Quasiconformal homeomorphisms and dynamics II: Structural stability implies hyperbolicity for Kleinian groups

by

DENNIS SULLIVAN

I.H.E.S., Bures-sur-Yvette, France

Contents

Introduction

Let $\Gamma_0 \subset \{z \mapsto (az+b)/(cz+d);\, a, b, c, d, \text{ in } \mathbf{C},\, ad-bc=1\} = PSl(2, \mathbf{C})$ be a finitely generated *non-solvable* subgroup of the orientation-preserving conformal transformations of the Riemann sphere $\bar{\mathbf{C}}$. Say that Γ_0 is *structurally stable* if all sufficiently near representations into $PSl(2, \mathbf{C})$ are injective—there are no new relations. Say that Γ_0 is *non-rigid* if there are arbitrarily close representations which are not conjugate in $PSl(2, \mathbf{C})$. Otherwise Γ_0 is *rigid*. If Γ_0 is discrete and the fundamental domain in hyperbolic 3-space is finite sided one says Γ_0 is *geometrically finite*. An element γ in $PSl(2, \mathbf{C})$ is *parabolic* iff trace $\gamma = a+d$ is ± 2. For torsion free groups we have,

THEOREM A. *A structurally stable $\Gamma_0 \subset PSl(2, \mathbf{C})$ is either rigid or it is a discrete geometrically finite group with no non-trivial parabolic elements.*

The condition *geometrically finite without parabolics* is equivalent (§ 9) in the context of discrete subgroups of $PSl(2, \mathbf{C})$ to an expanding property for the action of Γ_0

in its Poincaré limit set $\Lambda_0 \subset \bar{\mathbf{C}}$. Namely, for each $x \in \Lambda_0$ there is a $\gamma \in \Gamma_0$ so that $|\gamma' x| > 1$, in the spherical metric. This expanding property for Γ_0 on its limit implies a strong form of structural stability in the sense of differentiable dynamics. Let Γ_1 be the image of a representation of Γ_0 into C^1-diffeomorphisms of the sphere so that the generators of Γ_1 are sufficiently C^1 close to those of Γ_0. We allow torsion in

THEOREM B. *The representation of* Γ_0 *onto* Γ_1 *is also bijective. Moreover there is a minimal compact invariant set* $\Lambda_1 \subset \bar{\mathbf{C}}$ *for* Γ_1 *so that the action of* Γ_1 *on* Λ_1 *is topologically conjugate by a perturbation of the identity to the action of* Γ_0 *on* Λ_0, *its Poincaré limit set.*

The proof of Theorem A begins with a result about a holomorphic 1-parameter family of injective representations of an arbitrary non-solvable group Γ. One finds two possibilities:

THEOREM 1. *Either the* 1*-parameter family is*

(i) *trivial—all representation are conjugate, or it is*

(ii) *non trivial but all representations are discrete and canonically quasiconformally conjugate on their limit sets.*

Acknowledgement. Theorem 1 without the quasi-conformal part was proved independently and earlier by Bob Riley [8].

In the finitely generated case of (ii) we extend the conjugacies to quasi-conformal conjugacies in the entire sphere $\bar{\mathbf{C}}$. Combining this with the result, Sullivan [10], that with respect to quasi-conformal deformations the limit set of a finitely generated discrete group behaves as if it has measure zero, proves

THEOREM 2. *A non-trivial holomorphic* 1*-parameter family of injective representations of a finitely generated non-solvable group consists of quasi-conformally conjugate discrete groups with non-trivial domains of discontinuity on* $\bar{\mathbf{C}}$.

Quasi-conformal conjugacies turn up because they are unique on the limit set and so depend holomorphically on the parameter. In § 2 we make the following observation which arose in conversations with Ricardo Mañe, about the dynamics of rational maps of $\bar{\mathbf{C}}$.

THEOREM. *A homeomorphism* φ *of* $\bar{\mathbf{C}}$ *is quasi conformal iff* φ *is contained with the identity in a holomorphic* 1*-parameter family* φ_λ *of homeomorphisms of* $\bar{\mathbf{C}}$ (*for each* x *in* $\bar{\mathbf{C}}$, $\varphi_\lambda(x)$ *is holomorphic in* λ).

A corollary of the proof of Theorem A in the torsion free case is the following.

THEOREM C. *A neighborhood of a non-rigid, structurally stable group* $\Gamma_0 \subset PSl(2, \mathbf{C})$ *in the variety of representations up to conjugation is non-singular, has dimension* 3 (*Euler characteristic* Γ_0), *and consists of quasi-conformal conjugates of* Γ_0.

There are versions of Theorems A and C (but not B) allowing parabolics and keeping these fixed during perturbations (§ 8).

COROLLARY. *The geometrically finite torsion free Kleinian groups are precisely the quasi-conformally structurally stable groups in the sense of Bers* [3].

The proof of Theorem B that we offer has two parts. In the first part we give an abstract expanding-hyperbolicity axiom for a group action which implies this kind of *structural stability in the sense of differentiable dynamics*. In the second part we verify that all geometrically finite discrete groups without parabolics verify this hyperbolicity axiom.

In summary we have shown that either $\Gamma_0 \subset PSl(2, \mathbf{C})$ is geometrically rigid or that an algebraic structural stability for holomorphic perturbations implies a hyperbolicity property for the action of Γ_0 on its limit set—which in turn implies structural stability in the sense of differentiable dynamics. Thus in a natural sense of these words we have shown *structural stability is equivalent to hyperbolicity for finitely generated* non-solvable-non rigid subgroups of $PSl(2, \mathbf{C})$.

Open problems. The analogous problem *structural stability implies hyperbolicity* in the iteration theory of rational maps of $\bar{\mathbf{C}}$ to itself remains open and seems quite difficult. On the other hand for rational maps one knows the *structurally stable systems form an open and dense set* (see Mañe, Sad, Sullivan [7], Sullivan [12], and Sullivan-Thurston [13]). In the context of Kleinian groups we have now the opposite situation—we do not know structurally stable groups are dense (in discrete groups) but we do know by the paper here they must be hyperbolic. Here is the score card.

	Kleinian groups	Rational maps of $\bar{\mathbf{C}}$
Structural stability implies hyperbolicity	Yes (this paper)	? (verified numerically)
Structural stability is open and dense	? (main conjecture)	Yes (Mañe, Sad, Sullivan [7])

§1. Subgroups of $\{z \mapsto (az+b)/(cz+d)\}$, a, b, c, d complex $ad-bc=1\}$

Let Γ be a subgroup of $PSl(2, \mathbf{C})$. The following proposition and its elegant proof are certainly well known.

PROPOSITION. *After replacing* Γ *by a subgroup of index* 2 *if necessary, there are three possibilities:*

(i) Γ *is discrete.*

(ii) Γ *is solvable and conjugate to a subgroup of* $\{z \mapsto az+b\}$.

(iii) Γ *is* (a) *dense in* $PSl(2, \mathbf{C})$ *or* (b) *conjugate to a dense subgroup of* $PSl(2, \mathbf{R})$ *or* (c) *conjugate to a dense subgroup of* $SO(3, \mathbf{R})$.

Proof. Let G be the connected component of the identity of the topological closure $\bar{\Gamma}$ of Γ. Then G is a connected sub-Lie group of $PSl(2, \mathbf{C})$ and G is normal in $\bar{\Gamma}$. If G is trivial or all of $PSl(2, \mathbf{C})$ this is case (i) or case (iii)(a). Otherwise, G fixes a point in the hyperbolic space, a plane in the hyperbolic space, or a point at infinity of the hyperbolic space. Since G is normal in $\bar{\Gamma}$, these objects are also fixed by Γ (or by a subgroup of index two).

If one of the first two possibilities doesn't yield (iii)(b) or (iii)(c) consideration of G shows that we are in case (ii). The third possibility is obviously (ii). Q.E.D.

Remark. Of course the cases (i) and (ii), discrete or solvable are disjoint from those of (iii). The intersection of cases (i) and (ii) are most of the *elementary discrete groups* which are either solvable finite or solvable and contain $\mathbf{Z}$ or $\mathbf{Z}+\mathbf{Z}$ with small index. (A_5 is not included.)

Now let x and y be non-trivial conjugate elements of $PSl(2, \mathbf{C})$ with $y=z^{-1}xz$ and $\Gamma(x, y)$ the group generated by x and y. As usual, we think of $PSl(2, \mathbf{C})$ operating on the Riemann sphere.

COROLLARY. (i) *If x is hyperbolic, then $\Gamma(x, y)$ is not solvable iff x and y do not have a common fixed point.*

(ii) *If x is elliptic, and not of order two, then $\Gamma(x, y)$ is not solvable iff x and y do not have a common fixed point.*

Proof. Apply (ii) of the proposition for $\Gamma=\Gamma(x, y)$ and note $\Gamma(x^2, y^2)$ is contained in the subgroup of index two.

COROLLARY. *If x is hyperbolic and $\Gamma(x, y)$ is solvable then $\Gamma(x, y)$ is also Abelian iff x and y have the same fixed points.*

Proof. By (ii) of proposition, we are reduced to considering a subgroup of $\{z \mapsto az+b\}$. And we verify by inspection.

Conclusion. What z does to the fixed points of x (either elliptic or hyperbolic) is determined by the abstract group structure of $\Gamma(x, y)$, $y=z^{-1}xz$.

§ 2. Holomorphic families of homeomorphisms of $\bar{\mathbf{C}}$

For λ in $\{z: |z|<1\}$ let φ_λ be a homeomorphism of the sphere $\bar{\mathbf{C}}$ so that φ_0=identity and for each $x \in \bar{\mathbf{C}}$ $\varphi_\lambda(x)$ is complex analytic. We say that φ is contained in a holomorphic family with the identity.

THEOREM. *A homeomorphism $\varphi: \bar{\mathbf{C}} \to \bar{\mathbf{C}}$ is contained in a holomorphic family with the identity iff φ is quasi-conformal.*

Proof. (i) (Mañe, Sad, Sullivan [7].) If φ is contained in a holomorphic family with the identity (and φ is normalized to fix ∞) and x, y, z are three distinct points on the plane then

$$(x, y, z)(\lambda) = \frac{\varphi_\lambda(x)-\varphi_\lambda(y)}{\varphi_\lambda(z)-\varphi_\lambda(y)}$$

is an analytic function on $\{\lambda: |\lambda|<1\}$ omitting $0, 1, \infty$. By Schwarz's lemma $(x, y, z)(\lambda)$ is Lipschitz with constant ≤ 1 between the Poincaré metric on the λ-disk and the Poincaré metric on the sphere triply punctured at $(0, 1, \infty)$.

Since $(x, y, z)(\lambda)$ measures the shape of the triangle with vertices $(\varphi_\lambda(x), \varphi_\lambda(y), \varphi_\lambda(z))$ this control implies φ_λ is a quasi-conformal homeomorphism (definition § 2). This proves the first part.

(ii) (Ahlfors-Bers [2].) If φ is a quasi-conformal homeomorphism of $\bar{\mathbf{C}}$ (normalized to fix $0, 1, \infty$) the conformal distortion can be regarded as a bounded by 1 measurable complex valued function μ. For each λ in the unit disk of complex numbers, there is a unique normalized quasi-conformal homeomorphism φ_λ (depending holomorphically on λ) whose conformal distortion is $\lambda \cdot \mu$, Ahlfors-Bers [2]. By uniqueness φ_0=identity and $\varphi_1=\varphi$. This proves the second part.

§ 3. λ-lemma for holomorphic motions in $\bar{\mathbf{C}}$

Let A be a subset of $\bar{\mathbf{C}}$. For each $a \in A$ suppose there is an analytic function $\varphi(a, \lambda)$: $\{\lambda: |\lambda|<1\} \to \bar{\mathbf{C}}$ so that $\varphi(a, 0)=a$ and so that all the values $\varphi(a, \lambda)$ for $a \in A$ fixed are distinct.

If we regard φ as a function $A\times D\xrightarrow{\varphi}\bar{\mathbf{C}}$, the hypotheses above ("analytic", "distinct") imply surprisingly that φ is uniformly continuous, namely, if $\bar{A}$=closure A, we have the

λ-LEMMA. *There is a continuous map $\bar{A}\times D\xrightarrow{\varphi}\mathbf{C}$ extending $A\times D\xrightarrow{\varphi}\mathbf{C}$ which satisfies*

(i) *For fixed λ, $\varphi(\lambda,\cdot)$ is a quasi-conformal embedding of $\bar{A}$ into $\bar{\mathbf{C}}$.*
(ii) *For fixed $a\in\bar{A}$, $\varphi(a,\cdot)$ is analytic as a function on $\{\lambda:|\lambda|<1\}$.*

Explanation. If $X\subset\bar{\mathbf{C}}$ and $h:X\to\bar{\mathbf{C}}$ is a topological embedding, we say h is a quasi-conformal embedding if there is a constant K in $(0,\infty)$ so that for all x in X,

$$\lim_{\varepsilon\to 0}\sup_{(x,y)=\varepsilon}\left(\frac{\sup(h(x),h(y))}{\inf(hx,hy)}\right)<K$$

where $(\cdot,\cdot)$ denotes spherical distance and y is also in X.

If X is all of $\bar{\mathbf{C}}$, this definition yields the class of quasi-conformal homeomorphisms which have many equivalent characterizations (cf. Ahlfors book "Quasi-conformal homeomorphisms"). For general X the definition is not always strong enough. A correct and general definition of quasiconformal deformation of X will be given in Sullivan and Thurston "Holomorphic motions ..." to appear in Acta Math.

Proof of λ-lemma. See Mañe, Sad, Sullivan [7].

§4. Invariant line fields

A quasi-conformal homeomorphism φ of $\bar{\mathbf{C}}$ which conjugates a set of elements $\Gamma\subset PSl(2,\mathbf{C})$ to another set $\Gamma_\varphi\subset\varphi^{-1}\Gamma\varphi\subset PSl(2,\mathbf{C})$ must satisfy the following condition, a.e.: the *conformal distorsion* of φ is invariant by Γ. There is a converse—given by the *measurable Riemann mapping theorem.*

The conformal distorsion of φ is defined at almost all points of $\bar{\mathbf{C}}$ and at each of these is either zero or consists of a line in the tangent space at the point (the major axis of a homothetic family of ellipses) and a positive real number (the eccentricity of these ellipses). Invariance by $\gamma\in\Gamma$ means

(i) The zero set is a.e. invariant.
(ii) The tangent lines of x and γx correspond under the tangent map of γ, a.e.
(iii) The eccentricities at x and γx agree, a.e.

So we get from φ a Γ-invariant line field and a Γ-invariant function. The *measurable Riemann mapping* theorem provides a converse—a Γ-invariant line field and function determine a quasi-conformal conjugacy of Γ to another part $\varphi^{-1}\Gamma\varphi$ of $PSl(2, \mathbf{C})$. See Ahlfors-Bers [2].

We can say precisely which Γ have such non-trivial invariant line fields (invariant functions always exist, e.g. the constant function). Recall the three possibilities of § 1 for $\Gamma \subset PSL(2, \mathbf{C})$

(i) discrete,
(ii) solvable,
(iii) "dense".

The following slightly generalizes Sullivan [10].

THEOREM. *If Γ is finitely generated and not solvable then Γ admits a non-trivial invariant line field only when Γ is discrete and has a non-trivial domain of discontinuity. Moreover, the invariant line field is supported on the domain of discontinuity.*

Proof. Take a density point p of a set P of positive measure where the line field is approximately parallel (work in $\mathbf{C}$). In all cases of (iii) of § 1, we can suppose there is an approximate rotation $\gamma \in \Gamma$ by $\sim 90°$ say about a center arbitrarily near x. (In case (iii) (b) we can of course choose p not to lie on the invariant circle since density points have full 2-dimensional Lebesgue measure.) Clearly, the element γ does not preserve the field a.e. This proves there is no non-trivial invariant line field in this case. For discrete groups, case (i), we may suppose x belongs to the limit set of Γ. This case is more difficult and is treated in complete detail in Sullivan [10]. For the convenience of the interested reader, we sketch a more streamlined argument which was only indicated in a "note added to the proofs" there. Here are the steps

(i) There is no set of positive measure in the limit set which wanders ($|A|>0$, $\gamma A \cap A = \emptyset$, $\gamma \in \Gamma$). Otherwise, there would be an infinite dimensional space of different quasi-conformal conjugates of Γ constructed from the obvious L^∞-space of invariant line fields. The conjugacies are constructed using the measurable Riemann mapping theorem. The new groups are distinct because any homeomorphism commuting with Γ is the identity on the limit set.

(ii) By (i) there are elements γ_n in Γ of arbitrarily large norm in $PSl(2, \mathbf{C})$ which map a good part of P back to P and have $|\text{derivative}| \sim 1$. (This follows from discreteness of Γ, (i) and an abstract lemma from ergodic theory (K. Schmidt, "Cocycles in ergodic theory", Warwick notes).

(iii) Geometrically γ_n is up to a Euclidean motion of C an inversion in a circle with radius $r_n \to 0$.

The Euclidean motion maps an almost parallel field to an almost parallel field. By Lebesgue density considerations, a neighborhood of the inversion circle with r_n sufficiently small will intersect P nicely, (since $|\gamma_n'| \sim 1$) and the inversion will destroy the almost parallel property. For the missing details, see Sullivan [10].

§ 5. Holomorphic families of abstractly isomorphic subgroups (general case)

For λ in $\{z: |z|<1\} = D$ suppose $\Gamma_\lambda \subset PSl(2, C)$ is a subgroup *isomorphic* to a given abstract group Γ. The dependence on λ is holomorphic in the sense that for each γ in Γ the corresponding γ_λ varies holomorphically. We will construct all such families Γ_λ if Γ is not solvable.

First, suppose that for some $\lambda_0, \Gamma_{\lambda_0}$ is not a discrete subgroup which has a non-trivial domain of discontinuity on the sphere. In that case we have

THEOREM 1. *Either*

(i) *all the Γ_λ are conjugate by elements of $PSl(2, C)$ or,*

(ii) *all the Γ_λ are infinitely generated discrete groups which are quasi-conformally conjugate on $\bar{C}$. The conjugations are canonical and depend holomorphically on λ.*

COROLLARY. *If Γ is a finitely generated and the family Γ_λ is non-trivial (the Γ_λ are not all conjugate in $PSl(2, C)$) then each Γ_λ is a discrete subgroup of $PSl(2, C)$ with a non-trivial domain of discontinuity on $\bar{C}$.*

COROLLARY (of proof). *If some Γ_{λ_0} has a non-trivial domain of discontinuity, then all do and there are canonical conjugacies $\varphi_\lambda: \Lambda_{\lambda_0} \leftrightarrow \Lambda_\lambda$ between the actions on the limit sets depending holomorphically on λ.*

Proof of Theorem 1. Suppose first for some λ_0 and x in Γ, $x(\lambda_0)$ is hyperbolic. Consider the set F_λ of fixed points of conjugates of $x(\lambda)$ as λ varies near λ_0. Of course $x(\lambda)$ stays hyperbolic on a disk neighborhood U of λ_0. Also the coincidence or not of fixed points of two conjugates x_1 and x_2 is determined by the group structure of $\Gamma(x_1, x_2)$ (§ 1). This group structure stays constant as λ varies by our hypothesis.

We are thus in a position to apply the λ-lemma of § 3 to F_λ. We obtain a holomorphic family of quasi-conformal homeomorphisms $\varphi_\lambda: \bar{C} \leftrightarrow \bar{C}$ carrying F_{λ_0} to F_λ.

Note we have used the fact that $\bar{F}_{\lambda_0}$ is all of $\bar{\mathbf{C}}$, our hypotheses and § 1. Since each φ_λ is a conjugacy on F_{λ_0} (if $\gamma \in \Gamma, \gamma$(fixed point of x)=fixed point of $\gamma^{-1}x\gamma$) then φ_λ is a topological conjugacy on all of $\mathbf{C}$.

The derivative of φ_λ determines a measurable line field invariant by Γ. These do not exist for non-discrete groups or on limit sets of finitely generated groups, § 4. So the φ_λ are conformal unless Γ is an infinitely generated discrete group (with a wandering set of positive measure on its limit set [10]).

This proves Theorem 1 for λ in U. In case (i) $x(\lambda)$ stays hyperbolic up to ∂U and we can continue the argument trivially to cover all of D. In case (ii) Γ_λ stays discrete up to the boundary of U, Jorgensen [5]. Since Γ is not solvable there are hyperbolic elements and we can repeat the argument for a neighborhood of the boundary point. (In particular $x(\lambda)$ stayed hyperbolic up to the ∂U.) Continuing this way, we cover D and prove Theorem 2 under our initial assumption that for some λ_0 and some x in $\Gamma, x(\lambda_0)$ was hyperbolic.

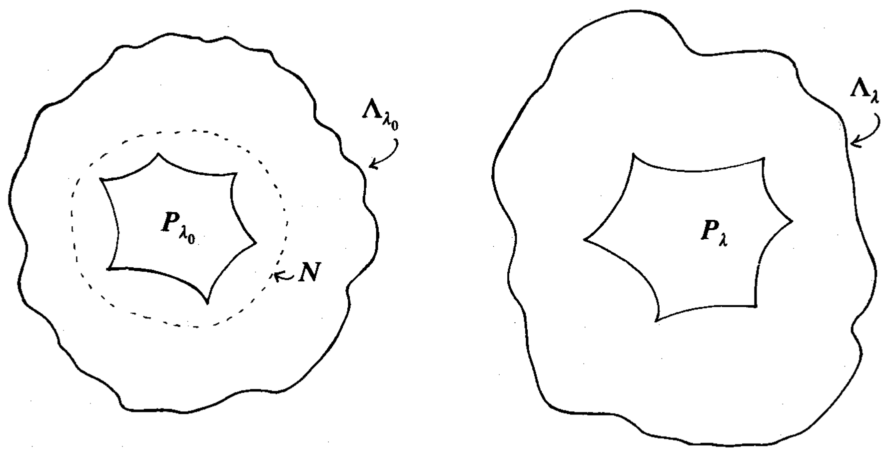
If not we must have by § 1, all the Γ_λ are conjugate to dense subgroups of $SO(3, \mathbf{R})$. Now we choose x in Γ of order >2 and apply the same argument to F_λ the set of fixed points of conjugates of $x(\lambda)$. Again F_λ is dense, moves holomorphically without collision, and defines quasi-conformal conjugacies which are actually conformal by § 4. This proves Theorem 1 in all cases.

§ 6. Holomorphic families of abstractly isomorphic groups (finitely generated case)

THEOREM 2. *If Γ is finitely generated, non-solvable and Γ_λ is a non-trivial holomorphic family of abstractly isomorphic groups then all the Γ_λ are quasi-conformally conjugate* (*on $\bar{\mathbf{C}}$*) *discrete groups with non-trivial domains of discontinuity* (*on $\bar{\mathbf{C}}$*).

Proof of Theorem 2. By the corollary to Theorem 1, § 5, all the Γ_λ are discrete with a non-trivial domain of discontinuity D_λ. Also the proof of Theorem 1 shows there are partial conjugacies $\varphi_\lambda: \bar{F}_{\lambda_0} \to \bar{F}_\lambda$ where F_λ is the set of fixed points of conjugates of some hyperbolic element $x(\lambda)$. But any such $\bar{F}_\lambda$ is just the limit set of Γ_λ. Thus we already have a canonical family of conjugacies between Γ_{λ_0} and the Γ_λ on their respective limit sets Λ_{λ_0} and Λ_λ.

Using the conjugacies φ_λ between limit sets (which start from the identity at $\lambda=\lambda_0$) we obtain a correspondence between the components of $\bar{\mathbf{C}}-\Lambda_{\lambda_0}$ and $\bar{\mathbf{C}}-\Lambda_\lambda$. This correspondence is compatible with the action of Γ. By the Ahlfors finiteness theorem



there are finitely many Γ orbits of components and the quotient of each by the stabilizer subgroup is a finite type Riemann surface R_λ (possibly branched). Ahlfors [1].

Let us first discuss the pure case when there is only one R which is compact and has no branched points. A somewhat subtle point (considered by Poincaré) can be dealt with more clearly in this case.

Let P_{λ_0} denote a finite sided polygon which serves as a fundamental domain for the action of Γ_0 on D_{λ_0} wtih side pairings $\gamma_1, \ldots, \gamma_n$ in Γ_0. Consider a small open neighborhood N of P_{λ_0} in $\bar{\mathbf{C}}$ and the identifications on N induced by $\gamma_1(\lambda), \ldots, \gamma_n(\lambda)$. For λ close to λ_0, we obtain a family of Riemann surfaces S_λ depending holomorphically on λ and homeomorphic to R_{λ_0}. This is so because the topological structure of R_{λ_0} is determined by a finite number of relations among the $\gamma_1(\lambda_0), \ldots, \gamma_n(\lambda_0)$ on the neighborhood N. Exactly these relations persist among the $\gamma_1(\lambda), \ldots, \gamma_n(\lambda)$ by our hypothesis.

For λ near λ_0 we may choose diffeomorphisms $S_\lambda \overset{\varphi_\lambda}{\leftrightarrow} R_{\lambda_0}$ depending holomorphically on λ (exercise). The cuts on R_{λ_0} defining P_{λ_0} may be carried back to S_λ by φ_λ and lifted to D_λ. We obtain a polygon P_λ varying holomorphically with side pairings $\gamma_1(\lambda), \ldots, \gamma_n(\lambda)$ topologically equivalent on N to P_{λ_0} and $\gamma_1(\lambda_0), \ldots, \gamma_n(\lambda_0)$.

Now for the subtle point. We claim P_λ is a fundamental domain for Γ_λ acting on D_λ. It is enough to show this for the component $\bar{D}_\lambda$ of D_λ containing P_λ and the corresponding stabilizer $\bar{\Gamma}_\lambda$. Now for a general fundamental polygon perturbation as discussed above the result we want is not true—the tiling generated by the perturbed polygon could cover the sphere infinitely many times. We are saved here by the fact that $\gamma(P_\lambda)\subset\bar{D}_\lambda$ for each γ in the stabilizer $\bar{\Gamma}_\lambda$ of $\bar{D}_\lambda$.

Formally, we regard P_λ and $\gamma_1(\lambda),\ldots,\gamma_n(\lambda)$ as determining a complex projective structure on the Riemann surface S_λ. There is an associated equivariant developing map d of some covering space $\tilde{S}_\lambda$ into $\bar{\mathbf{C}}$. (Intuitively the adjacent copies of P_λ spread out by words in $\gamma_1,\ldots,\gamma_n$.) By the above fact, $\gamma(P_\lambda)\subset\bar{D}_\lambda$ for γ in $\bar{\Gamma}_\lambda$, the developing map d is a covering of its image, Gunning [4]. For an easy proof consider the Poincaré metric on $\bar{D}_\lambda$ and the Poincaré metric on $\tilde{S}_\lambda$ making the developing map d into a local isometry. This shows d is an isometric immersion between complete connected manifolds $\tilde{S}_\lambda$ and $\bar{D}_\lambda$. Thus d is onto and a covering map.

Since d is equivariant, we may pass to the quotient getting an isometric immersion of S_λ ot R_λ. These have the same topological type, so this quotient map is an isomorphism. This proves P_λ is a fundamental domain for the action of $\bar{\Gamma}_\lambda$ on $\bar{D}_\lambda$.

Now we can use Γ_λ to extend the correspondence $R_{\lambda_0}\to S_\lambda$ to an equivariant bijective correspondence $D_{\lambda_0}\overset{\bar{\psi}_\lambda}{\leftrightarrow}D_\lambda$ depending holomorphically on λ. We can combine $\bar{\psi}_\lambda$ with the disjoint correspondence between limit sets $\Lambda_{\lambda_0}\overset{\varphi_\lambda}{\leftrightarrow}\Lambda_\lambda$. By § 3 these fit together to give a family of quasi conformal homeomorphisms. These are conjugations between Γ_{λ_0} and Γ_λ by construction. This proves Theorem 2 in this pure case for λ near λ_0.

For any λ in D choose an arc in D to λ_0. Compactness of the arc and the argument above now shows Γ_λ and Γ_{λ_0} are quasi-conformally conjugate. This completes the proof in the pure case.

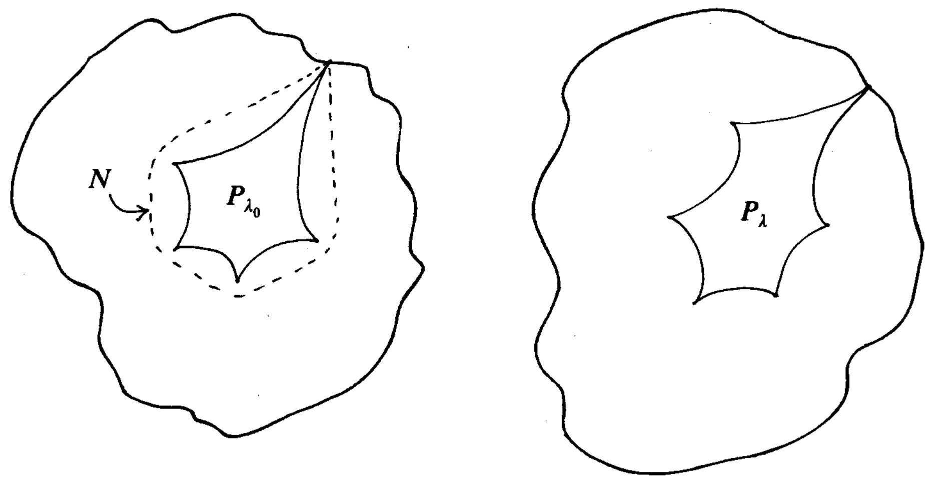
A finite number of cocompact torsion free components is treated in the same way.

Now consider adding branch points and cusps to R_{λ_0}. A subgroup of finite index Γ_t in Γ can be chosen to be normal and torsion free (Selberg lemma). A proof for Γ_t respecting the finite symmetries Γ/Γ_t will be a proof for Γ. In this way, we suppress branch points.

Considering Γ now to be torsion free, a puncture in R_{λ_0} corresponds to an infinite cyclic parabolic subgroup $\{\gamma_0(\lambda_0)\}\subset\bar{\Gamma}_{\lambda_0}$. Consider trace $\gamma_0(\lambda)$. This analytic function must be constantly 2, for otherwise elliptics of finite order would be created for λ near λ_0 contradicting our hypothesis. Thus, $\gamma_0(\lambda)$ stays parabolic for λ near λ_0.

Now we consider a fundamental domain P_{λ_0} for $\bar{\Gamma}_{\lambda_0}$ with cusps, a neighborhood N as in the figure and the perturbed fundamental domain P_λ defined as above.

The proof in the cocompact case can be said now in exactly the same way to treat



the case with punctures. The point is that the cusps move nicely because $\gamma_0(\lambda)$ stays parabolic. If a finite symmetry group is present, we make the constructions compatible with that symmetry. Q.E.D.

§ 7. Structural stability implies geometrically finite or rigid

Say that a finitely generated non-solvable subgroup $\Gamma_0 \subset PSl(2, \mathbf{C})$ is

non rigid if these are arbitrarily near non conjugate representations. Otherwise, it is *rigid,*

structurally stable if all sufficiently nearby representations are faithful i.e. injective,

convex cocompact if Γ_0 is discrete, has no parabolics, and has a finite sided fundamental domain in hyperbolic space.

Note. Convex cocompact for discrete groups is equivalent to an expanding or hyperbolicity property for the action of Γ_0 on the limit set $\Lambda \subset \bar{\mathbf{C}}$. For all x in Λ there is $\gamma \in \Gamma_0$ so that $|\gamma' x| > 1$ (spherical metric). It is also equivalent to the compactness of the intersection of a fundamental domain in hyperbolic space with the convex hull of the limit set. Sullivan [11] § 2, or § 9 of this paper.

THEOREM A. *A structurally stable $\Gamma_0 \subset PSl(2, C)$ is either rigid or convex cocompact.*

Proof. Suppose Γ_0 is not rigid. Then the algebraic variety of representations into $PSl(2, \mathbf{C})$ modulo conjugation has positive dimension at Γ_0. We can then have a non-trivial holomorphic family of subgroups Γ_λ, $\lambda \in \{z: |z|=1\}$ passing through Γ_0 and any nearby point (Bruhat, Cartan). These are all isomorphic (by structural stability). By Theorem 2, § 6, all the Γ_λ are discrete, quasi-conformally conjugate, with non-trivial domains of discontinuity on $\bar{\mathbf{C}}$.

Suppose for now that Γ_0 is torsion free. Let M_0 be the hyperbolic three-manifold $\mathbf{H}^3/\Gamma_0$ where $\mathbf{H}^3$ is hyperbolic 3-space. By Scott [9] Γ_0 is not only finitely presented but M_0 has a compact submanifold $C_0 \subset M_0$ of the same homotopy type as M_0. Denote the components of (M_0-interior C_0)$=E$ by $E_1, \ldots, E_n$. By excision and Mayer-Vietoris,

$$\bigoplus_i H_*(E_i, \partial E_i) = H_*(E, \partial E) = H_*(M_0, C_0) = 0.$$

Thus each of the E_i are homologically like cylinders on their boundaries ∂E_i—which are thus connected. We refer to the E_i as the ends of M_0.

Let d_i be 1 if ∂E_i is a torus and $3g-3$ if the genus g of $\partial E_i > 1$. Then Thurston (14, chapter 5] shows the dimenison D of the variety of representations modulo conjugacy at $\Gamma_0 \geq \Sigma_i d_i$. (The proof if there are no torii is just a matter of counting generators and relations. The proof for torii is also simple but ingenious.)

Now we derive an upper bound for the dimension D. Let S be the quotient of the domain of discontinuity by Γ_0. Then by the Ahlfors finiteness theorem [1], S is a finite union of compact Riemann surfaces with at most finitely many punctures. Each component S_i of S is hyperbolic in the sense that its Euler characteristic is negative, $-X_i = -$Euler characteristic of $S_i = (2g_i - 2) + p_i$ where g_i is the genus and p_i is the number of punctures. (This is true because some covering space of S_i is a region of $\bar{\mathbf{C}}$ with infinitely many frontier points.)

The quasiconformal deformations of Γ_0 all arise from deformations of S by Sullivan [10]. The deformations of S_i has dimension $(3g_i - 3) + p_i$ for $g_i = 0, 1, 2, 3, \ldots$. Thus we have $D = \Sigma_i (3g_i - 3 + p_i)$.

Recently Kulkarni and Shalen [6] have derived a purely topological result that implies the negative Euler characteristic of S is at most twice the negative Euler characteristic of Γ_0 (or M_0), i.e.

$$\sum_i (2g_i - 2) + p_i \leq -2(\text{Euler characteristic } M).$$

Combining all these relations we conclude no d_i can be 1 (i.e. no component of ∂M_0 is a torus) and all p_i are zero. Thus each component of S is compact and it also follows the Euler characteristics of ∂M_0 and S are equal. The homological product structure of the ends and the collar structure associated to cocompact domains of discontinuity then implies that each end of $M=\mathbf{H}^3/\Gamma$ defines one component of ∂M_0 and one component of S of the same genus.

It now follows that Γ_0 is geometrically finite without parabolics because after removing from M neighborhoods of the geometrically finite part the result is compact.

Acknowledgement. David Epstein kindly pointed out my earlier argument had a gap because of parabolics. This gap is neatly filled by the Kulkarni-Shalen inequality.

§ 8. Theorem A with fixed parabolics

There is also a relative version of Theorem A. Suppose Γ_0 is still non-rigid after fixing the traces of a finite number of elements to be 2. Because of non-rigidity by Theorem 2 Γ_0 is a discrete group. Assume now the above parabolic elements correspond to fixing a certain number of cusps of rank 2 and a certain number of rank one which are associated to punctures of the Riemann surface (domain of discontinuity of $\Gamma_0)/\Gamma_0$. Suppose finally that all sufficiently close representations with these fixed traces are injective.

THEOREM A′. *Then Γ_0 must be geometrically finite with no cusps other than the prescribed ones.*

Proof. The proof is the same as that of Theorem A.

§ 9. Structural stability in the sense of differentiable dynamics

We describe a structural stability theorem for group actions which applies to all convex cocompact discrete hyperbolic groups (=geometrically finite without parabolics). Let G be a finite symmetric set of generators for Γ ($g\in G \Leftrightarrow g^{-1}\in G$) acting smoothly on a manifold with a compact invariant set Λ. We suppose first that the action is *expanding* near Λ:

(i) For each g in G there is an open set Ug on which g expands the length of all tangent vectors by a factor $\lambda>1$. The Ug for $g\in G$ cover Λ.

The expanding property will allow us to code points of Λ by infinite sequences of elements of G. Our second assumption will imply this coding is essentially unique. Say two sequences $\{g_i\}$, $\{f_m\}$ in G are $\leq N$ apart if for all n there is an m (and vice versa) so that the *minimal* word length (in letters of G) of the element

$$(f_m \cdots f_2 f_1)(g_n \cdots g_2 g_1)^{-1}$$

is $\leq N$.

(ii) Now for any x in Λ consider any sequence $g_0, g_1, \ldots$ from G so that if $x_0=x$, $x_1=g_0x_0, \ldots, x_{n+1}=g_nx_n, \ldots$ we have x_n belongs to the expanding domain of g_{n+1} for $n=1,2,\ldots$. (Note $n=0$ is not included.) We say the action of Γ is *hyperbolic* if besides (i) we have there is an N so that for each x in Λ any two such sequences are $\leq N$ apart.

THEOREM I. *The action of any convex cocompact group is hyperbolic on its limit set in* $\bar{\mathbf{C}}$.

Now suppose we perturb the generators G slightly near Λ in the C^1 topology so that all the relations of Γ are still satisfied.

THEOREM II. *For any sufficiently small perturbation* Γ_1 *of a hyperbolic group action* Γ *there is a compact invariant set* Λ_1 *near* Λ *and a topological conjugacy between the action of* Γ *on* Λ *and* Γ_1 *on* Λ_1.

COROLLARY. *The group generated by the perturbed generators is abstractly isomorphic to* Γ. *That is, there are no new relations.*

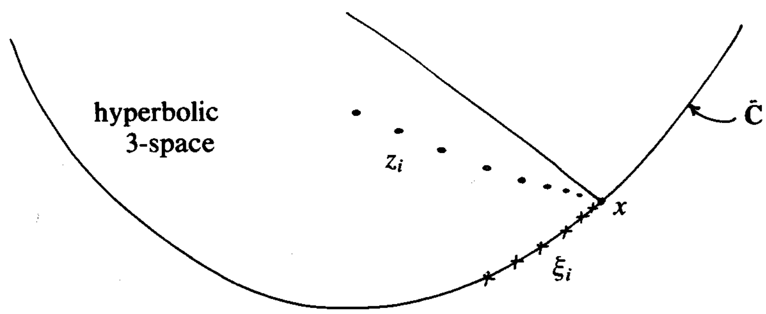
Proof of Theorem II. (i) Let $B(x, r)$ denote the ball of center x and radius r. Let $\varepsilon>0$ be smaller than the 1/2 Lebesgue number δ of the cover Ug of Λ. For each sequence $(g_0, g_1, \ldots)$ and $(x_0, x_1, \ldots)$ as above in (ii) with the additional assumption that $B(x_i, \delta)< U_{g_i}$, $i=0,1,2,\ldots$ consider the sequence of balls $B_i=B(x_i, \varepsilon)$. Let the generators G be C^1 perturbed (to $\bar{G}$) by small amounts to be determined below.

We will want first that if $h_i=g_i^{-1}$ $(\bar{h}_i=\bar{g}_i^{-1})$ then $\bar{h}_i$ compresses B_{i+1} well into the interior of B_i by a factor <1. This follows for sufficiently small C^1-perturbations because h_i compresses B_{i+1} concentrically into B_i by a definite factor less than one.

(ii) It follows that for n large $\bar{w}_n(B_{n+1})$ is an exponentially small ball about x (where $w_n=h_1\cdot h_2\cdot\ldots\cdot h_n$). *We will define our conjugacy* φ *by* $\varphi(x)=\bigcap_n \bar{w}_n(B_{n+1})$ *where* $\bar{w}_n=\bar{h}_1\cdot\bar{h}_2\cdot\ldots\cdot\bar{h}_n$.

(iii) φ *is well defined.*

Proof. If $(f_0, f_1, \ldots)$ is another sequence starting at x by our second assumption (ii) it is no more that N from $(g_0, g_1, \ldots)$. There are only finitely many elements in Γ of minimal word length $\leq N$ so there is an upper and lower bound on the distorsion of any tangent vector in Λ for them. If $B_1', B_2', \ldots$ denotes the balls corresponding to $(f_0, f_1, \ldots)$ and n is given there is an m and an element f from the finite set above carrying a central



part of B'_n into a central part of B_m with bounded distorsion (and vica versa). The perturbed generators satisfy the same relations which means f will also carry part of B'_n into B_m with bounded distortion for small enough perturbations. Thus $\bigcap_n \bar{\omega}_{n-1} B'_n = \bigcap_m \bar{\omega}_{m-1} B_m$, and these two choices determine the same $\varphi(x)$.

(ii) *φ is continuous.*

Proof. If x varies slightly, the balls B_i up to a fixed point $n+1$ (for a given code) only vary slightly. Thus $\varphi(x)$ in $\bar{\omega}_n(B_{n+1})$ only varies slightly.

(iii) *φ is a conjugacy.*

Proof. If x is Λ and $g_0 \in G$, let $y = g_0 x$. Choose a code $g_1, g_2, \ldots$ for y defining $\varphi(y)$. Choose a code $f_0, f_1, \ldots$ for x defining $\varphi(x)$. By our second assumption $g_0, g_1, \ldots$ is $\leq N$ from $f_0, f_1, \ldots$. (Note. This is why we omitted $n=0$ in the definition.) By the argument of (iii), $g_0^{-1}\varphi(y) = \varphi(x)$. Thus φ is a conjugacy.

(iv) *φ is injective.*

Proof (classic). $\varphi(x)$ is within ε of x because $\varphi(x)$ belongs to $B_1 = B(x, \varepsilon)$. The action of Γ on Λ is expansive. If two points are closer than δ, their distance apart can be expanded by the factor $\lambda > 1$. If $\varphi(x_1) = \varphi(x_2)$ we deduce distance $(x_1, x_2) \leq 2\varepsilon < \delta$ and $\varphi(\gamma x_1) = \varphi(\gamma x_2)$ for all γ. This is a contradiction. This proves Theorem II.

Proof of Theorem I. If Γ is a discrete group of hyperbolic isometries a fundamental domain in hyperbolic space may be constructed by removing those half spaces where $|\gamma' x| > 1$ (in the Euclidean metric on the ball) for all γ in Γ. This is the Dirichlet construction reinterpreted in the Euclidean metric. For Γ convex cocompact a finite number of these suffice and the resulting fundamental domain does not adhere to the limit set of Γ. Thus Γ is expanding on the limit set. The g in G correspond to the faces of the fundamental domain on the limit set. The Ug are the open disks on the sphere at infinity defined by the faces, cut down slightly.

For the second property, consider x in Λ and a sequence of group elements $g_0, g_1, \dots$ so that if $x_0=x,\ x_1=g_0x_0, \dots, x_{n+1}=g_nx_0, \dots$ then x_n belongs to Ug_n (cut down slightly by the Lebesgue number). Consider $w_n=h_1h_2\dots h_n$ where $h_i^{-1}=g_i$. Each w_n $(n>n_0)$ squeezes a ball of definite size $B(x_{n+1}, 1/2\delta)$ down around x by exponentially increasing with n faster.

Claim. This implies the sequence of points $z_0, z_1, \dots$ in hyperbolic space lies within a bounded distance of a geodesic heading towards x at infinity ($z_i=w_i$ (center of ball model)). To see this claim let ξ_i be the point at infinity behind z_i (as viewed from the center). Since w_i (center)$=z_i$, w_i consists of a rotation about the center followed by the hyperbolic element h_i with fixed point at ξ_i (and its antipode) which squeezes down around ξ_i the right amount to send the center to z_i. A look at the derivative of h_i shows how close ξ_i must be to x. This proves first that $\xi_i \to x$. This implies $z_i \to x$ since z_i does converge to infinity.

On the other hand if d_i=hyperbolic distance (center, z_i) then $d_i \leqslant$constant i because distance $(z_i, z_{i+1}) \leqslant$constant. Also $d_i \geqslant$constant i because of the statement above about w_i squeezing a ball down around x (exponentially).

Thus $i \to z_i$ is a discrete quasi-geodesic in hyperbolic space (the distance between any pair n, m of its points is comparable in ratio to $|n-m|$) and thus lies a bounded distance from a geodesic (for a reference see Thurston's discussion of Mostow rigidity [14, chapter 5]. This geodesic must converge to x because z_i does.

As a corollary, we deduce any two such sequences $z_i, \bar{z}_i$ are a bounded distance apart (the bounds are uniform in terms of the constants above) in hyperbolic space.

If we assume the center lies in the convex hull C of the limit set all the above discussion takes place in C. Since C/Γ is a compact manifold with boundary, an orbit of Γ in C has an induced metric equivalent in ratio to the minimal word length metric on Γ (observation of Milnor). Thus we have proven the uniqueness property for hyperbolicity. This proves Theorem I.

Remark. (Generalization of Theorem 1 to metric spaces.) The proof and formulation of the assertion *hyperbolicity implies structural stability* go through for expanding group actions on metric spaces. A sufficiently small perturbation means C^0 close but still expanding.

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