L^p and mean value properties of subharmonic functions on Riemannian manifolds

by

PETER LI[1] and RICHARD SCHOEN[2]

Purdue University
West Lafayette, IN, U.S.A.

University of California
Berkeley, CA, U.S.A.

§ 0. Introduction

Let M be a complete noncompact Riemannian manifold without boundary. We denote the Laplace operator with respect to the Riemannian metric by Δ. Consider the equation for harmonic functions,

$$\Delta f \equiv 0, \tag{0.1}$$

defined on M. To ensure a uniqueness property on equation (0.1) it is necessary to restrict f to lie in a suitable function space. Some of the most natural spaces are those consisting of L^p functions on M, denoted by $L^p(M)$, where integration is defined with respect to the Riemannian measure. In this setting, uniqueness of (0.1) means that if $f \in L^p(M)$ for some $0<p\leq\infty$, then f must be identically constant. We remark that when $p=\infty$ all constant functions satisfy (0.1) and belong to $L^\infty(M)$. On the other hand, for $0<p<\infty$, while all constant functions satisfy (0.1), they belong to $L^p(M)$ iff M has finite volume, unless the constant is zero.

For the sake of simplicity we say a manifold satisfies property $\mathcal{H}_p$ for $p\in(0,\infty]$ if every L^p harmonic function on M is constant. We also say that M satisfies property $\mathcal{S}_p$ if every nonnegative L^p subharmonic function on M is constant. Observing that the absolute value of a harmonic function is a nonnegative subharmonic function (in the weak sense), M satisfying $\mathcal{S}_p$ implies it also satisfies $\mathcal{H}_p$.

The first result towards understanding the uniqueness of (0.1) was due to

(1) Research supported in part by a Sloan Fellowship and an NSF grant, MCS81-07911.
(2) Research supported in part by an NSF grant, MCS-80-23356.

Greene–Wu [7]. In fact, they proved that if M is complete with nonnegative sectional curvature, then M satisfies property $\mathcal{S}_p$ for $p \in [1, \infty)$.

In 1975 Yau [14] showed that if M is complete and has nonnegative Ricci curvature, then M satisfies property $\mathcal{H}_\infty$. This result is in a way the best possible, since there exist infinitely many bounded harmonic functions on a simply connected manifold with sectional curvature identically -1. In fact, recent work of Sullivan [12], Anderson [1], and Anderson–Schoen [2] show and give a thorough understanding of the existence of bounded harmonic functions on a simply connected manifold with strongly negative curvature.

In [15] Yau showed that if $p \in (1, \infty)$ any complete Riemannian manifold satisfies property $\mathcal{S}_p$. He also proved that by only assuming completeness on M, any nonnegative L^p harmonic function must be constant for $p \in (0, 1)$. Up to that point, the case $p=1$ was completely unknown. Also, for $p \in (0, 1)$, without assuming nonnegativity of the harmonic function, uniqueness is still open.

It turns out that unless one imposes an addition hypothesis on the geometry of M, the property $\mathcal{S}_p$ (hence $\mathcal{H}_p$) is in general not valid for $p \in (0, 1]$. Indeed, in an unpublished preprint of Chung [5], he gave an example of a complete two-dimensional manifold with a nonconstant L^1 harmonic function. Later Sullivan provided examples of manifolds with nonconstant L^p harmonic functions for $p<1$ and sufficiently small.

Recently Garnett [6] showed that if M is complete and has bounded geometry, then M satisfies property $\mathcal{H}_1$. She also proved that on such manifolds any L^p harmonic function which is bounded from below must be constant if $p \in (0, 1)$.

The purpose of this article is to establish sharp conditions on the curvature of M to ensure the property $\mathcal{S}_p$ for the unknown cases $p \in (0, 1]$. In § 1 we proved a Poincaré inequality for geodesic balls on a manifold with Ricci curvature bounded from below by $-(n-1)k$, $k \geq 0$. Our proof is completely local but interior. As a corollary, a lower bound of the first Dirichlet eigenvalue of a geodesic ball is derived. This can be viewed as a local version of the λ_1 estimate for compact manifolds given in [10]. Combined with a Harnack-type inequality of Cheng–Yau [4], we prove a mean value inequality for nonnegative subharmonic functions on M.

In § 2 we apply the mean value inequality of § 1 to prove that if M is complete with

$$\mathrm{Ric}_M(x) \geq -c(1+r_0(x)^2)[\log(1+r_0(x))^2]^{-\alpha}, \quad c>0,$$

for all $x \in M$, where $r_0(x)$ is the distance function from some fixed point $x_0 \in M$, then M satisfies property $\mathcal{S}_1$.

We also prove that if M is complete and

$$\mathrm{Ric}_M(x) \geq -\delta(n)\, r_0(x)^{-2}$$

for all $x \in M$, where $\delta(n)>0$ is some sufficiently small constant depending only on the dimension of M, then M satisfies property $\mathcal{S}_p$ for $p \in (0, 1)$. We should point out that our argument for the cases $p \in (0, 1)$ relies on the fact that manifolds with the above Ricci curvature restriction must have infinite volume. This fact was first proved by Yau [15] for $\mathrm{Ric}_M \geq 0$, by Wu [13] for $\mathrm{Ric}_M(x) \geq -cr_0(x)^{-(2+\varepsilon)}$, and finally by Cheeger–Gromov–Taylor [3] for the above case.

We also observed that since our proof only utilized the mean value inequality, for manifolds satisfying either (i) simply connected with nonpositive sectional curvature, or (ii) both Ric_M and volume of unit balls are bounded from below, must satisfy property $\mathcal{S}_p$ for all $p \in (0, \infty)$. In particular, Garnett's theorems follow as a consequence of (ii).

Finally, in the last section we utilize Sullivan's method to construct two-dimensional examples of: (1) A manifold with sectional curvature $K(x) \sim -cr_0(x)^{2+\varepsilon}$ which property $\mathcal{H}_1$ (hence $\mathcal{S}_1$) is not valid. (2) A manifold with infinite volume which possesses nonnegative nonconstant L^1 harmonic functions. (3) A manifold with sectional curvature $K(x) \sim -cr_0(x)^{-2}$, for $c>2$, which property $\mathcal{H}_p$ (hence $\mathcal{S}_p$) is not valid for $p \in (0, 1)$.

We remark that examples (1) and (3) indicate the sharpness of our theorems (Theorem 2.4 and 2.5). Moreover, both examples are manifolds with finite volume, hence are probabilistically complete. This provides a counter-example to the speculations that all probabilistically complete manifolds have property $\mathcal{H}_1$. If the Ricci curvature of M behaves like $\mathrm{Ric}_M(x) \sim -cr_0(x)^2$, the question whether M satisfies property $\mathcal{S}_1$ lies on the borderline of the scoop of Theorem 2.4 and example 1, hence is yet to be answered. However, we feel that the answer should be affirmative. Example 2 indicates that Yau's assumption that $p \in (0, 1)$ is essential for the nonexistence of a nonnegative nonconstant L^p harmonic function.

Our method in § 1 and § 2, in general, yields estimates on the growth of nonnegative subharmonic functions with respect to its L^p norms and the lower bound of the Ricci curvature on any complete Riemannian manifold.

§ 1. Poincaré and mean value inequalities

In this section we prove a new version of the Poincaré inequality on a geodesic ball and then apply this inequality together with an estimate of Cheng–Yau [4] on harmonic functions to establish a rather sharp mean value inequality for nonnegative subharmonic functions. Throughout this paper M will denote a connected n-dimensional Riemannian manifold. For a point $x \in M$, the open geodesic ball of radius R centered at x will be denoted $B_R(x)$ or simply B_R if the center point is clear from the context. The Laplace operator on M will be denoted by Δ, i.e., Δf is the trace of the covariant Hessian of the function f.

Throughout this section M will be compact with (possibly empty) boundary and $k \geq 0$ will be a number such that the Ricci curvature of M satisfies

$$\mathrm{Ric}_M \geq -(n-1)k.$$

THEOREM 1.1. *Let $x_0 \in M$ and $R>0$. If $\partial M=\varnothing$ assume that the diameter D of M satisfies $D\geq 2R$. If $\partial M\neq\varnothing$, assume that the distance from x_0 to ∂M is at least $5R$. For every function Φ on $B_R(x_0)$ which vanishes on $\partial B_R(x_0)$ we have the Poincaré inequality*

$$\int_{B_R(x_0)} |\Phi|\, dV \leq c_1 \int_{B_R(x_0)} |\nabla\Phi|\, dV$$

where $c_1=R(1+\sqrt{k}\,R)^{-1} e^{2n(1+\sqrt{k}\,R)}$.

The following corollary is then a standard consequence of the Poincaré inequality.

COROLLARY 1.1. *Under the hypotheses of Theorem 1.1 for any $p \geq 1$ we have the inequality*

$$\int_{B_R(x_0)} |\Phi|^p\, dV \leq (pc_1)^p \int_{B_R(x_0)} |\nabla\Phi|^p\, dV$$

for Φ vanishing on $\partial B_R(x_0)$. In particular, the first Dirichlet eigenvalue of $B_R(x_0)$ satisfies

$$\lambda_1(B_R(x_0)) \geq (2R)^{-2}(1+\sqrt{k}\,R)^2 e^{-4n(1+\sqrt{k}\,R)}.$$

The corollary follows from the theorem by replacing $|\Phi|$ by $|\Phi|^p$ and using Hölder's inequality to get

$$\int_{B_R(x_0)} |\Phi|^p\, dV \leq pc_1 \int_{B_R(x_0)} |\Phi|^{p-1}|\nabla\Phi|\, dV \leq pc_1 \left(\int_{B_R} |\Phi|^p\right)^{(p-1)/p} \|\nabla\Phi\|_{L^p(B_R)}.$$

The corollary now follows.

Proof of Theorem 1.1. The hypotheses imply that the boundary of $B_{2R}(x_0)$ is not empty, so let $x_1 \in \partial B_{2R}(x_0)$. Let $r_1(x)$ denote the distance from x to x_1. Comparison theorems (see [8]) imply that

$$\Delta r_1 \leqslant \begin{cases} (n-1)\sqrt{k}\coth(\sqrt{k}\,r_1) & \text{if } k>0 \\ (n-1)r_1^{-1} & \text{if } k=0. \end{cases}$$

In either case we see that

$$\Delta r_1 \leqslant (n-1)r_1^{-1}+(n-1)\sqrt{k}. \tag{1.1}$$

Technically speaking this inequality holds only at points not on the cut locus of x_1; however, it is well known (see [15]) that the inequality effectively holds globally on M. For example, one can see that (1.1) holds in the distributional sense, i.e., if $\zeta \geqslant 0$ is a smooth function with compact support in M, then

$$\int_M r_1 \Delta\zeta\, dV \leqslant \int_M \left[(n-1)r_1^{-1}+(n-1)\sqrt{k}\right]\zeta\, dV.$$

We will use the inequality in this sense.

Next observe that the hypotheses on M imply that $B_{3R}(x_1)\cap\partial M=\varnothing$, and $B_R(x_0)\subset B_{3R}(x_1)-B_R(x_1)$. We have for $\alpha>0$ to be chosen

$$\Delta e^{-\alpha r_1}=\alpha e^{-\alpha r_1}(-\Delta r_1+\alpha).$$

Thus if we consider only points in $B_R(x_0)$ we have

$$\Delta e^{-\alpha r_1} \geqslant \alpha e^{-3\alpha R}(\alpha-(n-1)R^{-1}-(n-1)\sqrt{k}).$$

Setting $\alpha=n(R^{-1}+\sqrt{k})$ then gives on $B_R(x_0)$

$$\Delta e^{-\alpha r_1} \geqslant \alpha(R^{-1}+\sqrt{k})e^{-3\alpha R}.$$

Let Φ be any function on $B_R(x_0)$ vanishing on $\partial B_R(x_0)$. We multiply by $|\Phi|$ and integrate by parts

$$\alpha(R^{-1}+\sqrt{k})e^{-3\alpha R}\int_{B_R(x_0)} |\Phi|\, dV \leqslant \alpha\int_{B_R(x_0)} e^{-\alpha r_1}|\nabla\Phi|\, dV.$$

Since $r_1 \geqslant R$ on $B_R(x_0)$ we therefore have

$$\int_{B_R(x_0)} |\Phi|\, dV \leqslant R(1+\sqrt{k}\,R)^{-1}e^{2n(1+\sqrt{k}R)}\int_{B_R(x_0)} |\nabla\Phi|\, dV$$

which finishes the proof of Theorem 1.1.

We will now proceed to the mean value inequality. An important part of the proof is the following estimate of Cheng and Yau [4].

LEMMA 1.1. *Suppose M is a compact manifold with (possibly empty) boundary, and suppose $x_0 \in M$, $R>0$ are such that $B_R(x_0) \cap \partial M = \emptyset$. If h is a positive harmonic function on $B_R(x_0)$, then the following estimate holds*

$$\max_{x \in B_R(x_0)} (R - r_0(x)) |\nabla \log h|(x) \leq c_2(1 + \sqrt{k}\, R)$$

where c_2 depends only on n. Here $r_0(x)$ denotes distance from x_0 to x.

THEOREM 1.2. *Let M, x_0, R be as in Theorem 1.1. Suppose $v \geq 0$ is a subharmonic function defined on $B_R(x_0)$. There is a constant c_3 depending only on n such that for any $\tau \in (0, 1/2)$ we have*

$$\sup_{B_{(1-\tau)R}(x_0)} v^2 \leq \tau^{-c_3(1+\sqrt{k}\, R)} \fint_{B_R(x_0)} v^2\, dV$$

where $\fint_S f\,dV$ denotes the average value of f on the set S.

Proof. The result for any $R>0$ can be gotten from the case $R=1$ by rescaling the metric, so for the sake of notational simplicity we assume $R=1$. Let h be the harmonic function on $B_{1-2^{-1}\tau}(x_0)$ which agrees with v on the boundary. Then h is positive in $B_{1-2^{-1}\tau}(x_0)$ (unless v is identically zero, in which case the theorem is trivial). Since v is subharmonic we have $v \leq h$ in $B_{1-2^{-1}\tau}$. By Lemma 1.1 we have on $B_{1-\tau}(x_0)$

$$|\nabla \log h| \leq c_2(1+\sqrt{k})(1-r_0)^{-1}.$$

For any $x \in B_{1-\tau}(x_0)$ we can integrate along a minimizing geodesic from x_0 to x, hence concluding

$$\left|\log \frac{h(x)}{h(x_0)}\right| \leq c_2(1+\sqrt{k}) \int_0^{1-\tau} (1-s)^{-1}\, ds = c_2(1+\sqrt{k}) \log 1/\tau.$$

Thus for any two points $x, y \in B_{1-\tau}(x_0)$ we have

$$\frac{h(x)}{h(y)} = \frac{h(x)}{h(x_0)} \cdot \frac{h(x_0)}{h(y)} \leq \tau^{-2c_2(1+\sqrt{k})}.$$

In particular we have

$$\sup_{B_{1-\tau}(x_0)} v^2 \leqslant \sup_{B_{1-\tau}(x_0)} h^2 \leqslant \tau^{-4c_2(1+\sqrt{k})} \inf_{B_{1-\tau}(x_0)} h^2.$$

Thus we clearly have

$$\sup_{B_{1-\tau}} v^2 \leqslant \tau^{4c_2(1+\sqrt{k})} \fint_{B_{(1-\tau)}} h^2 \tag{1.2}$$

where $\fint_S f$ denotes the average value of f on S. The remainder of the proof consists in estimating the average value of h^2 by the average of v^2. We first note

$$\int_{B_{1-\tau}} h^2 \, dV \leqslant 2 \int_{B_{1-2^{-1}\tau}} (h-v)^2 \, dV + 2 \int_{B_1} v^2 \, dV \tag{1.3}$$

by the triangle inequality. Since $(h-v)$ vanishes on $\partial B_{1-2^{-1}\tau}(x_0)$ we can apply Corollary 1.1 to show

$$\begin{aligned}\int_{B_{(1-2^{-1}\tau)}} (h-v)^2 \, dV &\leqslant 4e^{4n(1+\sqrt{k})} \int_{B_{(1-2^{-1}\tau)}} |\nabla h - \nabla v|^2 \, dV \\ &\leqslant 8e^{4n(1+\sqrt{k})} \int_{B_{(1-2^{-1}\tau)}} (|\nabla h|^2 + |10v|^2) \, dV\end{aligned}$$

where we have used the triangle inequality. Since the Dirichlet integral of h is least among all functions which coincide with h on the boundary, we have

$$\int_{B_{1-2^{-1}\tau}} (h-v)^2 \, dV \leqslant 16e^{4n(1+\sqrt{k})} \int_{B_{1-2^{-1}\tau}} |\nabla v|^2 \, dV. \tag{1.4}$$

We now use the fact that v is subharmonic to estimate the Dirichlet integral of v in terms of the L_2 norm of v. We have for any Φ with compact support in $B_1(x_0)$

$$\int_{B_1} \Phi^2 v \Delta v \, dV \geqslant 0.$$

Integrating by parts we then get

$$\begin{aligned}\int_{B_1} |\nabla v|^2 \Phi^2 \, dV &\leqslant 2 \int_{B_1} \Phi v |\nabla v| \, |\nabla \Phi| \, dV \\ &\leqslant 2\left(\int_{B_1} \Phi^2 |\nabla v|^2 \, dV\right)^{1/2} \left(\int_{B_1} |\nabla \Phi|^2 v^2 \, dV\right)^{1/2}.\end{aligned}$$

Thus we have

$$\int_{B_1} |\nabla v|^2 \Phi^2 \, dV \leq 4 \int_{B_1} v^2 |\nabla \Phi|^2 \, dV.$$

Choosing Φ to be a function of r_0 which is one on $B_{1-2^{-1}\tau}(x_0)$, zero on $\partial B_1(x_0)$, and satisfying $|\nabla\Phi| \leq 2\tau^{-1}$ we get

$$\int_{B_{1-2^{-1}\tau}} |\nabla v|^2 \, dV \leq 16\tau^{-2} \int_{B_1} v^2 \, dV.$$

Combining this inequality with (1.2), (1.3), and (1.4)

$$\sup_{B_{(1-\tau)}} v^2 \leq c_4 \tau^{-c_4(1+\sqrt{k})} e^{4n(1+\sqrt{k})} \operatorname{Vol}(B_{1/2}(x_0))^{-1} \int_{B_1} v^2 \, dV \tag{1.5}$$

where we have used the fact that $\tau \leq 1/2$. To finish the proof we estimate the volume of B_1 in terms of the volume of $B_{1/2}$. Recall the bound for Δr^2

$$\begin{aligned} \Delta r^2 &\leq 2 + 2(n-1) r\sqrt{k} \coth(r\sqrt{k}) \\ &\leq 2(n + 2(n-1) r\sqrt{k}). \end{aligned}$$

Integrating over $B_t(x_0)$ and applying Stokes' theorem, we get

$$t \frac{d}{dt} \operatorname{Vol}(B_t(x_0)) \leq (n + 2(n-1) t\sqrt{k}) \operatorname{Vol}(B_t(x_0)).$$

Integrating over t from $[1/2, 1]$ we then have

$$\operatorname{Vol}(B_1(x_0)) \leq e^{2n(1+\sqrt{k})} \operatorname{Vol}(B_{1/2}(x_0)). \tag{1.6}$$

Combined with (1.5), we have completed the proof of Theorem 1.2.

§ 2. L^p harmonic and subharmonic functions

In this section we apply the mean value inequality to study growth properties of harmonic functions on complete Riemannian manifolds. Our first result shows that the L^p mean value inequality for any $p \in (0, 2]$ is a formal consequence of that given in Theorem 1.2.

THEOREM 2.1. *Let M, x_0, R be as in Theorem 1.1. Let $v \geq 0$ be a subharmonic function on $B_R(x_0)$. For any $p \in (0, 2]$ there is a constant c_5 depending only on n and p such that*

$$\sup_{B_{(1-\tau)R}(x_0)} v^p \leq \tau^{-c_5(1+\sqrt{k}R)} \mathrm{Vol}(B_R(x_0))^{-1} \int_{B_R(x_0)} v^p \, dV$$

for any $\tau \in (0, 1/2)$.

Proof. By Theorem 1.2 we have for any $\delta \in (0, 1/2]$, $\theta \in [1/2, 1-\delta]$

$$\sup_{B_{\theta R}} v^2 \leq \delta^{-c_3(1+\sqrt{k}R)} \fint_{B_{(\theta+\delta)R}} v^2 \, dV.$$

Since $\theta + \delta \geq 1/2$, this inequality implies

$$\sup_{B_{\theta R}} v^2 \leq \delta^{-c_3(1+\sqrt{k}R)} \mathrm{Vol}(B_{2^{-1}R})^{-1} \int_{B_{(\theta+\delta)R}} v^2 \, dV.$$

On the other hand we have

$$\begin{aligned} \int_{B_{(\theta+\delta)R}} v^2 \, dV &\leq \sup_{B_{(\theta+\delta)R}} v^{2-p} \int_{B_{(\theta+\delta)R}} v^p \, dV \\ &\leq \left(\sup_{B_{(\theta+\delta)R}} v^2 \right)^{1-p/2} \int_{B_R} v^p \, dV. \end{aligned}$$

If we set

$$M(\theta) = \sup_{B_{\theta R}} v^2$$

$$K = \mathrm{Vol}(B_{2^{-1}R})^{-1} \int_{B_R} v^p \, dV,$$

we have shown for any $\delta \in (0, 1/2]$, $\theta \in [1/2, 1-\delta]$

$$M(\theta) \leq K \delta^{-c_3(1+\sqrt{k}R)} M(\theta+\delta)^{1-p/2}.$$

Choosing $\theta_0 = 1 - \tau$ and $\theta_i = \theta_{i-1} + 2^{-i}\tau$ for $i = 1, 2, 3, \ldots$

$$M(\theta_{i-1}) \leq K_1 2^{ic_3(1+\sqrt{k}R)} M(\theta_i)^{\lambda}$$

where $\lambda = 1 - p/2$, $K_1 = K\tau^{-c_3(1+\sqrt{k}R)}$. Iterating we get

$$M(\theta_0) \leq K_1^{\sum_{i=1}^{j} \lambda^{i-1}} 2^{c_3(1+\sqrt{k}R) \sum_{i=1}^{j} i\lambda^{i-1}} M(\theta_j)^{\lambda^j}$$

for any $j \geq 1$. Letting j tend to infinity we get

$$M(\theta_0) \leq \tau^{-c_6(1+\sqrt{k}R)} \left[\operatorname{Vol}(B_{2^{-1}R})^{-1} \int_{B_R} v^p \, dV \right]^{2/p}$$

where c_6 depends only on n and p. This implies

$$\sup_{B_{(1-\tau)R}} v^p \leq \tau^{-2^{-1}pc_6(1+\sqrt{k}R)} \operatorname{Vol}(B_{2^{-1}R})^{-1} \int_{B_R} v^p \, dV.$$

The theorem now follows from (1.6) which shows

$$\operatorname{Vol}(B_R) \leq e^{n(1+\sqrt{k}R)} \operatorname{Vol}(B_{2^{-1}R}).$$

A theorem of Yau [15] shows that on any complete Riemannian manifold a harmonic function which lies in L^p for some $p \in (1, \infty)$ is necessarily constant. On the other hand, Greene–Wu [7] have shown that an L^1 harmonic function vanishes on a complete manifold of nonnegative sectional curvature. It turns out that the triviality of L^p harmonic functions for $p \in (0, 1]$ only holds under special geometric assumptions on M, which will be demonstrated by the counter-examples in § 3.

Definition. For $p \in (0, \infty]$, we say that a manifold satisfies property $\mathscr{H}_p$ if every L^p harmonic function on M is constant. We say that M satisfies property $\mathscr{S}_p$ if every nonnegative L^p subharmonic function on M is constant.

Since the absolute value of a harmonic function is subharmonic, we see that the validity of property $\mathscr{S}_p$ implies property $\mathscr{H}_p$ for any manifold M. Yau's theorem implies that every complete manifold satisfies property $\mathscr{S}_p$ for $p \in (1, \infty)$.

THEOREM 2.2. *The following two assertions hold.*

(a) *A complete simply connected manifold of nonpositive sectional curvature satisfies $\mathscr{S}_p$ for $p \in (0, \infty)$.*

(b) *A complete manifold of nonnegative Ricci curvature satisfies $\mathscr{S}_p$ for $p \in (0, \infty)$ and also satisfies $\mathscr{H}_p$ for $p \in (0, \infty]$.*

Proof. To prove (a) we observe that if M is complete and simply connected with nonpositive sectional curvature, then the mean value inequality

$$\sup_{B_{2^{-1}R}} v^2 \leq c_7 R^{-n} \int_{B_R} v^2 \, dV$$

holds for nonnegative subharmonic functions. This result is well known and its proof can be found in [8]. By varying the center of the ball and the radius this implies

$$\sup_{B_{(1-\tau)R}} v^2 \leq c_7 \tau^{-n} R^{-n} \int_{B_R} v^2\, dV$$

for any $\tau \in (0, 1/2)$. As in Theorem 2.1 we then get the mean value inequality for any $p \in (0, 2]$

$$\sup_{B_{2^{-1}R}} v^p \leq c_8 R^{-n} \int_{B_R} v^p\, dV$$

for a constant c_8 depending only on n and p. If v lies in L^p, we can then let R go to infinity to show that v is identically zero.

To establish assertion (b) we apply Theorem 2.1 with $k=0$ to get

$$\sup_{B_{2^{-1}R}} v^p \leq c_9 \operatorname{Vol}(B_R)^{-1} \int_{B_R} v^p\, dV$$

where c_9 depends only on n and p. A theorem of Yau [15] shows that for any complete manifold of nonnegative Ricci curvature and any $x \in M$,

$$\operatorname{Vol}(B_R(x)) \geq c_{10} R$$

where c_{10} is a positive constant depending on x and M. Therefore we can let $R \to \infty$ to show that M satisfies $\mathcal{S}_p$ for $p \in (0, \infty)$. The fact that M satisfies $\mathcal{H}_\infty$ is a theorem of Yau [14]. This completes the proof of Theorem 2.2.

THEOREM 2.3. *If M is a complete manifold whose Ricci curvature is bounded below by a negative constant and such that the volume of every unit geodesic ball in M has a positive lower bound, then M satisfies $\mathcal{S}_p$ for every $p \in (0, \infty)$.*

Proof. Since M automatically satisfies $\mathcal{S}_p$ for $p \in (1, \infty)$, we suppose $p \in (0, 1]$ and suppose v is a nonnegative L^p subharmonic function on M. For any point $x \in M$, Theorem 2.1 implies

$$v^p(x) \leq c_{11} \operatorname{Vol}(B_1(x))^{-1} \int_{B_1(x)} v^p\, dV$$

where c_{11} is independent of x. Since we are assuming a positive lower bound on $\operatorname{Vol}(B_1(x))$ independent of x, we see that there is a constant c_{12} so that

$$\sup_M v \leq c_{12}.$$

But this implies

$$\int_M v^2\,dV \leqslant c_{12}^{2-p}\int_M v^p\,dV < \infty$$

and $v \in L^2$. Therefore v is identically zero and Theorem 2.3 follows.

The case $p=1$ is a borderline case, and it turns out that very weak hypotheses guarantee property $\mathscr{S}_1$, although examples (see § 3) show that not every complete manifold enjoys property $\mathscr{S}_1$.

THEOREM 2.4. *Suppose M is a complete manifold and $x_0 \in M$. Let $r_0(x)$ denote distance to x_0, and assume*

$$\mathrm{Ric}_M(x) \geqslant -c_{13}(1+r_0(x)^2)\,[\log(1+r_0(x)^2)]^{-\alpha}$$

for all $x \in M$ and some $\alpha > 0$. Then M satisfies property $\mathscr{S}_1$.

Proof. Let v be a nonnegative L^1 subharmonic function on M. We first construct a sequence R_i tending to infinity such that

$$\lim_{i\to\infty}(\log R_i)\int_{B_{2R_i}-B_{R_i}} v\,dV = 0. \tag{2.1}$$

To construct such a sequence we define L_i by

$$L^i = 2^{2^i}, \quad i = 1,2,3,\ldots$$

and observe that $[L_i, L_{i+1}]$ is a disjoint union of 2^i intervals with length L_i and endpoints $a_{i,j}$ given by

$$a_{i,j} = 2^j L_i, \quad j = 0,1,\ldots,2^i.$$

Thus there is a j_0 with $1 \leqslant j_0 \leqslant 2^i$ such that if we set $R_i = j_0 L_i$ we have

$$\int_{B_{2R_i}-B_{R_i}} v\,dV \leqslant 2^{-i}\int_{B_{L_{i+1}}-B_{L_i}} v\,dV.$$

Next observe that $R_i \leqslant L_{i+1}$ and hence $\log R_i \leqslant 2^{i+1}\log 2$. Thus we have

$$(\log R_i)\int_{B_{2R_i}-B_{R_i}} v\,dV \leqslant 2\log 2\int_{B_{L_{i+1}}-B_{L_i}} v\,dV.$$

Inequality (2.1) now follows from the fact that v is integrable on M.

Let M_+ denote the set of $x \in M$ where $v(x) \geq e$, and let f_+ denote the positive part of a function f. Using the fact that v is subharmonic we have

$$\int_M \Phi^2 (\log\log v)_+^\alpha \, \Delta v \, dV \geq 0$$

where Φ has compact support. Integrating by parts

$$\int_{M_+} \Phi^2 (\log\log v)^{\alpha-1} (\log v)^{-1} v^{-1} |\nabla v|^2 dV \leq 2 \int_{M_+} \Phi (\log\log v)^\alpha |\nabla\Phi| \, |\nabla v| \, dV.$$

Applying the Schwarz inequality we get

$$\int_{M_+} \Phi^2 (\log\log v)^{\alpha-1} (\log v)^{-1} v^{-1} |\nabla v|^2 dV \leq 4 \int_{M_+} (\log\log v)^{\alpha+1} (\log v) v |\nabla\Phi|^2 dV.$$

For any $i=1,2,\ldots$ choose Φ to be a function of r_0 which is one on $B_{R_i}(x_0)$ and zero outside $B_{2R_i}(x_0)$. This gives

$$\int_{B_{R_i} \cap M_+} (\log\log v)^{\alpha-1} (\log v)^{-1} v^{-1} |\nabla v|^2 dV \leq \varepsilon(i) R_i^{-2} (\log R_i)^{-1} \sup_{B_{2R_i}} (\log\log v)^{\alpha+1} (\log v) \tag{2.2}$$

where we have used (2.1) and we denote by $\varepsilon(i)$ a term which tends to zero as i goes to infinity. Applying Theorem 2.1 in conjunction with the lower bound on the Ricci curvature of M inside B_{4R_i}, we get

$$\sup_{B_{2R_i}} v \leq e^{c(1+R_i^2(\log R_i)^{-\alpha})}$$

for some constant c. Using this in (2.2) we get

$$\int_{B_{R_i} \cap M_+} (\log\log v)^{\alpha-1} (\log v)^{-1} v^{-1} |\nabla v|^2 dV \leq \varepsilon(i).$$

Letting i go to infinity then shows that either v is constant or M_+ is empty, in which case $v \leq e$ on M. Thus v would be in L^2 and hence constant. In any case we have finished the proof of Theorem 2.4.

COROLLARY 2.1. *Suppose M is a complete manifold and $x_0 \in M$. Let $r_0(x)$ denote distance to x_0, and assume*

$$\mathrm{Ric}_M(x) \geq -c_{13}(1+r_0(x)^2)\left[\log(1+r_0(x)^2)\right]^{-\alpha}$$

for all $x \in M$ and some $\alpha>0$. Then the heat semi-group $e^{\Delta t}$ is the unique strongly continuous contractive semi-group on $L^1(M)$ with Δ as its infinitesimal generator.

Proof. To show that $e^{\Delta t}$ is a strongly continuous contractive semi-group with Δ as infinitesimal generator, following an argument of Strichartz in [11], it suffices to prove that there does not exist nontrivial L^∞ function satisfying

$$\Delta f = \lambda f$$

on M, for $\lambda>0$. To see this, we observe that both $e^{\Delta t}f$ and $e^{\lambda t}f$ are L^∞-solutions to the heat equation. Due to the assumption on the Ricci curvature, the volume growth of M must satisfy

$$\mathrm{Vol}(B_R(x_0)) \leq e^{CR^2}$$

for some postitive constant C. This condition fulfills the hypothesis of uniqueness theorem for L^∞-solutions in [9]. Hence

$$e^{\Delta t}f = e^{\lambda t}f.$$

However $e^{\Delta t}$ is contractive in $L^\infty(M)$, where $e^{\lambda t}f$ clearly grows exponentially in t. This gives a contradiction unless $f \equiv 0$.

To prove uniqueness of $e^{\Delta t}$, Strichartz's argument reduced to proving the non-existence of nontrivial L^1-solutions to the equation

$$\Delta f = \lambda f \quad \text{for } \lambda > 0.$$

If we let $v=|f|$, then v is a nonnegative subharmonic function. By Theorem 2.4, v, hence f, must be identically constant. However this is impossible unless $f \equiv 0$ because $\lambda>0$, and the corollary follows.

When $0<p<1$, Theorem 2.4 does not hold. In fact, examples (see § 3, example 3) show that M may not have property $\mathscr{H}_p$ for $0<p<1$ even if its curvature behaves like $-\alpha[(1-\alpha)r]^{-2}$ for $1/2<\alpha<1$. However, it turns out that this is the critical case for a manifold to satisfy property $\mathscr{S}_p$, hence $\mathscr{H}_p$, as the following theorem indicates.

THEOREM 2.5. *Suppose M is complete of dimension n. There exists a constant*

$\delta(n)>0$ depending only on n, such that, for some point $x_0\in M$, the Ricci curvature satisfies

$$\mathrm{Ric}_M(x)\geqslant -\delta(n)r_0^{-2}(x)$$

whenever the distance from x_0 to $x, r_0(x)$, is sufficiently large. Then M satisfies property $\mathcal{S}_p$ for any $p\in(0,\infty)$.

Proof. In view of Theorems 2.2 and 2.4, we only need to consider those cases when $p\in(0,1)$. We observe that since the arguments leading to Theorem 2.1 are local, the following local L^p mean value inequality holds

$$\sup_{B_{R/2}(x)} v^p\leqslant 2^{c_5(1+R\sqrt{k(x,5R)})}\,\mathrm{Vol}(B_R(x))^{-1}\int_{B_R(x)} v^p\,dV \tag{2.3}$$

(by setting $\tau=1/2$) for nonnegative subharmonic functions on $B_{5R}(x)$. Here the term $-(n-1)\,k(x,5R)$ denotes the lower bound of the Ricci curvature on $B_{5R}(x)$.

Our goal is to utilize (2.3) to show that v must vanish at infinity if $v\in L^p(M)$, $v\geqslant 0$, and subharmonic on M. In particular, this implies $v\in L^p(M)\cap L^\infty(M)$, hence $v\in L^2(M)$, and must be constant. In fact, by a theorem of Cheeger–Gromov–Taylor [3], under such hypothesis on the Ricci curvature, M must be of infinite volume and v must be identically zero.

Let $x\in M$ and consider a minimal geodesic γ joining x_0 to x with $\gamma(0)=x_0$ and $\gamma(T)=x$ where $T=r_0(x)$. Define a set of values $\{t_i\in[0,T]\}_{i=0}^k$ by $t_0=0$, $t_1=1+\beta,\dots,t_i=1+2\beta+2\beta^2+\dots+2\beta^{i-1}+\beta^i=2\,\Sigma_{j=0}^i\,\beta^j-1-\beta^i$, where $\beta>1$ to be chosen later, and $t_k=2\,\Sigma_{j=0}^k\,\beta^j-1-\beta^k$ is the largest such value with $t_k\leqslant T$. Clearly $\{\gamma(t_i)\}$ form a set of points $\{x_i\}$ with the property that $r(x_i;x_{i+1})=\beta^i+\beta^{i+1}$, $r_0(x_i)=t_i$ and $r(x_k,x)<\beta^k+\beta^{k+1}$. The set of geodesic balls $B_{R_i}(x_i)$ with $R_i=\beta^i$ cover $\gamma([0,2\,\Sigma_{j=0}^k\,\beta^j-1])$ and have disjoint interiors. We now claim that for a fixed $\beta>2/(2^{1/n}-1)>1$,

$$\mathrm{Vol}\,(B_{R_k}(x_k))\geqslant c_{14}\left(\frac{\beta^n}{(\beta+2)^n-\beta^n}\right)^k\mathrm{Vol}\,(B_1(x_0)). \tag{2.4}$$

In fact, this follows from the argument in [3] which proved M has infinite volume. However, for the sake of completeness, we will outline the proof of (2.4) again. For each $1\leqslant i\leqslant k$, a comparison theorem argument (see [3]) shows that

$$\begin{aligned}\mathrm{Vol}\,(B_{R_i}(x_i))&\geqslant T_i\,\mathrm{Vol}\,(B_{R_i+2R_{i-1}}(x_i)-B_{R_i}(x_i))\\&\geqslant T_i\,\mathrm{Vol}\,(B_{R_{i-1}}(x_{i-1}))\end{aligned}$$

where

$$T_i = \frac{\displaystyle\int_0^{R_i\sqrt{k(x_i,R_i+2R_{i-1})}} \sinh^{n-1} t\,dt}{\displaystyle\int_{R_i\sqrt{k(x_i,R_i+2R_{i-1})}}^{(r_i+2R_{i-1})\sqrt{k(x_i,R_i+2R_{i-1})}} \sinh^{n-1} t\,dt}.$$

Iterating this inequality, we conclude that

$$\operatorname{Vol}(B_{R_k}(x_k)) \geq \prod_{i=1}^{k} T_i \operatorname{Vol}(B_1(x_0)). \tag{2.5}$$

However, since $r_0(x_i)=2\ \Sigma_{j=0}^{i}\,\beta^j-1-\beta^i$ and $R_i=\beta^i$, the assumption on Ric_M yields

$$\begin{aligned}\sqrt{k(x_i,R_i+2R_{i-1})} &\leq \delta^{1/2}(n)\left(\sum_{j=0}^{i-2}\beta^j-1\right)^{-1}\\ &= \delta^{1/2}(n)\frac{\beta-1}{2\beta^{i-1}-\beta-1}\end{aligned}$$

for sufficiently large i. Since $\beta>2/(2^{1/n}-1)>1$ is fixed, the term

$$(R_i+2R_{i-1})\sqrt{k(x_i,R_i+2R_{i-1})} \leq \delta^{1/2}(n)\frac{\beta-1}{2\beta^{i-1}-\beta-1}(\beta_i+2\beta_{i-1})$$

can be made arbitrarily small by the smallness assumption on $\delta(n)$. Hence T_i has the following approximation

$$\begin{aligned}T_i &\sim \frac{R_i^n}{(R_i+2R_{i-1})^n-R_1^n}\\ &= \frac{\beta^n}{(\beta+2)^n\beta^n}\end{aligned}$$

by simply approximating $\sinh t$ with t. Combining with (2.5) gives (2.4).

In the case if $r(x_k,x)\leq R_k/10$ inequality (2.3) yields

$$\begin{aligned}v^p(x) &\leq \sup_{B_{R_{k/10}}(x_k)} v^p\\ &\leq 2^{c_5(1+(R_{k/5})\sqrt{k(x_k,R_k)})}\operatorname{Vol}(B_{R_{k/5}}(x_k))^{-1}\int_M v^p\,dV.\end{aligned}$$

However, the same argument that proved (1.6) gives

$$\mathrm{Vol}\,(B_{R_{k/5}}(x_k)) \geqslant e^{-c_{15}(1+R_k\sqrt{k(x_k,R_k)})}\,\mathrm{Vol}\,(B_{R_k}(x_k)).$$

Combined with the fact that $R_k\sqrt{k(x_k,R_k)}$ is bounded from above and (2.4), we have proved

$$v^p(x) \leqslant c_{16}\theta^k\,\mathrm{Vol}(B_1(x_0))^{-1}\int_M v^p\,dV \tag{2.6}$$

where

$$\theta = \frac{(\beta+2)^n-\beta^n}{\beta^n} < 1$$

for our choice of $\beta > 2/(2^{1/n}-1)$.

When $r(x_k,x) > R_k/10$, we observe that since $r(x_k,x) < R_k+R_{k+1}$, applying the same argument as above, we get

$$\begin{aligned}\mathrm{Vol}\,(B_{R_k/20}(x)) &\geqslant e^{-c_{17}(1+R_k+r(x_k,x))\sqrt{k(x,R_k+r(x_k,x))}}\,\mathrm{Vol}\,(B_{R_k/20}(x_k))\\ &\geqslant c_{18}\,\mathrm{Vol}\,(B_{R_k/20}(x_k)).\end{aligned}$$

Now, applying (2.3) to $B_{R_k/20}(x)$ and the above volume estimate, we deduce that (2.6) is still valid.

In any case, if $x\to\infty$, then $k\to\infty$. Hence by the fact that the $v\in L^p(M)$ and $\theta<1$, the right-hand side of (2.6) vanishes, thus proving our assertion and Theorem 2.5 follows.

§ 3. Counter-examples

In this section we will give three examples of manifolds which possess non-constant L^p-harmonic functions.

The first example is a manifold with finite volume. In particular, it is probabilistically complete (i.e., the life-time of most brownian motion is infinite). Moreover its sectional curvature decays like

$$-cr^{2+\varepsilon}, \quad c>0,$$

at infinity, and it possesses non-constant L^1-harmonic functions.

Example 2 is a manifold with two ends. Its curvature at one end behaves like

$$-cr^{2+\varepsilon}, \quad c>0,$$

where at the other end it is bounded. The first end is a cusp while the latter has infinite volume. This manifold possesses at least one non-constant positive L^1-harmonic function.

The third example is a class of manifolds with finite volumes, and curvature decay like

$$-cr^{-2}, \quad c>0,$$

at infinity. For each value of $0<p<1$, there are manifolds in this class which possess non-constant L^p-harmonic functions.

Example 1. Let M be a compact surface with arbitrary genus. Assume the metric on M around some point $0\in M$ is flat. Hence locally around 0 we can write the metric in polar coordinates as

$$ds_0^2=dt^2+t^2\,d\theta^2. \tag{3.1}$$

Consider the Green's function on M with the pole at $G(0,x)=f(x)$. By definition f is harmonic on $M-0$ with respect to the given metric ds^2. Let

$$ds^2=\varrho^2\,ds_0^2, \quad \varrho>0 \tag{3.2}$$

be a conformally changed metric on M. Since we are in dimension 2, the Laplacian Δ differs from the original Laplacian Δ_0 by a factor of $1/\varrho^2$, hence $\Delta f\equiv0$ on $M-0$. We will now choose ϱ so that $M-0$ is a complete manifold and $f\in L^1(M-0)$ with respect to ds^2.

Choose ϱ to be arbitrary outside a neighborhood of 0, and

$$\varrho(\theta,t)=\varrho(t)=t^{-1}(-\log t)^{-1}(\log(-\log t))^{-\alpha} \tag{3.3}$$

with $1/2<\alpha<1$, where (θ,t) are the flat polar coordinates system center at 0 with $0<t\leq1/2$.

To check completeness of this metric it suffices to evaluate the line integral

$$\int_0^{1/2}\varrho\,dt. \tag{3.4}$$

Infinite value of the above integral will ensure completeness. By a change of variable $u=\log(-\log t)$, this integral becomes

$$\int_T^{1/2} \varrho(t)\,dt = \int_{\log\log 2}^{\log\log\frac{1}{T}} u^{-\alpha}\,du$$
$$\sim \frac{1}{1-\alpha}\left(\log\log\frac{1}{T}\right)^{1-\alpha}, \quad \text{for } \alpha < 1, \tag{3.5}$$

as $T \to 0$. This proves completeness of ds^2.

The Green's function $f(x) = G(0, x)$ satisfies the estimate

$$|f(\theta, t)| \leq -\log t. \tag{3.6}$$

To verify that $f \in L^1(M-0)$, we need to evaluate the surface integral

$$\int_0^{1/2}\int_0^{2\pi} |f|\, t\varrho^2\,d\theta\,dt \leq 2\pi \int_0^{1/2}\left(\log\frac{1}{t}\right) t\varrho^2(t)\,dt$$
$$= 2\pi \int_{\log\log 2}^{\infty} u^{-2^{\alpha}}\,du$$
$$= 2\pi \left[\frac{u^{1-2\alpha}}{1-2\alpha}\right]_{\log\log 2}^{\infty}. \tag{3.7}$$

By the assumption that $\alpha > 1/2$, this value is finite and hence $f \in L^1(M-0)$.

We will now compute the curvature of ds^2 near 0. Standard computation shows that

$$K = \frac{-\Delta_0(\log\varrho)}{\varrho^2} = \frac{|\nabla_0\varrho|^2}{\varrho^4} - \frac{\Delta_0\varrho}{\varrho^3} = \frac{(\varrho')^2}{\varrho^4} - \frac{\varrho''}{\varrho^3} - \frac{\varrho'}{t\varrho^3}. \tag{3.8}$$

Differentiating ϱ with respect to t,

$$\varrho' = \left[\alpha t^{-1}\left(\log\frac{1}{t}\right)^{-1}\left(\log\log\frac{1}{t}\right)^{-1} + t^{-1}\left(\log\frac{1}{t}\right)^{-1} - t^{-1}\right]\varrho$$

and

$$\varrho'' = \left[\alpha\left(\log\frac{1}{t}\right)^{-1}\left(\log\log\frac{1}{t}\right)^{-1} + \left(\log\frac{1}{t}\right)^{-1} - 1\right]^2 t^{-2}\varrho$$
$$+\left[\alpha\left(\log\frac{1}{t}\right)^{-2}\left(\log\log\frac{1}{t}\right)^{-2} + \alpha\left(\log\frac{1}{t}\right)^{-2}\left(\log\log\frac{1}{t}\right)^{-1}\right.$$
$$\left. + \left(\log\frac{1}{t}\right)^{-2} - \alpha\left(\log\frac{1}{t}\right)^{-1}\left(\log\log\frac{1}{t}\right)^{-1} - \left(\log\frac{1}{t}\right)^{-1} + 1\right] t^{-2}\varrho$$
$$= \left[\alpha(\alpha+1)\left(\log\frac{1}{t}\right)^{-2}\left(\log\log\frac{1}{t}\right)^{-2} + 3\alpha\left(\log\frac{1}{t}\right)^{-2}\left(\log\log\frac{1}{t}\right)^{-1}\right.$$
$$\left. + 2\left(\log\frac{1}{t}\right)^{-2} - 3\alpha\left(\log\frac{1}{t}\right)^{-1}\left(\log\log\frac{1}{t}\right)^{-1} - 3\left(\log\frac{1}{t}\right)^{-1} + 2\right] t^{-2}\varrho.$$

Substituting into (3.8) yields

$$K=-t^{-2}\varrho^{-2}\left[\alpha\left(\log\frac{1}{t}\right)^{-2}\left(\log\log\frac{1}{t}\right)^{-2}+\alpha\left(\log\frac{1}{t}\right)^{-2}\left(\log\log\frac{1}{t}\right)^{-1}+\left(\log\frac{1}{t}\right)^{-2}\right]$$
$$=-\left(\log\log\frac{1}{t}\right)^{2\alpha}\left[\alpha\left(\log\log\frac{1}{t}\right)^{-2}+\alpha\left(\log\log\frac{1}{t}\right)^{-1}+1\right].$$

However, according to (3.5), the geodesic distance of ds^2 behaves like

$$r\sim\frac{1}{1-\alpha}\left(\log\log\frac{1}{t}\right)^{1-\alpha}.$$

Hence

$$K\sim-(1-\alpha)^{2\alpha/(1-\alpha)}r^{2\alpha/(1-\alpha)}\left[1+\frac{\alpha}{1-\alpha}r^{-1/(1-\alpha)}+\frac{\alpha}{(1-\alpha)^2}r^{-2/(1-\alpha)}\right].$$

The leading term as $r\to\infty$ is clearly

$$-(1-\alpha)^{2\alpha/(1-\alpha)}r^{2\alpha/(1-\alpha)},$$

which can be written as

$$K\sim-\left(\frac{2r}{4+\varepsilon}\right)^{2+\varepsilon} \tag{3.9}$$

by setting $\alpha=(2+\varepsilon)/(4+\varepsilon)$ for any $\varepsilon>0$.

Example 2. To construct a manifold which possesses a non-constant positive L^1-harmonic function we proceed as in Example 1. However, we modify the metric on the unit disk in $\mathbf{R}^2$. Choose the new metric to be

$$ds^2=\varrho^2\,ds_0^2$$

on $D^2=$ unit disk in $\mathbf{R}^2$, where ds_0^2 is the Euclidean metric. Pick ϱ as in (3.3) for $0<t\leq 1/2$, and

$$\varrho=(1-t)^{-1}\left(\log\frac{1}{1-t}\right)^{-\alpha},\quad \frac{1}{2}<\alpha\leq 1 \tag{3.10}$$

for $3/4\leq t<1$. The Green's function $G(0,x)$ in $\mathbf{R}^2$ with pole at the origin is given by a constant multiple of $-\log t=f(t)$. Hence f is a positive harmonic function on D^2 with metric ds^2. Clearly the computation shows ds^2 is complete and f is L^1 near the origin. Moreover, the curvature behaves as in (3.9) near 0. We only need to verify completeness by showing

$$\int_{3/4}^{1} \varrho\, dt = \infty$$

and $f \in L^1$, by checking

$$\int_{3/4}^{1} t\varrho^2(-\log t)\, dt < \infty.$$

Obviously, both conditions are satisfied by our choice of ϱ in (3.10). In particular, for example, the latter condition can be checked as follows:

$$\begin{aligned}\int_{3/4}^{1} t\varrho^2(-\log t)\, dt &\leq \int_{3/4}^{1} \frac{1}{(1-t)^2}\left(\log\frac{1}{1-t}\right)^{-2\alpha} \log\frac{1}{t}\, dt \\ &\leq 4\log\frac{4}{3}\int_{3/4}^{1} (1-t)^{-1}\left(\log\frac{1}{1-t}\right)^{-2\alpha} dt \\ &= 4\log\frac{4}{3}\int_{\log 4}^{\infty} u^{-2\alpha}\, du < \infty.\end{aligned}$$

Example 3. We choose

$$\varrho = t^{-1}\left(\log\frac{1}{t}\right)^{-\alpha}, \quad \alpha \leq 1 \tag{3.11}$$

in our construction of Example 1. The harmonic function will also be the same, namely, $f(x) = G(0, x)$. To verify $ds^2 = \varrho^2 ds_0^2$ is a complete metric is trivial. The condition for which $f \in L^p(M-0)$ is given by

$$\int_0^{1/2} \left(\log\frac{1}{t}\right)^p t\varrho^2\, dt < \infty. \tag{3.12}$$

The left-hand integral is

$$\int_0^{1/2} t^{-1}\left(\log\frac{1}{t}\right)^{-2\alpha}\left(\log\frac{1}{t}\right)^p dt.$$

As evaluated as before this is finite when $2\alpha - p > 1$. Hence to obtain non-constant L^p-harmonic functions on $M-0$ for any given $0 < p < 1$, we simply choose $\alpha > (1+p)/2$.

We will now compute the curvature K near 0 according to formulas (3.8) and (3.11). Differentiating ϱ, we get

$$\varrho' = t^{-1}\varrho\left[\alpha\left(\log\frac{1}{t}\right)^{-1} - 1\right]$$

and

$$\varrho''=t^{-2}\varrho\left[\alpha(\alpha+1)\left(\log\frac{1}{t}\right)^{-2}-3\alpha\left(\log\frac{1}{t}\right)^{-1}+2\right].$$

Therefore,

$$\begin{aligned}K&=-\alpha t^{-2}\varrho^{-2}(\log t)^{-2}\\&=-\alpha\left(\log\frac{1}{t}\right)^{-2(1-\alpha)}.\end{aligned}$$

However, the geodesic distance behaves like

$$\begin{aligned}r&\sim\int_t^{1/2}s^{-1}\left(\log\frac{1}{s}\right)^{-\alpha}ds\\&=\int_{\log 2}^{\log 1/t}u^{-\alpha}du\\&=\frac{\left(\log\frac{1}{t}\right)^{1-\alpha}-(\log 2)^{1-\alpha}}{1-\alpha}.\end{aligned}$$

Hence

$$K\sim-\alpha[(1-\alpha)r]^{-2}.$$

Note that when $\alpha=1$, this is the constant negative curvature metric. However, when $\alpha<1$, the curvature decays to 0 quadratically at infinity.

References

[1] ANDERSON, M., The Dirichlet problem at infinity for manifolds of negative curvature. To appear in *J. Differential Geom.*
[2] ANDERSON, M. & SCHOEN, R., Preprint.
[3] CHEEGER, J., GROMOV, M. & TAYLOR, M., Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds. *J. Differential Geom.*, 17 (1983), 15–53.
[4] CHENG, S. Y. & YAU, S. T., Differential equations on Riemannian manifolds and their geometric applications. *Comm. Pure Appl. Math.*, 28 (1975), 333–354.
[5] CHUNG, L. O., Existence of harmonic L^1 functions in complete Riemannian manifolds. Unpublished.
[6] GARNETT, L., Foliations, the ergodic theorem and brownian motion. Preprint.
[7] GREENE, R. E. & WU, H., Integrals of subharmonic functions on manifolds of nonnegative curvature. *Invent. Math.*, 27 (1974), 265–298.
[8] — *Function theory on manifolds which possess a pole.* Lecture Notes in Mathematics, 699. Springer-Verlag, Berlin–Heidelberg–New York (1979).

[9] KARP, L. & LI, P., The heat equation on complete Riemannian manifolds. Preprint.
[10] LI, P. & YAU, S. T., Estimates of eigenvalues of a compact Riemannian manifold. *Proc. Symp. Pure Math.*, 36 (1980), 205–239.
[11] STRICHARTZ, R., Analysis of the Laplacian on a complete Riemannian manifold. *J. Funct. Anal.*, 52 (1983), 48–79.
[12] SULLIVAN, D., Preprint.
[13] WU, H., On the volume of a noncompact manifold. *Duke Math. J.*, 49 (1982), 71–78.
[14] YAU, S. T., Harmonic functions on complete Riemannian manifolds. *Comm. Pure Appl. Math.*, 28 (1975), 201–228.
[15] — Some function-theoretic properties of complete Riemannian manifolds and their applications to geometry. *Indiana Univ. Math. J.*, 25 (1976), 659–670.

Received August 23, 1983
Received in revised form December 8, 1983