The Riemann-Roch theorem for complex spaces

by

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The subject of the present paper is the proof of the Riemann-Roch theorem for (possibly singular) complex spaces:

THE RIEMANN-ROCH THEOREM. *Denote by $K_0^{\mathrm{hol}}(M)$ the Grothendieck group of the category of all coherent sheaves on the complex space M and by $K_0^{\mathrm{top}}(M)$ the usual homology K-functor of the underlying topological space. Then there exists a group homomorphism $\alpha_M: K_0^{\mathrm{hol}}(M) \to K_0^{\mathrm{top}}(M)$, such that:*

(a) For M regular the restriction of the homomorphism α_M to the subgroup $K^0_{\mathrm{hol}}(M) \subset K_0^{\mathrm{hol}}(M)$, generated by the classes of all locally free sheaves, coincides with the natural morphism

$$K^0_{\mathrm{hol}}(M) \to K^0_{\mathrm{top}}(M) \approx K_0^{\mathrm{top}}(M)$$

attaching to each locally free sheaf on M the class of the corresponding vector bundle.

(b) If $f: M \to N$ is a proper morphism of complex spaces, and $f_!: K_0^{\mathrm{hol}}(M) \to K_0^{\mathrm{hol}}(N)$ is the direct image homomorphism, provided by Grauert's theorem, then the equality

$$f_* \alpha_M(\mathcal{L}) = \alpha_N(f_! \mathcal{L})$$

holds for any coherent sheaf $\mathcal{L}$ on M.

A detailed consideration of this form of the Riemann-Roch theorem, and its relation to the classical form of this theorem, due to Hirzebruch and Grothendieck, can be found in [4].

Originally, the R-R theorem was proven by F. Hirzebruch for algebraic vector bundles on a non-singular projective variety. Atiyah and Singer showed that the R-R theorem is a corollary of their Index theorem for differential operators; the R-R homomorphism attaches to any analytic fibre bundle E on the complex manifold M the element of the analytical K-group $\mathrm{Ell}\,(M)=K_0(M)$, generated by the Dolbeault complex of differential forms on M with coefficients in the bundle E, together with the representation of the algebra $C(M)$ of continuous functions by multiplication in every stage.

The first result, concerning coherent sheaves instead of vector bundles, appeared in the papers of Atiyah-Hirzebruch [2], [3]. In these papers the authors, using a real-analytic resolution of a coherent sheaf, construct the R-R homomorphism from $K_0^{\mathrm{hol}}(M)$ to $K^0(M)$ for a complex manifold M, and obtain as a consequence the relative version of the R-R theorem for embeddings of a complex manifold.

The essential step was made in the paper by Baum, Fulton, MacPherson where the R-R theorem was formulated in the form, cited above. There can be found a detailed explanation of the analogy between the algebraical and topological K-homology theories. However, the R-R theorem is proved for quasi-projective varieties, and the proof makes essential use of the existence of a locally free resolution for any coherent sheaf on the complex projective space.

In a series of recent papers (cf. [5]), O'Brian, Toledo, and Tong proved the R-R theorem in Hodge-Chern cohomology for coherent sheaves on a complex manifold. The proof is based on combinatorial "local–global" methods.

In his survey on K-theory in [1], M. F. Atiyah suggested that it would be desirable to have a proof of the R-R theorem in the form, cited above, based on operator-theoretical methods, and remaining valid on the larger category of all complex spaces. Such a proof is given in the present paper.

We start with some heuristic considerations. Let M be a complex manifold, and E an analytic vector bundle on M. In order to define the cohomology groups of M with coefficients in E, we can consider, instead of the Dolbeault complex, used in the Atiyah-Singer proof, the Čech complex $C_{\cdot}(M,\mathfrak{V},E)$ of alternating cochains of sections of E for a given Stein covering $\mathfrak{V}=\{U_i\}$ of M. We can replace this complex of Hilbert spaces $C_{\cdot}^{h}(M,\mathfrak{V},E)$, consisting of spaces of square-integrable sections of the bundle E; it can be proved that these two complexes are quasi-isomorphic. In order to obtain an element of the analytical K-group $\mathrm{Ell}(M)=K_0(M)$, one needs some more: a set of representations of the algebra of continuous functions $C(M)$ on each stage of the complex, such that the differentials of the complex are essentially splitting for these representations. To do this, choose the elements $\{U_i\}$ of the covering $\mathfrak{V}$ to be contractible and strongly pseudo-convex, and consider for any $U_\alpha=U_{i_1}\cap\ldots\cap U_{i_k}$ the algebra of all Toeplitz operators on U_α, i.e., the C^*-algebra of operators in $L^2(U_\alpha)$, generated by the operators of multiplication by the coordinate functions. This algebra is essentially commutative(1) and is a Fredholm representation of the algebra $C(bU_\alpha)$. Since the domain U_α is contractible, then, using Brown-Douglas-Fillmore theory, we can find a compact perturbation of this algebra, which is a usual (commutative) representation of the algebra $C(U_\alpha)$. Since the Toeplitz operators essentially commute with the operators of restriction, then, using an appropriate trivialisation of the bundle E on the domains U_α we obtain the desired element of the group $K_0(M)$; it can be proven that this element does not depend on the choice of compact perturbation of the algebra of Toeplitz operators. This gives an alternative proof of the Riemann-Roch theorem in the regular case.

Unfortunately, there does not exist an elaborated theory of Toeplitz operators in the singular case; in particular, it is not known if the operators of multiplication by coordinate functions are essentially normal in the spaces $H^2(U, \mathscr{L})$ of square-integrable sections of a coherent sheaf $\mathscr{L}$. Another difficulty comes from the fact that it is not known if the homology groups of a coherent sheaf can be computed by the use of cochains of square-integrable sections. All these difficulties can be avoided, if we try to construct, instead of the element $\alpha_M(\mathscr{L}) \in K_0(M)$, its Alexander dual, i.e. the corresponding element of the group $K^0(\mathbf{R}^{2n}, \mathbf{R}^{2n} \setminus M)$ for a suitable embedding of M in the Euclidean space $\mathbf{R}^{2n}$.

Let $T=(T_1, \dots, T_n)$ be an essentially commuting set of essentially normal operators, acting in a Hilbert space, $F \subset \mathbf{C}^n$ its joint essential spectrum, and ξ the corresponding element of the group $\operatorname{Ext}(F)$. In [1], M. F. Atiyah shows that its Alexander dual in $K^0(\mathbf{C}^n \setminus F)$ can be calculated by the use of Clifford matrices. It is well known that the notions of Clifford algebra and Koszul complex are closely related; hence, the same element of $K^0(\mathbf{C}^n \setminus F)$ can be defined by means of the parametrised Koszul complex of the operators $T_1, \dots, T_n$, which is a Fredholm complex of Hilbert spaces on $\mathbf{C}^n \setminus F$ depending continuously on a parameter. The Koszul complex is "more economical" than the Clifford algebra, since its definition does not involve the adjoint operators $T_1^*, \dots, T_n^*$. In particular, the above-mentioned algebra of Toeplitz operators corresponds to the parametrised Koszul complex of the operators of multiplication by the

(1) There is a gap in our consideration here. As far as is known to the author, there does not exist a proof of the fact that Toeplitz operators are essentially normal in the space $H^2(U)$, if U is an intersection of strongly pseudo-convex domains. However, this is well-known in the Banach spaces $H^\infty(U)$, and in our situation we can use Banach spaces instead of Hilbert spaces.

coordinate functions. This allows us to replace the Hilbert spaces of square-integrable sections by the Fréchet spaces of all sections; in this form, this complex makes sense even in the singular case. In order to obtain a topological equivalent of the operator-theoretical object constructed above, we have to attach together the Koszul complexes, corresponding to the various elements of the covering $\mathfrak{B}$. Suppose for the moment that M is a complex subspace of an open subset U in a Euclidean space; in this case, the canonical coordinate system on M can be used, the various Koszul complexes are well related and form a bicomplex. The corresponding total complex is Fredholm on U and exact off M, and defines the desired element of the group $K_M(U)=K_0(M)$. Using a slightly more complicated construction, the same idea can be realised in the general case.

Now we are going to describe the principal steps of our proof. In § 1 we consider parametrised complexes of Fréchet spaces in domains in Euclidean space. We shall call such a complex $\mathscr{C}$-, $\mathscr{E}$- or $\mathscr{O}$-exact, if the complex of continuous, smooth or holomorphic sections of the complex is exact. In contrast to the case of complexes of Banach spaces (see [11]), these conditions are mutually different and do not coincide with the condition of pointwise exactness of the complex. (In view of the criterion, proven in Corollary 1.3 in the text, the $\mathscr{C}$-exact complexes will be called uniformly exact.) Using this, we define the notions of uniformly Fredholm, $\mathscr{E}$-Fredholm and $\mathscr{O}$-Fredholm complex. The principal result of § 1 is that the space of uniformly Fredholm complexes can be used as a representative space for K-theory. Another result of § 1 is Lemma 1.8, a parametrised variant of the L. Schwartz theorem for compact perturbations of epimorphisms; we use this assertion as a substitute for Grauert's direct image theorem (a slight modification of this statement leads to a new proof of Grauert's theorem, which will be published separately).

In § 2 we recall the definition and properties of the parametrised Koszul complex of a commuting tuple of operators (cf. [11], and construct a R-R invariant for complex subspaces of open domains in $\mathbf{C}^n$. The general case is considered in § 3. In general, there does not exist an embedding of the complex space in a complex manifold; instead of this we show that any complex space M can be embedded in a suitable almost complex manifold $\tilde{M}$, and this embedding is stably unique (Lemma 3.1). Now, we can define the local Koszul complexes on $\tilde{M}$; using some standard algebraic machinery (Lemma 3.2), they can be attached together in a total complex. This completes the construction of the Riemann-Roch invariant. The rest of § 3 is devoted to a proof of the functorial property of this invariant. Finally, in § 4 we give some remarks and generalisations.

Note that in this way we obtain an infinite-dimensional free resolution of a given coherent sheaf $\mathscr{L}$. More exactly, we obtain a parametrised complex of Fréchet spaces on $\tilde{M}$, such that the complex of its smooth sections is quasi-isomorphic to the sheaf of smooth sections of $\mathscr{L}$, and the complex of almost-analytic sections is quasi-isomorphic to the sheaf $\mathscr{L}$. (See remark (1) of § 4.)

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0. Notations

Recall that a Fréchet space is a complete linear topological space with topology determined by a countable family of semi-norms $\|\cdot\|_n$, $n=1,2,\ldots$. A complex of Fréchet spaces is a system $X=\{X_i, d_i\}$, $i\in\mathbf{Z}$, where X_i are Fréchet spaces, and $d_i: X_i\to X_{i+1}$ are bounded linear operators. All complexes are assumed to be finite, i.e. $X_i=0$ for $|i|$ sufficiently large. Let U be a domain in a Euclidean space, and X and Y Fréchet spaces. An operator-valued function $d(\lambda): X\to Y, \lambda\in U$, will be called continuous, smooth or holomorphic on U, if for any $x\in X$ the Y-valued function $\lambda\mapsto d(\lambda)x$ is continuous, infinitely differentiable in the strong sense, or holomorphic, on the domain U. The parametrised complex $X_.(\lambda)=\{X_i, d_i(\lambda)\}$, $i\in\mathbf{Z}$, will be called continuous, smooth, or homomorphic respectively. Applying the Banach-Steinhaus theorem, one can see that in this case $d(\lambda)$ is uniformly bounded on any compact subset of U (in all cases) and all its derivatives in the strong operator topology exist (in the smooth or holomorphic case). Therefore, if $x(\lambda)$ is continuous, smooth or holomorhic respectively, then the Y-valued function $d(\lambda)x(\lambda)$ has the same property.

Let X be a Fréchet space. Denote by $\mathscr{C}X$, $\mathscr{E}X$ and $\mathscr{O}X$ the sheaves of germs of all continuous, smooth, and holomorphic X-valued functions on U. Let $X_.(\lambda)=\{X_i, d_i(\lambda)\}$ be a continuous complex of Fréchet spaces on U. Then by $\mathscr{C}X_.=\{\mathscr{C}X_i, d_i(\lambda)\}$ will be denoted the complex of sheaves of germs of all continuous sections of the complex $X_.(\lambda)$. The analogous notations, $\mathscr{E}X_.$ and $\mathscr{O}X_.$, will be adopted for the complexes of sheaves of germs of smooth and holomorphic sections (provided that the complex $X_.(\lambda)$ is smooth or holomorphic on U). Note that in the complexes used in our paper, all differentials are in fact linear functions of the parameter.

§ 1. Uniformly Fredholm complexes

Definition. The continuous complex of Fréchet spaces $X \xrightarrow{A(\lambda)} Y \xrightarrow{B(\lambda)} Z$, defined on the domain U, will be called *uniformly exact,* if for any compact subset F of U and for any natural number p there exists a natural number $q=q(p)$ and a constant $C=C(p)$, such that for any $\lambda \in F$ and $y \in Y$, satisfying $B(\lambda)y=0$, there exists $x \in X$, such that $A(\lambda)x=y$ and $\|x\|_p \leq C\|y\|_q$.

In other words, the complex is exact at any point of U, and the entities $q(p)$ and $C(p)$, provided by the Open mapping theorem, can be chosen to be locally independent of the parameter λ.

Example. Denote by X the Fréchet space of all infinitely smooth functions on the closed interval $[0, 1]$, vanishing at the origin with all derivatives, and let $A(t)=I-tD$, where D is the operator of differentiation. The complex $0 \to X \xrightarrow{A(t)} X \to 0$ is exact on the whole real axis. However, a simple calculation shows that it is not uniformly exact in any neighbourhood of zero.

The following assertion is a slight modification of Lemma 2.2 in [11]:

LEMMA 1.1. *Let the complex $X \xrightarrow{A(\lambda)} Y \xrightarrow{B(\lambda)} Z$ be uniformly exact on the domain U and $y(\lambda)$ be a continuous function with values in Y, satisfying $B(\lambda)y(\lambda)\equiv 0$. Then there exists a continuous function $x(\lambda)$ with values in X, such that $A(\lambda)x(\lambda)\equiv y(\lambda)$ on U.*

Proof. It is sufficient to prove the assertion on any compact subset F of U. We can suppose that for all $x \in X$, $n \in \mathbf{Z}_+$, we have $\|A(\lambda)x\|_n \leq \|x\|_n$, and for any $\lambda \in F$, $y \in \ker B(\lambda)$, $n \in \mathbf{Z}_+$ the equation $A(\lambda)x=y$ has a solution $x \in X$, satisfying $\|x\|_n \leq C_n\|y\|_{n+1}$. We shall construct a sequence $x_n(\lambda)$ of continuous X-valued functions, such that:

$$\|x_n(\lambda)\|_n \leq 2^{-n}, \quad \|y(\lambda)-A(\lambda)s_n(\lambda)\|_{n+2} \leq C_{n+1}^{-1} \cdot 2^{-n-1},$$

where $s_n(\lambda)$ denotes the nth partial sum of the series $\Sigma x_n(\lambda)$. Suppose that all $x_i(\lambda)$ are already constructed for $1 \leq i \leq n$, and give a construction for $x_{n+1}(\lambda)$. Let $r_n(\lambda)= y(\lambda)-A(\lambda)s_n(\lambda)$. Fix $\lambda \in F$. One can find an element $x_\lambda \in X$ such that $A(\lambda)x_\lambda=r_n(\lambda)$ and $\|x_\lambda\|_{n+1} \leq C_{n+1}\|r_n(\lambda)\|_{n+2} \leq 2^{-n-1}$. Using the continuity of the vector-functions $r_n(\lambda)$ and $A(\lambda)x$, on can find a neighborhood U_λ of λ in F such that for any $\lambda' \in U_\lambda$ one has $\|r_n(\lambda')-A(\lambda')x_\lambda\|_{n+3} \leq C_{n+2}^{-1} \cdot 2^{-n-2}$. Choose a finite covering $U_{\lambda_1}, \dots, U_{\lambda_k}$ of F, denote by $\{f_i\}, i=1, \dots, k$, a partition of unity, subordinated to the covering $\{U_{\lambda_i}\}$, and put

$$x_{n+1}(\lambda) = \sum_{i=1}^{k} f_i(\lambda) \cdot x_{\lambda_i}.$$

It is easy to check that all the required conditions for $x_{n+1}(\lambda)$ are satisfied.

Finally, we can put $x(\lambda)=\Sigma_{k=1}^{\infty} x_k(\lambda)$, and the proof of the lemma is completed.

COROLLARY 1.2. *Suppose that λ_0 is a point of F and the element $x_0 \in X$ satisfies $A(\lambda_0)x_0=y(\lambda_0)$. Then we can find a vector-function $x(\lambda)$, satisfying the requirements of the lemma and the supplementary conditions $x(\lambda_0)=x_0$,*

$$\sup_F \|x(\lambda)\|_n \leq 2\|x_0\|_n + 2C_n \sup_F \|y(\lambda)\|_{n+1}.$$

This follows immediately from the construction above.

Now, we are able to give the following characterisation of uniformly exact complexes:

COROLLARY 1.3. *The continuous complex $X_.(\lambda)=\{X_i, d_i(\lambda)\}$ is uniformly exact if and only if the corresponding complex of sheaves of continuous functions $\mathscr{C}X_.$ is exact.*

Proof. Suppose that the complex of sheaves $\mathscr{C}X_.$ is exact. Since the sheaves $\mathscr{C}X_i$ are soft, this implies that for any compact subset $F\subset U$ the complex $C(F, X_.)$ of continuous functions on F with values in $X_.$ is exact also. The system of semi-norms $\|x(\lambda)\|_p=\sup_{\lambda\in F}\|x(\lambda)\|_p$ determines a Fréchet structure in the spaces $C(F, X_k)$. Fix a number k. It follows from the Open mapping theorem that for any natural number p there exist a consant C and a natural number q, such that for any continuous X_k-valued function $y(\lambda)$ on F, satisfying $d_k(\lambda)y(\lambda)\equiv 0$, there exists a continuous X_{k-1}-valued function $x(\lambda)$, such that $d_{k-1}(\lambda)x(\lambda)=y(\lambda)$ and $\|x(\lambda)\|_p\leq C\|y(\lambda)\|_q$. Suppose that we have already proved that the complex $X_.(\lambda)$ is uniformly exact at the stage X_{k+1}. Let λ_0 be an arbitrary point of F and $y_0\in \ker d_k(\lambda_0)$. Using Corollary 1.3, we can find a continuous X_k-valued function $y(\lambda)$, such that $y(\lambda_0)=y_0$, $d_k(\lambda)y(\lambda)=0$, and $\|y(\lambda)\|_q\leq 2\|y_0\|_q$. Choosing $x(\lambda)$ as above, we obtain that $d_{k-1}(\lambda_0)x(\lambda_0)=y_0$ and $\|x(\lambda_0)\|_p\leq 2C\|y_0\|_q$. This means that $X_*(\lambda)$ is uniformly exact at X_k, and the proof is completed.

It will be useful for us to consider the classes of exact complexes, satisfying the same lifting property for smooth and for holomorphic vector-functions. Namely, the smooth (holomorphic) complex $X_.(\lambda)=\{X_k, d_k(\lambda)\}$ will be called *$\mathscr{E}$-exact ($\mathscr{O}$-exact)*, if the complex of sheaves $\mathscr{E}X_.$ ($\mathscr{O}X_.$) is an exact complex of sheaves.

LEMMA 1.4. *Any $\mathscr{E}$-exact complex is uniformly exact. Any $\mathcal{O}$-exact complex is $\mathscr{E}$-exact and uniformly exact.*

The proof uses the machinery of Koszul complexes and will be given in § 2.

Definition. Let $X_.(\lambda)$ and $Y_.(\lambda)$ be continuous complexes of Fréchet spaces, and $\varphi_.(\lambda): X_.(\lambda) \to Y_.(\lambda)$ be a continuous morphism of complexes. The morphism $\varphi_.(\lambda)$ will be called a *uniform quasi-isomorphism*, if its cone is a uniformly exact complex. An equivalent definition is the following one: $\varphi_.(\lambda)$ induces a quasi-isomorphism between the complexes of sheaves $\mathscr{C}X_.$ and $\mathscr{C}Y_.$.

In the same way one defines the notions of $\mathscr{E}$-quasi-isomorphism and $\mathcal{O}$-quasi-isomorphism.

The following lemma points out some elementary properties of the notions introduced above.

LEMMA 1.5. (a) *Let $\varphi(\lambda): X_.(\lambda) \to Y_.(\lambda)$ and $\psi_.(\lambda): Y_.(\lambda) \to Z_.(\lambda)$ be morphisms of complexes and $\zeta_.(\lambda) = \psi_.(\lambda) \circ \varphi_.(\lambda)$. Then if two of the morphisms $\zeta_.(\lambda), \psi_.(\lambda), \varphi_.(\lambda)$ are uniform quasi-isomorphisms, then the same is true for the third.*

(b) *Let $X_.(\lambda) = \{X_{i,j}, d'_{i,j}(\lambda), d''_{i,j}(\lambda)\}$ be a continuous bicomplex and $X_.(\lambda)$ be the corresponding total complex. Suppose that for any j the j-th row $X_{.,j}(\lambda)$ is a uniformly exact complex. Then the total complex $X_.(\lambda)$ is also uniformly exact.*

(c) *Let $X_.(\lambda) = \{X_i, d_i(\lambda)\}$, $X'_.(\lambda) = \{X'_i, d'_i(\lambda)\}$, be continuous complexes, and $\varphi(\lambda): X'_.(\lambda) \to X_.(\lambda)$ be a uniform quasi-isomorphism. Let $H_.(\lambda) = \{H_i, a_i(\lambda)\}$ be a continuous complex of finite-dimensional spaces and $\psi_.(\lambda): H_.(\lambda) \to X_.(\lambda)$ be a morphism of complexes. Then there exists a morphism $\psi'_.(\lambda): H_.(\lambda) \to X'_.(\lambda)$, such that $\psi_.(\lambda)$ is homotopic to $\varphi_.(\lambda) \circ \psi'_.(\lambda)$.*

Proof. The statements (a) and (b) are obvious. Let us prove (c). Denote by $\tilde{X}_.(\lambda) = \{\tilde{X}_i, \tilde{d}_i(\lambda)\}$ the cone of the morphism $\varphi_.(\lambda)$, and by $\tilde{\psi}_.(\lambda): H_.(\lambda) \to \tilde{X}_.(\lambda)$ the composition of the morphism $\psi_.(\lambda)$ and the natural embedding of $X_.(\lambda)$ in $\tilde{X}_.(\lambda)$. We have $\tilde{X}_i = X'_i \oplus X'_{i-1}$. Since the complex of sheaves $\tilde{X}_.$ is exact and H_i are finite-dimensional, standard sheaf-theoretical arguments show that the morphism of sheaves $\tilde{\psi}_.(\lambda)$ is homotopic to zero. So, there exists a series of continuous operator-valued functions, $\tilde{S}_i(\lambda): H_i \to \tilde{X}_{i-1}$, such that $\tilde{\psi}_i(\lambda) = \tilde{S}_{i+1}(\lambda) a_i(\lambda) - \tilde{d}_{i-1}(\lambda) \tilde{S}_i(\lambda)$. Denote by $\psi'_i(\lambda): H_i \to X'_i$, $S_i(\lambda): H_i \to X_{i-1}$ the components of the map $S_i(\lambda)$. Then the equality above is equivalent to:

$$\psi'_{i+1}(\lambda)\, a_i(\lambda) = d'_i(\lambda)\, \psi'_i(\lambda)$$

$$\psi_i(\lambda) - \varphi_i(\lambda)\, \psi'_i(\lambda) = d_i(\lambda)\, S_i(\lambda) + S_{i+1}(\lambda)\, a_i(\lambda).$$

In other words, $\psi'_.$ is a morphism of complexes, and $S_.(\lambda)$ defines a homotopy between $\psi_.(\lambda)$ and $\varphi_.(\lambda)\,\psi'_.(\lambda)$.

Of course, the same assertions are valid for $\mathscr{E}$- and $\mathcal{O}$-quasi-isomorphisms (in the latter case on Stein domains).

The statement (c) remains true in the case $H_.(\lambda)$ is a continuous complex of finite-dimensional vector bundles. In fact, we can construct $\psi'_.$ and $S_.(\lambda)$ locally and then use a partition of unity.

Definition. The continuous complex $X_.(\lambda)$, defined on the domain U, will be called *uniformly Fredholm,* if any point $\lambda_0 \in U$ has a neighborhood V, such that there exist on V a continuous complex $H_.(\lambda)$ of finite-dimensional spaces and a uniform quasi-isomorphism $\varphi_.(\lambda)\colon H_.(\lambda) \to X_.(\lambda)$.

In an analogous way we can define the notions of *$\mathscr{E}$-Fredholm* and *$\mathcal{O}$-Fredholm* complexes.

Example. Let U be a bounded domain in the complex plane $\mathbf{C}$, and denote by $X = H(U)$ the Fréchet space of all holomorphic functions on U. Let $A(\lambda) = M_z - \lambda I$, where M_z is the operator of multiplication by the coordinate function z, acting on X. Then the continuous complex $0 \to X \xrightarrow{A(\lambda)} X \to 0$ is Fredholm for all $\lambda \in \mathbf{C}$, but near the points of bU its index is not locally constant. It is not hard to see that this complex is uniformly Fredholm only on the domain $\mathbf{C}/bU$.

Note that any complex of Banach spaces, Fredholm at all points of U, is uniformly Fredholm.

Remark. Let $\mathscr{A} = \mathscr{C}$, $\mathscr{E}$ or $\mathcal{O}$. One can see from the definition that the parametrised complex $X_.(\lambda)$ is $\mathscr{A}$-Fredholm if and only if the complex of sheaves $\mathscr{A}X_.$ is a perfect complex in the category of all complexes of $\mathscr{A}$-modules in the sense of [12]. Suppose that all the sheaves of homologies $\mathscr{H}_i(\mathscr{A}X_.)$ are perfect, i.e. locally have a finite resolution of finitely generated free $\mathscr{A}$-modules. Then simple arguments (see [12], I, Lemma 4.15) show that the complex $\mathscr{A}X_.$ is also perfect and therefore $X_.(\lambda)$ is $\mathscr{A}$-Fredholm. In particular, the complex $X_.(\lambda)$ is $\mathcal{O}$-Fredholm if and only if all the sheaves $\mathscr{H}_i(\mathcal{O}X_.)$ are coherent.

The next lemmas show that the topological K-functor $K^0(U)$ of the topological space U can be defined by the use of uniformly Fredholm parametrised complexes of Fréchet spaces.

LEMMA 1.6. *Suppose that the continuous complex of Fréchet spaces $X_.(\lambda)$ is uniformly Fredholm on U and $F \subset U$ is a compact subset. Then there exist a continuous complex $H_.(\lambda)$ of finite-dimensional vector bundles on F and a uniform quasi-isomorphism $\varphi_.(\lambda): H_.(\lambda) \to X_.(\lambda)$. The class of the complex $H_.(\lambda)$ in the group $K^0(F)$ does not depend on the choice of $H_.(\lambda)$.*

Proof. Suppose that $H_i(X_.(\lambda))=0$ for any $i>n$ and $\lambda=F$. We shall construct a Euclidean space $\mathbf{C}^N$ and a linear operator $\psi(\lambda): \mathbf{C}^N \to X_n$, continuously depending on the parameter $\lambda \in F$, such that $d_n(\lambda)\psi(\lambda)\equiv 0$ and for any $\lambda \in F$ the image of $\psi(\lambda)$ generates the space of homologies $H_n(X_.(\lambda))$. In fact, take $\lambda^0 \in F$ and choose a continuous complex of finite-dimensional spaces $E_.(\lambda)$ and a continuous uniform quasi-isomorphism $\tau_.(\lambda): E_.(\lambda) \to X_.(\lambda)$, defined in a neighborhood of λ^0. One can take $E_.(\lambda)$ such that $E_i=0$ for $i>n$. Then the space E_n and the operator τ_n satisfy the conditions above in a neighborhood of λ^0. Multiplying $\tau_n(\lambda)$ by a continuous function with sufficiently small support, equal to one in a neighborhood of λ^0, we obtain an operator-function, defined on the whole of F. Since F is compact, we can choose a finite open covering $\{V_j\}$ of F, finite-dimensional spaces $\mathbf{C}^{N_j}$ and $\psi^j: \mathbf{C}^{N_j} \to X_n$ such that $d_n(\lambda)\psi^j(\lambda)=0$ for all $\lambda \in F$ and the image of $\psi^j(\lambda)$ generate $H_n(X_.(\lambda))$ for $\lambda \in V_j$. Taking for $\mathbf{C}^N$ the direct sum of all $\mathbf{C}^{N_j}$ and for $\psi(\lambda)$ the sum of $\psi^i(\lambda)$, we obtain the necessary.

Suppose that $X_i=0$ for $i<0$ or $i>n$. We shall construct $H_.(\lambda)=\{H_i, a_i(\lambda)\}$ and $\varphi_.(\lambda)$ by recurrence. Suppose that $H_i, a_i(\lambda), \varphi_i(\lambda)$ are already constructed for $i \geqq k+1$, and denote by $\tilde{X}_.(\lambda)=\{\tilde{X}_i, \tilde{d}_i(\lambda)\}$ the cone of $\varphi_.(\lambda)$. Then $\tilde{X}_.(\lambda)$ is uniformly Fredholm (see Step 1 in the proof of Lemma 1.7 below) and uniformly exact in the stages $>k$. If $k>0$, we apply the assertion above and obtain $\psi(\lambda): \mathbf{C}^N \to \tilde{X}_k$. Denote $H_k=\mathbf{C}^N$ and $a_k(\lambda), \varphi_k(\lambda)$ the projections of $\psi(\lambda)$ on H_{k+1} and X_k. When we reach the case $k=0$, the complex $\tilde{X}_.(\lambda)$ has only one non-zero homology group $\ker \tilde{d}_0(\lambda)$ and therefore the dimension of $\ker \tilde{d}_0(\lambda)$ is independent of λ. Denote by H_0 the corresponding vector bundle on F, and by $a_0(\lambda)$, $\varphi_0(\lambda)$ the components of the corresponding embedding of H_0 in $\tilde{X}_0(\lambda)$. The construction of $H_.(\lambda)$ and $\varphi(\lambda)$ is completed.

Let $H_.(\lambda)$, $H'_.(\lambda)$ be two complexes of vector bundles on F, uniformly quasi-isomorphic to $X_.(\lambda)$. As above, statement (c) of Lemma 1.5 shows that the complexes $H_.(\lambda)$ and $H'_.(\lambda)$ are quasi-isomorphic, and therefore determine the same element of the group $K^0(F)$.

We have proved that any uniformly Fredholm complex $X_.(\lambda)$ on U determines an element of the group $K^0(U)$ which will be denoted by $[X_.(\lambda)]$. Moreover, if the complex $X_.(\lambda)$ is exact off the closed subset M of U, then the element $[X_.(\lambda)]$ belongs to the group $K^0(U, U\setminus M)$. In such a way, the topological K-functor $K^0(U)$ (or $K^0(U, U\setminus M)$) coincides with the free abelian group, generated by the equivalence classes of uniformly Fredholm complexes of Fréchet spaces on U modulo the following relations:

(1) If the uniformly Fredholm complexes $X'_.(\lambda)$, $X''_.(\lambda)$ are uniformly quasi-isomorphic on U, then $[X'_.(\lambda)]=[X''_.(\lambda)]$.

(2) Suppose that the complexes $X'_.(\lambda)$, $X''_.(\lambda)$ are uniformly homotopic, i.e. there exists a complex $\tilde{X}_.(t,\lambda)$, uniformly Fredholm on $U\times[0,1]$ (and exact off $M\times[0,1]$) such that $\tilde{X}_.(0,\lambda)=X'_.(\lambda)$ and $\tilde{X}_.(1,\lambda)=X''_.(\lambda)$. Then $[X'_.(\lambda)]=[X''_.(\lambda)]$.

(3) If $X_.(\lambda)=X'_.(\lambda)\oplus X''_.(\lambda)$, then $[X_.(\lambda)]=[X'_.(\lambda)]+[X''_.(\lambda)]$.

The following assertions show that the group operation may be defined using exact sequences of complexes instead of direct sums.

Definition. Let $X'_.(\lambda)$, $X_.(\lambda)$, $X''_.(\lambda)$ be uniformly Fredholm complexes on U, and let $\varphi(\lambda): X'_.(\lambda)\to X_.(\lambda)$ and $\psi_.(\lambda): X_.(\lambda)\to X''_.(\lambda)$ be morphisms of complexes satisfying $\psi_.(\lambda)\varphi(\lambda)\equiv 0$. The short sequence of complexes

$$0\to X'_.(\lambda)\xrightarrow{\varphi_.(\lambda)} X_.(\lambda)\xrightarrow{\psi_.(\lambda)} X''_.(\lambda)\to 0$$

will be called *uniformly exact*, if the total complex of the corresponding bicomplex is uniformly exact on U.

In particular, this is satisfied, if for all numbers k the sequence

$$0\to X'_k\xrightarrow{\varphi_k(\lambda)} X_k\xrightarrow{\psi_k(\lambda)} X''_k\to 0$$

is uniformly exact (see Lemma 1.5 (c)).

LEMMA 1.7. *Suppose that*

$$0\to X'_.(\lambda)\xrightarrow{\varphi_.(\lambda)} X_.(\lambda)\xrightarrow{\psi_.(\lambda)} X''_.(\lambda)\to 0$$

is a uniformly exact sequence of uniformly Fredholm complexes. Then $[X_.(\lambda)]=[X'_.(\lambda)]+[X''_.(\lambda)]$.

Proof. Suppose that $X'_.(\lambda)$, $X_.(\lambda)$ are uniformly Fredholm complexes, and $\varphi_.(\lambda): X'_.(\lambda)\to X_.(\lambda)$ is a morphism. Then we shall prove that its cone $K\varphi_.(\lambda)$ is uniformly

Fredholm and $[K\varphi_{.}(\lambda)]=[X_{.}(\lambda)]-[X'_{.}(\lambda)]$. In fact, let $H'_{.}(\lambda)$, $H_{.}(\lambda)$ be complexes of vector bundles, and let $\alpha'_{.}(\lambda): H'_{.}(\lambda)\to X'_{.}(\lambda)$, $\alpha_{.}(\lambda): H_{.}(\lambda)\to X_{.}(\lambda)$ be uniform quasi-isomorphisms. Applying Lemma 1.5 (c), we obtain a morphism $\zeta_{.}(\lambda): H'_{.}(\lambda)\to H_{.}(\lambda)$, such that $\alpha_{.}(\lambda)\zeta(\lambda)$ is homotopically equivalent to $\varphi_{.}(\lambda)\alpha'_{.}(\lambda)$. Denote by $S_k(\lambda): H'_k\to X_{k-1}$ the corresponding homotopies. We can suppose that $\zeta_{.}(\lambda)$ is a monomorphism (if this is not satisfied, one may add a suitable exact complex to the complex $H_{.}(\lambda)$). Therefore, one can construct a sequence of operators $S_k(\lambda): H_k\to X_{k-1}$, such that $\tilde{S}_{.}(\lambda)\zeta(\lambda)=S_{.}(\lambda)$. Denote by $\tilde{\alpha}_{.}(\lambda): H_{.}(\lambda)\to X_{.}(\lambda)$ the morphism of complexes, obtained by modifying the morphism $\alpha_{.}(\lambda)$ by the homotopy $\tilde{S}_{.}(\lambda)$. Then we have $\tilde{\alpha}_{.}(\lambda)\tau(\lambda)\equiv\varphi(\lambda)\alpha'_{.}(\lambda)$. Denote by $K\varphi_{.}(\lambda)$, $K\tau_{.}(\lambda)$ the cones of the morphisms $\varphi_{.}(\lambda)$, $\tau_{.}(\lambda)$. It is easy to see that the pair $\alpha'_{.}(\lambda)$, $\tilde{\alpha}_{.}(\lambda)$ of quasi-isomorphisms determines a uniform quasi-isomorphism from the complex $K\tau_{.}(\lambda)$ to the complex $K\varphi_{.}(\lambda)$. The assertion is proved.

Let

$$0\to X'_{.}(\lambda)\xrightarrow{\varphi_{.}(\lambda)} X_{.}(\lambda)\xrightarrow{\psi_{.}(\lambda)} X''_{.}(\lambda)\to 0$$

be a uniformly exact sequence of complexes. This means that the morphism from the complex $K\varphi_{.}(\lambda)$ to the complex $X''_{.}(\lambda)$ induced from the morphism $\psi_{.}(\lambda)$ is a uniform quasi-isomorphism. Therefore, $[X''_{.}(\lambda)]=[K\varphi_{.}(\lambda)]=[X_{.}(\lambda)]-[X''_{.}(\lambda)]$, and the proof is completed.

It is well-known that if $X_{.}$, $Y_{.}$ are complexes of Fréchet spaces, and $K_{.}: X_{.}\to Y_{.}$ is a compact quasi-isomorphism, then both complexes $X_{.}$, $Y_{.}$ are Fredholm. This assertion is an immediate consequence of L. Schwartz's perturbation theorem and has been used in Cartan-Serre's proof of the finiteness theorem. We shall prove a parametrised variant of this assertion.

Lemma 1.8. *Suppose that $X_{.}(\lambda)=\{X_i, d_i(\lambda)\}$, and $Y_{.}(\lambda)=\{Y_i, s_i(\lambda)\}$ are continuous complexes of Fréchet spaces on the domain U, and $K_{.}: X_{.}(\lambda)\to Y_{.}(\lambda)$ is a uniform quasi-isomorphism of complexes, such that all K_i are compact operators. Then the complexes $X_{.}(\lambda)$, $Y_{.}(\lambda)$ are uniformly Fredholm on U.*

Proof. Step 1. Suppose that for some $\lambda^0\in U$ the complexes $X_{.}(\lambda^0)$, $Y_{.}(\lambda^0)$ are exact at the $(i+1)$st stage. We shall prove that on a sufficiently small neighborhood of λ^0 they are uniformly exact at the same stage. In fact, fix the seminorms $\|\cdot\|_p$ on X_i and $\|\cdot\|_r$ on Y_{i-1}. There exist natural numbers q, s and a constant C, such that for any λ close to λ^0 and for any pair $x\in X_{i+1}$, $y\in Y_i$, satisfying $d_{i+1}(\lambda)x=0$, $s_i(\lambda)y=K_{i+1}x$, there exist $z\in X_i$, $t\in Y_{i-1}$, such that $d_i(\lambda)z=x$, $K_iz+s_{i-1}(\lambda)t=y$, and $\|z\|_p+\|t\|_r\le C(\|x\|_q+\|y\|_s)$.

SUBLEMMA. *There exist a constant C' and a neighborhood V of λ^0 such that for any $\lambda \in V$ and $x \in \ker d_{i+1}(\lambda)$ with $\|x\|_q \leq 1$ there exists an element $z \in X_i$, satisfying $\|z\|_p \leq C'$ and $\|x-d_i(\lambda)z\|_q \leq 1/2$.*

Proof. Suppose that the assertion is not true. Then there exists a sequence of points $\lambda^n \to \lambda^0$ and a sequence of elements $x_n \in \ker d_{i+1}(\lambda^n)$ with $\|x_n\|_q \leq 1$ such that for any $z \in X_i$ with $\|z\|_p \leq n$ we have $\|x_n - d_i(\lambda^n)z\|_q > 1/2$. Let $\lambda^n \to \lambda^0$ be a sequence of points of U and $x_n \in X_{i+1}$ be a sequence of elements, such that $d_{i+1}(\lambda^n)x_n=0$ and $\|x_n\|_q \leq 1$. Replacing the sequence x_n by a subsequence, one can find an element $x_0 \in Y_{i+1}$, such that $K_{i+1}x_n \to x_0$. There exist a constant D and a natural number m (not depending on x_0), and an element $y_0 \in Y_i$, satisfying $s_i(\lambda^0)y_0 = x_0$, $\|y_0\|_s \leq D\|x_0\|_m$. Take constants $D_{m,q}$ such that $\|K_j x\|_m \leq D_{m,q}\|x\|_q$ for any number j and element $x \in X_j$. Then $\|x_0\|_m \leq D_{m,q}$ and $\|y_0\|_s \leq D \cdot D_{m,q}$. Now $s_i(\lambda^n)y_0 - K_{i+1}x_n \underset{n}{\to} 0$ and there exist sequences of elements $u_n \in Y_i$, $v_n \in X_{i+1}$ such that $u_n \to 0$, $v_n \to 0$ and

$$s_i(\lambda^n)u_n - K_{i+1}v_n = s_i(\lambda^n)y_0 - K_{i+1}x_n.$$

This implies that there exists a sequence $\{z_n\}$, $z_n \in X_i$, such that $\|z_n\|_p \leq C' = C(1+D\cdot D_{m,q})$ and $x_n - v_n = d_i(\lambda^n)z_n$. Then $x_n - d_i(\lambda^n)z_n \underset{n}{\to} 0$ and we obtain a contradiction.

Fix $\lambda \in V$ and $x \in \ker d_{i+1}(\lambda)$ with $\|x\|_q \leq 1$. Iterating the construction, stated in the sublemma, we obtain a sequence $\tilde{z}_n \in X_i$, such that $\|\tilde{z}_n\|_p \leq 2C'$ and $\|x - d_i(\lambda)\tilde{z}_n\|_q \underset{n}{\to} 0$. Put $\tilde{y}_n = K_i \tilde{z}_n$. One sees that the sequence $\tilde{y}_n$ has a limit point $\tilde{y} \in Y_i$, satisfying $s_i(\lambda)\tilde{y} = K_{i+1}x$ and $\|\tilde{y}\|_s \leq 2C \cdot D_{s,p}$. Applying again the fact that $K_.$ is a uniform quasi-isomorphism, one obtains the existence of an element $z \in X_i$ such that $d_i(\lambda)z = x$ and $\|z\|_p \leq C'' = C(1+C'\cdot D_{s,p})$. The proof of step 1 is completed.

Step 2. We shall construct a continuous complex $H_.(\lambda) = \{H_i, a_i(\lambda)\}$ of finite-dimensional spaces, defined on some neighborhood of λ^0, and a uniform quasi-isomorphism $\varphi_.(\lambda): H_.(\lambda) \to X_.(\lambda)$. Suppose that H_i, $a_i(\lambda)$, and $\varphi_i(\lambda)$ are already constructed for $i \geq k+1$, and denote by $\tilde{X}_.(\lambda) = \{\tilde{X}_i, \tilde{d}_i(\lambda)\}$ the cone of the morphism $\varphi(\lambda)$, determined for stages $\geq k$. Since the complex $X_.(\lambda^0)$ is Fredholm, the space

$$\ker d_k(\lambda^0)/\operatorname{im} d_{k-1}(\lambda^0)$$

is finite-dimensional, and one can choose elements $x_1, \dots, x_m \in \ker d_k(\lambda^0)$, forming a basis of this space. Since the complex $\tilde{X}_.(\lambda)$ is uniformly exact at the $(k+1)$st stage, applying Corollary 1.2, one can construct continuous functions $x_j(\lambda)$, $j=1,\dots,m$, with values in $\tilde{X}_k$, such that $d_k(\lambda)x_j(\lambda) \equiv 0$ and $x_j(\lambda^0) = x_j$. Put $H_k = \mathbf{C}^m$ and denote by $x(\lambda)$ the map from H_k to $\tilde{X}_k$, determined by the functions $x_1(\lambda), \ldots, x_m(\lambda)$, and let $a_k(\lambda)$, $\varphi_k(\lambda)$ be the projections of $x(\lambda)$ on H_{k+1} and X_k. Now the complex $H_.(\lambda)=\{H_i, a_i(\lambda)\}$ and the morphism of complexes $\varphi_.(\lambda)=\{\varphi_k(\lambda)\}$ are defined also for $i=k$, and it follows from the construction that $\varphi_.(\lambda^0)$ is a quasi-isomorphism at the stages $\geq k$. If X_k is the first non-zero stage of the complex $X_.(\lambda)$, then we put $H_i=0$ for $i<k$. Then $\varphi_.(\lambda^0)$ is a quasi-isomorphism at all the stages.

Let $\psi_.(\lambda)=K_.\circ\varphi_.(\lambda)$, and let $\tilde{Y}(\lambda)$ be the cone of the morphism $\psi_.(\lambda)\colon H_.(\lambda)\to Y_.(\lambda)$. Let $\tilde{K}_.\colon \tilde{X}_.(\lambda)\to\tilde{Y}_.(\lambda)$ be the morphism of complexes, determined by the morphism $K_.$ and the identical map of $H_.(\lambda)$ into itself. Since $K_.$ is a uniform quasi-isomorphism, $\tilde{K}_.$ is also. Applying the assertion proved in step 1, we obtain that $X_.(\lambda)$ is uniformly exact, i.e. $\varphi_.(\lambda)$ is a uniform quasi-isomorphism.

§ 2. Koszul complexes

In this paragraph we shall describe a construction of an infinite-dimensional free resolution of a coherent sheaf in $\mathbf{C}^n$, using the notion of Koszul complex.

Denote by Λ_n the free anticommutative algebra with n generators $s_1, \ldots, s_n$, and by Λ_n^p the space of all its homogeneous elements of degree p. Let $a=(a_1, \ldots, a_n)$ be an n-tuple of elements of a commutative algebra A. Recall that a *Koszul complex* for the n-tuple a in A is the complex $K_.(a, A)=\{K_i(a, A), d_i\}$, where $K_i(a, A)=\Lambda_n^i\otimes A$, and $d_i\colon K_i(a, A)\to K_{i+1}(a, A)$ is the operator of exterior multiplication by the element $\Sigma_{i=1}^n a_i s_i$. If X is a linear space and $a=(a_1, \ldots, a_n)$ an n-tuple of commuting operators, acting in X, then a Koszul complex for the n-tuple a in X will be defined as the complex $K_.(a, X)=K_.(a, A)\oplus_A X$, where A is the algebra of operators, generated by the operators of a. For our purposes, it will be useful to recall the inductive definition of the Koszul complexes. If the set a consists of only one operator a_1, then $K_.(a, X)$ coincides with the complex $0\to X\xrightarrow{a_1} X\to 0$. Let $a'=(a_1, \ldots, a_{n-1})$, $a=a'\cup\{a_n\}$ be sets of commuting operators. Since a_n commutes with the operators of the set a', the multiplication by a_n determines an endomorphism of the complex $K_.(a', X)$. It is not hard to see that the complex $K_.(a, X)$ coincides with the cone of this endomorphism (up to a change of enumeration). Let a, b be sets of commuting operators in X and $b\subset a$. Then the inductive construction shows that there exist a natural embedding of $K_.(b, X)$ in the complex $K_.(a, X)$ and a natural projection of $K_.(a, X)$ on $K_.(b, X)$. If there exists a commutative n-tuple $p_1, \ldots, p_n$ of operators in X, which commute with the operators of a, such that $\Sigma_{i=1}^n p_i a_i=I$; then the complex $K_.(a, X)$ is homotopically trivial.

Let $\lambda=(\lambda_1,\dots,\lambda_n)$ be a point of $\mathbf{C}^n$. Denote by $K_.(a,X)(\lambda)$ the Koszul complex for the n-tuple of operators $a-\lambda I=(a_1-\lambda_1 I,\dots,a_n-\lambda_n I)$. This parametrised Koszul complex was used by J. L. Taylor in [11] in his definition of joint spectra of several commuting operators. If X is a Fréchet space, the complex $K_.(a,X)(\lambda)$ is a continuous (and holomorphic) complex of Fréchet spaces of $\mathbf{C}^n$.

Let $X_.(\mu)=\{X_i, d_i(\mu)\}$ be a continuous complex of Fréchet spaces on the domain U and $a=(a_1,\dots,a_n)$ be an n-tuple of commuting endomorphisms of the complex $X(\mu)$. (This means that for each i, $a_i=\{a_{i,j}\}_{j=1}$, $a_{i,j}$ are continuous linear operators, acting in the space X_j, and $a_{i_1,j}$ commutes with $a_{i_2,j}$.) Then each differential $d_j(\mu)$ induces a morphism of complexes $d_{j,.}(\mu)\colon K_.(A,X_j)(\lambda)\to K_.(a,X_{j+1})(\lambda)$. Denote the parametrised bicomplex constructed in this way by $K_.(a,X_.)(\lambda,\mu)$, where $(\lambda,\mu)\in\mathbf{C}^n\times U$, and let $KX_.(a)(\lambda,\mu)$ be the corresponding total complex.

LEMMA 2.1. *Suppose that $X_.(\mu)$ is uniformly exact or uniformly Fredholm on the domain U. Then the same is true for the complex $KX_.(a)(\lambda,\mu)$ on $\mathbf{C}^n\times U$.*

Proof. If $X_.(\mu)$ is uniformly exact on U, the assertion follows from Lemma 1.5 (b). If $X_.(\mu)$ is uniformly Fredholm, one uses the inductive definition of Koszul complex and applies several times the assertion of step 1 in the proof of Lemma 1.7.

Denote by M the support of the uniformly Fredholm complex $X_.$ in U and by M' the support of $KX_.(a)(\lambda,\mu)$ in $\mathbf{C}^n\times U$. Let $[X_.(\mu)]$ and $[KX_.(a)(\lambda,\mu)]$ be the corresponding elements of the groups $K_M(U)\approx K_0(M)$ and $K_{M'}(\mathbf{C}^n\times U)\approx K_0(M')$. Suppose that $a_{i,j}-\lambda_i I$ are invertible for $|\lambda_i|$ sufficiently large.

LEMMA 2.2. (a) $[X_.(\mu)]=[KX_.(0)(\lambda,\mu)]$, *where* 0 *is the n-tuple consisting of zero operators. (In this case $M'=M$.)*

(b) $[X_.(\mu)]=p_*[KX_.(a)(\lambda,\mu)]$, *where p is the projection of M' on M, induced by the projection of $\mathbf{C}^n\times U$ on U.*

Proof. (a) If $H_.(\mu)$ is a complex of vector bundles, uniformly quasi-isomorphic to $X_.(\mu)$, then $KH_.(0)(\lambda,\mu)$ is uniformly quasi-isomorphic to $KX_.(0)(\lambda,\mu)$ (see Lemma 2.1). Since $[KH_.(0)(\lambda,\mu)]$ coincides with the element $i_![H_.(\lambda)]$, where $i_!\colon K_M U\to(\mathbf{C}^n\times U)$ is the Thom-Gysin isomorphism, the assertion (a) follows.

(b) Note that the uniformly Fredholm complexes $KX_.(0)(\lambda,\mu)$ and $KX_.(a)(\lambda,\mu)$ are uniformly homotopic on $\mathbf{C}^n\times U$, the homotopy being given by the complex $KX_.(ta)(\lambda,\mu)$, $0\leqslant t\leqslant 1$. Let B be a closed ball in $\mathbf{C}^n$, such that $M'\subset B\times M=B'$. Therefore in the group $K_{B'}(\mathbf{C}^n\times U)$ we have

$$[KX_.(a)(\lambda,\mu)] = [KX_.(0)(\lambda,\mu)] = i_![X_.(\lambda)]$$

and the assertion is proved.

Now, we are able to prove Lemma 1.4.

Proof of Lemma 1.4. We shall prove that any $\mathscr{E}$-exact complex on a domain U is uniformly exact. The case of an $\mathcal{O}$-exact complex can be treated in an analogous way. It is sufficient to consider the case when U is the product of n copies of the interval $(0,1)$. Let X be a Fréchet space. Denote by $D(U,X)$ the space of smooth X-valued functions on U, and let a_i, $1=i,\dots,n$, be the operator of multiplication by the ith coordinate function, acting in $D(U,X)$. Denote by $K_.(a,D(U,X))(t)$ for $t=(t_1,\dots,t_n)$ the Koszul complex for the operators a_1-t_1I, a_n-t_nI, and let $\Delta(t): D(U,X)\to X$ be the evaluation at the point t. The first step of the proof is the following: the complex

$$K_.(a,D(U,X))(t) \overset{\Delta(t)}{\to} X\to 0$$

is uniformly exact on U. We shall use an induction by n. For $n=1$, the complex

$$0\to D(U,X) \overset{a-tI}{\to} D(U,X) \overset{\Delta(t)}{\to} X\to 0 \tag{*}$$

is uniformly exact on the domain $U=(0,1)$; this follows from the fact that any smooth X-valued function $f(s)$, vanishing at the point $t\in(0,1)$, can be represented in the form $f(s)=(s-t)g(s)$. In the general case, denote by U' the product of $n-1$ copies of the interval $(0,1)$, $U=U'\times(0,1)$, $a=(a_1,\dots,a_{n-1})$. Note that $D((0,1),D(U',X))=D(U,X)$. As in (*), we obtain the uniformly exact complex (for $t_n\in(0,1)$)

$$0\to D(U,X) \xrightarrow{a_n-t_nI} D(U,X) \xrightarrow{\Delta(t_n)} D(U',X)\to 0$$

which induces a uniformly exact sequence of complexes on U:

$$0\to K_.(a',D(U,X))(t) \xrightarrow{a_n-t_nI} K_.(a',D(U,X))(t) \xrightarrow{\Delta(t_n)} K_.(A',D(U',X))(t)\to 0.$$

Recalling the inductive definition of the Koszul complexes, we obtain that $\Delta(t_n)$ indues a uniform quasi-isomorphism of the complex $K_.(a,D(U,x))(t)$ with $K_.(a',D(U',X))(t)$, compatible with the evaluation morphism $\Delta(t)$. Then step 1 of the proof is completed.

Now, let $X_.(t)=\{X_i, d_i(t)\}$ be an $\mathscr{E}$-exact complex on the domain U. Let a be, as above, the tuple of operators of multiplication by the coordinate functions in the spaces $D(U,X_i)$, and denote by $KD_.(t)$ the corresponding Koszul complex, i.e. the total complex of the bicomplex $K_.(a, D(U, X))(t)$. The first step of the proof and Lemma 1.5 (b) shows that the parametrised complexes $X_.(t)$ and $KD_.(t)$ are uniformly quasi-isomorphic on U. On the other hand, since all the rows of the bicomplex $K_.(a, D(U, X_.))(t)$ are constant complexes (the differentials do not depend on the parameter t), and any exact constant complex is uniformly exact, Lemma 1.5 (b) shows that the total complex $KD_.(t)$ is uniformly exact. The lemma is proved.

Now, we begin the construction of the main object of the paper in the case of a Euclidean space. Let U be a Stein domain in the space $\mathbf{C}^m$ with coordinates $z_1, \ldots, z_n$, and $\mathcal{L}$ be a coherent analytic sheaf, defined on some neighborhood of U. Denote by $\mathcal{E}\mathcal{L}$ the corresponding coherent sheaf of $\mathcal{E}$-modules: $\mathcal{E}\mathcal{L}=\mathcal{E}\oplus_{\mathcal{O}}\mathcal{L}$, here $\mathcal{E}$ is the sheaf of germs of smooth functions on $\mathbf{C}^n$. The space $\Gamma_U(\mathcal{L})$, of all sections of $\mathcal{L}$ on U has a canonical structure of Fréchet space. Let $a=(a_1, \ldots, a_n)$ be the n-tuple of operators of multiplication by $z_1, \ldots, z_n$, acting on the space $\Gamma_U(\mathcal{L})$. In order to abbreviate the notations we shall denote the corresponding parametrised Koszul complex $K_.(a, \Gamma_U(\mathcal{L}))(\lambda)$ by $K_.(U, \mathcal{L})(\lambda)$.

If $\mathcal{L}$ is a sheaf on the topological space M and U is an open subset of M, then we shall denote by $\mathcal{L}_U$ the sheaf on M, associated to the present $V \mapsto \Gamma_{V\cap U}(\mathcal{L})$.

LEMMA 2.3. *The complexes of sheaves $\mathcal{O}K_.(U, \mathcal{L})$ and $\mathcal{E}K_.(U, \mathcal{L})$ are quasi-isomorphic on $\mathbf{C}^n$ to the sheaves $\mathcal{L}_U$ and $\mathcal{E}\mathcal{L}_U$ respectively.*

Proof. Denote $U'=\mathbf{C}^n\times U$, and let $\lambda=(\lambda_1, \ldots, \lambda_n)$ be the coordinate function on $\mathbf{C}^n$, and $z=(z_1, \ldots, z_n)$ on U. Let p and q be the projections of U' on $\mathbf{C}^n$ and U respectively. Put $\mathcal{L}'=q^*\mathcal{L}$. Then it is easy to see that the sheaf $\mathcal{O}\Gamma_U(\mathcal{L})$ coincides with the direct image $p_*\mathcal{L}'$ of the sheaf $\mathcal{L}'$. Denote by Δ the diagonal $\{(\lambda, z): \lambda=z\}$ in U', and by $K_.(\lambda-z, \mathcal{L}')$ the Koszul complex of the operators of multiplication by $\lambda_1-z_1, \ldots, \lambda_n-z_n$ in the sheaf $\mathcal{L}'$. Then the complex of sheaves

$$K_.(\lambda-z, \mathcal{L}')\to\mathcal{O}_\Delta\otimes\mathcal{L}'\to 0$$

is exact. In fact, this follows from the fact that the complex $K_.(\lambda-z, \mathcal{O}_{U'})\to\mathcal{O}_\Delta\to 0$ is exact and that the sheaves $\mathcal{O}_\Delta$ and $\mathcal{L}'$ are Tor-independent on U'. It is easy to see that the direct image of the complex $K_.(\lambda-z, \mathcal{L}')$ under the projection $p: U'\to\mathbf{C}^n$ is equal to the complex $\mathcal{O}K_.(U, \mathcal{L})$ and that the direct image of the sheaf $\mathcal{L}'\otimes\mathcal{O}_\Delta$ coincides with the sheaf $\mathcal{L}_U$. Therefore we obtain the exact complex $\mathcal{O}K_.(U, \mathcal{L})\to\mathcal{L}_U\to 0$. Multiplying this complex by the sheaf $\mathcal{E}$ and recalling the theorem of Malgrange (see [8]), which asserts that the sheaves $\mathcal{L}$ and $\mathcal{E}$ are Tor-independent as $\mathcal{O}$-modules, we obtain the exact complex of sheaves $\mathcal{E}K_.(U, \mathcal{L})\to\mathcal{E}\mathcal{L}_U\to 0$.

Remark. Let $U=V\times W$, where $V\subset\mathbf{C}^n$, $W\subset\mathbf{C}^n$ are Stein domains, and $\mathscr{L}$ be a coherent sheaf on U. Denote by $K_.(V,\mathscr{L})(\lambda)$ the Koszul complex of the operator of multiplication by the coordinate functions $z_1,\dots,z_n$ on V, acting in the space $\Gamma_U(\mathscr{L})$. The same arguments as above show that the complexes $\mathscr{O}K_.(V,\mathscr{L})\to\pi_*\mathscr{L}\to 0$ and $\mathscr{E}K_.(V,\mathscr{L})\to\mathscr{E}\pi_*\mathscr{L}\to 0$ are exact, where π is the projection of U on V.

We shall need to use for technical purposes the following generalisation of Lemma 2.3. Let $f(t)=(f_1(t),\dots,f_n(t))$ be an n-tuple of continuous scalar functions on the interval $[0,1]$ and denote by $K_.(f(t)a-\lambda,U)$ the Koszul complex for the operators $f_1(t)a_1-\lambda_1 I,\dots,f_n(t)a_n-\lambda_n I$ in the space $\Gamma_U(\mathscr{L})$ defined on $\mathbf{C}^n\times[0,1]$ (here, as above, a is the n-tuple of operators of multiplication by the coordinate functions, and λ_i are coordinate functions on $\mathbf{C}^n$). Denote by $\hat{\mathscr{O}}K_.(f(t)a-\lambda,U)$ (resp. $\hat{\mathscr{E}}K_.(f)t)a-\lambda,U)$) the complex of sheaves of all sections of this complex, which are holomorphic (resp. smooth) with respect to λ and continuous with respect to t. Denote by $\mathscr{C}\mathscr{L}$ the sheaf over the space $U\times[0,1]$, such that the sections of this sheaf over a subspace of the type $V\times(\alpha,\beta)$ are all continuous vector-functions on (α,β) with values in the Fréchet space $\Gamma_V(\mathscr{L})$. Replacing $\mathscr{L}$ by $\mathscr{E}\mathscr{L}$, we obtain a definition of the sheaf $\mathscr{C}\mathscr{E}\mathscr{L}$. Denote by $p\colon U\times[0,1]\to\mathbf{C}^n\times[0,1]$ the map, determined by the formula

$$p(\lambda_1,\dots,\lambda_n,t)=(f_1(t)\lambda_1,\dots,f_n(t)\lambda_n,t).$$

LEMMA 2.3′. *The complexes of sheaves*

$$\hat{\mathscr{O}}K_.(f)(t)a-\lambda,U)\to p_*\mathscr{C}\mathscr{L}\to 0 \quad\text{and}\quad \hat{\mathscr{E}}K_.(f(t)a-\lambda,U)\to p_*\mathscr{C}\mathscr{E}\mathscr{L}\to 0$$

are exact on $\mathbf{C}^n\times[0,1]$.

The proof is the same as that of Lemma 2.3. We need only to remark that if we denote by $\hat{\Delta}$ the subspace of $U'\times[0,1]$, defined by the equations $f_i(t)z_i=\lambda_i$, $i=1,\dots,n$, and by $\hat{\mathscr{O}}_{\hat{\Delta}}$ the sheaf of all functions on $\hat{\Delta}$, holomorphically depending on λ,z and continuously on t, then $\hat{\Delta}$ is homeomorphic to $U\times[0,1]$, $\hat{\mathscr{O}}_{\hat{\Delta}}\otimes\mathscr{L}'$ is isomorphic to $\mathscr{C}\mathscr{L}$, $\hat{\mathscr{O}}_{\hat{\Delta}}\otimes\mathscr{E}\mathscr{L}$ to $\mathscr{C}\mathscr{E}\mathscr{L}$ and the natural projection of $\hat{\Delta}$ on $\mathbf{C}^n\times[0,1]$ coincides with the above defined map p.

Let U be an *arbitrary* domain in $\mathbf{C}^n$, $\mathscr{L}$ a coherent sheaf on U, and let $\mathfrak{V}=\{U_i\}$, $i\in I$ be an open bounded locally finite Stein covering of U. As usual, for a given finite subset $\alpha=(i_1,\dots,i_k)\subset I$ denote $|\alpha|=k$, $U_\alpha=U_{i_1}\cap U_{i_k}$, $C_k(U,\mathfrak{V},\mathscr{L})=\bigoplus_{|\alpha|=k}\Gamma_{U_\alpha}(\mathscr{L})$, and let $C_.(U,\mathfrak{V},\mathscr{L})=\{C_k(U,\mathfrak{V},\mathscr{L}),\delta_k\}$ be the standard cochain complex of the covering $\mathfrak{V}$. Let $z_1,\dots,z_n$ be the coordinate functions in $\mathbf{C}^n$. Then the operators of multiplication by

$z_1, \ldots, z_n$ act as endomorphisms in the complex $C_{\cdot}(U, \mathfrak{V}, \mathscr{L})$. The corresponding Koszul complex on $\mathbf{C}^n$ will be denoted by $KC_{\cdot}(U, \mathfrak{V}, \mathscr{L})(\lambda)$.

LEMMA 2.4. *The complex of sheaves $\mathscr{O}KC_{\cdot}(U, \mathfrak{V}, \mathscr{L})$ is quasi-isomorphic to the sheaf $\mathscr{L}$ on U.*

Proof. Denote by $\mathscr{C}_{\cdot}(U, \mathfrak{V}, \mathscr{L})$ the canonical alternating resolution of $\mathscr{L}$ relative to $\mathfrak{V}$ (see [6], §3, chapter B). We have $\mathscr{C}_k(U, \mathfrak{V}, \mathscr{L}) = \bigoplus_{|\alpha|=k} \mathscr{L}_{U_\alpha}$. Lemma 2.3 shows that the complex $\mathscr{O}KC_{\cdot}(U, \mathfrak{V}, \mathscr{L})$ is quasi-isomorphic to $\mathscr{C}_{\cdot}(U, \mathfrak{V}, \mathscr{L})$. On the other hand, the complex $\mathscr{C}_{\cdot}(U, \mathfrak{V}, \mathscr{L})$ is quasi-isomorphic to $\mathscr{L}$, and the assertion of the lemma is proved.

Note that the same arguments imply immediately that the complex of sheaves $\mathscr{E}KC_{\cdot}(U, \mathfrak{V}, \mathscr{L})$ is quasi-isomorphic to the sheaf $\mathscr{E}\mathscr{L}$.

COROLLARY 2.5. *The complex $KC_{\cdot}(U, \mathfrak{V}, \mathscr{L})$ is $\mathscr{O}$- and $\mathscr{E}$-Fredholm on the domain U.*

Let $\mathfrak{V} = \{U_i\}$, $i \in I$, and $\mathfrak{V}' = \{U'_{i'}\}$, $i' \in I'$ be two locally finite Stein coverings of the domain U such that $\mathfrak{V}$ is a refinement of $\mathfrak{V}'$, and let $\theta: I \to I'$ be a refinement mapping (i.e. $U_i \subset U'_{\theta(i)}$ for any $i \in I$). Denote by $C_{\cdot}(\theta): C_{\cdot}(U, \mathfrak{V}, \mathscr{L}) \to C_{\cdot}(U, \mathfrak{V}, \mathscr{L})$ the corresponding morphism of cochain complexes, and by $\mathscr{C}_{\cdot}(\theta): \mathscr{C}_{\cdot}(U, \mathfrak{V}', \mathscr{L}) \to \mathscr{C}_{\cdot}(U, \mathfrak{V}, \mathscr{L})$ the corresponding morphism of complexes of sheaves. Then the morphism $C_{\cdot}(\theta)$ induces a morphism of the parametrised Koszul complex

$$KC_{\cdot}(\theta): KC_{\cdot}(U, \mathfrak{V}', \mathscr{L})(\lambda) \to KC_{\cdot}(U, \mathfrak{V}, \mathscr{L})(\lambda).$$

It is easy to see that the corresponding morphism $\mathscr{O}KC_{\cdot}(\theta)$ of the complexes of sheaves of holomorphic sections is associated with the morphism $\mathscr{C}_{\cdot}(\theta)$, and, therefore, is a quasi-isomorphism. The same is true of the morphism $\mathscr{E}KC_{\cdot}(\theta)$ of sheaves of smooth sections. Then we obtain:

COROLLARY 2.6. *The morphism of complexes $KC_{\cdot}(\theta)$ is an $\mathscr{O}$- and an $\mathscr{E}$-quasi-isomorphism on the domain U.*

Now we are able to define the Riemann-Roch invariant for coherent sheaves on open domains in $\mathbf{C}^n$. Namely, if $\mathscr{L}$ is a coherent sheaf on $U \subset \mathbf{C}^n$, we put $\alpha_U(\mathscr{L}) = [KC_{\cdot}(U, \mathfrak{V}, \mathscr{L})(\lambda)]$, where $\mathfrak{V}$ is a Stein covering of U. Corollary 2.6 shows that $\alpha_U(\mathscr{L})$ does not depend on the choice of $\mathfrak{V}$, and the functorial property of this invariant is a simple consequence of Lemma 2.2 (b).

§ 3. Proof of the Riemann-Roch theorem

If we attempt to reproduce the construction of the previous paragraph in the case of an arbitrary complex space, we meet two principal obstacles. The first is that, in general, complex spaces are not embeddable in complex manifolds. The second is that there is no global coordinate system on a general complex space, and it is not clear how to combine the Koszul complexes corresponding to the various local coordinate systems.

In order to overcome the first obstacle, we shall introduce a class of embeddings of complex spaces in smooth manifolds, which are suitable for our purposes.

Let M be a complex space. As usual, a triple (W, φ, U) will be called a chart on M, if W is an open subset of M, U is a bounded Stein domain in $\mathbf{C}^n$, and $\varphi\colon W\to U$ is a closed holomorphic embedding. A locally finite covering $\mathfrak{W}=\{W_i\}, i\in I$, will be called an atlas on M, if any W_i is an element of a chart (W_i, φ_i, U_i). Likewise, for any finite subset $\alpha=(i_1,\dots,i_k)\subset I$ there exists a chart $(W_\alpha, \varphi_\alpha, U_\alpha)$ with $W_\alpha=W_{i_1}\cap\dots\cap W_{i_k}$, φ_α a restriction, for instance, of φ_{i_1}, and U_α a suitable open subset of U_{i_1}.

For any chart (W_i, φ_i, U_i) there exists an ideal J_i in the sheaf of germs of holomorphic functions $\mathcal{O}_{U_i}$ on U_i such that φ_i^* defines an isomorphism between the factor-sheaf $\mathcal{O}_{U_i}/J_i$ and the restriction $\mathcal{O}_{W_i}$ of the structure sheaf $\mathcal{O}_M$ of the complex space M to the open subset W_i. Take $i,j\in I$ with $W_{i,j}=W_i\cap W_j\neq\emptyset$, and let U_i^j be an open subset of U_i such that $\varphi_i^{-1}(U_i^j)=W_{i,j}$. The holomorphic map $h\colon U_i^j\to U_j$ will be called a *connecting holomorphic map*, if h^* maps J_j in J_i, and the induced map of $\mathcal{O}_{U_j}/J_j$ in $\mathcal{O}_{U_i}/J_i$ coincides with the isomorphism $(\varphi_i^*)^{-1}\circ\varphi_j^*$. It is easy to prove that a connecting holomorphic function always exists, and if h, h' are two connecting functions, then all the components of the difference $h-h'$ (considered as an n-dimensional vector-function on U_i^j) belong to the ideal J_i.

Fix an atlas $\mathfrak{W}=\{(W_i, \varphi_i, U_i)\}$, $i\in I$, on the complex space M. For any $i\in I$ denote by $\mathscr{E}_{U_i}$ the sheaf of germs of smooth functions on U_i, by $\mathscr{E}J_i$ the ideal $J_i\otimes_{\mathcal{O}_{U_i}}\mathscr{E}_{U_i}$ in $\mathscr{E}_{U_i}$, and by $\mathscr{E}_{W_i}$ the sheaf $\mathscr{E}_{U_i}/\mathscr{E}J_i$. It follows from Malgrange's theorem that the sheaf $\mathscr{E}J_i$ consists of germs of all functions of the type $\Sigma_{m=1}^s f_m\cdot g_m$, where $f_1,\dots,f_s$ are the generators of the sheaf J_i and $g_1,\dots,g_s$ are arbitrary smooth functions on U_i. Denote by $\tilde{\mathcal{O}}_{U_i}$ the direct sum in $\mathscr{E}_{U_i}$ of its subsheaves $\mathcal{O}_{U_i}$ and $\mathscr{E}J_i$. Then $\tilde{\mathcal{O}}_{U_i}$ is a sheaf of rings, and $\mathscr{E}J_i$ is an ideal in $\tilde{\mathcal{O}}_{U_i}$. We have $\tilde{\mathcal{O}}_{U_i}/\mathscr{E}J_i=\mathcal{O}_{W_i}$.

Definition. Let (W, φ, U) and (W', φ', U') be two charts on M with $W\subset W'$. The infinitely smooth map $\psi\colon U\to U'$ will be called *admissible*, if ψ^* transfers $\tilde{\mathcal{O}}_{U'}$, in $\tilde{\mathcal{O}}_U$, $\mathscr{E}J_{U'}$ in $\mathscr{E}J_U$, and the induced map of $\tilde{\mathcal{O}}_{U'}/\mathscr{E}J_{U'}$ in $\tilde{\mathcal{O}}_U/\mathscr{E}J_U$ coincides with the map $(\varphi^*)^{-1}\circ(\varphi')^*$.

Equivalent definition. The map ψ is admissible if for any holomorphic connecting function $h\colon U\to U'$ all the components of the vector function $\psi-h$ belongs to the ideal $\mathscr{E}J_U$.

In fact, suppose that ψ satisfies the first definition. Then all the components of the vector-function ψ belong to $\tilde{\mathcal{O}}_U$ and therefore ψ can be represented in the form $\psi=h+g$, where h is holomorphic and g has the components from $\mathscr{E}J_U$. It is easy to see that h is a holomorphic connecting map. Conversely, suppose that ψ has the form $\psi=h+g$, h and g as above. Then all the components of ψ belong to $\tilde{\mathcal{O}}_U$ and therefore ψ^* maps $\mathcal{O}_{U'}$ in $\tilde{\mathcal{O}}_U$. In order to prove that ψ satisfies the conditions of the first definition, it is sufficient to show that ψ^* maps $\mathscr{E}J_{U'}$ in $\mathscr{E}J_U$. Malgrange's theorem shows that it is sufficient to check this at the level of formal series. Let f be one of the generators of J_U. Then the formal series of the difference $f\circ\psi-f\circ h$ belongs to the ideal, generated by the formal series of the components of $g=\psi-h$, and therefore to $\mathscr{E}J_U$. Since h is connecting, then $f\circ h$ belongs to J_U and $f\circ\psi$ belongs to $\mathscr{E}J_U$, which proves the equivalence of the definitions.

Definition. An *almost complex embedding* of the complex space M in the smooth manifold $\tilde{M}$ is determined by an atlas $\mathfrak{B}$ on M, a closed embedding $\varrho\colon M\to\tilde{M}$ of the underlying topological space of M in $\tilde{M}$, and a set of C^∞-diffeomorphisms $\psi_i\colon U_i\to\tilde{U}_i$ of U_i to open subsets $\tilde{U}_i$ of $\tilde{M}$ such that $\psi_i|_{W_i}=\varrho|_{W_i}$ and for any $i,j\in I$ the connecting map $\psi_{i,j}=\psi_j^{-1}\circ\psi_i$, defined on $\psi_i^{-1}(\tilde{U}_i\cap\tilde{U}_j)$, is admissible.

It is easy to see that in that case the system of sheaves $\psi_{i*}(\tilde{\mathcal{O}}U_i)$, defined on $\tilde{U}_i$, $i\in I$, determines a globally defined sheaf of rings on $\tilde{M}$, which will be denoted by $\mathcal{O}_{\tilde{M}}$. In the same way, the system of sheaves $\psi_{i*}(\mathscr{E}J_i)$ determines a globally defined sheaf $\mathscr{E}J_M$ of ideals in $\mathcal{O}_{\tilde{M}}$. Then the factor sheaf $\mathcal{O}_{\tilde{M}}/\mathscr{E}J_M$ is isomorphic to the structure sheaf $\mathcal{O}_M$ of the complex space M and the isomorphism is given by the mapping ϱ_*. Another important sheaf connected with an almost complex embedding is the sheaf $\mathscr{E}_M=\mathscr{E}_{\tilde{M}}/\mathscr{E}J_M$, and the mapping ψ_{i*} defines an isomorphism between $\mathscr{E}_{W_i}$ and the restriction of $\mathscr{E}_M$ on $\tilde{U}_i$.

Let $\varrho\colon M\to\tilde{M}$ be an almost complex embedding. We shall prove that the tangent bundle $T(\tilde{M})$ of $\tilde{M}$ has an almost complex structure. In fact, it follows from the definition, that for any i,j we have $\bar{\partial}\psi_{i,j}(\lambda)=0$ for all points $\lambda\in\varphi_i(W_i)$. Therefore, the maps $d\psi_i$ transfer the complex structure from $T(U_i)$ to $T(\tilde{U}_i)|_{\varrho(M)\cap\tilde{U}_i}$, and determine a globally defined complex structure on the restriction of $T(\tilde{M})$ to $\varrho(M)$. It is easy to see that this complex structure can be extended to some neighborhood of $\varrho(M)$ in $\tilde{M}$. Shrinking $\tilde{M}$, if it is necessary, we obtain the assertion.

Definition. Let $(\varrho, \mathfrak{V}, \{\psi_i\})$ and $(\varrho', \mathfrak{V}', \{\psi'_{i'}\})$ be two almost complex embeddings of M in the manifold $\tilde{M}$ with $\varrho=\varrho'$. We will say that these embeddings are *equivalent*, if the set $(\varrho, \mathfrak{V}\cup\mathfrak{V}', \{\psi_i\}\cup\{\psi_{i'}\})$ is again an almost complex embedding.

One can prove that two almost complex embeddings are equivalent if and only if they determine the same sheaves $\mathcal{O}_{\tilde{M}}$ and $\mathscr{E}J_M$ on $\tilde{M}$.

Let us note some particular cases of equivalence of almost complex embeddings.

—The equivalence class of almost complex embedding does not depend on the choice of the covering $\{W_i\}$. More precisely, if the covering $\{W_{i'}\}$ of M is a refinement of $\{W_i\}$, and φ'_i, U'_i are the corresponding restrictions of φ_i and U_i, then we obtain an equivalent embedding.

—The equivalence class of an almost complex embedding depends only on the germ of the manifold $\tilde{M}$ near its closed subspace $\varrho(M)$. In fact, if we replace the domains U_i by arbitrary small neighborhoods of $\varphi_i(W_i)$ in U_i, and the diffeomorphisms φ_i by their restrictions, we obtain an equivalent embedding.

—Let $\{h_i\}$ be a set of admissible diffeomorphic mappings of the domains U_i onto themselves. Then, replacing ψ_i by $\psi_i\circ h_i$, we obtain an equivalent embedding.

To prove this, we shall use the following

SUBLEMMA. *Let (W, φ, U) be a chart, and suppose that h is an admissible diffeomorphism of U into itself. Then the inverse map h^{-1} is admissible also.*

Proof. Consider first the case when W is regular. One can assume that W is embedded in U as the coordinate subspace $z_i=\ldots z_k=0$. The components of the map h have the form:

$$h_i(z)=\sum_{j=1}^{k} z_j\cdot a_{ij}(z), \quad i=1,\ldots,k,$$

$$h_i(z)=z_i+\sum_{j=1}^{k} z_j\cdot b_{ij}(z), \quad i=k+1,\ldots,n,$$

where $a_{ij}(z), b_{ij}(z)$ are smooth functions and the matrix $\{a_{ij}(z)\}$ is non-degenerate near W. Solving the equations with respect to z_i, we obtain the statement.

In the general case denote by $f_1,\ldots,f_k$ the generators of the corresponding ideal J. Then the components of h have the form:

$$h_i(z) = z_i + \sum_{j=1}^{k} f_j(z)\cdot a_{ij}(z), \quad i = 1, \dots, n, \quad a_{ij}(z) \text{ smooth}.$$

Put $U' = U \times B$, where B is a sufficiently small open ball in $\mathbf{C}^k$, and denote by (z, λ) the coordinates on U'. Let J' be the ideal in $\mathcal{O}_{U'}$, generated by the functions $f_j(z) - \lambda_j$, $j = 1, \dots, k$. Then the space W', determined by $\mathcal{O}_{W'} = \mathcal{O}_{U'}/J'$, is regular. Denote by $h'(z, \lambda)$ the map from U' to U', determined by the formula:

$$h_i'(z, \lambda) = z_i + \sum_{j=1}^{k} (f_j(z) - \lambda_j)\, a_{ij}(z), \quad i = 1, \dots, n,$$

$$h_{n+m}'(z, \lambda) = \lambda_m, \quad m = 1, \dots, k.$$

Then h' is a diffeomorphism near $U \times \{0\}$, and is admissible with respect to W', and therefore its inverse $(h')^{-1}$ is admissible. Since h^{-1} coincides with the restriction of $(h')^{-1}$ to $U \times \{0\}$, then it is admissible also.

Let $\varrho: M \to \tilde{M}, \varrho': M \to \tilde{M}'$ be two almost complex embeddings with corresponding atlases $\mathfrak{V}, \mathfrak{V}'$ and corresponding mappings $\psi_i, \psi_{i'}'$.

Definition. The infinitely smooth map $h: \tilde{M} \to \tilde{M}'$ will be called a *morphism* from the almost complex embedding ϱ to the almost complex embedding ϱ', if for any $i \in I, i' \in I'$ the map $(\psi_{i'}')^{-1} \circ h \circ \psi_i$ is an admissible map from the open subset $\psi_i^{-1}(\tilde{U}_i \cap \tilde{U}_{i'}')$ of U_i to $U_{i'}'$.

Remark 1. If h is a morphism of almost complex embeddings, then the morphism of sheaves h^* sends the sheaf $\mathcal{O}_{\tilde{M}'}$, to $\mathcal{O}_{\tilde{M}}$ and $\mathcal{E}J_M'$ to $\mathcal{E}J_M$.

Remark 2. For an arbitrary pair of almost complex embeddings $\varrho: M \to \tilde{M}$, $\varrho': M \to \tilde{M}'$ there exists a morphism of almost complex embeddings $h: \tilde{M} \to \tilde{M}'$. In fact, let $\mathfrak{V} = \{(W_i, \varphi_i, U_i)\}$, $\{\psi_i\}$, $i \in I$, and $\mathfrak{V}' = \{(W_{i'}', \varphi_{i'}', U_{i'}')\}$, $\{\psi_{i'}'\}$, $i' \in I'$, be the corresponding atlases and diffeomorphisms. Without loss of generality one can assume that $I' = I$ and $W_i = W_i'$. Put $V_k = \bigcup_{i=1}^{k} \tilde{U}_i$. We shall assume that we have already defined a morphism $h_k: V_k \to \tilde{M}'$ of almost complex embeddings, and we shall show that it can be extended to V_{k+1}. Let f and g be two non-negative infinitely smooth functions, defined on V_{k+1}, such that $f = 1$ on some neighborhood of $V_{k+1} \setminus \tilde{U}_{k+1}$, $g = 1$ on some neighborhood of $V_{k+1} \setminus V_k$, and $f + g = 1$. Put $\tilde{f} = \psi_{k+1}^* f$ and $\tilde{g} = \psi_{k+1}^* g$. Denote

$$\tilde{h} = (\psi_{k+1}')^{-1} \circ h_k \circ \psi_{k+1}.$$

Then $\tilde{h}'$ is an admissible mapping from $\psi_{k+1}^{-1}(\tilde{U}_{k+1} \cap V_k)$ to U_{k+1}'. Let $\tilde{h}''$ be an arbitrary admissible mapping from U_{k+1} to U_{k+1}'. Then it is easy to see that the mapping

$\tilde{h}=\tilde{f}\cdot\tilde{h}'+\tilde{g}\cdot\tilde{h}''$ is also an admissible mapping from U_{k+1} to U'_{k+1}. Define the mapping h_{k+1} on $\tilde{U}_{k+1}$ by the formula $h_{k+1}=\psi_{k+1}\circ\tilde{h}\circ(\psi_{k+1})^{-1}$, and on $V_{k+1}\setminus\tilde{U}_{k+1}$ by the formula $h_{k+1}=h_k$. Then it is easy to see that h_{k+1} is a smooth mapping and defines a morphism of almost complex embeddings from V_{k+1} to M'.

Definition. The atlas $\mathfrak{B}'=\{(W'_i, \varphi'_i, U'_i)\}$ will be called a *modification* of the atlas $\mathfrak{B}=\{(W_i, \varphi_i, U_i)\}$, if for any i we have $W'_i=W_i$ and there exists a closed regular embedding $e_i\colon U_i\to U'_i$ such that e_i is a holomorphic connecting map.

Definition. The almost complex embedding $\varrho'\colon M\to\tilde{M}'$, corresponding to the atlas $\mathfrak{B}'$, will be called a *modification* of the almost complex embedding $\varrho\colon M\to\tilde{M}$, corresponding to the atlas $\mathfrak{B}$, if the atlas $\mathfrak{B}'$ is a modification of the atlas $\mathfrak{B}$, and there exists a regular embedding of C^∞-manifolds $e\colon\tilde{M}\to\tilde{M}'$, such that for any $i\in I$ the diagram

$$\begin{array}{ccc} U_i & \xrightarrow{e_i} & U'_i \\ \downarrow{\scriptstyle\psi_i} & & \downarrow{\scriptstyle\psi'_i} \\ \tilde{M} & \xrightarrow{e} & \tilde{M}' \end{array}$$

is commutative.

Denote by (λ,μ) the coordinate system on U'_i, where $\lambda=(\lambda_1,\dots,\lambda_n)$, $\mu=(\mu_1,\dots,\mu_k)$, $n=\dim U_i$, $k+n=\dim U'_i$. Without loss of generality one can assume that $e_i(U_i)$ coincides with the subspace of U'_i defined by the equation $\mu=0$. Denote by p_i the projection of U'_i on $e_i(U_i)$. If the ideal J_i in the ring $\mathcal{O}_{U_i}$ is generated by the functions $f_1,\dots,f_m$, then the ideal J'_i in $\mathcal{O}_{U'_i}$ is generated by the functions $p_i^*f_1,\dots,p_i^*f_m,\mu_1,\dots,\mu_k$, and therefore the same functions generate the ideal $\mathscr{E}J'_i$ as an $\mathscr{E}_{U'_i}$-module.

Let the almost complex embedding $\varrho'\colon M\to\tilde{M}'$ be a modification of the almost complex embedding $\varrho\colon M\to\tilde{M}$, and $e\colon\tilde{M}\to\tilde{M}'$ be the corresponding embedding. It is easy to see that the complex structure of the tangent bundle $T(\tilde{M})$ of $\tilde{M}$ coincides with the complex structure, induced from $T(\tilde{M}')$ (at least, at the points of $\varrho(M)$), and, therefore, the conormal bundle $T(\tilde{M}')/T(\tilde{M})$ of $e(\tilde{M})$ in $\tilde{M}'$ has a complex structure at the points of $\varrho(M)$. The following assertion shows that the equivalence class of a modification is uniquely determined by this bundle.

LEMMA 3.1. (a) *Let E be a complex vector bundle on M, and $\varrho\colon M\to\tilde{M}$ be an almost complex embedding. Then there exists a modification $\varrho'\colon M\to\tilde{M}'$ of ϱ such that* *the restriction of the conormal bundle $T(\tilde{M}')/T(\tilde{M})$ on $\varrho(M)$ is isomorphic to E.*

(b) *Suppose that the conormal bundles, corresponding to two modifications of a given almost complex embedding, are isomorphic as complex vector bundles on M. Then these modifications are isomorphic.*

Proof. (a) One can extend E up to a complex vector bundle on $\tilde{M}$. Let $A_i: \tilde{U}_i \times \mathbf{C}^k \to E$ be a trivialisation of the vector bundle E on the domain $\tilde{U}_i$, and let $A_{i,j}(\lambda)=A_j^{-1}\circ A_i$ be a corresponding connecting function. Then $A_{i,j}(\lambda)$ is a smooth function on $\tilde{U}_i \cap \tilde{U}_j$ with values in $GL(k,\mathbf{C})$. Denote $U_i'=U_i\times\mathbf{C}^k$, $W_i'=W_i$ and $\varphi_i'=\varphi_i\times\{0\}$. Then the atlas $\mathfrak{V}'=\{(W_i', \varphi_i', U_i')\}$ is a modification of the atlas $\mathfrak{V}=\{(W_i, \varphi_i, U_i)\}$. As holomorphic connecting maps for the atlas $\mathfrak{V}'$ we can take $\varphi_{i,j}'(\lambda,\mu)=\varphi_{i,j}'(\lambda)\times\{0\}$, where $\psi_{i,j}: U_i^j\to U_i$ are holomorphic connecting maps for the atlas $\mathfrak{V}$, and λ is a point of U_i and $\mu=(\mu_1, \ldots, \mu_k)$ a point of the space $\mathbf{C}^k$. As it was pointed out above, for any i the ideal $J_i'\subset\mathcal{O}_{U_i'}$ is generated by the generators of the ideal J_i and by the coordinate functions $\mu_1, \ldots, \mu_k$. Now, denote by $\tilde{M}'$ the total space of the vector bundle E, and let $e: \tilde{M}\to\tilde{M}'$ be an embedding of $\tilde{M}$ as a zero section in $\tilde{M}'$. Define the maps $\psi_i': U_i'\to\tilde{M}'$ by the formula $\psi_i'=A_i\circ(\psi_i\times\mathrm{id})$. We have only to check that the connecting maps $\psi_{i,j}'=(\psi_j')^{-1}\circ\psi_i'$ are admissible. In fact we have

$$\psi_{i,j}'(\lambda,\mu)=\psi_{i,j}(\lambda)\times\langle A_{i,j}(\psi_i(\lambda)),\mu\rangle,$$

and therefore

$$\psi_{i,j}'(\lambda,\mu)-\varphi_{i,j}'(\lambda,\mu)=[\psi_{i,j}(\lambda)\times\langle A_{i,j}(\lambda),\mu\rangle-\psi_{i,j}(\lambda)\times\{0\}]+[\psi_{i,j}(\lambda)\times\{0\}-\varphi_{i,j}(\lambda)\times\{0\}].$$

Both the summands on the right are vector-functions with components, belonging to $\mathscr{E}J_i'$. Therefore, since $\varphi_{i,j}'$ are admissible, then so are $\psi_{i,j}'$.

(b) Let $\varrho'': M\to\tilde{M}''$ be the modification of the almost complex embedding $\varrho: M\to\tilde{M}$, constructed in the proof of (a), and $\varrho': M\to\tilde{M}'$ be another modification of this almost complex embedding with conormal bundle, equal to E. Let $\mathfrak{V}'$ and $\{\psi_i'\}$ be the atlas and the set of local homeomorphisms, corresponding to ϱ'. We shall construct an isomorphism of almost complex embeddings $h: \tilde{M}'\to\tilde{M}''$. First, using the same argument as in Remark 2 above, one can construct a retraction from $\tilde{M}'$ to $\tilde{M}$, i.e. a morphism $p: \tilde{M}'\to\tilde{M}$ of almost complex embeddings such that $p\circ e=\mathrm{Id}_{\tilde{M}}$. Denote by $\tilde{p}_i$ the retraction from U_i' to $e_i(U_i)$ induced by the retraction p, i.e. $\tilde{p}_i=\psi_i^{-1}\circ p\circ\psi_i'$. The map $\tilde{p}_i$ is admissible. Define the map $h_i: U_i'\to U_i'$ by the formula $h_i(\lambda,\mu)=(\tilde{p}_i(\lambda,\mu),\mu)$. The map h_i is equal to the identity on $e_i(U_i)$ and sends the fiber of p_i over λ^0 to the coordinate subspace $\{\lambda=\lambda^0\}$ and is therefore a diffeomorphism (in a neighborhood of

$e_i(U_i)$). Then h_i^{-1} is an admissible diffeomorphism. Now we can replace ψ_i' by $\tilde{\psi}_i' = \psi_i' \circ h_i^{-1}$. Then $\tilde{\psi}_i'$ sends any subspace $\{\lambda = \lambda^0\}$ of U_i' to the fiber of the map p over the point $\psi_i(\lambda^0)$.

One sees from the conditions of the lemma that one can fix an isomorphism F of complex vector bundles on $\tilde{M}$ between E and the subbundle $T(p)$ of the tangent bundle to $\tilde{M}'$ at the points of $e(\tilde{M})$, consisting of the tangential vectors of the fibers of the map p. Define the trivialisations A_i': $U_i \times \mathbf{C}^k \to E$ by the formula $A_i' = F^{-1} \circ (d_\mu \tilde{\psi}_i')|_{\mu=0}$. (We here identify $U_i \times \mathbf{C}^k$ with the normal bundle $T(p_i)$ to $e_i(U_i)$ in U_i'.) Consider U_i' as a neighborhood of $U_i \times \{0\}$ in $U_i \times \mathbf{C}^K$. Then $A_i^{-1} \circ A_i'$ is an admissible diffeomorphism from U_i' to U_i''. Put $\tilde{h}_i = \psi_i'' \circ A_i^{-1} \circ A_i' \circ (\tilde{\psi}_i')^{-1}$. Then $\tilde{h}_i$ is a morphism of almost complex embeddings, acting from $\tilde{U}_i'$ to $\tilde{U}_i''$. The differential of $\tilde{h}_i$ at the points of $e(\tilde{U}_i)$ induces a map F^{-1} of tangent bundles. For any $\lambda \in \tilde{U}_i' \cap \tilde{U}_j'$ the images $\tilde{h}_i(\lambda), \tilde{h}_j(\lambda)$ belong to the same fibre of the fibre bundle $\tilde{M}''$. Let $\{f_i\}, i \in I$ be a smooth partition of unity on $\tilde{M}$, subordinated to the covering $\tilde{U}_i'$. Put $h = \Sigma_{i \in I} f_i \cdot \tilde{h}_i$. Then h is defined on the whole of $\tilde{M}'$. The differential of h again induces a map F^{-1} of tangent bundles and therefore h is a diffemorphism near $e(\tilde{M})$. It follows from the sublemma that for any i the map $(\psi_i'')^{-1} \circ h \circ \tilde{\psi}_i'$ is an admissible diffeomorphism as well as its inverse. Therefore h and h^{-1} are morphisms of almost complex embeddings, which proves the part (b).

Now we are able to prove the existence and uniqueness (in an appropriate sense) of almost complex embeddings.

LEMMA 3.2. (a) *Any complex space with finite atlas has an almost complex embedding in some Euclidean space* $\mathbf{R}^{2N}$.

(b) *If ϱ and ϱ' are two almost complex embeddings of the complex space M, then some modification of ϱ is equivalent to some modification of ϱ'.*

Proof. First, note that if $\varrho: M \to \tilde{M}$ is an almost complex embedding, then there exists a modification $\varrho': M \to \tilde{M}'$ of ϱ such that $\tilde{M}'$ is diffeomorphic to a domain in $\mathbf{R}^{2N}$ and the tangent bundle of $\tilde{M}'$ is trivial as a complex bundle. In fact, denote by E the tangent bundle of $\tilde{M}$, and choose a complex bundle E' on $\tilde{M}$ such that $E \oplus E'$ is trivial as a complex bundle. Then for suitable m and N there exists a closed regular embedding $i: \tilde{M} \to \mathbf{R}^{2N}$ such that the normal bundle of $i(\tilde{M})$ is isomorphic (as a real bundle) to the direct sum of E' and the trivial bundle $\tilde{M} \times \mathbf{C}^m$. Denote by $\tilde{M}'$ the modification of $\tilde{M}$ with conormal bundle equal to $E' \oplus (\tilde{M} \times \mathbf{C}^m)$. Then $\tilde{M}'$ is diffeomorphic to a neighborhood of $i(\tilde{M})$ in $\mathbf{R}^{2N}$ and $T(\tilde{M}')$ is trivial.

Let us prove the statement (b). Suppose that $\varrho: M \to \tilde{M}$ and $\varrho': M \to \tilde{M}'$ are almost complex embeddings, $\mathfrak{B} = \{(W_i, \varphi_i, U_i)\}$ and $\mathfrak{B}' = \{(W_i', \varphi_i', U_i')\}$ the corresponding atlases and ψ_i and ψ_i' the corresponding local diffeomorphisms. As above, we can assume that $I=I'$, and $W_i=W_i'$. Denote by $\mathfrak{V}''$ the atlas $\{(W_i'', \varphi_i'', U_i'')\}$, where $W_i''=W_i$, $\varphi_i''=\varphi_i\times\varphi_i'$, $U_i''=U_i\times U_i'$, by ϱ'' the embedding of M in the manifold $\tilde{M}''=\tilde{M}\times\tilde{M}'$, defined by $\varrho''=\varrho\times\varrho'$, and by ψ_i'' the diffeomorphism $\psi_i\times\psi_i'$ from U_i'' to $\tilde{M}''$. It is easy to check that ϱ'' is an almost complex embedding of M in $\tilde{M}''$. We shall prove that ϱ'' is a modification simultaneously of ϱ and ϱ'. Choose for any $i\in I$ a connecting holomorphic mapping $h_i\colon U_i\to U_i'$, and a morphism of almost complex embeddings $h\colon\tilde{M}\to\tilde{M}'$ (see Remark 2 above). Denote by $e_i\colon U_i\to U_i''=U_i\times U_i'$ and $e\colon\tilde{M}\to\tilde{M}''=\tilde{M}\times\tilde{M}'$ the regular embeddings, determined by the graphs of h_i and h respectively. We have only to assure the relation of commutativity $\psi_i''\circ e_i=e\circ\psi_i$; in order to obtain this relation, we shall modify the mappings ψ_i''. Define the mapping $\tilde{h}_i\colon U_i\to U_i'$ by the formula $\tilde{h}_i=(\psi_i')^{-1}\circ h\circ\psi_i$. Then $\tilde{h}_i$ is admissible and therefore all the components of the vector-function $\tilde{h}_i(\lambda)-h_i(\lambda)$ on U_i belong to $\mathscr{E}J_i$. Denote by λ the coordinates on U_i, by μ the coordinates on U_i', and by (λ,μ) the coordinates on U_i''. Define the mapping $s_i\colon U_i''\to U_i''$ by the formula $s_i(\lambda,\mu)=(\lambda,\mu+\tilde{h}_i(\lambda)-h_i(\lambda))$. Then s_i is an admissible diffeomorphism and maps the graph of h_i on the graph of $\tilde{h}_i$. Replacing ψ_i'' by $\tilde{\psi}_i''=\psi_i''\circ s_i$, we obtain the relation $\tilde{\psi}_i''\circ e_i=e\circ\psi_i$. Hence, ϱ'' is a modification of ϱ. The same reasons show that ϱ'' is a modification of ϱ'. Statement (b) is proved. Note that the conormal bundle of $e(\tilde{M})$ in $\tilde{M}''$ is isomorphic to the tangent bundle of $\tilde{M}'$.

Proof of (a). Let M', M'' be open subsets of the complex space M, and suppose that $\varrho'\colon M'\to\tilde{M}'$, $\varrho''\colon M''\to\tilde{M}''$ are almost complex embeddings. We shall prove that the space $M'''=M'\cup M''$ has an almost complex embedding in a smooth manifold $\tilde{M}'''$. We can suppose that the tangent bundles of $\tilde{M}'$ and $\tilde{M}''$ are trivial as complex bundles (in the opposite case we can make suitable modifications). Fix the open subsets $N'\subset\tilde{M}'$, $N''\subset\tilde{M}''$ such that $N'\cap\varrho'(M')=\varrho(M'\cap M'')$, $N''\cap\varrho''(M'')=\varrho(M'\cap M'')$. Then, as in the proof of (b), the almost complex embedding $\varrho'\times\varrho''\colon M'\cap M''\to N'\times N''$ is a modification of both $\varrho'\colon M'\cap M''\to N'$ and $\varrho''\colon M'\cap M''\to N''$.

The conormal bundle of N' in $N'\times N''$ is trivial and can therefore be extended up to a bundle on $\tilde{M}'$. Denote the corresponding modification of $\tilde{M}'$ by $\tilde{N}'$. Then there exists an isomorphism $\alpha(\lambda)$ of almost complex embeddings from $N'\times N''$ to an open neighborhood of the image of $M'\cap M''$ in $\tilde{N}'$. Similarly, one can construct a modification $\tilde{N}''$ of $\tilde{M}''$ and an isomorphism $\beta(\lambda)$ from $N'\times N''$ to an open subset of $\tilde{N}''$. Denote by $\tilde{M}'''$ the manifold, obtained from the disjoint union of $\tilde{N}'$ and $\tilde{N}''$ by identification of all pairs of points $\lambda_1\in\tilde{N}',\lambda_2\in\tilde{N}''$, satisfying $\alpha^{-1}(\lambda_1)=\beta^{-1}(\lambda_2)$. It is easy to see that the natural embedding of $M'''=M'\cup M''$ in $\tilde{M}'''$ is an almost complex embedding. It remains to prove that the smooth manifold $\tilde{M}$ is Hausdorf. In fact, it is easy to see that any two point, belonging to the image of M, can be separated. Then, replacing $\tilde{M}$ by a sufficiently small neighborhood of the image of M, we obtain a Hausdorf manifold.

Let $\mathfrak{B}=\{(W_i, \varphi_i, U_i)\}$, $i=1,\ldots,s$, be an atlas on M. Put $M_k=W_1 \cup \ldots \cup W_k$. Suppose that M_k has an almost complex embedding in the manifold $\tilde{M}_k$. Then we can put $M'=M_k$, $\tilde{M}'=\tilde{M}_k$, $M''=W_{k+1}$, $\tilde{M}''=U_{k+1}$, and the above considerations show that M_{k+1} has an almost complex embedding also. Using induction on k, we complete the proof of the statement (a).

In the rest of the paper we shall assume that the complex space M has a finite atlas. The general case can easy be obtained by passing to the limit.

Let $\varrho\colon M\to\tilde{M}$ be an almost complex embedding. As it was pointed out above, the maps $(\psi_i^{-1})^*$ determine isomorphisms between the restrictions $\mathcal{O}_{W_i}$ of the structure sheaf $\mathcal{O}_M$ of M and the sheaf $\mathcal{O}_{\tilde{M}}/\mathcal{E}J_M$ on $\tilde{M}$, and between the sheafs $\mathcal{E}_{W_i}=\mathcal{E}_{U_i}\otimes\mathcal{O}_{W_i}$ and the sheaf $\mathcal{E}_M=\mathcal{E}_{\tilde{M}}/\mathcal{E}J_M$. More generally, let $\mathcal{L}$ be a coherent sheaf of $\mathcal{O}_M$-modules on the space M, and denote by $\mathcal{L}_i$ the coherent sheaf $(\varphi_i)_*(\mathcal{L})$ on U_i. Note that J_i has zero action on $\mathcal{L}_i$ and therefore $\mathcal{L}_i$ can be considered as an $\tilde{\mathcal{O}}_{U_i}$-module. The maps $(\psi_i^{-1})^*$ then define isomorphisms between the sheafs $\mathcal{L}_i$ and the sheaf $\varrho_*(\mathcal{L})$ on $\tilde{U}_i$. If we denote by $\mathcal{E}\mathcal{L}$ the sheaf $\mathcal{E}_{\tilde{M}}\otimes_{\mathcal{O}_{\tilde{M}}}\varrho_*\mathcal{L}$ on $\tilde{M}$, then the maps $(\psi_i^{-1})^*$ also define isomorphisms between the $\mathcal{E}_{U_i}$-module $\mathcal{E}\mathcal{L}_i$ and the restriction of the $\mathcal{E}_{\tilde{M}}$-module $\mathcal{E}\mathcal{L}$ on U_i, which agree with the action of $(\psi_i^{-1})^*$ from $\mathcal{E}_{U_i}$ to $\mathcal{E}_{\tilde{M}}$.

In the next pages, for any coherent sheaf $\mathcal{L}$ on M, we shall construct an $\mathcal{E}$-Fredholm parametrised complex of Fréchet spaces on $\tilde{M}$ such that the corresponding complex of sheaves of germs of smooth sections is quasi-isomorphic to the sheaf $\mathcal{E}\mathcal{L}$.

Without loss of generality we can assume that $\tilde{M}$ is a domain in $\mathbf{R}^{2n}$ and that the diffeomorphisms $\psi_i\colon U_i\to\tilde{U}_i$ can be extended to some neighborhood of $\tilde{U}_i$ in $\mathbf{C}^n$. Then the diffeomorphism $\psi_i^{-1}\colon \tilde{U}_i\to U_i$ can be extended up to a smooth map $\tau_i\colon \tilde{M}\to\mathbf{C}^n$ such that $\tau_i^{-1}(\tilde{U}_i)=U_i$. Recall that from Lemma 2.3 we have the exact complex of sheaves $\mathcal{E}K_.(U_i, \mathcal{L}_i)\to(\mathcal{L}_i)_{U_i}\to 0$ on $\mathbf{C}^n$. Denote by $K_.(\tilde{U}_i, \mathcal{L})(\lambda)$ the smooth complex of Fréchet spaces on $\tilde{M}$ which is obtained as an inverse image of the complex $K_.(U_i, \mathcal{L}_i)(\lambda)$ on $\mathbf{C}^n$ via the map τ_i. Then on $\tilde{M}$ the complex of sheaves $\mathcal{E}K_.(\tilde{U}_i, \mathcal{L})\to(\mathcal{E}\mathcal{L})_{U_i}\to 0$ is exact. More generally, for any finite subset $\alpha=(i_1,\ldots,i_k)\subset I$ one can define in the same way the parametrised complex $K_.(\tilde{U}_\alpha, \mathcal{L})(\lambda)$ on $\tilde{M}$ such that the complex of sheaves $\mathcal{E}K_.(\tilde{U}_\alpha, \mathcal{L})\to(\mathcal{E}\mathcal{L})_{\tilde{U}_\alpha}\to 0$ is exact (we put $\tilde{U}_\alpha=\tilde{U}_{i_1}\cap\ldots\cap\tilde{U}_{i_k}$). Now we shall describe an algebraic construction, which will be used in order to overcome the second obstruction, i.e. to attach all the complexes $K_.(\tilde{U}_\alpha, \mathcal{L})(\lambda)$ together.

Let I be a finite set of indexes and S a simplicial scheme of subsets of I. Recall that a *simplicial system of rings* $\mathscr{A}_S$ on S is a system of rings $A_\alpha, \alpha \in S$, and ring homomorphisms $R_{\alpha,\beta}: A_\alpha \to A_\beta$, defined for any pair $\alpha, \beta \in S, \alpha \subset \beta \subset I$, such that for any triple $\alpha, \beta, \gamma \in S, \alpha \subset \beta \subset \gamma \subset I$, we have $R_{\alpha\gamma} = R_{\alpha\beta} \circ R_{\beta\gamma}$. We define *a simplicial system of $\mathscr{A}_S$-modules* to be a system of A_α-modules L_α and homomorphisms $r_{\alpha,\beta}: L_\alpha \to L_\beta$ lying over the ring homomorphisms $R_{\alpha,\beta}$, and satisfying the equality $r_{\alpha,\gamma} = r_{\alpha,\beta} \circ r_{\beta,\gamma}$.

Now we shall define the notion of simplicial system of complexes of modules. Let $\mathscr{A}_S$ be a simplicial system of rings, and let there be given for any $\alpha \in S$ a complex $L_{\cdot,\alpha} = \{L_{m,\alpha}, d_{m,\alpha}\}$ of A_α-modules. Denote by $L_{\cdot,k} = \{L_{m,k}, d_{m,k}\}$ the complex $L_{\cdot,k} = \oplus_{|\alpha|=k} L_{\cdot,\alpha}$. Suppose that for any two subsets α and β of I such that $\alpha, \beta \in S, \alpha \subset \beta$ and $|\beta| - |\alpha| = n > 0$ there exists a homomorphism

$$r_{m,\alpha,\beta}: L_{m,\alpha} \to L_{m-n+1,\beta}$$

lying over the ring homomorphism $R_{\alpha,\beta}$. Put $\tilde{L}_p = \oplus_{m+k=p} L_{m,k}$, and let $\tilde{d}_p: \tilde{L}_p \to \tilde{L}_{p+1}$ be the operator determined by the set of homomorphisms $r_{m,\alpha,\beta}$ with $m + |\alpha| = p$ and by the differentials of the complex $L_{\cdot,\alpha}$.

Definition. The system $\{L_{\cdot,\alpha}, r_{m,\alpha,\beta}\}$ will be called a *simplicial system of complexes*, if for all p we have $\tilde{d}_p \circ \tilde{d}_{p-1} = 0$. The complex $\tilde{L}_\cdot = \{\tilde{L}_p, \tilde{d}_p\}$ will be called the *cochain complex* of this system.

Let $\{L_{\cdot,\alpha}, r_{m,\alpha,\beta}\}$ and $\{L'_{\cdot,\alpha}, r'_{m,\alpha,\beta}\}$ be two simplicial systems of complex of $\mathscr{A}_S$-modules. Suppose that for any pair $\alpha, \beta \in S$, such that $\alpha \subset \beta$ and $|\beta| - |\alpha| = n \geqslant 0$, there is given a homomorphism $\varphi_{m,\alpha,\beta}: L_{m,\alpha} \to L'_{m-n,\beta}$ lying over the ring homomorphism $R_{\alpha,\beta}$.

Definition. The system of mappings $\varphi_{m,\alpha,\beta}$, will be called a *morphism* from the simplicial system $\{L_{\cdot,\alpha}, r_{m,\alpha,\beta}\}$ to the simplicial system $\{L'_{\cdot,\alpha}, r'_{m,\alpha,\beta}\}$, if the corresponding mappings $\tilde{\varphi}_p: \tilde{L}_p \to \tilde{L}'_p$ determine a morphism from the complex $\tilde{L}_\cdot$ to the complex $\tilde{L}'_\cdot$.

LEMMA 3.3. *Let S be a simplicial scheme of subsets of the finite set I, $\mathscr{A}_S$ be a simplicial system of rings on S, $\mathscr{L}_S = \{\mathscr{L}_\alpha\}$ be a simplicial system of $\mathscr{A}_S$-modules, and let for any $\alpha \in S$ the complex $L_{\cdot,\alpha} = \{L_{m,\alpha}, d_{m,\alpha}\}, m \leqslant 0$ be a free resolution of the A_α-module $\mathscr{L}_\alpha$.*

(a) *There exists a system of morphisms $r_{m,\alpha,\beta}: L_{m,\alpha} \to L_{m-n+1,\beta}$ such that $\{L_{\cdot,\alpha}, r_{m,\alpha,\beta}\}$ is a simplicial system of complexes, and a quasi-isomorphism $\tilde{\varepsilon}_\cdot$ from* *the corresponding cochain complex $\tilde{L}_.$ to the cochain complex $\tilde{\mathscr{L}}_.$ of the simplicial system $\mathscr{L}_S$.*

(b) *Let $\mathscr{L}'_S$ be another simplicial system of $\mathscr{A}_S$-modules, and $\psi=\{\psi_\alpha\}, \psi_\alpha: \mathscr{L}_\alpha \to \mathscr{L}'_\alpha$ be a morphism of simplicial systems of $\mathscr{A}_S$-modules. Let $L'_{.,\alpha}$ be a free resolution of $\mathscr{L}'_\alpha$ and $\{L'_{.,\alpha}, r'_{m,\alpha,\beta}\}$ be the corresponding simplicial system of complexes. Then there exists a morphism $\varphi=\{\varphi_{m,\alpha,\beta}\}$ from the simplicial system $\{L_{.,\alpha}, r_{m,\alpha,\beta}\}$ to the simplicial system $\{L'_{.,\alpha}, r'_{m,\alpha,\beta}\}$, such that the diagram of morphisms of complexes*

$$\begin{array}{ccc} \tilde{L}_. & \xrightarrow{\tilde{\varphi}_.} & \tilde{L}'_. \\ {\scriptstyle \tilde{\varepsilon}_.}\downarrow & & \downarrow{\scriptstyle \tilde{\varepsilon}'_.} \\ \tilde{\mathscr{L}}_. & \xrightarrow{\tilde{\psi}_.} & \tilde{\mathscr{L}}'_. \end{array}$$

is commutative.

Proof. (a) Set $L_{.,k}=\bigoplus_{|\alpha|=k} L_{.,\alpha}$ and $A_k=\bigoplus_{|\alpha|=k} A_\alpha$. Then $L_{.,k}$ is a free resolution of the A_k-module $\tilde{\mathscr{L}}_k=\bigoplus_{|\alpha|=k}\mathscr{L}_\alpha$. Let $\varepsilon_k: L_{.,k}\to\tilde{\mathscr{L}}_k$ be the corresponding quasi-isomorphism. We have to construct a series of maps $r_{m,n,k}: L_{m,k}\to L_{m-n+1,k+n}$. Let $\tilde{\mathscr{L}}_.=\{\tilde{L}_k, \delta_k\}$ and $\tilde{A}_.=\{\tilde{A}_k, \Delta_k\}$ be the cochain complexes of the simplicial systems $\mathscr{L}_S$ and $\mathscr{A}_S$, and denote by $\tilde{\mathscr{L}}^s_.$ the truncated complex $0\to\tilde{\mathscr{L}}_s\to\tilde{\mathscr{L}}_{s+1}\to\dots$. Denote by $\tilde{L}^s_p$ the direct sum of all $L_{m,k}$ with $k\geq s, m\geq 0, m+k=p$. For negative s with $|s|$ sufficiently large we have $\tilde{L}^s_p=\tilde{L}_p$.

Suppose that we have already constructed the mappings $r_{m,n,k}$ for all $m, n\geq 0$ and $k\geq s$, such that:

(a_s) Put $\tilde{L}^s_.=\{\tilde{L}^s_p, d^s_p\}$, where $d^s_p: \tilde{L}^s_p\to\tilde{L}^s_{p+1}$ is the operator determined by the mappings $r_{m,n,k}$, $m+k=p, k\geq s$, and by the differentials of the complexes $L_{.,k}$. Then $\tilde{L}^s_.$ is a complex, and the series of morphisms $\varepsilon_k, k\geq s$, determine a quasi-isomorphism $\tilde{\varepsilon}^s: \tilde{L}^s_.\to\tilde{\mathscr{L}}^s_.$.

Now, we shall define the mappings $r_{m,n,s-1}$ such that the property (a_{s-1}) holds. The differential $\delta_{s-1}: \tilde{\mathscr{L}}_{s-1}\to\tilde{\mathscr{L}}_s$ can be interpreted as a morphism from the one-term complex $\tilde{\mathscr{L}}_{s-1}$ to the complex $\tilde{\mathscr{L}}^s_.$; the cone of this morphism coincides with the complex $\tilde{\mathscr{L}}^{s-1}_.$. Using the standard construction of the covering morphism of resolutions, we can find a morphism of complexes $r^{s-1}_.: L_{.,s-1}\to\tilde{L}^s_.$, such that the diagram

$$\begin{tikzcd}
L_{\cdot,s-1} \arrow[r, "r_{\cdot}^{s-1}"] \arrow[d, "\varepsilon_{s-1}"'] & \tilde{L}_{\cdot}^{s} \arrow[d, "\tilde{\varepsilon}_{\cdot}^{s}"] \\
\tilde{\mathscr{L}}_{s-1} \arrow[r, "\delta_{s-1}"] & \tilde{\mathscr{L}}_{\cdot}^{s}
\end{tikzcd}$$

is commutative. Define the mappings $r_{m,n,s-1}$ (or, equivalently, the mappings $r_{m,\alpha,\beta}$ with $|\alpha|=s-1$) to be equal to the corresponding entry of the operator r_m^{s-1}, multiplied by $(-1)^m$. Then the complex $\tilde{L}_{\cdot}^{s-1}$ coincides with the cone of the morphism $r_{\cdot}^{s-1}$, and the quasi-isomorphisms ε_s, $\tilde{\varepsilon}_{\cdot}^{s}$ determine a quasi-isomorphism from $\tilde{L}_{\cdot}^{s-1}$ to the cone of δ_{s-1}, i.e. to the complex $\tilde{\mathscr{L}}_{\cdot}^{s-1}$. Therefore, the condition (a_{s-1}) is satisfied. Using an induction on s, we obtain a proof of the statement (a).

In order to prove (b), we can apply the same arguments, taking instead of the complex $\tilde{\mathscr{L}}_{\cdot}$ the cone of the morphism of the complexes $\psi_{\cdot}: \tilde{\mathscr{L}}_{\cdot} \to \tilde{\mathscr{L}}'_{\cdot}$.

Remark 1. One sees that if the morphism $\psi_{\cdot}$ is a quasi-isomorphism, then the same is true for the morphism $\tilde{\varphi}_{\cdot}$. Moreover, one can prove that if $\psi_{\cdot}$ is an isomorphism, then $\tilde{\varphi}_{\cdot}$ is a homotopical equivalence.

Remark 2. The proof of Lemma 3.2 holds without modification when $\mathscr{A}_S, \mathscr{L}_S, L_{\cdot,\alpha}$ are systems of sheaves on a topological space; we shall need this generalisation.

Let M be a complex space and $\mathscr{L}$ a coherent sheaf on M. Now, we are going to describe the construction of the Riemann-Roch invariant $\alpha_M(\mathscr{L})$. Let $\varrho: M \to \tilde{M}$ be an almost complex embedding, and let $\mathscr{E}\mathscr{L}$ and $\mathscr{E}_M = \mathscr{E}\mathscr{O}_M$ be the sheaves on $\tilde{M}$ introduced above. Let $\mathfrak{V} = \{(M_i, \varphi_i, U_i)\}, i \in I$, be the corresponding atlas on M. Denote by S the nerve of the covering $\{W_i\}$, by A_α, $\alpha \in S$ the sheaf $\mathscr{E}\Gamma_{W_\alpha}(\mathscr{O}_M)$ of germs of smooth functions on the manifold $\tilde{M}$ with values in the Fréchet algebra $\Gamma_{W_\alpha}(\mathscr{O}_M)$, and by $R_{\alpha,\beta}: A_\alpha \to A_\beta$ the morphism of sheaves, induced by the restriction map from $\Gamma_{W_\alpha}(\mathscr{O}_M)$ to $\Gamma_{W_\beta}(\mathscr{O}_M)$. We obtain a simplicial system of rings A_S. Denote by $\mathscr{E}\mathscr{L}_\alpha$ the restriction $\mathscr{E}\mathscr{L}_{\tilde{U}_\alpha}$ of the sheaf $\mathscr{E}\mathscr{L}$ to the open set $\tilde{U}_\alpha$, and take the complex $L_{\cdot,\alpha}$ to be equal to the complex $\mathscr{E}K_{\cdot}(\tilde{U}_\alpha, \mathscr{L})$. All the conditions of Lemma 3.2 are satisfied, and we can include the complexes $\mathscr{E}K_{\cdot}(\tilde{U}_\alpha, \mathscr{L})$ in a certain simplicial system of complexes of sheaves. The simplicial complex of this system will be denoted by $\mathscr{E}KC_{\cdot}(M, \mathfrak{V}, \mathscr{L})$.

In this case the mappings $r_{m,\alpha,\beta}$ coming from Lemma 3.2, can be described more explicitly. Recall that for any m, $0 \leq -m \leq n$, and $\alpha \in S$, the space $K_m(\tilde{U}_\alpha, \mathscr{L})$ is a direct sum of $\binom{n}{-m}$ copies of the space $\Gamma_{W_\alpha}(\mathscr{L})$. It is easy to see that the maps $r_{m,\alpha,\beta}$ have the form:

$$r_{m,\alpha,\beta}(\lambda) = F_{m,\alpha,\beta}(\lambda) \cdot R_{\alpha,\beta}$$

where $F_{m,\alpha,\beta}(\lambda)$ is a smooth function of the parameter $\lambda \in \tilde{M}$ whose values are operators of multiplication by a matrix of dimension $\binom{n}{-m} \times \binom{n}{-m-k}$ $(k=|\beta|-|\alpha|)$ with entries in the Fréchet algebra $\Gamma_{W_\beta}(\mathcal{O}_M)$. Therefore, the mappings $r_{m,\alpha,\beta}(\lambda)$ determine a simplicial system of complexes of Fréchet spaces, depending smoothly on the parameter $\lambda \in \tilde{M}$, whose cochain complex will be denoted by $KC_.(M, \mathfrak{V}, \mathscr{L})(\lambda)$. One can see that the complex of sheaves $\mathscr{E}KC_.(M, \mathfrak{V}, \mathscr{L})$ is really the complex of germs of smooth functions on $\tilde{M}$ with values in the parametrised complex $KC_.(M, \mathfrak{V}, \mathscr{L})(\lambda)$.

Remark. The mappings $r_{m,\alpha,\beta}$ can be chosen to be independent of the coherent sheaf $\mathscr{L}$ on M. In fact, if we construct the maps $r_{m,\alpha,\beta}$ for the structure sheaf $\mathcal{O}_M$, then it is easy to show that the same maps are suitable for an arbitrary $\mathscr{L}$.

Recall that, as was pointed out in Lemma 3.3, the complex of sheaves $\mathscr{E}KC_.(M, \mathfrak{V}, \mathscr{L})$ is quasi-isomorphic to the cochain complex of the simplicial system $\mathscr{E}\mathscr{L}_S$, or, in other words, to the canonical resolution of the sheaf $\mathscr{E}\mathscr{L}$ relative to the covering $\{\tilde{U}_i\}$. Malgrange's theorem asserts that the sheaf $\mathscr{E}\mathscr{L}$ is $\mathscr{E}$-perfect on $\tilde{M}$; since the complex $\mathscr{E}KC_.(M, \mathfrak{V}, \mathscr{L})$ is quasi-isomorphic to the sheaf $\mathscr{E}\mathscr{L}$, this implies that the parametrised complex of Fréchet spaces $KC_.(M, \mathfrak{V}, \mathscr{L})(\lambda)$ is $\mathscr{E}$-Fredholm on M. This complex is exact off the closed subset $\varrho(M)$ of $\tilde{M}$: therefore, as was shown in § 1, it determines an element $\alpha_M(\mathscr{L})$ of the group $K^0(\tilde{M}, \tilde{M} \setminus \varrho(M)) = K_0(M)$.

In order to obtain the Riemann-Roch theorem, we have to prove the following statements:

LEMMA 3.4. (a) *The element $\alpha_M(\mathscr{L})$ does not depend on the choices involved.*

(b) *If $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$ is an exact sequence of coherent sheaves on M, then $\alpha_M(\mathscr{G}) = \alpha_M(\mathscr{F}) + \alpha_M(\mathscr{H})$.*

(c) *If M is a complex manifold, and $\mathscr{L}$ a locally free sheaf, then $\alpha_M(\mathscr{L})$ is the Poincaré dual to the class of the underlying vector bundle in the group $K^0(M)$.*

(d) *If $f: M \to N$ is a proper morphism of complex spaces, then $f_*(\alpha_M(\mathscr{L})) = \alpha_N(f_!\mathscr{L})$.*

Proof. (a) Remark 1 after Lemma 3.3 shows that the equivalence class of the parametrised complex $KC_.(M, \mathfrak{V}, \mathscr{L})(\lambda)$ does not depend on the choice of connecting maps $r_{m,\alpha,\beta}$ and of Koszul complexes $K_.(U_i, \mathcal{L})(\lambda)$ (which are determined by the choice of coordinate system of the domain U_i).

Statement (b) of Lemma 3.3 shows that the equivalence class of the complex $KC_.(M, \mathfrak{V}, \mathcal{L})(\lambda)$ does not depend on the choice of the covering $\{W_i\}$ (see the proof of Corollary 2.6).

Finally, we shall prove that the element $\alpha_M(\mathcal{L})$ does not depend on the choice of the atlas and of the almost complex embedding of M. Let $\mathfrak{V}=\{W_i, \varphi_i, U_i)\}$, $i\in I$, and $\mathfrak{V}'=\{W_i', \varphi_i', U_i')\}$, $i\in I'$, be two atlases of M, and let $\varrho: M\to\tilde{M}$, $\varrho': M\to\tilde{M}'$ be corresponding almost complex embeddings. Taking into account the independence on the covering, proved above, and Lemma 3.2 (b), one can assume that $I=I'$, $W=W_i'$, and that $\mathfrak{V}'$ is a modification of $\mathfrak{V}$ and ϱ' is the corresponding modification of ϱ. It is easy to see that the complex $K_.(U_i', \mathcal{L})(\lambda)$ coincides with the Koszul-Thom transformation of the complex $K_.(U_i, \mathcal{L})(\lambda)$. As was shown above, the manifold $\tilde{M}'$ can be identified with the total space of a certain complex vector bundle on $\tilde{M}$. Then the Koszul-Thom transformation of the complex $KC_.(M, \mathfrak{V}, \mathcal{L})(\lambda)$ on $\tilde{M}$ can be considered as a cochain complex of a simplicial system of complexes, formed by the complexes $K_.(U_i', \mathcal{L})(\lambda)$ on $\tilde{M}'$. The Thom isomorphism theorem shows that this complex determines the same element of the group $K_0(M)$ as the complex $KC_.(M, \mathfrak{V}, \mathcal{L})(\lambda)$ on $\tilde{M}$.

(b) This follows immediately from Lemma 1.7.

(c) In this case we can take $\tilde{M}=M$. The complex of sheaves $\mathscr{E}KC_.(M, \mathfrak{V}, \mathcal{L})$ is quasi-isomorphic on M to the sheaf $\mathscr{E}\mathcal{L}$. Using Lemma 1.4, we obtain that the parametrised complex of Fréchet spaces $KC_.(M, \mathfrak{V}, \mathcal{L})(\lambda)$ is uniformly quasi-isomorphic to the vector bundle corresponding to the locally free sheaf $\mathcal{L}$. The assertion is proved.

(d) The proof consists of two steps.

Step 1. Consider the case when N is a subspace of a Stein domain V in $\mathbf{C}^n$. Let $\mathfrak{V}=\{W_i, \varphi_i', U_i')\}$ be an atlas on M, $\varrho': M\to\tilde{M}'$ be an almost complex embedding, and $\psi_i': U_i'\to\tilde{M}'$ the corresponding diffeomorphisms. Put $U_i=V\times U_i'$, $\varphi_i=(f, \varphi_i')$, $\tilde{M}=V\times\tilde{M}'$, $\varrho=(f, \varrho')$, $\psi_i=\mathrm{Id}\times\psi_i'$. Then the atlas $\mathfrak{V}=\{(W_i, \varphi_i, U_i)\}$ with the embedding $\varrho: M\to\tilde{M}$ and the diffeomorphisms ψ_i form an almost complex embedding of M in $\tilde{M}$ such that the restriction of the projection of $\tilde{M}$ on V to $\varrho(M)$ coincide with the map f. We shall assume that $\tilde{M}'$ is a domain in $\mathbf{R}^{2m}=\mathbf{C}^m$. Since the map f is proper, we can assume that $\tilde{M}=V\times\mathbf{C}^m$. The diffeomorphisms ψ_i can be extended up to diffeomorphisms from $\hat{U}_i=V\times\mathbf{C}^m$ to $\tilde{M}=V\times\mathbf{C}^m$ such that ψ_i is equal to the identity off the set $V\times B$, where B

is a sufficiently large closed ball in $\mathbf{C}^m$. Put $\tau_i=\psi_i^{-1}$. Choose a coordinate system $z_1,\dots,z_n$ on V, and $w_1^\alpha,\dots,w_m^\alpha$ on U_α for any $\alpha\subset I$. Denote by $a_1,\dots,a_n,b_1,\dots,b_m$ the operators of multiplication by $z_1,\dots,z_n$, $w_1^\alpha,\dots,w_m^\alpha$ in the Fréchet space $\Gamma_{W_\alpha}(\mathscr{L})$.

For any points $\lambda=(\lambda_1,\dots,\lambda_n)\in V$, $\mu=(\mu_1,\dots,\mu_m)\in\mathbf{C}^m$ and $t\in[0,1]$ denote by $\hat{K}_\cdot(U_\alpha,\mathscr{L}_\alpha)(\lambda,\mu,t)$ the Koszul complex for the tuple of operators $\lambda_1-a_1,\dots,\lambda_n-a_n,\mu_1-t\cdot b_1,\dots,\mu_m-t\cdot b_m$, acting in the space $\Gamma_{W_\alpha}(\mathscr{L})$. Denote by $\mathscr{E}\hat{K}_\cdot(U_\alpha,\mathscr{L}_\alpha)$ the complex of sheaves of functions on $\hat{U}_\alpha\times[0,1]$ with values in the spaces $\hat{K}_j(U_\alpha,\mathscr{L}_\alpha)$ which depend smoothly on $(\lambda,\mu)\in U_\alpha$ and continuously on the parameter $t\in[0,1]$. We shall describe the sheaves of homologies of the complex $\mathscr{E}\hat{K}_\cdot(U_\alpha,\mathscr{L}_\alpha)$. Denote by $\mathscr{C}\mathscr{E}\mathscr{L}_\alpha$ the sheaf on $U_\alpha\times[0,1]$, consisting of germs of all sections of the sheaf $\mathscr{E}\mathscr{L}_\alpha$ on U_α, depending continuously on the parameter $t\in[0,1]$. Denote by $\mathscr{E}\hat{\mathscr{L}}_\alpha$ the direct image of the sheaf $\mathscr{C}\mathscr{E}\mathscr{L}_\alpha$ under the map $p_\alpha\colon U_\alpha\times[0,1]\to\hat{U}_\alpha\times[0,1]$, defined by the formula $p_\alpha(\lambda,\mu,t)=(\lambda,t\mu,t)$. The same arguments as in Lemma 2.3'. show that the complex of sheaves $\mathscr{E}\hat{K}_\cdot(U_\alpha,\mathscr{L})\to\widehat{\mathscr{E}\mathscr{L}_\alpha}\to 0$ is exact on $U_\alpha\times[0,1]$.

Our next objective is to attach all the complexes $\hat{K}_\cdot(U_\alpha,\mathscr{L})(\lambda,\mu,t)$ together. For any $\alpha\subset I$ denote by $\hat\psi_\alpha$ the map from $\hat{U}_\alpha\times[0,1]$ to $\tilde{M}\times[0,1]$, defined by the formula:

$$\hat\psi_\alpha(\lambda,\mu,t)=(\lambda,t\cdot\psi_\alpha'(t^{-1}\mu),t),\quad\text{for } t>0$$

$$\hat\psi_\alpha(\lambda,\mu,0)=(\lambda,\mu,0).$$

The maps $\hat\psi_\alpha$ are bijective, continuous and depend smoothly on λ and μ for fixed $t\in[0,1]$. Denote by $\mathscr{E}\tilde{\mathscr{L}}_\alpha$ the direct image of the sheaf $\widehat{\mathscr{E}\mathscr{L}_\alpha}$ by the map $\hat\psi_\alpha$ and by $\tilde{K}_\cdot(\tilde{U}_\alpha,\mathscr{L})(\lambda,\mu,t)$ the image in $\tilde{M}\times[0,1]$ of the complex $\hat{K}_\cdot(U_\alpha,\mathscr{L})(\lambda,\mu,t)$ by $\hat\psi_\alpha^{-1}$.

We shall show that the sheaves $\mathscr{E}\tilde{\mathscr{L}}_\alpha$ form a simplicial system of sheaves on $\tilde{M}\times[0,1]$. Denote by $\mathscr{C}\mathscr{E}\mathscr{L}$ the sheaf on $\tilde{M}\times[0,1]$, consisting of families of germs of sections of the sheaf $\mathscr{E}\mathscr{L}$, depending continuously on the parameter $t\in[0,1]$, and by $p\colon\tilde{M}\times[0,1]\to\tilde{M}\times[0,1]$ the map, defined by the formula $p(\lambda,\mu,t)=(\lambda,t\mu,t)$. Then the diagram

$$\begin{tikzcd}
U_\alpha\times[0,1] \arrow[r, "p_\alpha"] \arrow[d, "\psi_\alpha\times\mathrm{Id}"'] & \hat{U}_\alpha\times[0,1] \arrow[d, "\hat\psi_\alpha"] \\
\tilde{M}\times[0,1] \arrow[r, "p"] & \tilde{M}\times[0,1]
\end{tikzcd}$$

is commutative. Denote $\mathscr{C}\mathscr{E}\tilde{\mathscr{L}}_\alpha=(\psi_\alpha\times\mathrm{Id})_*\mathscr{C}\mathscr{E}\mathscr{L}$. Then the sheaf $\mathscr{C}\mathscr{E}\tilde{\mathscr{L}}_\alpha$ coincides with the restriction of the sheaf $\mathscr{C}\mathscr{E}\mathscr{L}$ to $\tilde{U}_\alpha\times[0,1]$. Therefore, the sheaves $\mathscr{C}\mathscr{E}\tilde{\mathscr{L}}_\alpha$ form a simplicial system of sheaves on $\tilde{M}\times[0,1]$, and the same is true for the sheaves $\mathscr{E}\tilde{\mathscr{L}}_\alpha=p_*\mathscr{C}\mathscr{E}\tilde{\mathscr{L}}_\alpha$, endowed with the system of morphisms $\tilde{R}_{\alpha,\beta}$, induced from the restriction morphisms of sheaves $\mathscr{C}\mathscr{E}\tilde{\mathscr{L}}_\alpha$. Note that the specialisation of the simplicial system of sheaves $\mathscr{E}\tilde{\mathscr{L}}_\alpha$ on the space $\tilde{M}\times\{1\}$ coincides with the simplicial system $\mathscr{E}\mathscr{L}_\alpha$ (used above in the construction of the complex $KC_.(m,\mathfrak{B},\mathscr{L})$, and on $M\times\{0\}$ with the simplicial system $i_*f_*\mathscr{E}\mathscr{L}_\alpha$, where i is the embedding of V in $\tilde{M}$ as the subspace $V\times\{0\}$.

Since the complexes of sheaves $\mathscr{E}\hat{K}_.(\tilde{U}_\alpha,\mathscr{L})$ are quasi-isomorphic to $\mathscr{E}\tilde{\mathscr{L}}_\alpha$ for all $\alpha\subset I$, we can apply to them the construction of Lemma 3.3. We obtain a parametrised complex $\widehat{KC}_.(M,\mathfrak{B},\mathscr{L})(\lambda,\mu,t)$ on the space $\tilde{M}\times[0,1]$. If we denote by $\mathscr{E}\widehat{KC}_.(M,\mathfrak{B},\mathscr{L})$ the complex of sheaves on $\tilde{M}\times[0,1]$ consisting of all sections of this complex, depending smoothly on $(\lambda,\mu)\in\tilde{M}$ and continuously on $t\in[0,1]$, then the complex $\mathscr{E}\widehat{KC}_.(M,\mathfrak{B},\mathscr{L})$ is quasi-isomorphic to the sheaf $p_*\mathscr{C}\mathscr{E}\mathscr{L}$ on $\tilde{M}\times[0,1]$. It is easy to see that the restriction of the complex $\widehat{KC}_.(M,\mathfrak{B},\mathscr{L})(\lambda,\mu,t)$ to the space $\tilde{M}\times\{1\}$ is in fact the complex $KC_.(M,\mathfrak{B},\mathscr{L})(\lambda,\mu)$, and determines the element $\alpha_M(\mathscr{L})$ of $K_0(M)$. Consider the restriction of $\widehat{KC}_.(M,\mathfrak{B},\mathscr{L})(\lambda,\mu,t)$ to the space $\tilde{M}\times\{0\}$. Denote by $\mathscr{C}_.(M,\mathfrak{B},\mathscr{L})$ the canonical alternating resolution of $\mathscr{L}$ relative to the covering $\mathfrak{B}$, and by $C_.(M,\mathfrak{B},\mathscr{L})$ the complex of global sections of $C_.(M,\mathfrak{B},\mathscr{L})$ on M, i.e. the cochain complex for the covering $\mathfrak{B}$ and the sheaf $\mathscr{L}$. Then it follows from Lemma 3.3 (b) that the restriction of the complex $\widehat{KC}_.(M,\mathfrak{B},\mathscr{L})$ to $\tilde{M}\times\{0\}$ is $\mathscr{E}$-quasi-isomorphic to the Koszul complex for the operators $\lambda_1I-a_1,\dots,\lambda_nI-a_n,\mu_1I,\dots,\mu_mI$, acting in the complex of Fréchet spaces $C_.(M,\mathfrak{B},\mathscr{L})$. In fact, the sheaves of germs of smooth sections of both complexes are quasi-isomorphic to the complex of sheaves $i_*f_*\mathscr{C}(M,\mathfrak{B},\mathscr{L})$ (for the latter this follows from the remark following Lemma 2.3). On the other hand, the generalised direct image $f_!\mathscr{L}$ of the sheaf $\mathscr{L}$ under the morphism f, considered as an element of the derived category of the category of complexes of sheaves on V, can be represented by the complex $f_*\mathscr{C}(M,\mathfrak{B},\mathscr{L})$. Therefore, the parametrised Koszul complex for the operators $a_1,\dots,a_n$ of multiplication by the coordinate functions of V, acting in the complex $C_.(M,\mathfrak{B},\mathscr{L})$, determines the element $\alpha_V(f_!\mathscr{L})$ of the group $K_0(V)$. Using Lemma 2.1 (a), we can see that the parametrised Koszul complex of operators $a_1,\dots,a_n,0,\dots,0$, considered above, determines an element of the group $K^0(V\times\mathbf{C}^m,V\times(\mathbf{C}^m\setminus 0))$, corresponding under the Thom isomorphism to the element $\alpha_V(f_!\mathscr{L})$, and therefore the image of this element in the group $K_0(V)$ under the natural projection of $V\times\mathbf{C}^m$ on V coincides with the element $\alpha_V(f_!\mathscr{L})$. In order to complete the proof, it remains to show that the complex $\widehat{KC}_.(M,\mathfrak{B},\mathscr{L})(\lambda,\mu,t)$ determines a uniform homotopy between these complexes, i.e. to show that this complex is uniformly Fredholm on the space $\tilde{M}\times[0,1]$. Let V_1 be a Stein domain such that $\bar{V}_1\subset V$, and $\mathfrak{V}_1$ is a refinement of the covering $\mathfrak{V}$. Put $M_1=f^{-1}(V_1)$, $\tilde{M}_1=V_1\times\tilde{M}'$. Then there exist restriction morphisms

$$R_{V,V_1}\colon \widehat{KC}_.(M,\mathfrak{V},\mathscr{L})(\lambda,\mu,t)\to\widehat{KC}_.(M_1,\mathfrak{V},\mathscr{L})(\lambda,\mu,t),$$

and

$$R_{\mathfrak{V},\mathfrak{V}_1}\colon \widehat{KC}_.(M_1,\mathfrak{V},\mathscr{L})(\lambda,\mu,t)\to\widehat{KC}_.(M_1,\mathfrak{V}_1,\mathscr{L})(\lambda,\mu,t),$$

It is easy to see that R_{V,V_1} and $R_{\mathfrak{V},\mathfrak{V}_1}$ induce quasi-isomorphisms of corresponding complexes of germs of vector-functions on $M_1\times[0,1]$, depending smoothly on $(\lambda,\mu)\in\tilde{M}_1$ and continuously on $t\in[0,1]$. Applying Lemma 1.4, we obtain that R_{V,V_1} and $R_{\mathfrak{V},\mathfrak{V}_1}$ are uniform quasi-isomorphisms on $M_1\times[0,1]$. Applying Lemma 1.8 to their composition, we complete the proof of Step 1.

Step 2. The general case. We start with an auxiliary definition, Let $\mathscr{L}$ be a coherent sheaf on the complex space M, and let $\mathfrak{M}_1,\dots,\mathfrak{M}_k$ be Stein coverings of M, where $\mathfrak{M}_s=\{W_i^s\}, i\in I_s$. For any k-tuple $\beta=(\beta_1,\dots,\beta_k)$, where β_s is a subset of I_s, put

$$W_\beta=W^1_{\beta_1}\cap\ldots\cap W^k_{\beta_k}\quad\text{and}\quad |\beta|=\beta_1+\ldots+|\beta_k|.$$

Let

$$\mathscr{C}_i(M,\mathfrak{M}_1,\dots,\mathfrak{M}_k,\mathscr{L})=\bigoplus_{|\beta|=i}\mathscr{L}_{W_\beta},$$

and let $\mathscr{C}_.(M,\mathfrak{M}_1,\dots,\mathfrak{M}_k,\mathscr{L})$ be the complex of sheaves, consisting of the sheaves $\mathscr{C}_i(M,\mathfrak{M}_1,\dots,\mathfrak{M}_k,\mathscr{L})$ and the standard co-boundary operators. More precisely, the complexes of sheaves $\mathscr{C}_.(M,\mathfrak{M}_1,\dots,\mathfrak{M}_k,\mathscr{L})$ on M can be defined by induction on k. When $k=1$, this is the alternating resolution, used above. The complex $\mathscr{C}_.(M,\mathfrak{M}_1,\dots,\mathfrak{M}_k,\mathfrak{M}_{k+1},\mathscr{L})$ can be defined as the total complex of the bicomplex

$$\mathscr{C}_.(M,\mathfrak{M}_{k+1},\mathscr{C}_.(M,\mathfrak{M}_1,\dots,\mathfrak{M}_k,\mathscr{L})).$$

It is easy to see that the definition does not depend on the enumeration of the coverings $\mathfrak{M}_1,\dots,\mathfrak{M}_k$. For any two numbers s,k, $s\leq k$, there exists a natural quasi-isomorphism

$$r_{s,k}\colon \mathscr{C}_.(M,\mathfrak{M}_1,\dots,\mathfrak{M}_s,\mathscr{L})\mapsto\mathscr{C}_.(M,\mathfrak{M}_1,\dots,\mathfrak{M}_k,\mathscr{L});$$

it is easy to see that $r_{s,k}$ does not depend of the enumeration and that for any $p\leq s\leq k$ we have $r_{s,k}\circ r_{p,s}=r_{p,k}$. In particular, all the complexes $\mathscr{C}_.(M,\mathfrak{M}_1,\dots,\mathfrak{M}_k,\mathscr{L})$ are quasi-isomorphic to the sheaf $\mathscr{L}$. We shall denote by $C_{.}(M, \mathfrak{M}_1, \dots, \mathfrak{M}_k, \mathscr{L})$ the complex of Fréchet spaces, consisting of all global sections of $\mathscr{C}_{.}(M, \mathfrak{M}_1, \dots, \mathfrak{M}_k, \mathscr{L})$.

Now, let $f\colon M \to N$ be a proper morphism. Let $\varrho''\colon N \to \tilde{N}$ and $\varrho''\colon M \to \tilde{M}'$ be almost complex embeddings. Put $\tilde{M} = \tilde{M}' \times \tilde{N}$ and $\varrho = (\varrho', \varrho'' \circ f)$; then ϱ is an almost complex embedding of $\tilde{M}'$ in $\tilde{M}$ and the projection of $\tilde{M}$ on $\tilde{N}$ extends the mapping f. Let $\mathfrak{B} = \{(W_i, \varphi_i, U_i)\}, i \in I$ be an atlas on N. Let $M_\alpha = f^{-1} W_\alpha$ and fix for any $i \in I$ an atlas $\mathfrak{M}_i = \{(W_{i,j}, \varphi_{i,j}, U_{i,j})\}, j \in I_i$ on M_i. For any $\alpha \subset I$ such that $i \in \alpha$ the restriction of this atlas to M_α will be denoted again by $\mathfrak{M}_i$. As in the proof of Step 1, we can assume that the domains $U_{i,j}$ have the form $U_{i,j} = U'_{i,j} \times U_i$, and that the projection of $U_{i,j}$ on U_i agrees with the mapping f. Denote by $\mathfrak{M}$ the atlas on M, consisting of the union of all atlases $\mathfrak{M}_i, i \in I$, and for any $\alpha = (i_1, \dots, i_k) \subset I$ set

$$\mathscr{C}_{.}(M_\alpha, \mathfrak{M}_\alpha, \mathscr{L}) = \mathscr{C}_{.}(M, \mathfrak{M}_{i_1}, \dots, \mathfrak{M}_{i_k}, \mathscr{L}).$$

As was pointed out above, there exists for any two tuples $\alpha, \beta, \alpha \subset \beta$, a morphism of complexes $r_{\alpha,\beta}\colon \mathscr{C}_{.}(M_\alpha, \mathfrak{M}_\alpha, \mathscr{L}) \to \mathscr{C}_{.}(M_\beta, \mathfrak{M}_\beta, \mathscr{L})$, and for any $\alpha \subset \beta \subset \gamma$ we have $r_{\alpha,\beta} \circ r_{\beta,\gamma} = r_{\alpha,\gamma}$. Therefore the complexes $\mathscr{C}_{.}(M_\alpha, \mathfrak{M}_\alpha, \mathscr{L})$ form a simplicial system of complexes. It is easy to see that the cochain complex of this system is equal to the complex $\mathscr{C}_{.}(M, \mathfrak{M}, \mathscr{L})$. For any $\alpha \subset I$ denote by $z_1^\alpha, \dots, z_n^\alpha$ a coordinate system on the domain U_α. Denote by $KC_{.}(U_\alpha, \mathfrak{M}_\alpha, \mathscr{L})(\lambda)$ the parametrised Koszul complex of operators of multiplication by $z_1^\alpha, \dots, z_n^\alpha$ in the complex $C_{.}(M_\alpha, \mathfrak{M}_\alpha, \mathscr{L})$, and by $KC_{.}(\tilde{U}_\alpha, \mathfrak{M}_\alpha, \mathscr{L})(\lambda)$ the corresponding complex on the domain $\tilde{U}_\alpha \subset \tilde{N}$. We can apply Lemma 3.3 to these complexes: denote by $KC_{.}(N, \mathfrak{M}, \mathscr{L})(\lambda)$ the cochain complex of the corresponding simplicial system of complexes. It is easy to see that this complex determines the same element of the group $K_0(N)$ as the complex $KC_{.}(N, \mathfrak{B}, f_! \mathscr{L})(\lambda)$.

For any tuples $\alpha = (i_1, \dots, i_k) \subset I$ and $\beta = (\beta_1, \dots, \beta_k)$ with $\beta_j \subset I_{i_j}$, $j = 1, \dots, k$, we can construct in the same manner as in Step 1 mappings

$$\psi_{\alpha,\beta}(\lambda, \mu, t)\colon U_{\alpha,\beta} \times [0,1] \to \tilde{M} \times [0,1],$$

sheaves $\mathscr{E}\mathscr{L}_{\alpha,\beta}$, and parametrised complexes $\widehat{KC}_{.}(\tilde{U}_{\alpha,\beta}, \mathscr{L})(\lambda, \mu, t)$ on $\tilde{M} \times [0,1]$. Applying Lemma 3.2 again, we obtain a parametrised complex of Fréchet spaces $\widehat{KC}_{.}(M, \mathfrak{M}, \mathscr{L})(\lambda, \mu, t)$ on the space $\tilde{M} \times [0,1]$. It is easy to check that the restriction of this complex to the space $\tilde{M} \times \{0,1\}$ coincides with $KC_{.}(M, \mathfrak{M}, \mathscr{L})(\lambda)$, and its restriction to $\tilde{M} \times \{0\}$ is homotopically equivalent to the Koszul complex of the operators $\mu_1, I, \dots, \mu_m, I$, acting on the complex $KC_{.}(N, \mathfrak{M}, \mathscr{L})(\lambda)$. The same arguments as were used in the proof of Step 1 permit us to complete the proof in the general case.

§ 4. Remarks

(1) Let (W, φ, U) be a chart on the complex space M, and $\mathcal{O}_W=\mathcal{O}_U/J$. Denote by $\tilde{\mathcal{O}}_U$ the sheaf $\tilde{\mathcal{O}}_U=\mathcal{O}_U+J\otimes\mathcal{E}_U$. The sections of this sheaf will be called almost holomorphic functions on U. In the same way, one can define the complex $\tilde{\mathcal{O}}K_.(U, \mathcal{L})$ of sheaves of almost holomorphic sections of the complex $K_.(U, \mathcal{L})(\lambda)$. It is easy to prove that the complex of sheaves $\tilde{\mathcal{O}}K_.(U, \mathcal{L})\to\mathcal{L}\to 0$ is exact on U.

Let $\varrho\colon M\to\tilde{M}$ be an almost complex embedding and $\psi_i\colon U_i\to\tilde{U}_i$ the corresponding diffeomorphisms. Since the connecting maps $\psi_i^{-1}\circ\psi_j$ are almost holomorphic by definition, we can define on $\tilde{M}$ the globally determined sheaf $\tilde{\mathcal{O}}_M$ of almost holomorphic functions (and, similarly, of almost holomorphic vector-functions). Having this, we may attempt to perform the above construction, using almost holomorphic functions instead of smooth ones. The only point where this is not possible is in extending the maps $\tau_i=\psi_i^{-1}$ away from $\tilde{U}_i$; however, the maps $r_{m,\alpha,\beta}$ can be chosen to be almost holomorphic near $\tilde{U}_\alpha$. As a result of the construction, we obtain a parametrised complex $KC_.(M, \mathfrak{V}, \mathcal{L})(\lambda)$ on M, such that on any sufficiently small open subset V of M the complex $KC_.(M, \mathfrak{V}, \mathcal{L})(\lambda)$ can be represented as a direct sum of two parametrised complexes $K'_.(\lambda), K''_.(\lambda)$. The complex $K'_.(\lambda)$ is smooth and $\mathcal{E}$-exact on V; the complex $K''_.(\lambda)$ is almost holomorphic on V, and the corresponding complex of almost holomorphic sections $\tilde{\mathcal{O}}K''_.$ is quasi-isomorphic to the sheaf $\mathcal{L}$.

(2) Let us fix the almost complex embedding $M\to\tilde{M}$ and connecting mappings $r_{m,\alpha,\beta}$ (see the remark following Lemma 3.2). Then the correspondence $\alpha_M\colon\mathcal{L}\to KC_.(M, \mathfrak{V}, \mathcal{L})(\lambda)$ defines an exact functor α_M from the category of all coherent sheaves on M to the category of all uniformly Fredholm parametrised complexes of Fréchet spaces on $\tilde{M}$, exact off M. More precisely, any short exact sequence of sheaves transforms into a uniformly exact short sequence of parametrised complexes. Therefore, this functor determines a homomorphism of higher K-groups; $K_*^{\mathrm{hol}}(M)\to K_*^{\mathrm{top}}(M)$, where $K_*^{\mathrm{hol}}(M)$ denote the algebraic K-theory of the category of all coherent sheaves on M, defined in [9], (see also [10]). Moreover, if we fix all auxiliary entities, used in the proof of functoriality (Lemma 3.4 (d)), then we can see that the parametrised complex $\widehat{KC}_.(M, \mathfrak{V}, \mathcal{L})(\lambda, \mu, t)$ (realising the homotopy between the complexes $KC_.(M, \mathfrak{M}, \mathcal{L})(\lambda, \mu)$) and $KC_.(N, \mathfrak{M}, \mathcal{L})(\lambda, \mu)$) can be considered as an exact functor of $\mathcal{L}$. On the other hand, it is easy to see that the natural quasi-isomorphism between $KC_.(N, \mathfrak{M}, \mathcal{L})(\lambda, \mu)$ and $KC_.(N, \mathfrak{V}, f_!\mathcal{L})(\lambda, \mu)$ is a natural transformation of functors (if $KC_.(N, \mathfrak{M}, \mathcal{L})(\lambda, \mu)$ is considered as a bicomplex, then $KC_.(N, \mathfrak{V}, f_!\mathcal{L})(\lambda, \mu)$ coincides with the total complex of the bicomplex consisting of

the homology groups of the rows of $KC_{.}(M, \mathfrak{M}, \mathscr{L})(\lambda, \mu)$). This natural transformation gives us an equivalence between the functors $f_* \circ \alpha_M$ and $\alpha_N \circ f_!$, i.e. a Riemann-Roch theorem for higher K-groups.

(3) Let N be a complex manifold and let M be a precompact domain in N. Denote by $\mathfrak{V} = \{U_i\}, i \in I$ a locally finite covering of M by strongly-pseudoconvex domains, and by $\mathscr{L}$ a holomorphic vector bundle on N. Denote by $C^h_{.}(M, \mathfrak{V}, \mathscr{L})$ the cochain complex of square-integrable sections of $\mathscr{L}$ on the elements of $\mathfrak{V}$. We shall assume that the Toeplitz operators on the domains U_α are essentially normal (see the note in the introduction). In that case, denote by ξ_k the element of the Brown-Douglas-Fillmore group Ext, corresponding to the algebra of the Toeplitz operator in the Hilbert space $C^h_k(M, \mathfrak{V}, \mathscr{L})$, and let $\xi = \Sigma(-1)^k \xi_k$ (the covering $\mathfrak{V}$ may be chosen in such a way that the complex $C^h_{.}(M, \mathfrak{V}, \mathscr{L})$ is finite). ξ is an element of the group

$$\operatorname{Ext}\left(\overline{\bigcup_{i \in I} bU_i}\right);$$

however, it can be proven that in fact ξ belongs to the image of the group $\operatorname{Ext}(bM)$ under the natural embedding. On the other hand, we can construct, as above, the complex $KC^h_{.}(M, \mathfrak{V}, \mathscr{L})(\lambda)$, replacing in the complex $KC_{.}(M, \mathfrak{V}, \mathscr{L})(\lambda)$ Fréchet spaces by Hilbert spaces. This complex is Fredholm off bM. Denote by $[\mathscr{L}_M]$ the element of the group $K^0(N \setminus bM)$, generated by the vector bundle equal to $\mathscr{L}$ on M and zero off $\bar{M}$. It is easy to see that the element of the group $K^0(N \setminus bM)$ determined by the complex $KC^h_{.}(M, \mathfrak{V}, \mathscr{L})(\lambda)$ coincides with $[\mathscr{L}_M]$. (In fact, this element coincides with the image of $\alpha_M(\mathscr{L})$ under the map $\partial: K_0(M) \to K_{-1}(bM) \approx K^0(N \setminus bM)$.) The following index theorem is valid:

The element $\xi \in \operatorname{Ext}(bM)$ is dual to the class of the complex $KC^h(M, \mathfrak{V}, \mathscr{L})(\lambda)$, i.e. to the element $[\mathscr{L}_M]$.

When bM is smooth a standard calculation shows that:

$$\operatorname{ch}_*(\xi) = (\operatorname{ch}^*(\mathscr{L}) \cup \operatorname{td}(bM)) \cap [bM],$$

i.e. we obtain a generalisation of Boutet de Monvel's index theorem.

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