U}(\mathcal{A}))$ of characteristic orbits. It is conceivable at this stage that there may be examples where there are only countably many orbits although $\Lambda(G, N, \mathcal{U}(\mathcal{A}))$ is uncountable. (1)

§ 2.2. The groupoid approach

Let $H=\ker(\alpha|_{\mathcal{A}})$ and $K=G/H$. The commutativity of G together with ergodicity implies that the action α gives rise naturally to a free ergodic action of K on $\mathcal{A}$. The injectivity of $\mathcal{M}$ together with ergodicity implies the decomposition of $\mathcal{M}$:

$$\mathcal{M}=\mathcal{N}\otimes\mathcal{A}$$

with $\mathcal{N}$ a semi-finite injective factor, [5]. Representing $\mathcal{A}$ on a standard measure space $\{X, \mu\}$ as $\mathcal{A}=L^\infty(X, \mu)$, one can "split" the action α of G on $\mathcal{M}$ as a field of actions of H on $\mathcal{N}$ and a free ergodic action of K on $\{X, \mu\}$. Since G need not be the direct sum of H and K, this type of decomposition of α does not make complete sense. However, the commutativity of K implies the hyperfiniteness of the ergodic transformation group $\{\mathcal{A}, K, \alpha\}$ [11], [18], which yields a principal hyperfinite measured groupoid. Furthermore, our problem is by its nature cohomological. It is well-known that the cohomological behavior of ergodic transformation groups depends only on the orbit structure of the system, i.e. the measured groupoid. These general observations motivate us to look at the groupoid associated to the system $\{\mathcal{A}, G, \alpha\}$. As will be seen in the next section, the above "splitting" of the action α becomes real on the groupoid level, which enables us to handle successfully the linking of "H-part" and "K-part" in the later sections.

Let us begin by recalling some of the definitions from the theory of measured groupoids [8], [20]. Since we handle only those measured groupoids coming from

(1) K. Schmidt showed in a private conversation that this never happen. Namely $\Lambda(G, N\mathcal{U}(\mathcal{A}))/\mathrm{Aut}_G(\mathcal{U}(\mathcal{A})$ is not countably separated if $\Lambda(G, N, \mathcal{U}(\mathcal{A}))$ is not.

ergodic actions of a countable discrete group on a standard measure space, we restrict ourselves to the following type of measured groupoids.

$$\mathcal{G} \text{ is a small category with inverses where the source and range maps are denoted by } s \text{ and } r; \tag{2.2.1}$$

$$\mathcal{G} \text{ is equipped with a standard Borel structure with respect to which all relevant maps and sets are Borel;} \tag{2.2.2}$$

$$\text{The objects } X=\mathcal{G}^{(0)} \text{ carry a } \sigma\text{-finite Borel measure } \mu. \tag{2.2.3}$$

$$\text{For each } x \in X,\ r^{-1}(x)=\mathcal{G}^x \text{ is countable;} \tag{2.2.4}$$

$$\text{The measure } m \text{ on } \mathcal{G} \text{ obtained by integrating the counting measure on } \mathcal{G}^x \text{ with respect to } \mu \text{ is quasi-invariant under the map: } \gamma \in \mathcal{G} \mapsto \gamma^{-1} \in \mathcal{G}. \tag{2.2.5}$$

When $\gamma \in \mathcal{G}$ has $s(\gamma)=x$ and $r(\gamma)=y$, we will write this as $\gamma: x \to y$.

When we say that a statement is true almost everywhere in $\mathcal{G}$, we refer to the measure m on $\mathcal{G}$. By the countability condition for $\mathcal{G}^x$, (2.2.4), this means that one can find a saturated Borel conull set X' in X such that the statement is true for every $\gamma \in \mathcal{G}'=r^{-1}(X')=s^{-1}(X')$. We shall then work with this smaller measured groupoid $\mathcal{G}'$ instead of $\mathcal{G}$, which will not affect the rest of the discussion. We shall make use of this replacement freely without any further comment whenever almost everywhere equations arise in $\mathcal{G}$.

Definition 2.2.6. An *action* α of $\mathcal{G}$ on a von Neumann algebra $\mathcal{N}$ is a Borel homomorphism: $\gamma \in \mathcal{G} \mapsto \alpha_\gamma \in \mathrm{Aut}(\mathcal{N})$ from $\mathcal{G}$ into $\mathrm{Aut}(\mathcal{N})$, i.e., $\alpha_{\gamma_1\gamma_2}=\alpha_{\gamma_1}\alpha_{\gamma_2}$ whenever $s(\gamma_1)=r(\gamma_2)$. Two actions α and β of $\mathcal{G}$ on $\mathcal{N}$ are called *conjugate* if there exists a Borel map $\theta: x \in X \mapsto \theta_x \in \mathrm{Aut}(\mathcal{N})$ such that

$$\alpha_\gamma = \theta_{r(\gamma)}^{-1}\beta_\gamma\theta_{s(\gamma)}$$

for almost every $\gamma \in \mathcal{G}$.

Remark. We could have used the terminology "cocycle" and "cohomologous" for action and conjugate, respectively. We use these words because of our intention to associate actions of groupoids with actions of groups. Hence the following:

Definition 2.2.7. Let α be an action of $\mathcal{G}$ on $\mathcal{N}$. A *unitary cocycle* for α is a Borel map $u: \gamma \in \mathcal{G} \mapsto u_\gamma \in \mathcal{U}(\mathcal{N})$ such that

$$u_{\gamma_1\gamma_2} = u_{\gamma_1} \alpha_{\gamma_1}(u_{\gamma_2})$$

whenever $s(\gamma_1)=r(\gamma_2)$. If this is the case, the map: $\gamma \in \mathcal{G} \mapsto \mathrm{Ad}(u_\gamma) \cdot \alpha_\gamma$ defines a new action of $\mathcal{G}$ on $\mathcal{N}$, called the *perturbed* action of α by u. Two actions α and β of $\mathcal{G}$ on $\mathcal{N}$ are *cocyle conjugate* if there is a unitary cocycle u for α such that β is conjugate to the perturbed action of α by u.

Now, if $\mathcal{N}$ is a factor, let

$$\mathcal{M} = \mathcal{N} \otimes \mathcal{A}, \quad \mathcal{A} = L^\infty(X, \mu),$$

with $\{X, \mu\}$ a standard measure space. Let α be an action of a countable discrete abelian group G on $\mathcal{M}$. The restriction of α to $\mathcal{A}$ gives rise to an action of G on $\{X, \mu\}$ as a non-singular transformation group such that

$$(\alpha_g f)(x) = f(g^{-1}x), \quad f \in \mathcal{A},\ g \in G,\ x \in X.$$

We assume ergodicity for $\{\mathcal{A}, G, \alpha\}$.

Definition 2.2.8. The *auxiliary groupoid* for the action α is the groupoid $\mathcal{G}_\alpha = G \times X$ where $\mathcal{G}_\alpha^{(0)} = X$, $s(g,x)=x$, $r(g,x)=gx$ and $(g,hx)(h,x)=(gh,x)$, $g,h \in G$, $x \in X$. The Borel structure in $\mathcal{G}_\alpha$ is the product Borel structure and the measure on $\mathcal{G}$ is the product of the counting measure on G and the original measure μ on X. The *auxiliary action*, modulo G-automorphisms of $\{X, \mu\}$ of $\mathcal{G}_\alpha$ on $\mathcal{N}$ is defined by:

$$\alpha_{(g,x)}(a(x)) = (\alpha_g(a))(gx), \quad a \in \mathcal{M} = L^\infty(X, \mathcal{N}, \mu). \tag{2.2.9}$$

By changing on sets of measure zero, we may assume that

$$\alpha_{(g,hx)} \cdot \alpha_{(h,x)} = \alpha_{(gh,x)}, \quad g,h \in G,\ x \in X.$$

Properties of actions can now be phrased in terms of properties of their auxiliary actions. We establish a dictionary:

PROPOSITION 2.2.10. *Let α and β be two actions of G on $\mathcal{M}$ such that $\{\mathcal{A}, G, \alpha\} = \{\mathcal{A}, G, \beta\}$. The auxiliary groupoids may then be identified and the actions are:*

(i) *Conjugate under* $\mathrm{Aut}(\mathcal{M}/\mathcal{A})$, *the group of center fixing automorphisms, if and only if their auxiliary actions are,*

(ii) *Cocycle conjugate under* $\mathrm{Aut}(\mathcal{M}/\mathcal{A})$ *if and only if their auxiliary actions are.*

Proof. (i) Suppose the two actions are conjugate under $\mathrm{Aut}(\mathcal{M}/\mathcal{A})$, say $\theta\alpha_g\theta^{-1}=\beta_g$. We may suppose that θ is of the form:

$$\theta(a)(x)=\theta_x(a(x)), \quad a\in\mathcal{M},$$

with $x\in X\to\theta_x\in\mathrm{Aut}(\mathcal{N})$ a Borel map. Writing out $\theta\alpha_g\theta^{-1}=\beta_g$ in terms of the auxiliary actions, one sees that they are conjugate.

The converse is now clear.

(ii) If α and β are cocycle conjugate, say $\mathrm{Ad}(u_g)\cdot\alpha_g=\theta\beta_g\theta^{-1}$ for some cocycle u for α and $\theta\in\mathrm{Aut}(\mathcal{M}/\mathcal{A})$, then we set $u_\gamma=u_g(x)$ for $\gamma=(g,x)\in\mathcal{G}_\alpha$. The cocycle identity for $\{u_g\}$ implies (2.2.7) for $\{u_\gamma\}$ for almost every γ. The cocycle conjugacy of the auxiliary actions follows immediately. The converse is also easy. Q.E.D.

Thus, actions of groups may be classified by classifying actions of groupoids. This gives a heuristic proof that the characteristic invariant space depends only on the groupoind. This will be taken up in the next section. For the moment, we want to show one great advantage of the groupoid approach, on which we commented in Chapter 1. In general, if G is an abelian group and H is a subgroup, we don't have a splitting $G=H\times G/H$. But if β is an ergodic action of G on a measure space $\{X,\mu\}$ (1), and $H=\ker\beta$, we *can* write the groupoid $\mathcal{G}_\beta=G\times X$ as $H\times(G/H\times X)$, and now $K=G/H$ acts freely on X, so the groupoid $\mathcal{K}=K\times X$, $K=G/H$, is a principal groupoid, i.e. the graph of an equivalence relation.

LEMMA 2.2.11 (Splitting lemma for groupoids). *Let β be an ergodic action of a countable discrete abelian group G on a measure space $\{X,\mu\}$. If $H=\ker\beta$ and $K=G/H$, then there is an isomorphism of measured groupoids:*

$$M\colon \mathcal{G}_\beta=G\times X\mapsto H\times(K\times X),$$

where K acts on X freely via β and

$$M(h,x)=(h,1,x), \quad h\in H,\ x\in X.$$

Furthermore, $K\times X$ is a principal ergodic hyperfinite measured groupoid, henceforth written $\mathcal{K}$.

Proof. That $K=G/H$ acts freely on X follows from ergodicity and commutativity. Thus we identify $K\times X=\mathcal{K}$ with the equivalence relation induced on X by β. The map

(1) We mean by a measure space always a standard measure space.

$\pi: (g, x) \in \mathcal{G}_\beta = G \times X \mapsto (\dot{g}, x) \in \mathcal{K} = K \times X$, with $\dot{g} = gH \in K$, is a Borel groupoid homomorphism with "kernel" H. We shall construct a section for it.

First, let $k \in K \mapsto g(k) \in G$ be a section of the quotient map: $g \in G \mapsto \dot{g} \in K = G/H$. For each $\gamma = (k, x) \in \mathcal{K}$ set $g_\gamma = g(k) \in G$ and

$$h(\gamma_1, \gamma_2) = g_{\gamma_1} g_{\gamma_2} g_{\gamma_1\gamma_2}^{-1},$$

for every composable pair $(\gamma_1, \gamma_2) \in \mathcal{K}^{(2)}$. Writing $\gamma_1 = (k_1, k_2 x)$ and $\gamma_2 = (k_2, x)$, we have

$$h(\gamma_1, \gamma_2) = g(k_1)\, g(k_2)\, g(k_1 k_2)^{-1} \in H.$$

Therefore, the function $h: \mathcal{K}^{(2)} \mapsto H$ is a measurable 2-cocycle on the hyperfinite ergodic principal groupoid $\mathcal{K}$, whose second cohomology is trivial for any coefficient group, [12]. Hence h is a coboundary, i.e. there is a Borel function $k: \gamma \in \mathcal{K} \mapsto k_\gamma \in H$ such that

$$h(\gamma_1, \gamma_2) = k_{\gamma_1} k_{\gamma_2} k_{\gamma_1\gamma_2}^{-1}, \quad \text{a.e. } (\gamma_1, \gamma_2) \in \mathcal{K}^{(2)}.$$

We then set $\bar{g}_\gamma = g_\gamma k_\gamma^{-1}$, $\gamma \in \mathcal{K}$, and obtain a Borel homomorphism: $\gamma \in \mathcal{K} \mapsto \bar{g}_\gamma \in G$ such that

$$\pi(\bar{g}_\gamma, s(\gamma)) = \gamma, \quad \gamma \in \mathcal{K}.$$

Now, the map M defined by

$$M(g, x) = (g\bar{g}_\gamma^{-1}, \gamma) \in H \times \mathcal{K}, \quad \gamma = \pi(g, x),$$

is an isomorphism of $\mathcal{G}_\beta$ onto $H \times \mathcal{K}$.

It is easy to show that this map M maps the measure class of $\mathcal{G}$ into the class of the product measure on $H \times \mathcal{K}$. Q.E.D.

In the principal groupoid $\mathcal{K} = K \times X$, a K-automorphism of $\mathcal{K}$ means, by definition, a non-singular transformation T of $\{X, \mu\}$ commuting with the action of K. Let $\mathrm{Aut}_K(\mathcal{K})$ denote the group of K-automorphisms of $\mathcal{K}$. Now, putting the previous discussion, Proposition 2.2.10 and Lemma 2.2.11, together, we have proved the following:

THEOREM 2.2.12. *Every centrally ergodic action of G on $\mathcal{M} = \mathcal{N} \otimes \mathcal{A}$ determines, up to $\mathrm{Aut}_K(\mathcal{K})$, an action of $H \times \mathcal{K}$ on $\mathcal{N}$. Two actions α and β of G with the same action on $\mathcal{A}$ are conjugate (resp. cocycle conjugate) if and only if the corresponding actions $\bar{\alpha}$ and $\bar{\beta}$ of $H \times \mathcal{K}$ are also up to the action of $\mathrm{Aut}_K(\mathcal{K})$ on $H \times \mathcal{K}$.*

Note that $X = \mathcal{K}^{(0)}$ can be either $\{1, 2, \ldots, n\}$ for $n = 1, 2, \ldots, \infty$ or $[0, 1]$.

Let us not forget that the groupoid $\mathcal{K}$ itself depends on the actions of G on the center $\mathcal{A}$. All hyperfinite ergodic principal groupoids, including type III cases, occur from trace preserving actions of G as will be shown in the next section.

§ 2.3. The characteristic invariant for groupoids

We saw in the last section that centrally ergodic actions of groups correspond to actions of groupoids on factors. This implies that the cohomological formalism for the characteristic invariant in the groupoid language must be worked out.

Let $\mathcal{G}$ be a measured groupoid described by (2.2.1)–(2.2.5). We assume ergodicity for $\mathcal{G}$, i.e. all saturated subsets in $X=\mathcal{G}^{(0)}$ are either null or conull. A Borel subgroupoid $\mathfrak{N}$ of $\mathcal{G}$ is said to be normal if

$$\text{The set } \mathfrak{N}^{(0)} \text{ of units of } \mathfrak{N} \text{ coincides with } X; \tag{2.3.1}$$

$$s(\eta)=r(\eta) \text{ for every } \eta\in\mathfrak{N}; \tag{2.3.2}$$

$$\text{For every pair } (\gamma,\eta)\in\mathcal{G}\times\mathfrak{N}\cap\mathcal{G}^{(2)},\ \gamma\eta\gamma^{-1}\in\mathfrak{N}. \tag{2.3.3}$$

The first condition (2.3.1) follows from (2.3.2) and (2.3.3) by the ergodicity of $\mathcal{G}$. Once again, we will freely alter sets of measure zero.

A normal subgroupoid $\mathfrak{N}$ of $\mathcal{G}$ is then nothing but a measurable field: $x\in X\mapsto N(x)\subset\mathcal{G}_x^x=\{\gamma\in\mathcal{G}: x=s(\gamma)=r(\gamma)\}$ of subgroups such that

$$\gamma N(x)\gamma^{-1}=N(y),\quad \gamma: x\to y.$$

We fix a normal subgroupoid $\mathfrak{N}$ of $\mathcal{G}$ and a Polish abelian group A, written additively. We denote by $Z(\mathcal{G},\mathfrak{N},A)$ the abelian group of all classes, modulo null sets, of pairs (λ,μ) of Borel functions, $\lambda:\{(\gamma,\eta)\in\mathcal{G}\times\mathfrak{N}:(\eta,\gamma)\in\mathcal{G}^{(2)}\}\to A$ and $\mu:\mathfrak{N}^{(2)}\to A$ such that

$$\lambda(\gamma,\eta)=0=\mu(\gamma,\eta)\quad \text{if either } \gamma \text{ or } \eta\in\mathcal{G}^{(0)}; \tag{2.3.4}$$

$$\mu(\eta_1,\eta_2)+\mu(\eta_1\eta_2,\eta_3)=\mu(\eta_1,\eta_2\eta_3)+\mu(\eta_2,\eta_3),\quad (\eta_1,\eta_2,\eta_3)\in\mathfrak{N}^{(3)}; \tag{2.3.5}$$

$$\begin{gathered}\lambda(\gamma,\eta_1\eta_2)-\lambda(\gamma,\eta_1)-\lambda(\gamma,\eta_2)=\mu(\eta_1,\eta_2)-\mu(\gamma^{-1}\eta_1\gamma,\gamma^{-1}\eta_2\gamma),\\ (\eta_1,\eta_2)\in\mathfrak{N}^{(2)} \text{ and } (\eta_1,\gamma)\in\mathcal{G}^{(2)};\end{gathered} \tag{2.3.6}$$

$$\lambda(\gamma_1\gamma_2\eta)=\lambda(\gamma_1,\eta)+\lambda(\gamma_2,\gamma_1^{-1}\eta\gamma_1),\quad (\gamma_1,\gamma_2)\in\mathcal{G}^{(2)},\ (\eta,\gamma_1)\in\mathcal{G}^{(2)}; \tag{2.3.7}$$

$$\lambda(\eta_1,\eta_2)=\mu(\eta_1,\eta_1^{-1}\eta_2\eta_1)-\mu(\eta_2,\eta_1),\quad (\eta_1,\eta_2)\in\mathfrak{N}^{(2)}. \tag{2.3.8}$$

If $\sigma:\mathfrak{R}\to A$ is a Borel function with $\sigma(X)=\{0\}$, we define

$$\begin{cases}(\delta_1\sigma)(\gamma,\eta)=\sigma(\eta)-\sigma(\gamma^{-1}\eta\gamma);\\ (\delta_2\sigma)(\eta_1,\eta_2)=\sigma(\eta_1\eta_2)-\sigma(\eta_1)-\sigma(\eta_2).\end{cases}\tag{2.3.9}$$

Then the pair $\delta\sigma=(\delta_1\sigma,\delta_2\sigma)$ satisfies (2.3.4)–(2.3.8) and we let $B(\mathcal{G},\mathfrak{R},A)$ be the image $\{\delta\sigma\}$ in $Z(\mathcal{G},\mathfrak{R},A)$. (Note that (2.3.4)–(2.3.8) are formally identical to (1.2.1)–(1.2.5) of [14].) We then define

$$\Lambda(\mathcal{G},\mathfrak{R},A)=Z(\mathcal{G},\mathfrak{R},A)/B(\mathcal{G},\mathfrak{R},A).$$

Since it is tedious to check all the formulas (2.3.4)–(2.3.8), it is desirable to identify a member of $\Lambda(\mathcal{G},\mathfrak{R},A)$ with a mathematical object which is more conceptually manageable. Let $(\lambda,\mu)\in Z(\mathcal{G},\mathfrak{R},A)$. Consider $\tilde{\mathfrak{R}}=A\times\mathfrak{R}$ with the cartesian product Borel structure. Let

$$\tilde{\mathfrak{R}}^{(2)}=\{((a,\eta),(b,\zeta)):(a,b)\in A\times A,\ (\eta,\zeta)\in\mathfrak{R}^{(2)}\}$$

and set

$$(a,\eta)(b,\zeta)=(a+b+\mu(\eta,\zeta),\eta\zeta),\quad (\eta,\zeta)\in\mathfrak{R}^{(2)},\ (a,b)\in A^2.\tag{2.3.10}$$

We then obtain an exact sequence of standard Borel groupoids:

$$X\to A\times X\to\tilde{\mathfrak{R}}\to\mathfrak{R}\to X.\tag{2.3.11}$$

Furthermore, the above exact sequence can be viewed as a functor from the category of $\mathfrak{R}$ with Ad as morphisms into the measurable category of exact sequences:

$$0\to A\to\tilde{N}(x)\to N(x)\to 1,\quad x\in X.$$

The function λ then gives rise to a functor from $\mathcal{G}$ to $\tilde{\mathfrak{R}}$ which extends the above functor from $\mathfrak{R}$ to $\tilde{\mathfrak{R}}$ by the following:

$$\alpha_\gamma((a,\gamma^{-1}\eta\gamma))=(a+\lambda(\gamma,\eta),\eta),\quad (\eta,\gamma)\in\mathcal{G}^{(2)}.\tag{2.3.12}$$

Conversely, any measurable functor from $\mathcal{G}$ to an exact sequence of standard Borel groupoids $\tilde{\mathfrak{R}}$, (2.3.11), extedning the natural functor from $\mathfrak{R}$ to $\tilde{\mathfrak{R}}$, corresponds uniquely to the cohomology class of (λ,μ) in $\Lambda(\mathcal{G},\mathfrak{R},A)$. Note that this corresponds to the "crossed module formulation" in [13], [15].

PROPOSITION 2.3.13. *Let β be an ergodic action of the countable discrete abelian group G on $\mathcal{A}=L^\infty(X,\mu)$ coming from a non-singular action of G on X: $x\in X\mapsto gx\in X$*

and $N \subset H = \ker\beta$. Let $\mathcal{G} = G \times X$ be the auxiliary groupoid and $\mathfrak{N} = N \times X \subset \mathcal{G}$. Then $\mathfrak{N}$ is a normal subgroupoid of $\mathcal{G}$ and there is a natural isomorphism between $\Lambda(\mathcal{G}, \mathfrak{N}, \mathbf{T})$ and $\Lambda(G, N, \mathcal{U}(\mathcal{A}))$.

Proof. Given $\lambda: G \times N \to \mathcal{U}(\mathcal{A})$, and $\mu: N \times N \to \mathcal{U}(\mathcal{A})$ with $(\lambda, \mu) \in Z(G, N, \mathcal{U}(\mathcal{A}))$, we represent them by Borel functions $\lambda(g, h, \cdot)$ and $\mu(h, k, \cdot)$ on $X = \mathcal{G}^{(0)}$ with values in $\mathbf{T}$. For each $\gamma = (g, g^{-1}x)$ and $\eta = (h, x)$, we set $\lambda(\gamma, \eta) = \lambda(g, h, x)$, and $\mu(\eta_1, \eta_2) = \mu(h_1, h_2, x)$ for $\eta_1 = (h_1, x)$ and $\eta_2 = (h_2, x)$. Then (2.3.4)–(2.3.9) follow immediately almost everywhere from (2.1.1)–(2.1.5). The groups $B(G, N, \mathcal{U}(\mathcal{A}))$ and $B(\mathcal{G}, \mathfrak{N}, \mathbf{T})$ correspond in a similar way. The rest of the discussion is just routine, so we leave it to the reader. Q.E.D

Let $\mathcal{N}$ be a factor. The group $\mathrm{Aut}(\mathcal{N})$ is a Polish group with respect to the pointwise convergence topology on the predual. The unitary group $\mathcal{U}(\mathcal{N})$ is also a Polish group with respect to the strong operator topology. The map: $u \in \mathcal{U}(\mathcal{N}) \mapsto \mathrm{Ad}(u) \in \mathrm{Int}(\mathcal{N})$ is a continuous homomorphism with kernel $\mathbf{T}$. The quotient group $\mathcal{U}(\mathcal{N})/\mathbf{T}$ is then a Polish group which is continuously and bijectively mapped to $\mathrm{Int}(\mathcal{N})$. Therefore, $\mathrm{Int}(\mathcal{N})$ is a Borel subgroup of $\mathrm{Aut}(\mathcal{N})$ even if it is not closed. By definition, an action of $\mathcal{G}$ on $\mathcal{N}$ is a Borel homomorphism α of $\mathcal{G}$ into $\mathrm{Aut}(\mathcal{N})$. We set $\mathfrak{N}_\alpha = \alpha^{-1}(\mathrm{Int}\,\mathcal{N}) \cap \{\gamma \in \mathcal{G}: s(\gamma) = r(\gamma)\}$. It follows that $\mathfrak{N}_\alpha$ is a normal subgroupoid of $\mathcal{G}$. Since the quotient map: $u \in \mathcal{U}(\mathcal{N}) \mapsto \dot{u} \in \mathcal{U}(\mathcal{N})/\mathbf{T}$ admits a Borel cross-section, we may choose a Borel function: $\eta \in \mathfrak{N}_\alpha \mapsto u_\eta \in \mathcal{U}(\mathfrak{N})$ such that

$$\alpha_\eta = \mathrm{Ad}(u_\eta), \quad \eta \in \mathfrak{N}, \quad u_\eta = 1 \text{ if } \eta \in \mathfrak{N}^{(0)}.$$

the characteristic invariant χ_α is defined as the class in $\Lambda(\mathcal{G}, \mathfrak{N}_\alpha, \mathbf{T})$ of the pair (λ, μ) determined by the following:

$$\alpha_\gamma(u_{\gamma^{-1}\eta\gamma}) = \lambda(\gamma, \eta)\, u_\eta, \quad (\eta, \gamma) \in \mathcal{G}^{(2)}; \tag{2.3.14}$$

$$u_{\eta_1} u_{\eta_2} = \mu(\eta_1, \eta_2)\, u_{\eta_1\eta_2}, \quad (\eta_1, \eta_2) \in \mathfrak{N}^{(2)}. \tag{2.3.15}$$

PROPOSITION 2.3.16. *Let $\mathcal{N}$ be a factor and $\mathcal{M} = \mathcal{N} \otimes \mathcal{A}$ with $\mathcal{A} = L^\infty(X, \mu)$. If $\alpha: G \mapsto \mathrm{Aut}(\mathcal{M})$ is a centrally ergodic action of the abelian group G, then the characteristic invariant χ_α of α corresponds to the characteristic invariant of the auxiliary action of $\mathcal{G}_\alpha$ under the isomorphism of* (2.3.13).

Proof. Note first that $\mathfrak{N}_\alpha = N_\alpha \times X$. A choice u_h with $\mathrm{Ad}\, u_h = \alpha_h$, $h \in N_\alpha$, gives a choice u_η for $\eta \in \mathfrak{N}_\alpha$. The rest is just a formal calculation, which we leave to the reader. Q.E.D.

The following result corresponds to the splitting lemma for groupoids at the cohomology level.

LEMMA 2.3.17 (Cohomolgy splitting lemma). *If $\mathcal{K}$ is an ergodic principal measured groupoid, H is a countable discrete abelian group, N a subgroup and if $\mathcal{G}=H\times\mathcal{K}$ and $\mathfrak{N}=N\times X$ with $X=\mathcal{K}^{(0)}$, then there exists an isomorphism:*

$$H^1(\mathcal{K},\hat{N})\oplus\Lambda(H,N,\mathbf{T})\cong\Lambda(\mathcal{G},\mathfrak{N},\mathbf{T}),$$

which is given on cocycles by

$$\varepsilon\oplus\lambda\mapsto(\tilde{\lambda},\tilde{\mu})\in Z(\mathcal{G},\mathfrak{N},\mathbf{T}),$$

where $\varepsilon\colon\mathcal{K}\mapsto\hat{N}$ is a homomorphism, i.e. a cocycle, $\lambda\in\Lambda(H,N,\mathbf{T})$, (see Remark 2.1.12), and

$$\tilde{\lambda}((h,\gamma),(k,y))=\langle\varepsilon(\gamma),k\rangle\,\lambda(h,k),\quad y=r(\gamma),\ h\in H,\ k\in N;$$

$$\tilde{\mu}((h,x),(k,x))=\mu(h,k),\quad h,k\in N,\ x\in X,$$

μ being determined by λ as in Remark 2.1.12.

Proof. Each $\lambda\in Z(H,N,\mathbf{T})$ determines a group extension:

$$1\to\mathbf{T}\to\tilde{N}\to N\to 1 \tag{*}$$

such that if $k\in N\mapsto u(k)\in\tilde{N}$ is a cross-section of the above extension, then $u(k_1)\,u(k_2)=\mu(k_1,k_2)\,u(k_1k_2)$ and $\lambda(k_1,k_2)=\mu(k_1,k_2)\,\mu(k_2,k_1)^{-1}$. Furthermore, λ specifies an action of H on $\tilde{N}$ by $h\cdot u(k)=\lambda(h,k)\,u(k)$, $h\in H$, $k\in N$. The cohomology class $[\lambda]$ of λ in $\Lambda(H,N,\mathbf{T})$ is in one-to-one correspondence, up to conjugacy equivalence, with the above extension (*) equipped with an action of H, which extends the natural action of N on $\tilde{N}$.

Now, let $\varepsilon\in H^1(\mathcal{K},\hat{N})$ and $\lambda\in\Lambda(H,N,\mathbf{T})$. We then have a short exact sequence (*). We let $\mathcal{G}$ act on (*) as follows:

$$\alpha_{(h,\gamma)}(u(k))=\langle\varepsilon(\gamma),k\rangle\,\lambda(h,k)\,u(k).$$

Viewing $\tilde{N}(x)=\tilde{N}$ and $N(x)=N$, $x\in X$, we see immediately that $x\in X\mapsto\tilde{N}(x)$ and $(h,\gamma)\in H\times\mathcal{K}\mapsto\alpha_{(h,\gamma)}$ is a functor; hence it gives a member $(\tilde{\lambda},\tilde{\mu})$ of $\Lambda(\mathcal{G},\mathfrak{N},\mathbf{T})$.

Conversely, suppose $(\tilde{\lambda},\tilde{\mu})\in Z(\mathcal{G},\mathfrak{N},\mathbf{T})$. We then have an exact sequence of standard Borel groupoids associated with $(\tilde{\lambda},\tilde{\mu})$:

$$X\to\mathbf{T}\times X\to\tilde{\mathfrak{N}}\to\mathfrak{N}\to X,$$

on which $\mathscr{G}$ acts, extending the natural action of $\mathfrak{N}$ on $\tilde{\mathfrak{N}}$. Since $\mathfrak{N}=N\times\mathscr{K}$, the above exact sequence can be viewed as a measurable field of exact sequences:

$$1\to\mathbf{T}\to\tilde{N}(x)\to N\to 1,$$

on which the groupoid $\mathscr{G}$ acts. If $\gamma\in\mathscr{K}: x\mapsto y$, then we have a commutative diagram:

$$\begin{tikzcd}
& & \tilde{N}(x) \arrow[dd, "\alpha_{(1,\gamma)}"] \arrow[dr] & & \\
1 \arrow[r] & \mathbf{T} \arrow[ur] \arrow[dr] & & N \arrow[r] & 1. \\
& & \tilde{N}(y) \arrow[ur] & &
\end{tikzcd}$$

Therefore, the cohomology class $[\mu_x]$ of $\{\tilde{\mu}((h,x),(k,x)): h,k\in N\}$ and $[\mu_y]$ of $\{\tilde{\mu}((h,y),(k,y)): h,k\in N\}$ must agree whenever $x\sim y$. Thus the associated antisymmetric bicharacter $\{\lambda_x\}$ on $N\times N$ is constant on $\mathscr{K}$-orbits in X. By ergodicity, one has $\lambda_x=\lambda_y$ for every $x,y\in X$. Let λ be this common bicharacter on $N\times N$ and μ be a 2-cocycle on N whose corresponding bicharacter is λ. Since $\mu_x^{-1}\mu\in B^2(N,\mathbf{T})$ and $x\mapsto\mu_x^{-1}\mu$ is a Borel function, and δ_2 is an open homomorphism from the compact group $C^1(N,\mathbf{T})=\{\sigma\in\mathbf{T}^N: \sigma(1)=1\}$ onto $B^2(N,\mathbf{T})$, there exists a Borel function: $x\in X\mapsto\sigma_x\in C^1(N,\mathbf{T})$ such that $\mu=\mu_x\,\delta_2(\sigma_x)$, $x\in X$. We consider σ as a $\mathbf{T}$-valued Borel function on $\mathfrak{N}$. Replacing $(\tilde{\lambda},\tilde{\mu})$ by $(\tilde{\lambda},\tilde{\mu})\,\delta(\sigma)$, we may assume that

$$x\in X\mapsto\tilde{\mu}((h,x),(k,x))\in\mathbf{T},\quad h,k\in N,$$

is constant. Therefore, the groupoid $\mathscr{G}=H\times\mathscr{K}$ acts on the constant exact sequence

$$1\to\mathbf{T}\to\tilde{N}\to N\to 1$$

determined by μ. Hence, if $\{u(k): k\in N\}$ is a cross-section of the above exact sequence, then

$$\alpha_{(h,\gamma)}(u(k))=\tilde{\lambda}((h,\gamma),(k,y))\,u(k),$$

for $y=r(\gamma)$, $h\in H$, $k\in N$ and $\gamma\in\mathscr{K}$ is indeed the action of $\mathscr{G}$ on the exact sequence:

$$X\to\mathbf{T}\times X\to\tilde{\mathfrak{N}}\to\mathfrak{N}\to 1.$$

If $\gamma=x\in\mathscr{G}^{(0)}$, then

$$\alpha_{(h,x)}(u(k))=\tilde{\lambda}((h,x),(k,x))\,u(k),\quad h\in H,\ k\in N.$$

If $\gamma=(y,x)\in\mathcal{H}$, then

$$\alpha_{(1,\gamma)}\alpha_{(h,x)}\alpha_{(1,\gamma)}^{-1}=\alpha_{(h,y)},\quad h\in H; \tag{2.3.18}$$

$$\alpha_{(1,\gamma)}(u(k))=\tilde{\lambda}((1,\gamma),(k,y))\,u(k),\quad k\in N. \tag{2.3.19}$$

Therefore, it is now easy to see that the map: $(k,\gamma)\in N\times\mathcal{H}\mapsto\tilde{\lambda}((1,\gamma),(k,r(\gamma))\in\mathbf{T}$ is a bicharacter, which determines a Borel homomorphism $\varepsilon:\gamma\in\mathcal{H}\mapsto\hat{N}$ such that

$$\tilde{\lambda}((1,\gamma),(k,r(\gamma))=\langle\varepsilon(\gamma),k\rangle.$$

Furthermore, (2.3.18) implies that if $\gamma:x\to y$, then

$$\tilde{\lambda}((h,x),(k,x))=\tilde{\lambda}((h,y),(k,y)).$$

By ergodicity, $x\in X\mapsto\tilde{\lambda}((h,x),(k,x))$ is a constant $\lambda(h,k)$. Now, we have

$$\tilde{\lambda}((h,\gamma),(k,r(\gamma)))=\langle\varepsilon(\gamma),h\rangle\,\lambda(h,k)$$

and $(\lambda,\mu)\in Z(H,N,\mathbf{T})$. Q.E.D.

The above cohomology splitting lemma, Lemma 2.3.17, and Proposition 2.3.13, together with the groupoid splitting lemma (2.2.11) allow us to decompose $\Lambda(G,N,\mathcal{U}(\mathcal{A}))$ into a direct sum of $H^1(G/H,H^1(N,\mathcal{U}(\mathcal{A}))$ and $\Lambda(H,N,\mathbf{T})$ as follows:

PROPOSITION 2.3.18. *Let β be an ergodic action of the discrete abelian group G on $\mathcal{A}=L^\infty(X,\mu)$ and $H=\ker\beta$. If N is a subgroup of H, then there are natural maps $i:H^1(G/H,H^1(N,\mathcal{U}(\mathcal{A})))\to\Lambda(G,N,\mathcal{U}(\mathcal{A}))$ and $\pi:\Lambda(G,N,\mathcal{U}(\mathcal{A}))\to\Lambda(H,N,\mathbf{T})$ such that the sequence:*

$$1\to H^1(G/H,H^1(N,\mathcal{U}(\mathcal{A})))\xrightarrow{i}\Lambda(G,N,\mathcal{U}(\mathcal{A}))\xrightarrow{\pi}\Lambda(H,N,\mathbf{T})\to 1$$

is exact. The sequence is also split.

We leave the proof to the reader.

The natural question is then: What is the size of $\Lambda(G,N,\mathcal{U}(\mathcal{A}))$? The group $\Lambda(G,N,\mathbf{T})$ is a (separable) compact abelian group, so it is certainly possible to compute in a given situation. On the other hand, $H^1(G/H,H^1(N,\mathcal{U}(\mathcal{A})))$ presents non-type I phenomena. This group is often the quotient of a Polish group by a dense subgroup and is studied in [25]. In the special case where $\mathcal{A}$ is atomic, Shapiro's lemma proves that $H^1(G/H,H_1(N,\mathcal{U}(\mathcal{A})))$ vanishes.

We are now in a position to show that all values of the characteristic invariant are realized by the kind of actions we are interested in. Let $\mathcal{N}$ be a semi-finite injective factor, (type I or type II), and let $\mathcal{A}=L^\infty(X,\mu)$ where $\{X,\mu\}$ is a standard σ-finite measure space, and β be an ergodic action of G on $\mathcal{A}$. Let $N\subset H=\ker\beta$ be a subgroup.

THEOREM 2.3.19. *For every $\chi\in\Lambda(G,N,\mathcal{U}(\mathcal{A}))$ there is an action α of G on $\mathcal{M}=\mathcal{N}\otimes\mathcal{A}$, which admits an invariant faithful semi-finite normal trace τ on $\mathcal{M}$, such that $\alpha|_{\mathcal{A}}=\beta$, $N_\alpha=N$ and $\chi_\alpha=\chi$. The restrictions on β and χ are the following: if $\mathcal{N}$ is of type* I *then $H=N$ and $\{\mathcal{A},\beta\}$ is not of type* III *and if $\mathcal{N}$ is finite then $\{\mathcal{A},\beta\}$ is not of type* III.

Proof. The necessity of the above restriction for the type I case follows from $\mathrm{Aut}(\mathcal{N})=\mathrm{Int}(\mathcal{N})$ if $\mathcal{N}$ is of type I.

To prove the theorem, we will use the description of the Λ group given in (2.3.13) and (2.3.17) together with (2.2.11). So let us first see what the characteristic invariant means in these terms:

Let $\mathcal{K}=K\times X$, $K=G/H$, be the hyperfinite groupoid. If $\alpha\colon\mathcal{G}=H\times\mathcal{K}\mapsto\mathrm{Aut}(\mathcal{N})$ is an action and $\mathfrak{N}_\alpha=N\times X$, we may choose, by (2.3.17), a 2-cocycle $\mu\colon N\times N\to\mathbf{T}$ and a Borel map: $x\in X\mapsto u_x(k)\in\mathcal{U}(\mathcal{N})$, $k\in N$, such that

$$\alpha_{(k,x)}=\mathrm{Ad}(u_x(k)),\quad k\in N,\ x\in X; \tag{2.3.20}$$

$$u_x(k_1)\,u_x(k_2)=\mu(k_1,k_2)\,u_x(k_1k_2),\quad k_1,k_2\in N. \tag{2.3.21}$$

One may also suppose that there exist $\lambda\in Z(H,N,\mathbf{T})$ and $\varepsilon\in H^1(\mathcal{K},\hat{N})$ such that

$$\alpha_{(h,\gamma)}(u_x(k))=\langle\varepsilon(\gamma),k\rangle\,\lambda(h,k)\,u_y(k),\quad \gamma\colon x\to y. \tag{2.3.22}$$

The characteristic invariant of the action α corresponds to the pair (ε,λ) under the isomorphism of (2.3.17). For future reference, we note here that simply by changing the choice of $u_x(k)$ appropriately we may suppose that μ takes any value in its cohomology class, as does ε.

From the point of view of this theorem, ε, λ and μ are given and we must construct the action α and $\{u_x(k)\colon k\in N\}$.

By [16], if $\mathcal{N}$ is of type II, there is a trace preserving action θ of H on $\mathcal{N}$ with $\chi_\theta=\lambda$. If $\mathcal{N}$ is of type I, this is also true via a projective representation since $N=H$, [24]. We choose $\{w_k\colon k\in N\}\subset\mathcal{U}(\mathcal{N})$ such that

$$w_{k_1}w_{k_2}=\mu(k_1,k_2)\,w(k_1k_2),\quad k_1,k_2\in N;$$

$$\theta_h(w_k)=\lambda(h,k)\,w_k,\quad h\in H,\ k\in N.$$

By a crossed product construction, we choose a unitary representation: $k\in N\mapsto v_k\in\mathcal{N}$ and an action σ of $\hat{N}$ on $\mathcal{N}$ such that $\sigma_s(v_k)=\langle s,k\rangle v_k$ for $(s,k)\in\hat{N}\times N$. By [2, Lemma 2.3.12], the representation v can be extended to a unitary representation of H into $\mathcal{N}$ which we denote again by v.

Since $\mathcal{N}\cong\mathcal{N}\otimes\mathcal{N}$ in all cases, it suffices to define an action on $\mathcal{N}\otimes\mathcal{N}$. Define

$$\alpha'_{(h,\gamma)}=\theta_h\otimes\sigma_{\varepsilon(\gamma)}\operatorname{Ad}v_k,\quad h\in H,\ \gamma\in\mathcal{K}. \tag{2.3.23}$$

Clearly, $\mathfrak{N}_\alpha=N\times X$ and for $(k,x)\in\mathfrak{N}_\alpha$ we set

$$u_x(k)=w_k\otimes v_k,\quad k\in N,\ x\in X.$$

With $\{\alpha',u_x(\cdot)\}$, (2.3.21) and (2.3.22) are satisfied.

Finally, we must fix up the action of $H\times\mathcal{K}$ so that the corresponding action of G preserves a faithful semi-finite normal trace on $\mathcal{M}$. Note that there is a homomorphism $\varrho:\mathcal{K}\to\mathbf{R}^*_+$ defined by the Radon-Nikodym derivative, so that if $\gamma=(gH,x)$ then $\varrho(\gamma)=(d\mu\circ g/d\mu)(x)$, where μ is the measure on X fixed to give $\mathcal{A}=L^\infty(X,\mu)$. If $\mathcal{N}$ is of type I or of type II_1, then the only way for the action of G to preserve a measure is for ϱ to be a coboundary, i.e. there is a measure equivalent to μ which is invariant under β. In this case, the model action α' already constructed will do for α.

If $\mathcal{N}$ is of type II_∞, then it is possible for ϱ to be non-trivial. Let $\{\theta_t\}$ be a one parameter automorphism group of $\mathcal{N}$ such that $\tau\cdot\theta_t=e^{-t}\tau$ for a faithful semi-finite normal trace τ on $\mathcal{N}$, [28]. We then define the action α of $H\times\mathcal{K}$ on $\mathcal{N}\otimes\mathcal{N}$ by

$$\alpha_{(h,\gamma)}=\alpha'_{(h,\gamma)}\otimes\theta_{\log\varrho(\gamma)},\quad h\in H,\ \gamma\in\mathcal{K}. \tag{2.3.24}$$

The resulting action of G will preserve the trace on $\mathcal{M}$ given by integrating the trace τ on $\mathcal{N}$ with respect to the measure μ, and of course it has the same characteristic invariant as α'. Q.E.D.

Remark. If $\mathcal{N}$ is of type I_n, $n<+\infty$, the proof of Theorem 2.3.19 shows there is an action of G on $\mathcal{N}\otimes\mathcal{A}$ with characteristic invariant represented by the pair (ε,λ) whenever

(i) There is a projective unitary representation of H in $\mathcal{N}_1$ whose 2-cocycle μ satisfies $\lambda(h,k)=\mu(h,k)\mu(k,h)^{-1}$;

(ii) There is a unitary representation: $h\to v_h$ of H in $\mathcal{N}_2$ and an action σ of $\hat{H}$ on $\mathcal{N}_2$ such that $\sigma_s(v_h)=\langle s,h\rangle v_h$;

(iii) $\mathcal{N}=\mathcal{N}_1\otimes\mathcal{N}_2$.

These conditions may be relaxed somewhat to obtain necessary and sufficient conditions but we shall not be too much concerned with this case, as it does not arise as the crossed product of a type II factor by a compact abelian group.

§ 2.4. Model actions with a specified property

We now come to what is probably the most subtle point of this paper. To ease the reader's task in following the argument, we shall make a digression to explain what is going on in a much simpler context, that of outer actions of discrete groups on factors.

Suppose $\mathcal{P}$ is a factor and G is a group, and suppose we could show that any two actions α and β of G on $\mathcal{P}$ are conjugate modulo $\mathrm{Int}(\mathcal{P})$. How might one show that any two actions are cocycle conjugate? Certainly we know that, after conjugation, there are unitaries $\{u_g\}$ such that $\alpha_g = \mathrm{Ad}\, u_g \beta_g$. The problem is that $\{u_g\}$ is not necessarily a β-cocycle. Indeed, $u_g \beta_g(u_h) = \mu(g, h)\, u_{gh}$ for some $\mathbf{T}$-valued 2-cocycle μ and the cohomology class of μ is an obstruction to any further attempts to make α and β cocycle conjugate involving only inner automorphisms. There is, however, a method in some cases to overcome these difficulties. Suppose that for some explicitly constructed model action $m: G \to \mathrm{Aut}(\mathcal{P})$ we could exhibit, for any 2-cocycle μ, an automorphism θ of $\mathcal{P}$ with

$$\theta m_g \theta^{-1} = \mathrm{Ad}\, v_g m_g;$$

$$v_g m_g(v_h) = \overline{\mu(g, h)}\, v_{gh}.$$

Now we can compare an arbitrary action α to the model action m. We know that, after a preliminary conjugation of α, there are unitaries u_g with $\alpha_g = \mathrm{Ad}\, u_g m_g$ and $u_g m_g(u_h) = \mu(g, h)\, u_{gh}$. But we may conjugate by the above θ to obtain $\theta \alpha_g \theta^{-1} = \mathrm{Ad}(\theta(u_g) v_g)\, m_g$. But now $g \mapsto \theta(u_g) v_g$ is indeed a cocycle and we are through. (This technique was used in § 6.3 of [14] to solve a similar problem.)

In the context of this paper, the group "G" of the previous discussion has been replaced by the groupoid $\mathcal{G} \cong H \times \mathcal{K}$, so we expect to have to deal with its second cohomology. The Künneth formula suggests that this will have three parts: $H^2(\mathcal{K})$, $H^2(H)$ and $H^1(\mathcal{K}) \otimes H^1(H)$. The first two will be dealt with in § 2.5 using the Bures-Connes-Krieger-Sutherland cohomology lemma, Appendix, and Ocneanu's theorem, [23], respectively. To handle the last part, we will follow the procedure outlined above and in this section we shall construct the model actions m with given characteristic invariant. We begin by treating the case where the subgroup N of H is trivial. This is, of course, only relevant when $\mathcal{N}$ is of type II.

So let $\alpha: H \times \mathcal{K} \mapsto \mathrm{Aut}(\mathcal{P})$ be an action on the factor $\mathcal{P}$ with $\mathfrak{N}_\alpha = \{1\} \times X$, $X = \mathcal{K}^{(0)}$.

We define $\mathscr{C}_\alpha$ to be the set of all Borel bicharacters $\varphi: H\times\mathscr{H}\mapsto\mathbf{T}$, i.e. Borel homomorphisms in both variables, and for which there is a Borel map: $x\in X\mapsto\theta_x\in\mathrm{Aut}(\mathscr{P})$ such that

$$\theta_y\alpha_{1,\gamma}\theta_x^{-1}=\mathrm{Ad}\,u_\gamma\alpha_{1,\gamma},\quad(\gamma: x\to y,\ \gamma\mapsto u_\gamma \text{ a cocycle});\tag{2.4.1}$$

$$\theta_x\alpha_{h,x}\theta_x^{-1}=\mathrm{Ad}\,(v_{h,x})\,\alpha_{h,x},\quad(x\in X,\ h\mapsto v_{h,x} \text{ a cocycle});\tag{2.4.2}$$

$$u_\gamma\alpha_{1,\gamma}(v_{h,x})=\varphi(h,\gamma)\,v_{h,y}\,\alpha_{h,y}(u_\gamma),\quad\gamma: x\to y.\tag{2.4.3}$$

PROPOSITION 2.4.4. *The set $\mathscr{C}_\alpha$ is a group and a cocycle conjugacy invariant for the action α, i.e. if β is cocycle conjugate to α then $\mathscr{C}_\alpha=\mathscr{C}_\beta$.*

Proof. Let $\varphi_i\in\mathscr{C}_\alpha$, $i=1,2$, and $\{\theta_x^i, u_\gamma^i, v_{h,x}^i\}$ be the associated objects in (2.4.1)–(2.4.3). Set

$$\theta_x=(\theta_x^1)^{-1}\theta_x^2,\quad u_\gamma=(\theta_y^1)^{-1}(u_\gamma^2)^*u_\gamma^1,\quad\gamma: x\to y$$

$$v_{h,x}=(\theta_x^1)^{-1}(v_{h,x}^2)^*v_{h,x}^1,\quad\varphi(h,\gamma)=\overline{\varphi_1(h,\gamma)}\,\varphi_2(h,\gamma).$$

It is a straightforward calculation to check (2.4.1)–(2.4.3) for φ, θ, u and v. Hence $\varphi\in\mathscr{C}_\alpha$. Thus $\mathscr{C}_\alpha$ is a group.

Suppose that there is a Borel map $\sigma: x\in X\mapsto\sigma_x\in\mathrm{Aut}(\mathscr{P})$ such that

$$\sigma_y(\mathrm{Ad}\,(w_{h,\gamma})\,\alpha_{h,\gamma})\,\sigma_x^{-1}=\beta_{h,\gamma},\quad\gamma: x\to y.$$

with an α-cocycle w. Let $\varphi\in\mathscr{C}_\alpha$ with associated $\{\theta,u,v\}$. Set

$$\begin{cases}\bar\theta_x=\sigma_x\theta_x\sigma_x^{-1},\quad \bar u_\gamma=\sigma_y(\theta_y(w_{1,\gamma})\,u_\gamma\,w_{1,\gamma}^*),\\ \bar v_{h,x}=\sigma_x(\theta_x(w_{h,x})\,v_{h,x}\,w_{h,x}^*),\end{cases}\tag{2.4.5}$$

where $\gamma\in\mathscr{H}: x\to y$. We then have

$$\begin{aligned}\bar\theta_y\beta_{1,\gamma}\bar\theta_x^{-1}&=(\sigma_y\theta_y\sigma_y^{-1})(\sigma_y\,\mathrm{Ad}\,(w_{1,\gamma})\,\alpha_{1,\gamma}\sigma_x^{-1})(\sigma_x\theta_x\sigma_x^{-1})^{-1}\\&=\sigma_y\,\mathrm{Ad}\,(\theta_y(w_{1,\gamma}))\,\mathrm{Ad}\,(u_\gamma)\,\alpha_{1,\gamma}\sigma_x^{-1}\\&=\mathrm{Ad}\,(\sigma_y(\theta_y(w_{1,\gamma})\,u_\gamma\,w_{1,\gamma}^*))\,\sigma_y(\mathrm{Ad}\,(w_{1,\gamma})\,\alpha_{1,\gamma})\,\sigma_x^{-1}\\&=\mathrm{Ad}\,(\bar u_\gamma)\,\beta_{1,\gamma};\end{aligned}$$

similarly

$$\bar{\theta}_x \beta_{h,x} \bar{\theta}_x^{-1} = \mathrm{Ad}(\bar{v}_{h,x}) \beta_{h,x};$$

$$\bar{u}_\gamma \beta_{1,\gamma}(\bar{v}_{h,x}) = \varphi(h,\gamma)\, \bar{v}_{h,x} \beta_{h,x}(\bar{u}_\gamma).$$

The cocycle properties of $\bar{u}$ and $\bar{v}$ are also easily checked. Hence $\varphi \in \mathscr{C}_\beta$ with associated $\{\bar{\theta}, \bar{u}, \bar{v}\}$. By symmetry, we get $\mathscr{C}_\beta \subset \mathscr{C}_\alpha$, so that $\mathscr{C}_\alpha = \mathscr{C}_\beta$. Q.E.D.

We now define our first model action $\varkappa$ on an injective factor $\mathscr{R}$ of type II_1 by

$$\varkappa_{h,\gamma} = \alpha_h \otimes \mathrm{id} \quad \text{on } \mathscr{R} \otimes \mathscr{R} \cong \mathscr{R}. \tag{2.4.6}$$

where α is an outer action of H, specified by the next theorem:

THEOREM 2.4.7. *There exists a properly outer action α of H such that $\mathscr{C}_\varkappa$ contains all bicharacters on $H \times \mathscr{K}$.*

Proof. Let φ be an arbitrary Borel bicharacter on $H \times \mathscr{K}$. We note first that φ may be regarded as a cocycle on $\mathscr{K}$ taking values in $\hat{H}$. The first step of the proof for $\varphi \in \mathscr{C}_\varkappa$ will be to reduce to the case where φ takes values in some countable dense subgroup H_0 of $\hat{H}$. The effect of perturbing φ by a coboundary as a member of $Z(\mathscr{K}, \hat{H})$ is absorbed by the perturbation of u and v by the coboundary of the cochain which is used to perturb φ. Thus φ belongs to $\mathscr{C}_\varkappa$ if and only if any of its perturbations by coboundaries belongs to $\mathscr{C}_\varkappa$. Hence it suffices to show that any Borel cocycle on $\mathscr{K}$ with values in $\hat{H}$ is cohomologous to one with values in H_0. This is, however, an immediate consequence of the Bures-Connes-Krieger-Sutherland cohomology lemma for hyperfinite ergodic groupoids, see Appendix.

Now, suppose that $q: \mathscr{K} \mapsto H_0$ is a Borel homomorphism of $\mathscr{K}$ into the discrete countable group H_0. We shall construct an action τ of $H \times \mathscr{K}$ on $\mathscr{R}$ with $q^{-1} \in \mathscr{C}_\tau$ and then show that τ is cocycle conjugate to $\varkappa$. This is sufficient by (2.4.4).

Begin by choosing an outer action of H_0 on $\mathscr{R}$, [1, 22, 27], and let ϱ be the dual action restricted to H on the crossed product $\mathscr{R} \rtimes H_0$ which is isomorphic to $\mathscr{R}$ by [5]. Let $p \in H_0 \mapsto a_p \in \mathscr{U}(\mathscr{R} \rtimes H_0)$ be the unitary representation of H_0 in the crossed product so that $\varrho_h(a_p) = \langle h, p \rangle\, a_p$, $p \in H_0$, $h \in H \subset \hat{H}_0$. Now, set up the following system:

$$\mathscr{N} = \prod_{n \in \mathbf{Z}}^{\otimes} \mathscr{R}_n, \quad \mathscr{R}_n = \mathscr{R} \rtimes H_0;$$

$$\alpha_h = \prod_{n \in \mathbf{Z}}^{\otimes} \varrho_{h,n}, \quad \varrho_{h,n} = \varrho_h;$$

$$\beta_p = \mathrm{id} \otimes \prod_{n\geq 1}^{\otimes} \mathrm{Ad}\,(a_p)_n,$$

$$v_p = \left(\prod_{n\leq -1}^{\otimes} 1_n\right) \otimes a_p \otimes \left(\prod_{n\geq -1}^{\otimes} 1_n\right), \quad 1_n = 1.$$

$$\sigma = \text{shift to the left on } \mathcal{N}.$$

The following properties of the above system are easily verified:

$$\alpha_h \beta_p = \beta_p \alpha_h, \quad h \in H,\ p \in H_0; \tag{2.4.8}$$

$$\alpha_h \beta_p \text{ is outer except when } h = 1 \text{ and } p = 1; \tag{2.4.9}$$

$$\alpha_h(v_p) = \langle h, p \rangle\, v_p; \tag{2.4.10}$$

$$\sigma \alpha_h \sigma^{-1} = \alpha_h; \tag{2.4.11}$$

$$\sigma \beta_p \sigma^{-1} = \mathrm{Ad}\,(v_p) \beta_p. \tag{2.4.12}$$

Now define the action τ of $H\times\mathcal{K}$ on $\mathcal{N}$ ($\cong\mathcal{R}$) by $\tau_{h,\gamma}=\alpha_h\beta_{q(\gamma)}$. Then τ is an action since α and β commute and q is a homomorphism. Moreover, if we set $\theta_x=\sigma$ for all $x\in X=\mathcal{K}^{(0)}$, then

$$\theta_y \tau_{1,\gamma} \theta_x^{-1} = \mathrm{Ad}\,(v_{q(\gamma)})\, \tau_{1,\gamma} \quad \text{by (2.4.12);}$$

$$\theta_x \tau_{h,x} \theta_x^{-1} = \tau_{h,x} \quad \text{by (2.4.11),}$$

and $\gamma\in\mathcal{K}\mapsto v_{q(\gamma)}$ is a cocycle since $p\mapsto v_p$ is a cocycle for β. By (2.4.10), we have

$$\tau_{h,y}(v_{q(\gamma)}) = \alpha_h(v_{q(\gamma)}) = \langle h, q(\gamma) \rangle\, v_{q(\gamma)}.$$

Hence by the definition of $\mathscr{C}_\tau$, if we set

$$\varphi(h,\gamma) = \overline{\langle (h, q(\gamma)) \rangle},$$

then $\varphi\in\mathscr{C}_\tau$.

Now, we want to show that τ is cocycle conjugate to the action $\varkappa$ defined by (2.4.6). By (2.4.8) and (2.4.9), we may apply Ocneanu's theorem [23] to conclude that the action of $H\times H_0$ defined by α and β is cocycle conjugate to the action: $(h,p)\in H\times H_0\mapsto\alpha_h\otimes\beta_p$ on $\mathcal{N}\otimes\mathcal{N}$. This immediately shows that the action τ is cocycle conjugate to the action: (h,γ): $H\times\mathcal{K}\to\alpha_h\otimes\beta_{q(\gamma)}$ on $\mathcal{N}\otimes\mathcal{N}$. Since $\mathrm{Int}\,(\mathcal{N})$ is a dense

Borel subgroup of the Polish group Aut $(\mathcal{N})$, the cohomology lemma Appendix, implies that there exist Borel function: $x \in X \mapsto \sigma_x \in \mathrm{Aut}(\mathcal{N})$ and $\gamma \in \mathcal{H} \mapsto u_\gamma \in \mathcal{U}(\mathcal{N})$ such that

$$\sigma_y \beta_{q(\gamma)} \sigma_x^{-1} = \mathrm{Ad}(u_\gamma).$$

Since the second cohomology of $\mathcal{H}$ in $\mathbf{T}$ vanishes (a fact we already used in § 2.2), u_γ may be supposed to be a homomorphism of $\mathcal{H}$ into $\mathcal{U}(\mathcal{N})$. Thus, finally the action τ is cocycle conjugate to $\varkappa$. Q.E.D.

We can now define the model action with arbitrary characteristic invariant. Let $\chi \in \Lambda(H \times \mathcal{H}, N \times \mathcal{H}^{(0)}, \mathbf{T})$ be given. Let δ be the model action with $\chi = \chi_\delta$ defined in (2.3.19). Then define the action m of $H \times \mathcal{H}$ by

(a) $m = \delta$ if $\mathcal{N}$ is of type I;

(b) $m = \delta \otimes \varkappa$ otherwise,

where $\varkappa$ is the action of $H/N \times \mathcal{H}$ defined by (2.4.6) on $\mathcal{R}$. Since $\mathcal{N} \cong \mathcal{N} \otimes \mathcal{R} \cong \mathcal{R}$, $\varkappa$ is a trace preserving action on $\mathcal{N}$, so that m gives rise to a trace preserving action of G on $\mathcal{M} = \mathcal{A} \otimes \mathcal{N}$, $\mathcal{A} = L^\infty(X, \mu)$. Moreover, by (2.4.7) and (2.4.4), m has the property that if q is any homomorphism from $\mathcal{H}$ to $N^\perp = (H/N)^\wedge \subset \hat{H}$, there is a Borel function: $x \in X \mapsto \theta_x \in \mathrm{Aut}(\mathcal{N})$ such that

$$\theta_y m_{1,\gamma} \theta_x^{-1} = \mathrm{Ad}(u_\gamma) m_{1,\gamma} \quad \text{for a cocycle } u_\gamma; \tag{2.4.13}$$

$$\theta_x m_{h,x} \theta_x^{-1} = \mathrm{Ad}(v_{h,x}) m_{h,x} \quad \text{for a cocycle } v_{h,x}; \tag{2.4.14}$$

$$u_\gamma m_{1,\gamma}(v_{h,x}) = \langle h, q(\gamma) \rangle v_{h,y} m_{h,y}(u_\gamma), \quad \text{where } \gamma: x \to y. \tag{2.4.15}$$

§ 2.5. Proof of Theorem 2.1.14

By (ii) of (2.1.14), there is an automorphism ϱ of $\mathcal{A}$ which ties up the characteristic invariants χ_α of α and χ_β of β. By central ergodicity and semi-finiteness, $\mathcal{M} \cong \mathcal{N} \otimes \mathcal{A}$ with $\mathcal{N}$ a factor, so ϱ extends to an automorphism (also called ϱ) of $\mathcal{M}$. The auxiliary actions of α and $\varrho \beta \varrho^{-1}$ have the same characteristic invariant by (2.3.16). By (2.2.12) it suffices to show that these two groupoid actions are cocycle conjugate. We shall do this by showing that any two actions α and β of $H \times \mathcal{H}$ on $\mathcal{N}$ with $\mathfrak{N}_\alpha = \mathfrak{N}_\beta = N \times \mathcal{H}^{(0)}$ and $\chi_\alpha = \chi_\beta$ are cocycle conjugate to the model actions of § 2.4 with the same characteristic invariant.

To begin, note that to each $x \in X = \mathcal{H}^{(0)}$ there corresponds, in a Borel measurable fashion, an action α_x of H on $\mathcal{N}$ and by (2.3.22) they are almost all cocycle conjugate to the restriction of the model action. Furthermore, this restriction does not depend on x, and we shall write it simply $h \mapsto m_h$. By the von Neumann measurable cross section theorem, we may choose a Borel function: $x \to \tau_x \in \mathrm{Aut}(\mathcal{N})$ such that $\tau_x \alpha_{h,x} \tau_x^{-1} = \mathrm{Ad}(v_{h,x}) m_h$ for almost every x and a Borel family $\{v_{h,x}\}$ of m-cocycles. Set for $(h, \gamma) \in H \times \mathcal{K}$

$$\sigma_{h,y} = \tau_y \alpha_{h,\gamma} \tau_x^{-1}, \quad \gamma: x \to y. \tag{2.5.1}$$

It suffices to show that σ and m are cocycle conjugate, and we already know that their restrictions to $H \times X$ differ by a cocycle.

Now, let $\sigma_\gamma = \sigma_{1,\gamma}$ and $m_\gamma = m_{1,\gamma}$. By the above, we have $\sigma_{h,x} = \mathrm{Ad}(v_{h,x}) m_h$, so

$$\sigma_\gamma m_h \sigma_\gamma^{-1} = \mathrm{Ad}(\sigma_\gamma(v^*_{h,x}) v_{h,y}) m_h, \tag{2.5.2}$$

and for each $\gamma: x \to y$, the map: $h \to \sigma_\gamma(v^*_{h,x}) v_{h,y}$ is a cocycle for the action m of H on $\mathcal{N}$. Thus the map: $\gamma \in \mathcal{K} \to \sigma_\gamma$ defines a homomorphism of the principal hyperfinite ergodic groupoid $\mathcal{K}$ into the group of all automorphisms which commute, up to a cocycle, with m. We would like to apply the cohomology lemma again here, so we must construct a complete metric for this group. It is no use taking the metric it inherits as a subgroup of $\mathrm{Aut}(\mathcal{N})$ as it is not complete in this metric. For these reasons, we make the following definitions.

Let β be an action of the countable discrete group F on the separable factor $\mathcal{P}$. Let $\mathrm{Aut}^F(\mathcal{P})$ be the set of all pairs $(\theta, \{u_t\})$ where $\theta \in \mathrm{Aut}(\mathcal{P})$ and $t \in F \mapsto u_t \in \mathcal{U}(\mathcal{P})$ is a cocycle for β such that

$$\theta \beta_t \theta^{-1} = \mathrm{Ad}(u_t) \beta_t, \quad t \in F. \tag{2.5.3}$$

With multiplication:

$$(\theta_1, \{u_t\})(\theta_2, \{v_t\}) = (\theta_1 \theta_2, \{\theta_1(v_t) u_t\}), \tag{2.5.4}$$

$\mathrm{Aut}^F(\mathcal{P})$ is a group. Also, $\mathrm{Aut}^F(\mathcal{P})$ is a closed subgroup of the Polish group $\mathrm{Aut}(\mathcal{P}) \cdot \mathcal{U}(\mathcal{P})^F$, where $\mathrm{Aut}(\mathcal{P}) \cdot \mathcal{U}(\mathcal{P})^F$ is the semi-direct product of $\mathrm{Aut}(\mathcal{P})$ and $\mathcal{U}(\mathcal{P})^F$, the group of all functions of F into $\mathcal{U}(\mathcal{P})$ equipped with the product topology. With this relative topology $\mathrm{Aut}^F(\mathcal{P})$ is a Polish group and the natural projection: $(\theta, \{u_t\}) \mapsto \theta$ is continuous. The group $\mathrm{Aut}^F(\mathcal{P})$ may appear artificial with the above definition. But, if F is abelian, then it is just the group of all automorphisms of the crossed product $\mathcal{P} \rtimes_\beta F$ which commute with the dual action.

Now, suppose F is abelian. Then $\hat{F}$ sits in a natural way inside $\mathrm{Aut}^F(\mathcal{P})$ as elements of the form: $(\mathrm{id}, \{p(t)\})$, $p \in \hat{F}$. (This corresponds to the dual action.) Thus $\hat{F}$ is a closed central subgroup of $\mathrm{Aut}^F(\mathcal{P})$, and the quotient group, which we shall write $\mathrm{Aut}_F(\mathcal{P})$, is Polish. Let $\pi\colon \mathrm{Aut}^F(\mathcal{P}) \mapsto \mathrm{Aut}_F(\mathcal{P})$ be the quotient map.

Another normal subgroup of $\mathrm{Aut}^F(\mathcal{P})$ is $\mathrm{Int}(\mathcal{P})$, where $\mathrm{Ad}(u)$ defines the element $(\mathrm{Ad}(u), \{u\beta_t(u^*)\})$ of $\mathrm{Aut}^F(\mathcal{P})$. We want to apply the cohomology lemma, Appendix, to this subgroup, so we need the next result, which follows from Ocneanu's stability theorem [23]. For it we specialize to the case $\mathcal{P}=\mathcal{N}$, a semi-finite injective factor. We recall that $\theta \in \mathrm{Aut}(\mathcal{N})$ is approximately inner if and only if it preserves a semi-finite normal trace on $\mathcal{N}$. Set

$$\overline{\mathrm{Int}}^F(\mathcal{N}) = \text{the closure of } \mathrm{Int}(\mathcal{N}) \text{ in } \mathrm{Aut}^F(\mathcal{N}) \tag{2.5.5}$$

$$\overline{\mathrm{Int}}_F(\mathcal{N}) = \pi(\overline{\mathrm{Int}}^F(\mathcal{N})).$$

LEMMA 2.5.6. *Let P be the closed subgroup of* $\mathrm{Aut}^F(\mathcal{N})$ *consisting of pairs* $(\theta, \{u_h\})$ *such that θ preserves the trace on $\mathcal{N}$. Then $\pi(\mathrm{Int}(\mathcal{N}))$ is a dense normal Borel subgroup of the closed subgroup $\pi(P)$.*

Proof. The normality of $\pi(\mathrm{Int}(\mathcal{N}))$ is immediate. The group $\mathcal{U}(\mathcal{N})/\mathbf{T}$ is a Polish group and the map: $\dot{u} \in \mathcal{U}(\mathcal{N})/\mathbf{T} \to (\mathrm{Ad}(u), \{u\beta_t(u^*)\})$ is a continuous injective homomorphism into $\mathrm{Aut}_F(\mathcal{N})$, whose image is $\pi(\mathrm{Int}(\mathcal{N}))$, so $\pi(\mathrm{Int}(\mathcal{N}))$ is Borel. Also $\pi(\mathrm{Int}(\mathcal{N})$ is a subgroup of P, so $\pi(\mathrm{Int}(\mathcal{N})) \subset \pi(P)$. Since P contains $\hat{F}$, $\pi(P)$ is closed. Thus there only remains the density of $\pi(\mathrm{Int}(\mathcal{N}))$ in $\pi(P)$.

We will show that for any $z \in \pi(P)$ there is a $(\theta, \{u_t\})$ in $\pi^{-1}(z)$ and a sequence of elements of $\mathrm{Int}(\mathcal{N})$ converging to $(\theta, \{u_t\})$ in $\mathrm{Aut}_F(\mathcal{N})$.

For this let $N = \beta^{-1}(\mathrm{Int}\,\mathcal{N}) \subseteq F$ and choose for each $h \in N$, a unitary a_h with $\beta_h = \mathrm{Ad}(a_h)$. Let $\mu\colon N \times N \to \mathbf{T}$ be the 2-cocycle defined by

$$a_h a_k = \mu(h,k)\, a_{hk}, \quad h, k \in N.$$

If $(\theta, \{v_t\})$ is an arbitrary element of $\pi^{-1}(z)$ then $\theta\beta_h\theta^{-1} = \mathrm{Ad}\, v_h \beta_h$, $h \in N$, so

$$\theta(a_h) = c_h v_h a_h$$

for some function $c\colon N \to \mathbf{T}$. Moreover

$$\begin{aligned} c_h c_k \mu(h,k)\, v_{hk}\, a_{hk} &= c_h c_k v_h \beta_h(v_k)\, a_h a_k \\ &= c_h v_h c_k v_k v_k \\ &= \theta(a_h)\,\theta(a_k) \\ &= \theta(a_h a_k) \end{aligned}$$

$$= \theta(\mu(h, k) a_{hk})$$

$$= c_{hk} \mu(h, k) v_{hk} a_{hk}$$

so that $c_{hk} = c_h c_k$ for $h, k \in N$. Therefore c is a character of N which may be extended to a character $\bar{c}$ of F. If $u_t = \bar{c}_t v_t$ then
$(\theta, \{u_t\}) \in \pi^{-1}(z)$ and

$$\theta(a_h) = \mu_h a_h, \quad h \in N. \tag{2.5.7}$$

Our next task is to find a sequence y_n in $\mathcal{U}(\mathcal{N})$ such that $\theta = \lim \mathrm{Ad}(y_n)$ and $u_t = \lim y_n \beta_t(y_n^*)$ for all $t \in F$.

Let ω be a free ultrafilter on $\mathbf{N}$, A_ω the subalgebra of $l^\infty(\mathbf{N}, \mathcal{N})$ of all ω-centralizing sequences and $\mathcal{I}_\omega$ the ideal of A_ω of all sequences tending $*$-strongly to zero along the ultrafilter ω. Let $\mathcal{N}_\omega$ be the von Neumann algebra $A_\omega / \mathcal{I}_\omega$, see [3].

By hypothesis θ preserves the trace on $\mathcal{N}$ so $\theta \in \overline{\mathrm{Int}}\, \mathcal{N}$ and we may choose a sequence $\{w_n\}$ of unitaries with $\theta = \lim_{n \to \infty} \mathrm{Ad}\, w_n$. Then since $\theta \beta_t \theta^{-1} = \mathrm{Ad}\, u_t \beta_t$ we have

$$\lim_{n \to \infty} \mathrm{Ad}\,(w_n \beta_t(w_n^*)) \beta_t = \mathrm{Ad}\ u_t \beta_t, \quad t \in F,$$

so that the sequence $\{u_t^* w_n \beta_t(w_n^*)\}$ is centralizing. Let X_t be the unitary in $\mathcal{N}_\omega$ given by $\{u_t^* w_n \beta_t(w_n^*)\}$. Then a calculation, using the fact that $\{u_t^* w_n \beta_t(w_n^*)\}$ is central, shows that $t \mapsto X_t$ is a cocycle for the action β on $\mathcal{N}_\omega$. But if $h \in N$,

$$u_h^* w_n \beta_h(w_n^*) = u_h^* w_n a_h w_n^* a_h^*$$

which tends σ-strongly* to $u_h^* \theta(a_h) a_h^* = 1$ by (2.5.7). Thus $X_h = 1$ for $h \in N$. But the action β of F on $\mathcal{N}_\omega$ factors through an action of $Q = F/N$ and the above condition shows precisely that $\{X_t\}$ defines a cocycle for this action of Q. Moreover, we know, by [7] and [4], that the action of Q is free on $\mathcal{N}_\omega$. Thus, by [23], $\{X_t\}$ is a coboundary, i.e., there exists a unitary $Y \in \mathcal{N}_w$ such that $X_t = Z^* \beta_t(Z)$, $t \in F$. This means that if $\{z_n\}$ is a representing sequence of Z which consists of unitaries, then $\{z_n\}$ is ω-centralizing and

$$\lim_{n \to \omega} \{z_n^* \beta_t(z_n) - u_t^* w_n \beta_t(w_n^*)\} = 0$$

in the σ-strong* topology. Hence, if we set $y_n = z_n w_n$, then

$$\theta = \lim_{n \to \omega} \mathrm{Ad}\,(y_n),$$

$$u_t = \lim_{n\to\omega} y_n \beta_h(y_n^*) \text{ in the } \sigma\text{-strong topology,}$$

where to conclude the last equality one uses the fact that $\lim_{n\to\omega}[u_t, z_n^*]=0$. By choosing a subsequence of $\{y_n\}$, one can conclude the existence of a sequence $\{y_n\}$ in $\mathcal{U}(\mathcal{N})$ such that

$$\theta = \lim_{n\to\infty} \mathrm{Ad}(y_n), \quad u_t = \lim_{n\to\infty} y_n \beta_t(y_n^*), \quad t \in F,$$

as required. Q.E.D.

Having proved Lemma 2.5.6, we pick up the proof of 2.1.14. By formula (2.5.2), we define a map:

$$\Phi_\sigma: \gamma \in \mathcal{H} \mapsto \pi(\sigma_\gamma, \sigma_\gamma(v_{h,x}^*)\, v_{h,y}) \in \mathrm{Aut}_H(\mathcal{N}), \quad \gamma: x \to y. \tag{2.5.8}$$

If $\gamma_1 \in \mathcal{H}: y \to z$ and $\gamma_2 \in \mathcal{H}: x \to y$, then

$$(\sigma_{\gamma_1}, \sigma_{\gamma_1}(v_{h,y}^*)\, v_{h,z})(\sigma_{\gamma_2}, \sigma_{\gamma_2}(v_{h,x}^*)\, v_{h,y}) = (\sigma_{\gamma_1\gamma_2}, \sigma_{\gamma_1\gamma_2}(v_{h,x}^*)\, v_{h,z}),$$

so that Φ_σ is a homomorphism of $\mathcal{H}$ into $\mathrm{Aut}_H(\mathcal{N})$.

The model action m also defines a homomorphism $\Phi_m: \mathcal{H} \to \mathrm{Aut}_H(\mathcal{N})$ via $\Phi_m(\gamma) = \pi((m_{1,\gamma}, 1))$. By assumption, both the model action m and σ transform the trace on $\mathcal{N}$ in exactly the same way, because both of them preserve a trace on $\mathcal{M} = \mathcal{N} \otimes \mathcal{A}$. The two actions σ_γ and $m_{1,\gamma}$ are thus equal modulo $\overline{\mathrm{Int}}(\mathcal{N})$. Therefore, we have, by (2.5.6),

$$\Phi_\sigma(\gamma) \equiv \Phi_m(\gamma) \bmod \overline{\pi(\mathrm{Int}(\mathcal{N}))}. \tag{2.5.9}$$

Thus, the cohomology lemma, Appendix, yields that there exist Borel maps $W: \gamma \in \mathcal{H} \to W_\gamma \in \pi(\mathrm{Int}(\mathcal{N}))$ and $\Theta: x \in X = \mathcal{H}^{(0)} \to \Theta_x \in \pi(P)$ such that

$$W_\gamma \Phi_m(\gamma) = \Theta_y \Phi_\sigma(\gamma) \Theta_x^{-1}, \quad \gamma: x \to y.$$

Let

$$W_\gamma = \pi((\mathrm{Ad}(w_\gamma), \{w_\gamma m_h(w_\gamma^*)\})) \in \pi(\mathrm{Int}(\mathcal{N})),$$

$$\Theta_x = \pi(\theta_x, \{u_{h,x}\}) \in \pi(P).$$

We then have

$$\begin{aligned} \mathrm{Ad}(w_\gamma)\, m_{1,\gamma} &= \theta_y \sigma_y \theta_x^{-1}, \quad \gamma: x \to y; \\ \theta_x m_h \theta_x^{-1} &= \mathrm{Ad}(u_{h,x})\, m_h \end{aligned} \tag{2.5.10}$$

for a measureable field $\{u_{h,x}\}$ of cocycles. By the hyperfiniteness of $\mathcal{K}$, one can readjust $\{w_\gamma\}$ so that $\gamma \in \mathcal{K} \mapsto w_\gamma \in \mathcal{U}(\mathcal{N})$ becomes a cocycle for $\{m_\gamma\}$.

We set

$$\delta_{h,\gamma} = \theta_y \sigma_{h,y} \theta_x^{-1}, \quad \gamma: x \to y, \; h \in H. \tag{2.5.11}$$

It follows that the action δ of $H \times \mathcal{K}$ is conjugate to the original action α and that

$$\begin{aligned} \delta_{1,\gamma} &= \mathrm{Ad}(w_\gamma) m_{1,\gamma}, \quad \gamma \in \mathcal{K} \\ \delta_{h,x} &= \mathrm{Ad}(w_{h,x}) m_h, \quad h \in H, \; x \in X, \end{aligned} \tag{2.5.12}$$

with

$$w_{h,x} = \theta_x(v_{h,x}) u_{h,x}. \tag{2.5.13}$$

It is easy to show that $\{w_{h,x}\}$ is a cocycle for $\{m_h\}$. The problem remaining is that the pair $\{w_\gamma, w_{h,x}\}$ does not necessarily extend to an m-cocycle of $H \times \mathcal{K}$. Note, however, that it does modulo scalars. In fact, since

$$\delta_{1,\gamma} \delta_{h,x} \delta_{1,\gamma}^{-1} = \delta_{h,y}, \quad \gamma: x \to y,$$

we have

$$(\mathrm{Ad}(w_\gamma) m_{1,\gamma})(\mathrm{Ad}(w_{h,x}) m_h)(m_{1,\gamma}^{-1} \mathrm{Ad}(w_\gamma^*)) = \mathrm{Ad}(w_{h,y}) m_h,$$

and since $\mathcal{N}$ is a factor, there exist scalars $\varphi(h,\gamma)$ such that

$$w_\gamma m_\gamma(w_{h,x}) = \varphi(h,\gamma) w_{h,y} m_h(w_\gamma). \tag{2.5.14}$$

LEMMA 2.5.15. *The function φ on $H \times \mathcal{K}$ is a bicharacter on $H \times \mathcal{K}$, and hence gives rise to a Borel homomorphism p of $\mathcal{K}$ into $\hat{H}$ such that $\varphi(h,\gamma) = \langle h, p(\gamma) \rangle$. The cohomology class of φ is independent of the choice of the cocycles $\{w_\gamma\}$ and $\{w_{h,x}\}$. In particular, it is an obstruction to the inner conjugacy of δ and m.*

The proof is a straightforward calculation, so we leave it to the reader.

LEMMA 2.5.16. *One may choose the $\{w_{h,x}\}$ in such a way that $\varphi(h,\gamma)=1$ if $h \in N$.*

Proof. Let ε and μ be as in (2.3.17), i.e. $m_h = m_{h,x} = \mathrm{Ad}\, a_h$ for $h \in N$ and

$$m_\gamma(a_h) = \langle h, \varepsilon(\gamma) \rangle a_h, \quad h \in N; \tag{2.5.17}$$

$$a_h a_k = \mu(h,k) a_{hk}, \quad h, k \in N. \tag{2.5.18}$$

Since m and δ have the same characteristic invariant, one may choose unitaries $b_{h,x}$ with $\delta_{h,x}=\mathrm{Ad}(b_{h,x})$, $h\in N$, such that

$$\delta_{1,\gamma}(b_{h,x})=\langle h,\varepsilon(\gamma)\rangle\, b_{h,y},\quad \gamma: x\to y; \tag{2.5.19}$$

$$b_{h,x}\, b_{k,x}=\mu(h,k)\, b_{hk,x}. \tag{2.5.20}$$

But $\delta_{h,x}=\mathrm{Ad}(w_{h,x}\,a_h)$, so there are scalars $\{c_{h,x}\}$ with

$$b_{h,x}=c_{h,x}\, w_{h,x}\, a_h.$$

Since $\{w_{h,x}\}$ is a cocycle, (2.5.18) and (2.5.20) imply that $h\mapsto c_{h,x}$ is a character of N. Since $\hat{N}=\hat{H}/N^{\perp}$, one can choose a Borel map: $x\in X\to\bar{c}_x\in\hat{H}$ so that $c_{h,x}=\langle h,\bar{c}_x\rangle$, $h\in N$. Replacing $w_{h,x}$ by $\langle h,\bar{c}_x\rangle\, w_{h,x}$, which continues to be a cocycle, we will have

$$b_{h,x}=w_{h,x}\, a_h,\quad h\in N. \tag{2.5.21}$$

We now have, for $h\in N$,

$$\begin{aligned}
w_\gamma\, m_\gamma(w_{h,x}) &= w_\gamma\, m_\gamma(b_{h,x}\, a_h^*) \quad \text{by (2.5.21)}\\
&= w_\gamma\, m_\gamma(b_{h,x})\, w_\gamma^* w_\gamma\, m_\gamma(a_h^*)\\
&= \delta_{1,\gamma}(b_{h,x})\, \overline{\langle h,\varepsilon(\gamma)\rangle}\, w_\gamma\, a_h^* \quad \text{by (2.5.17), (2.5.12),}\\
&= b_{h,y}\, w_\gamma\, a_h^* \quad \text{by (2.5.19)}\\
&= (b_{h,y}\, a_h^*)(a_h\, w_\gamma\, a_h^*) = b_{h,y}\, a_h^* m_h(w_\gamma)\\
&= w_{h,y}\, m_h(w_\gamma) \quad \text{by (2.5.21).}
\end{aligned}$$

So, we have $\varphi(h,\gamma)=1$, $h\in N$. Q.E.D.

Lemma 2.5.16 shows that there exists a homomorphism $p:\gamma\in\mathscr{K}\mapsto N^{\perp}\subset\hat{H}$ such that

$$\varphi(h,\gamma)=\langle h,p(\gamma)\rangle,\quad h\in H,\ \gamma\in\mathscr{K}.$$

We now apply Theorem 2.4.7 in the form (2.4.13)–(2.4.15). Thus, we may choose a Borel function: $x\mapsto v_x\in\mathrm{Aut}(\mathscr{N})$ such that

$$\begin{cases} v_y\, m_{1,\gamma}\, v_x^{-1}=\mathrm{Ad}(z_\gamma)\, m_{1,\gamma}, & \gamma: x\to y;\\ v_x\, m_{h,x}\, v_x^{-1}=\mathrm{Ad}(z_{h,x})\, m_{h,x}, & \end{cases} \tag{2.5.22}$$

for cocycles $\{z_\gamma\}$ and $\{z_{h,x}\}$ such that

$$z_\gamma m_{1,\gamma}(z_{h,x}) = \overline{\langle h, p(\gamma)\rangle}\, z_{h,y}\, m_{h,y}(z_\gamma). \tag{2.5.24}$$

We then set

$$\begin{cases} w'_\gamma = v_y(w_\gamma)\, z_\gamma, & \gamma: x \to y; \\ w'_{h,x} = v_x(w_{h,x})\, z_{h,x}. \end{cases} \tag{2.5.23}$$

We are now ready to complete the proof of Theorem 2.1.14 as follows. First, (2.5.12) and (2.5.22) together yield

$$\begin{cases} v_y\, \delta_{1,\gamma}\, v_x^{-1} = \mathrm{Ad}\,(w'_\gamma)\, m_\gamma; \\ v_x\, \delta_{h,x}\, v_x^{-1} = \mathrm{Ad}\,(w'_{h,x})\, m_{h,x}. \end{cases} \tag{2.5.24}$$

Secondly, $\{w'_\gamma, w'_{h,x}\}$ satisfies the formula:

$$w'_\gamma m_\gamma(w'_{h,x}) = w'_{h,y}\, m_{h,y}(w'_\gamma) \tag{2.5.25}$$

by the following calculation:

$$\begin{aligned} w'_\gamma m_\gamma(w'_{h,x}) &= w'_\gamma m_\gamma(v_x(w_{h,x})\, z_{h,x}) \\ &= [w'_\gamma m_\gamma(v_x(w_{h,x}))\, w'^*_\gamma]\, [w'_\gamma m_\gamma(z_{h,x})] \\ &= v_y(w_\gamma\, m_{1,\gamma}(w_{h,x})\, w^*_\gamma w_\gamma)\, z_\gamma\, m_\gamma(z_{h,x}) \\ &= \varphi(h,\gamma)\, v_y(w_{h,y}\, m_{h,y}(w_\gamma))\, \overline{\langle h, p(\gamma)\rangle}\, z_{h,y}\, m_{h,y}(z_\gamma) \quad (2.5.24) \\ &= v_y(w_{h,y}\, m_{h,y}(w_\gamma))\, z_{h,y}\, m_{h,y}(z_\gamma) \\ &= v_y(w_{h,y})\, z_{h,y}\, m_{h,y}(v_y(w_\gamma))\, m_{h,y}(z_\gamma) \quad (2.5.22) \\ &= w'_{h,y}\, m_{h,y}(w'_\gamma). \end{aligned}$$

Since $\{w'_\gamma\}$ and $\{w'_{h,x}\}$ are both cocycles respectively for $\{m_\gamma\}$ and $\{m_{h,x}\}$, formula (2.5.25) finally shows that

$$w'_{h,\gamma} = w'_\gamma m_\gamma(w'_{h,x}), \quad \gamma: x \to y, \tag{2.5.26}$$

defines a cocycle for the action m of $H \times \mathscr{K}$. Furthermore, (2.5.24) and (2.5.26) together yield the final conjugation

$$v_y\, \delta_{h,\gamma}\, v_x^{-1} = \mathrm{Ad}\,(w'_{h,\gamma})\, m_{h,\gamma}.$$

This completes the proof of Theorem 2.1.14. Q.E.D.

Chapter 3. Compact abelian groups

§ 3.1. Stable conjugacy

As stated in Chapter 1, the problem of deciding when two actions of a compact abelian group A are conjugate will be broken into two parts, the first of which will be stable conjugacy: two actions α and β of A on the von Neumann algebras $\mathcal{M}_1$ and $\mathcal{M}_2$ are called *stably conjugate* when the actions $\alpha\otimes\varrho$ and $\beta\otimes\varrho$ are conjugate on $\mathcal{M}_1\otimes\mathcal{B}(L^2(A))$ and $\mathcal{M}_2\otimes\mathcal{B}(L^2(A))$ respectively, where ϱ is the action $t\mapsto \operatorname{Ad} u_t$ on $\mathcal{B}(L^2(A))$, $\{u_t\}$ being the regular representation of A, $(u_t\xi)(s)=\xi(st)$ for $\xi\in L^2(A)$.

From now on G will be the dual of A, so that G is a countable discrete abelian group.

PROPOSITION 3.1.1. *Actions α and β of A on $\mathcal{M}_1$ and $\mathcal{M}_2$ are stably conjugate iff the dual actions $\hat{\alpha}$ and $\hat{\beta}$ of G on $\mathcal{M}_1\times_\alpha A$ and $\mathcal{M}_2\times_\beta A$ are cocycle conjugate.*

Proof. Suppose α and β are stably conjugate. Then there is an isomorphism $\theta\colon \mathcal{M}_1\otimes\mathcal{B}(L^2(A))\mapsto\mathcal{M}_2\otimes\mathcal{B}(L^2(A))$ such that $\theta(\alpha_t\otimes\varrho_t)\theta^{-1}=\beta_t\otimes\varrho_t$. But by [28] we know that $\alpha\otimes\varrho$ and $\beta\otimes\varrho$ are conjugate to the second dual actions of α and β respectively. Thus we may view θ as an isomorphism $\theta\colon (\mathcal{M}_1\rtimes_\alpha A)\rtimes_{\hat{\alpha}} G\mapsto(\mathcal{M}_2\rtimes_\beta A)\rtimes_{\hat{\beta}} G$ such that $\theta\hat{\hat{\alpha}}_t\theta^{-1}=\hat{\hat{\beta}}_t$. But this means that θ sends the spectral subspaces of $\hat{\hat{\alpha}}$ onto those of $\hat{\hat{\beta}}$ and we deduce the existence of an isomorphism $\theta'\colon \mathcal{M}_1\rtimes_\alpha A\mapsto\mathcal{M}_2\rtimes_\beta A$ and a $\hat{\beta}$ cocycle $\{v_g\}$ such that $\theta'\hat{\alpha}_g=\operatorname{Ad} v_g\hat{\beta}_g\theta'$, i.e. $\hat{\alpha}$ and $\hat{\beta}$ are cocycle conjugate.

If $\hat{\alpha}$ and $\hat{\beta}$ are cocycle conjugate, it is easy to construct an explicit isomorphism of $(\mathcal{M}_1\rtimes_\alpha A)\rtimes_{\hat{\alpha}} G$ onto $(\mathcal{M}_2\rtimes_\beta A)\rtimes_{\hat{\beta}} G$ conjugating the second dual actions. As above this means that $\alpha\otimes\varrho$ and $\beta\otimes\varrho$ are conjugate. Q.E.D.

Definition 3.1.2. Let $\mathfrak{S}$ be the set of all stable conjugacy classes of actions of A on von Neumann algebras $\mathcal{M}$. We define the *dual invariant* $\partial(\alpha)$ for an action α to be its stable conjugacy class.

We want to use the results of Chapter 2 to parametrize $\mathfrak{S}$ for actions of A on injective semifinite factors. For this we need to know two things. First that the crossed product is semifinite and injective, and second that the dual action of G on $\mathcal{Z}(\mathcal{M}\rtimes_\alpha A)$ is ergodic. The first fact follows from [5] and the compactness of A. The second follows from duality.

We summarize with a theorem.

THEOREM 3.1.3. *Two centrally ergodic actions α and β of A on semifinite injective von Neumann algebras* $\mathcal{M}_1$ and $\mathcal{M}_2$ *are stably conjugate iff* $\partial(\alpha)=\partial(\beta)$. *This condition is equivalent to*

(a) *The actions $\hat{\alpha}$ and $\hat{\beta}$ of G on $\mathcal{Z}(\mathcal{M}_1 \rtimes_\alpha A)$ and $\mathcal{Z}(\mathcal{M}_2 \rtimes_\beta A)$ are conjugate.*

(b) *The characteristic invariants of $\hat{\alpha}$ and $\hat{\beta}$ are the same.*

Proof. This follows immediately from (2.1.11) and Proposition 3.1.1. Q.E.D.

To enumerate all the cases covered by Theorem 3.1.3 would be tedious. So we shall make several comments.

(3.1.4) Since the action of G on $\mathcal{Z}(\mathcal{M} \rtimes_\alpha A)$ is ergodic, and by the classification in [5] of injective factors, $\mathcal{M} \rtimes_\alpha A$ must be $L^\infty(X,\mu)\otimes\mathcal{N}$ where $\mathcal{N}$ is an injective semifinite factor and either

(a) $X=\{1,2,\dots,m\}$, $m=1,2,\dots$

(b) $X=\mathbf{N}$

(c) $X=[0,1]$.

The possibilities for $\mathcal{N}$ are similar: either

(i) $\mathcal{N}\doteq M_n(\mathbf{C})$, $n=1,2,\dots$

(ii) $\mathcal{N}=\mathcal{B}(l^2(\mathbf{N}))$

(iii) $\mathcal{N}=\mathcal{R}$

(iv) $\mathcal{N}=\mathcal{R}_{0,1}$ $(=\mathcal{R}\otimes\mathcal{B}(l^2(\mathbf{N}))$.

Any combination of X and $\mathcal{N}$ may be obtained by an appropriate choice of $\mathcal{M}$, A and the action α. If A is fixed there are restrictions. It is easy to imagine, but hard to spell out, what happens in the type I_n case. As Katayama noted in [17], cases (i) and (iii) are excluded if there is a sequence $\{t_n\}$ in A, $t_n \to \mathrm{id}$, such that α_{t_n} is outer.

(3.1.5) It should not be forgotten that the principal hyperfinite groupoid $\mathcal{K}$ can be of type III. This can only ocur in the combination (c), (iv) of (3.1.4). Even for actions of $\mathbf{T}$ on $\mathcal{R}$ the action of $\mathbf{Z}$ on the center of the crossed product can be of type III_λ for any $\lambda \in [0,1]$ and any type III_0 action can occur. To explicitly construct such an example, take an arbitrary non-singular transformation T. The Radon-Nikodym derivative gives a homomorphism from the associated principal groupoid into $\mathbf{R}_*^+$ whose effect may be neutralized by baking the appropriate trace scaling action of the groupoid on $\mathcal{R}_{0,1}$. This gives an action of $\mathbf{Z}$ on an algebra of the form $\mathcal{M}=L^\infty([0,1])\otimes\mathcal{R}_{0,1}$ which preserves a faithful normal semifinite trace. The crossed product $\mathcal{M}\rtimes\mathbf{Z}$ is thus of type II_∞ and reducing the dual action by a finite projection in $\mathcal{M}$ gives the appropriate action of $\mathbf{T}$ on $\mathcal{R}$.

(3.1.6) There is already a well known invariant for abelian group actions: Connes' Γ-spectrum defined in [2]. It must appear somewhere in our scheme. Indeed by Theorem 3.2 of [2] the Γ-spectrum is the subgroup H of G defined in Chapter 2. It is thus part of the action of G on $\mathcal{Z}(\mathcal{M} \rtimes_\alpha A)$.

(3.1.7) Two actions may easily be stably conjugate without acting on the same algebra. For instance the action on $\mathcal{R}$ described in (3.1.5) is stably conjugate to the non-reduced dual action on $\mathcal{R}_{0,1}$.

(3.1.8) There are reasons for being interested in actions on factors. Here central ergodicity always holds but not all values of the characteristic invariant can occur. For $\mathcal{M}$ to be a factor is the same as for $\mathcal{M} \otimes \mathcal{B}(L^2(A))$ to be one so by duality, those characteristic invariants occurring for actions on factors are determined by (2.1.10). That is the action σ described on $\mathcal{A}_\mu N$ must be ergodic. In general this condition is not easy to handle but in the case where $H=G$ (the prime case of [24]), this condition corresponds simply to the non-degeneracy of the map defined by λ from G to $\hat{N}$.

(3.1.9) Closely related to stable conjugacy of actions of compact groups is cocycle conjugacy. The definition is the same as for discrete groups except that cocycles are continuous. Cocycle conjugacy implies conjugacy of the dual actions and hence stable conjugacy. The converse implication does not always hold. For instance, an ergodic action may have the same characteristic invariant as a non-ergodic one which implies stable conjugacy. But they cannot be cocycle conjugate since ergodic actions are stable ([24]) and cocycle conjugate actions are actually conjugate. Stable conjugacy and cocycle conjugacy coincide when the fixed point algebra is large enough, e.g. for finite groups on $\mathcal{R}$.

(3.1.10) It may seem unsatisfactory that the invariants of the classification are only defined after forming the crossed product. In fact it is possible to define them by a close examination of the spectral subspaces of the original action. One uses the Anzai skew product construction [20] to reconstruct the center of the crossed product and once this is done the characteristic invariant may be identified by examining the action of A on the relative commutant of the fixed point algebra $\mathcal{M}^A$.

(3.1.11) If A is finite, both A and $\hat{A}$ are discrete and we know by [14] that the characteristic invariant of the action itself is a complete invariant for stable conjugacy. This implies that in this case the characteristic invariant for the action and its dual determine each other. We leave it to the reader to work out the details.

(3.1.12) The group structure of the characteristic invariant is reflected by the fact that one may define an operation on stable conjugacy classes of actions, using the tensor product of two actions. If one restricts to actions having conjugate actions on the center $\mathcal{A}$ of the crossed product, and the same subgroup N as defined in § 2.1, this product defines a group structure isomorphic to $\Lambda(G, N, \mathcal{U}(\mathcal{A}))$.

(3.1.13) The ergodic case ($\mathcal{M}^A=\mathbf{C}$) was treated in [24]. In terms of the possibilities of (3.1.4), the only cases that can occur are (a) with (i) and (ii). The invariant of [24] is of course the characteristic invariant. When possibility (ii) occurs, the algebra $\mathcal{M}$ can actually be of type II. This is the case for some ergodic $\mathbf{T}^2$ actions, e.g. the "irrational rotation algebra."

(3.1.14) It is possible to identify the subgroup of A acting by inner automorphisms. It is related to the point spectrum of the action σ of A on $\mathcal{A}_\mu N$ defined in § 2.1. Indeed for actions on factors (i.e., σ is ergodic) the subgroup is exactly the point spectrum for σ. Typically it is not a closed subgroup of A, which is not too surprising since $\operatorname{Int}\mathcal{R}$ is not a closed subgroup of $\operatorname{Aut}\mathcal{R}$.

§ 3.2. The inner invariant

As in Definition 3.1.2 let $\mathfrak{S}$ be the set of all stable conjugacy classes of centrally ergodic actions of A on injective semifinite von Neumann algebras. For each $s\in\mathfrak{S}$ consider the set of all von Neumann algebras $\mathcal{M}$ which admit actions α with $\partial(\alpha)=s$. For each such pair $(s, \mathcal{M})$ choose some (model) action m of A on $\mathcal{M}$ with $\partial(m)=s$. Let p be the projection $\int_A 1\otimes u_t\,dt$ in $\mathcal{M}\otimes\mathcal{B}(L^2(A))$ where $t\mapsto u_t$ is the regular representation and dt is Haar measure on A. Let $\mathcal{X}_{s,\mathcal{M}}$ be the set of all projections in $\mathcal{M}\rtimes_m A$ (which will be identified with the fixed point algebra for the action $m\otimes\varrho$ on $\mathcal{M}\otimes\mathcal{B}(L^2(A))$, $\varrho_t=\operatorname{Ad} u_t$) of the form $\theta(p)$ where $\theta\in\operatorname{Aut}\mathcal{M}\otimes\mathcal{B}(L^2(A))$. The group of all such θ which commute with $m\otimes\varrho$ acts on $\mathcal{X}_{s,\mathcal{M}}$. Let $\mathcal{P}_{s,\mathcal{M}}$ be the orbit space for this action.

If α is an action of A on $\mathcal{M}$ with $\partial(\alpha)=s$, there is an automorphism θ of $\mathcal{M}\otimes\mathcal{B}(L^2(A))$ with $\theta(\alpha_t\otimes\varrho_t)\theta^{-1}=m_t\otimes\varrho_t$. Since p is fixed by $\alpha\otimes\varrho$, $\theta(p)\in\mathcal{M}\rtimes_m A$. Moreover if θ' is any other automorphism conjugating $\alpha\otimes\varrho$ and $m\otimes\varrho$ then $\theta^{-1}\theta'$ commutes with $m\otimes\varrho$. Thus the orbit of $\theta(p)$ in $\mathcal{P}_{s,\mathcal{M}}$ is well defined.

Definition 3.2.1. If $\alpha\colon A\to\operatorname{Aut}\mathcal{M}$ is a centrally ergodic action of A on the semifinite von Neumann algebra $\mathcal{M}$ with $\partial(\alpha)=s$, the *inner invariant* $\iota(\alpha)$ is the orbit of $\theta(p)$ in $\mathcal{P}_{s,\mu}$ where θ is some automorphism conjugating $\alpha\otimes\varrho$ and $m\otimes\varrho$.

PROPOSITION 3.2.2. *Let α and β be two actions of A on $\mathcal{M}$. Then α and β are conjugate iff $\partial(\alpha)=\partial(\beta)$ and $\iota(\alpha)=\iota(\beta)$.*

Proof. The necessity of the conditions is obvious.

Suppose $\partial(\alpha)=\partial(\beta)=s$. Since p is a minimal projection in the relative commutant of $\mathcal{M}\otimes\mathbf{C}$, α and β are conjugate to the restrictions of $\alpha\otimes\varrho$ and $\beta\otimes\varrho$ to $p\mathcal{M}\otimes\mathcal{B}(L^2(A))p$ respectively (note that $\alpha_t\otimes\varrho_t(p)=p$ for $t\in A$). But by the definition of $\iota(\alpha)$, if e and f are projections in $\mathcal{X}_{s,\mathcal{M}}$ representing $\iota(\alpha)$ and $\iota(\beta)$ then α and β are conjugate to the restrictions of $m\otimes\varrho$ to $e\mathcal{M}\otimes\mathcal{B}(L^2(A))e$ and $f\mathcal{M}\otimes\mathcal{B}(L^2(A))f$, respectively. Q.E.D.

Theorem 3.1.3 and Proposition 3.2.2 together prove Theorem 1.

In the spirit of § 3.1, rather than trying to give an exhaustive list of all cases, we shall make some illustrative comments.

(3.2.3) If A is finite and $\mathcal{M}=\mathcal{R}$, we ought to retrieve the space of [14]. Let us describe how this happens. Let m be a model action of A on $\mathcal{R}$ and let $a\mapsto u_a$ be the implementing unitary representation in the crossed product so that elements of $\mathcal{R}\rtimes_m A$ are sums $\Sigma_{t\in A}x_t u_t$, $x_t\in\mathcal{R}$. Then $p=(1/|A|)\Sigma_{t\in A}u_t$ and s is given by some element of $\Lambda(A,N,\mathbf{T})$ for some subgroup N of A. The space $\mathcal{X}_{p,m}$ is the set of all projections in $\mathcal{R}\rtimes_m A$ which are equivalent to p via automorphisms of $\mathcal{R}\otimes\mathcal{B}(L^2(A))$. Also if tr is the trace on $\mathcal{R}$, the restriction of the normalized trace on $\mathcal{R}\otimes\mathcal{B}(L^2(A))$ to $\mathcal{R}\rtimes_m A$ is given by $\mathrm{Tr}(\Sigma_{t\in A}a_t u_t)=\mathrm{tr}(a_1)$. Thus $\mathcal{X}_{p,m}$ is the set of all projections q in $\mathcal{R}\rtimes_m A$ which satisfy $\mathrm{Tr}(q)=1/|A|$. We must decide when two projections e and f are in the same orbit under the action of the group of automorphisms of $\mathcal{R}\otimes\mathcal{B}(L^2(A))$ which commute with $m\otimes\varrho$. An obvious (normal) subgroup of this group is the group of inner automorphisms of $\mathcal{R}\rtimes_m A$. Thus $\mathcal{P}_{s,\mathcal{R}}$ will be a quotient of the set of equivalence classes of projections $e\in\mathcal{R}\rtimes_m A$ with $\mathrm{Tr}(e)=|a|^{-1}$. This set is a simplex whose vertices are indexed by minimal projections in the center of $\mathcal{R}\rtimes_m A$. Other automorphisms of $\mathcal{R}\rtimes_m A$ are given by the dual action. Since the center of $\mathcal{R}\rtimes_m A$ can be identified with $(\mathbf{C}_\mu N)^G$ (see [14]), we conclude that $\mathcal{P}_{s,\mathcal{R}}$ is the quotient of a simplex by an action of $H^1(N,\mathbf{T})$ coming from permutations of the vertices. This ties up with [14].

(3.2.4) It can easily happen that the space $\mathcal{P}_{s,\mathcal{M}}$ is reduced to a point. This is the case in the examples of (3.2.3) when $(\mathbf{C}_\mu N)^G=\mathbf{C}$ and more generally for prime actions with large fixed point algebras (see [16]).

Appendix: Bures-Connes-Krieger-Sutherland cohomology theorem

In his lecture notes, Sutherland states the following important result attributing to Connes and Krieger:

THEOREM. *Let $\mathcal{G}$ be a hyperfinite measured groupoid, G a Polish group and ϱ_1, ϱ_2 be Borel homomorphisms of $\mathcal{G}$ into G such that*

$$\varrho_1 = \varrho_2 \mod \bar{H},$$

where H is a normal Borel subgroup of G and $\bar{H}$ means the closure of H. Then there exist Borel measurable maps $h: \mathcal{G} \mapsto H$, $P: X = \mathcal{G}^{(0)} \mapsto \bar{H}$ such that

$$\varrho_2(\gamma) = h(\gamma) P(r(\gamma)) \varrho_1(\gamma) P(s(\gamma))^{-1}, \quad \gamma \in \mathcal{G},$$

where r and s denote the range and the source maps of $\mathcal{G}$ onto X, respectively.

Note, by definition a hyperfinite measured groupoid means a Borel equivalence relation $\mathcal{G} \subset X \times X$, where $\mathcal{G} = \bigcup_{k=1}^{\infty} \mathcal{G}_k$ and each $\mathcal{G}_k$ is a finite equivalence relation of type I_{n_k} in the sense that every orbit has precisely n_k points.

His proof uses however a lemma which requires a two sided invariant complete metric on the group G. Since we do not want to have such a restriction, we present a modified version of the proof. In fact, we do not need to change it a lot. Simply, we replace small positive numbers ε and δ by small measurable functions $\varepsilon(\cdot)$ and $\delta(\cdot)$ in Sutherland's proof.

LEMMA 1. *Let $\mathcal{G}\,(=\mathcal{G}_k)$ be a type I_n equivalence relation on X, and ϱ_1, ϱ_2, G, and H be as above. Let d be a complete metric on G compatible with the topology. If $x \in X \mapsto \varepsilon(x) > 0$ is a Borel measurable function, then there are Borel measurable functions $h: \mathcal{G} \mapsto H$, $P: X \mapsto H$ such that*

(a) $\varrho_2(\gamma) = h(\gamma) P(r(\gamma)) \varrho_1(\gamma) P(s(\gamma))^{-1}, \quad \gamma \in \mathcal{G}$;

(b) $d(P(x), 1) < \varepsilon(x)$, $x \in X$;

(c) $P \equiv 1$ *on some section for* $\mathcal{G}$.

Proof. By assumption, we can choose measurable sets $\{A(j): 0 \leq j \leq n-1\}$ such that $X = \bigcup_{j=0}^{n-1} A(j)$ and each $A(j)$ is a section for $\mathcal{G}$.

Set $P(x) = 1$ for every $x \in A(0)$. We then define h on $s^{-1}(A(0))$ as follows:

For $\gamma = (y, x) \in \mathcal{G}$, $x \in A(0)$, consider

$$H(\gamma) = \{h \in H: d(h^{-1} \varrho_2(\gamma) \varrho_1(\gamma)^{-1}, 1) < \varepsilon(x)\}.$$

By assumption, $H(\gamma)$ is a non-empty Borel subset of H for each $\gamma \in s^{-1}(A(0))$. By the von Neumann measurable cross-section theorem, there exists a measurable function $h: \gamma \in s^{-1}(A(0)) \mapsto h(\gamma)$. We remove a null set N from $A(0)$ so that h is Borel measurable on $s^{-1}(A(0)\setminus N)$. Remove the saturation $\mathscr{G}(N)$ of N from X and replace $\mathscr{G}$ by $\mathscr{G}\cap(X\setminus N)\times(X\setminus N)$. Furthermore, if $\gamma=(x,x)$, $x\in A(0)$, then we can choose $h(\gamma)=1$. We then have a Borel function h on $s^{-1}(A(0))$. We set:

$$P(y)=h(\gamma)^{-1}\varrho_2(\gamma)\varrho_1(\gamma)^{-1},\quad y=r(\gamma),\quad \gamma\in s^{-1}(A(0)).$$

Now, P is defined everywhere in X and h is defined on $s^{-1}(A(0))$ and

$$\varrho_2(\gamma)=h(\gamma)P(r(\gamma))\varrho_1(\gamma)P(s(\gamma))^{-1},\quad \gamma\in s^{-1}(A(0)).$$

We then extend h to all of $\mathscr{G}$ in two stages: First, if $\gamma\in r^{-1}(A(0))$, then $\gamma^{-1}\in s^{-1}(A(0))$, so we set

$$h(\gamma)=P(r(\gamma))\varrho_1(\gamma)P(s(\gamma))^{-1}h(\gamma^{-1})^{-1}P(s(\gamma))\varrho_1(\gamma)^{-1}P(r(\gamma))^{-1}.$$

Second, every $\gamma\in\mathscr{G}$ admits a unique decomposition $\gamma=\gamma_1\gamma_2$ where $\gamma_1\in r^{-1}(A(0))$, $\gamma_2\in s^{-1}(A(0))$. So we set

$$h(\gamma)=h(\gamma_1)P(r(\gamma_1))\varrho_1(\gamma_1)P(s(\gamma_1))^{-1}h(\gamma_2)P(s(\gamma_1))\varrho_1(\gamma)^{-1}P(r(\gamma))^{-1}.$$

It is now routine to check that h and P have the required property.

LEMMA 2. *Let $\mathscr{G}, X, \varrho_1, \varrho_2, G, N$ and $\varepsilon(\cdot)>0$ be as in the previous lemma, and let h and P be as in the conclusion of Lemma 1. Suppose that $\mathscr{H}$ $(=\mathscr{G}_{k+1})\supset\mathscr{G}$ is a type I_m equivalence relation. If $\delta>0$ is given, then there exist Borel maps $k\colon\mathscr{H}\mapsto H$, $Q\colon X\mapsto\bar{H}$ such that*

(a) $\varrho_2(\gamma)=k(\gamma)Q(r(\gamma))\varrho_1(\gamma)Q(s(\gamma))^{-1},\quad \gamma\in\mathscr{H}$,
(b) $k(\gamma)=h(\gamma),\quad \gamma\in\mathscr{G}$,
(c) $d(Q(x),P(x))<\delta,\quad x\in X$.

Proof. Let $\{A(j)\colon 0\leqslant j\leqslant n-1\}$ be as in the proof of Lemma 1; we assume that $P(x)\equiv 1$, $x\in A(0)$.

Let $\mathscr{H}_0=\mathscr{H}|_{A(0)}=\mathscr{H}\cap[A(0)\times A(0)]$. For each $x_0\in A(0)$, we have only finitely many x's such that $(x,x_0)\in\mathscr{G}$. Set

$$B(x_0)=\{g\in G\colon \sup_{(x,x_0)\in\mathscr{G}} d(P(x)\varrho_1(x,x_0)g\varrho_1(x,x_0)^{-1},P(x))\geqslant\delta\}$$

$$\varepsilon_0(x)=\operatorname{dist}(1,B(x_0))>0.$$

It follows that $\varepsilon_0(\cdot)$ is a measurable function on X. Throwing a saturated null set away from X, we may assume that ε_0 is a Borel function on X. We apply Lemma 1 to $\mathscr{H}_0$ and ε_0 to find Borel functions $Q_0: A(0) \mapsto \bar{H}$ and $k_0: \mathscr{H}_0 \mapsto H$ such that

$$\varrho_2(\gamma) = k_0(\gamma)\, Q_0(r(\gamma))\, \varrho_1(\gamma)\, Q_0(s(\gamma))^{-1}, \quad \gamma \in \mathscr{H}_0;$$

$$d(Q_0(y), 1) < \varepsilon_0(y), \quad y \in A(0).$$

For every $(y,x) \in \mathscr{H}$, there exist uniquely $y_0, x_0 \in A(0)$ such that (y, y_0), $(x, x_0) \in \mathscr{G}$ and $(y_0, x_0) \in \mathscr{H}$ we then set, for every $(y,x) \in \mathscr{H}$,

$$\begin{aligned} k(y,x) = {} & h(y,y_0)\, P(y)\, \varrho_1(y,y_0)\, k_0(y_0,x_0)\, \varrho_1(y,y_0)^{-1} P(y)^{-1} \\ & \times P(y)\, \varrho_1(y,y_0)\, Q_0(y_0)\, \varrho_1(y_0,x_0)\, Q_0(x_0)^{-1} h(x_0,x) \\ & \times Q_0(x_0)\, \varrho_1(y_0,x_0)^{-1} Q_0(y_0)^{-1} \varrho_1(y,y_0)^{-1} P(y)^{-1}; \end{aligned}$$

$$Q(x) = P(x)\, \varrho_1(x,x_0)\, Q_0(x_0)\, \varrho_1(x,x_0)^{-1}.$$

We note that $Q(x_0) = Q_0(x_0)$ for $x_0 \in A(0)$. Furthermore, $(y,x) \in \mathscr{G}$ if and only if $y_0 = x_0$, so that we have, by a direct computation, $h(y,x) = k(y,x)$ for every $(y,x) \in \mathscr{G}$. Hence k extends h. Also, another direct computation shows that

$$\varrho_2(\gamma) = k(\gamma)\, Q(r(\gamma))\, \varrho_1(\gamma)\, Q(s(\gamma))^{-1}, \quad \gamma \in \mathscr{H}.$$

Finally, we have, since $d(Q_0(x_0), 1) < \varepsilon(x_0)$,

$$d(Q(x), P(x)) = d(P(x)\, \varrho_1(x,x_0)\, Q_0(x_0)\, \varrho_1(x,x_0)^{-1}, P(x)) < \delta$$

as required. Q.E.D.

Proof of the theorem. Let $\mathscr{G} = \bigcup_{k=1}^{\infty} \mathscr{G}_k, \varrho_1, \varrho_2, G$ and H be as in the theorem. Applying Lemma 1 first and then Lemma 2 inductively, we find sequences of Borel maps $h_k: \mathscr{G}_k \mapsto H$, $P_k: X \mapsto \bar{H}$ such that

(a) $\varrho_2(\gamma) = h_k(\gamma)\, P_k(r(\gamma))\, \varrho_1(\gamma)\, P_k(s(\gamma))^{-1}, \quad \gamma \in \mathscr{G}_k;$
(b) $h_{k+1}(\gamma) = h_k(\gamma), \quad \gamma \in \mathscr{G}_k;$
(c) $d(P_k(x), P_{k+1}(x)) < 2^{-k}, \quad x \in X.$

Set $h(\gamma) = h_k(\gamma)$ for $\gamma \in \mathscr{G}_k$. By (b), h is well-defined and a Borel function. Set $P(x) = \lim_{k\to\infty} P_k(x)$, which exists by (c). Again P is a Borel map. From (a), we get

$$\varrho_2(\gamma) = h(\gamma)\, P(r(\gamma))\, \varrho_1(\gamma)\, P(s(\gamma))^{-1}, \quad \gamma \in \mathscr{G}.$$

Q.E.D.

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Received June 16, 1982
Received in revised form December 20, 1983