Homotopy classes in Sobolev spaces and the existence of energy minimizing maps

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Introduction

Consider the problem of finding maps $f: M \to N$ that are stationary for an energy functional such as $\int_M |Df|^p$ (where M and N are compact riemannian manifolds and $p \geq 1$). Such maps may be found by minimizing the functional, but if we minimize among all maps from M to N, then the minimum is 0 and is attained only by constant maps. Thus in order to find nontrivial stationary maps, we would like to use the topology of M and N to define classes of maps from M to N in which we can minimize the functional. For instance one could try to minimize among maps in a given homotopy class, but this is not possible in general (unless $p > \dim M$), since a minimizing sequence of mappings in one homotopy class can converge (in the appropriate weak topology) to a map in another homotopy class. However, in this paper we show that it is possible to minimize among maps f whose restrictions to a lower dimensional skeleton of (a triangulation of) M belong to a given homotopy class.

To state the results precisely we need to refer to certain Sobolev spaces and norms. We will assume without loss of generality that M and N are submanifolds of euclidean spaces $\mathbf{R}^m$ and $\mathbf{R}^n$, respectively, and we let $\mathrm{Lip}(M, N)$ denote the space of lipschitz maps from M to N. We define $L^{1,p}(M, \mathbf{R}^n)$ to be the space of all functions $f \in L^p(M, \mathbf{R}^n)$ such that there exist functions

$$f_i \in \mathrm{Lip}(M, \mathbf{R}^n) \quad \text{and} \quad g \in L^p(M, \mathrm{Hom}(\mathbf{R}^m, \mathbf{R}^n))$$

satisfying

$$\|f_i - f\|_p + \|Df_i - g\|_p \to 0.$$

In this case we say that g is a *weak derivative* of f, and we let

$$\|f\|_{1,p}=\left(\int_M|f|^p\right)^{1/p}+\left(\int_M|g|^p\right)^{1/p}.$$

(One can show that any two weak derivatives of f are equal almost everywhere, so $\|f\|_{1,p}$ is well-defined.) We also define:

$$L^{1,p}(M,N)=\{f\in L^{1,p}(M,\mathbf{R}^n)\colon f(x)\in N \text{ for every } x\in M\}.$$

Finally, we let $H^{1,p}(M,N)$ be the weak-bounded closure of $\mathrm{Lip}(M,N)$ in $L^{1,p}(M,N)$. That is, $f\in H^{1,p}(M,N)$ if and only if there exists a sequence $f_i\in\mathrm{Lip}(M,N)$ such that

$$\|f_i-f\|_p\to 0$$

$$\|Df_i\|_p \quad \text{is bounded (independently of } i)$$

$$f(x)\in N \quad \text{for every} \quad x\in M.$$

These spaces have the following nice compactness property [GT, Theorem 7.22]. If f_i is a sequence of maps in $L^{1,p}(M,N)$ or $H^{1,p}(M,N)$ with $\|f_i\|_{1,p}$ bounded, then there is an f in $L^{1,p}(M,N)$ or $H^{1,p}(M,N)$, respectively, such that $\|f_i-f\|_p\to 0$.

Recall that the *d-homotopy type* of a continuous map from M to N is the homotopy class of its restriction to the d-dimensional skeleton of M. Our main results may now be stated.

THEOREM 2.1. *Let d be the greatest integer strictly less than p. For each $K<\infty$, there is an $\varepsilon>0$ such that if $f_1,f_2\in\mathrm{Lip}(M,N)$, $\|f_1-f_2\|_p<\varepsilon$ and $\|Df_i\|_p\leq K$, then f_1 and f_2 have the same d-homotopy type.*

Consequently each $f\in H^{1,p}(M,N)$ has a well-defined d-homotopy type, and d-homotopy types are preserved by bounded weak convergence. Thus for each continous map g from M to N, there is a map that minimizes $\int_M|Df|^p$ among all maps $f\in H^{1,p}(M,N)$ having the same d-homotopy type as g.

Similarly for $L^{1,p}(M,N)$ we have

THEOREM 3.4. *Let $d=[p-1]$ be the greatest integer less than or equal to $p-1$. Then each $f\in L^{1,p}(M,N)$ has a well-defined d-homotopy type, and d-homotopy types are preserved by bounded weak convergence. Furthermore, for each continuous map g*

from the $(d+1)$*-skeleton* M^{d+1} *of* M *into* N*, there is a map that minimizes* $\int_M |Df|^p$ *among all* $f \in L^{1,p}(M,N)$ *having the same d-homotopy type as* g*.*

More generally, we can replace $\int_M |Df|^p$ in Theorems 2.1 and 3.4 by any other functional $\int_M Q(x, f(x), Df(x))\,dx$ that is lower semicontinuous with respect to bounded weak convergence in $L^{1,p}$ and such that $|A|^p \le c(1+Q(x,y,A))$ for each $x \in M$, $y \in N$, and linear map A from $\mathrm{Tan}_x M$ to $\mathrm{Tan}_y N$.

Although the values of d that occur in Theorems 2.1 and 3.4 are different, in each case the result is in some sense optimal. For instance let $1<p\le 2$ and let $f_i\colon \partial B^3 \to \partial B^3$ be a sequence of conformal diffeomorphisms that converge almost everywhere to a constant map. Then $\int_M |Df_i|^p$ is uniformly bounded and the f_i, each of which is 2-homotopic to the identity map, converge weakly to a constant map, which is not 2-homotopic to the identity. This shows that d may not be replaced by $d+1$ in Theorem 2.1. Similarly, although the map $f\colon B^2 \to \partial B^2$ defined by $f(x)=x/|x|$ is in $L^{1,p}(B^2, \partial B^2)$ for $1 \le p < 2$, curves that are homotopic in B^2 can have images (under f) that are not homotopic in ∂B^2. This shows that d may not be replaced by $d+1$ in Theorem 3.4.

Note that these theorems imply that each f in $L^{1,p}(M,N)$ or in $H^{1,p}(M,N)$ induces homomorphisms of the k-dimensional homology groups (with any coefficients) for $0 \le k \le d$. Likewise for $0 \le k \le d$, f determines a conjugacy class of homomorphisms of the k-dimensional homotopy groups. These homomorphisms (or conjugacy classes of homomorphisms) are preserved by bounded weak convergence.

The regularity theory developed by R. Schoen and K. Uhlenbeck [SU1] for $p=2$ and extended by R. Hardt and F. H. Lin [HL] to general $p>1$ applies to the minimizers $f \in L^{1,p}(M,N)$ given by Theorem 3.4. In particular, such an f is Hölder-continuous outside of a closed singular set $Z \subset M$ with Hausdorff $(m-p)$-dimensional measure 0. (Indeed, for the particular functional $\int_M |Df|^p$, the Hausdorff dimension of Z is at most $m-[p]-1$.)

On the other hand, there are no known partial regularity theorems for the minimizers $f \in H^{1,p}(M,N)$ given by Theorem 2.1. (The proofs of the regularity theorems mentioned above do not readily generalize to $H^{1,p}(M,N)$ because they involve comparison maps that may not lie in $H^{1,p}(M,N)$.)

In case M is a manifold with nonempty boundary, one can also find nontrivial stationary maps by minimizing the functional subject to Dirichlet boundary conditions (indeed, this is the setting considered in [SU1] and [HL]). Thas is, if for a given lipschitz map $\varphi\colon \partial M \to N$ there exists any $f \in L^{1,p}(\partial M, N)$ with boundary values or trace (in the Sobolev space sense) φ, then there exists an $f \in L^{1,p}(M,N)$ that minimizes $\int_M |Df|^p$ among all such maps. That raises the question: for which $\varphi \in \mathrm{Lip}(\partial M, N)$ is there an $f \in L^{1,p}(M,N)$ with boundary trace φ? Our third main result (Theorem 4.1) gives the simple answer: φ is the trace of a map in $L^{1,p}(M,N)$ if and only if it can be extended to a continuous map from $\partial M \cup M^{[p]}$ to N.

The question of when a map $\varphi: \partial M \to N$ is the trace of some $f \in H^{1,p}(M,N)$ is a more difficult one and is not answered in this paper. Again, a regularity theory of such maps is lacking.

We remark that there is a third Sobolev space $W^{1,p}(M,N)$, which is defined to be the strong (i.e., $\|\cdot\|_{1,p}$) closure of $\mathrm{Lip}(M,N)$ in $L^{1,p}(M,N)$. This space lacks nice compactness properties and therefore is not suitable for finding minima of energy functionals. But it is well suited for questions about the infima of energy functionals in homotopy classes of smooth (or lipschitz maps). We have [W]:

THEOREM. *Let $d=[p]$ be the greatest integer less than or equal to p. Then each $f \in W^{1,p}(M,N)$ has a well-defined d-homotopy type, and these d-homotopy types are preserved by $\|\cdot\|_{1,p}$ convergence. Furthermore, the infimum of $\int_M |Df|^p$ among all lipschitz maps homotopic to a given map g depends only on the d-homotopy type of g. In particular, the infimum is zero if and only if g has the d-homotopy type of a constant map.*

Some cases of the results presented here were already known. In particular, R. Schoen and S. T. Yau [SY] proved that an $L^{1,p}$ map from a p-dimensional manifold to any manifold determines a conjugacy class of homomorphisms from $\pi_{p-1}(M)$ to $\pi_{p-1}(N)$, R. Schoen and K. Uhlenbeck [SU2] proved that an $L^{1,p}$ map induces homomorphisms $f^{\#}: H^k(N,R) \to H^k(M,R)$ on the real cohomology groups for $0 \leq k \leq [p-1]$, and F. Burstall [B] showed that $L^{1,2}$ maps determine conjugacy classes of homomorphisms of the fundamental groups.

The organization of this paper is as follows. Section 1 contains some basic definitions and lemmas. Sections 2 and 3 are about $H^{1,p}$ and $L^{1,p}$ maps, respectively, and are independent of each other. Section 4 is about the dirichlet problem for $L^{1,p}$ maps and depends on Section 3.

1. Preliminaries

Throughout this paper, M and N are compact riemannian manifolds. We will assume without loss of generality that M and N are submanifolds of euclidean spaces $\mathbf{R}^m$ and

$\mathbf{R}^n$, respectively. More generally, N need not be compact, or even be a manifold: the proofs only require that N be a closed subset of $\mathbf{R}^n$ and that there be a retraction of an ε-neighborhood of N onto N.

We will also assume that M has been triangulated.

If X is a polyhedral complex, we let X^k denote the k-dimensional skeleton of X. We say that a d-dimensional polyhedral complex X is a *regular polyhedral complex* if it is the union of its d-dimensional cells and if for every connected open set $U \subset X$, the set $U \setminus X^{(d-2)}$ is also connected. Note in particular that the d-skeleton of a manifold is regular.

If X is a polyhedral complex or an open subset of $\mathbf{R}^m$, we define the spaces $L^{1,p}(X,\mathbf{R}^n)$, $L^{1,p}(X,N)$, and $H^{1,p}(X,N)$ exactly as $L^{1,p}(M,\mathbf{R}^n)$, $L^{1,p}(M,N)$, and $H^{1,p}(M,N)$ were defined in the introduction. It is essential that elements of these spaces be thought of as maps rather than as equivalence classes of maps. That is, we do not identify maps that differ on a set of measure 0 (even though the $\|\cdot\|_{1,p}$ norm of their difference is 0). The reason is that we sometimes refer to the restriction of an $f \in L^{1,p}(M,\mathbf{R}^n)$ to a low dimensional subset of M; if f were merely an equivalence class of maps this would not be well defined.

If f is a continuous map, we let $[f]$ denote its homotopy class.

The following two basic theorems will be used repeatedly.

MORREY-TYPE THEOREM 1.1. *Let X be a regular d-dimensional polyhedral complex, $d<p$, and $0<\gamma<1-d/p$. For every $\varepsilon>0$, there is a $C(\varepsilon)<\infty$ such that if $f: X \to \mathbf{R}$ is lipschitz, then*

$$|f|_{0,\gamma} \leq \varepsilon\|Df\|_p + C(\varepsilon)\|f\|_p. \tag{1}$$

Consequently if $f \in L^{1,p}(X)$, then f is equal almost everywhere to a $C^{0,\gamma}$ function that satisfies (1).

Proof. First note that (1) holds for f that are compactly supported in $\Omega \subset \mathbf{R}^d$. For if not, then there exists a sequence $f_k \in C_0^\infty(\Omega)$ such that

$$\begin{aligned} 1 &= \|f_k\|_{0,\gamma} \\ &\geq \varepsilon\|Df_k\|_p + (k+\varepsilon)\|f_k\|_p \\ &= \varepsilon\|f_k\|_{1,p} + k\|f_k\|_p. \end{aligned} \tag{2}$$

Since $L_0^{1,p}(\Omega)$ embeds compactly in $C^{0,\gamma}(\bar{\Omega})$ (cf. [GT, Theorem 7.17]) and since (by (2))

$\|f_k\|_{1,p} \leq \varepsilon^{-1}$, it follows that there is a subsequence of f_k that converges in $C^{0,\gamma}$ to a limit f. But then letting $k \to \infty$ in (2) gives $|f|_{0,\gamma}=1$ and $\|f\|_p=0$, a contradiction.

Now consider the case where X is the unit d-dimensional cube $[0,1]^d$. By reflecting we can extend a lipschitz function f defined on X to be a lipschitz function F defined on $X'=[-1,2]^d$. Let $h: X' \to [0,1]$ be a C^∞ function on X' that is 1 on X and that vanishes on $\partial X'$. Then

$$\begin{aligned} |f|_{0,\gamma} &\leq |h \cdot F|_{0,\gamma} \\ &\leq \varepsilon \|D(h \cdot F)\|_p + C(\varepsilon) \|h \cdot F\|_p \end{aligned}$$

(because $h \cdot F$ is compactly supported)

$$\begin{aligned} &\leq \varepsilon \|DF \cdot h\|_p + \varepsilon \|F \cdot Dh\|_p + C(\varepsilon) \|h \cdot F\|_p \\ &\leq \varepsilon \|DF\|_p + (\varepsilon \sup |Dh| + C(\varepsilon)) \|F\|_p \\ &= 3^{d/p}(\varepsilon \|Df\|_p + (\varepsilon \sup |Dh| + C(\varepsilon)) \|f\|_p) \end{aligned}$$

as desired.

Now let X be any regular d-dimensional polyhedral complex. We may by subdividing ([W, p. 129]) assume that X consists of cubes $Q_1, Q_2, \ldots, Q_k$ rather than simplices. Since (as we have just seen) (1) holds on each cube,

$$|f(x)| + \frac{|f(x)-f(y)|}{\operatorname{dist}(x,y)^\gamma}$$

is bounded by the right hand side of (1) provided $x \neq y$ and x and y lie in the same cube Q_i. More generally this is true provided x and y belong to distinct cubes Q_i and Q_j that have a common $(d-1)$-dimensional face, since $Q_i \cup Q_j$ is then bilipschitz equivalent to a single cube. But if x and y are any two points in $X \setminus X^{d-2}$, then they are joined by a path in $X \setminus X^{d-2}$ whose length is $\leq 2 \operatorname{dist}(x,y)$ (because X is regular). The conclusion follows immediately. □

FUBINI-TYPE LEMMA 1.2. *Let h be a lipschitz map from a compact polyhedral complex X to an open subset U* of $\mathbf{R}^m$ *and let $\delta = \operatorname{dist}(h(X), \partial U)$. For every $k \geq 1$, there is a $c = c(\delta, k)$ such that if $F_1, \ldots, F_k \in L^1(U)$, then*

$$\int_{x \in X} |F_i(h(x)+v)|\, dx < \infty \quad (1 \leq i \leq k) \tag{1}$$

for almost all $v \in B^m(\delta)$, *and*

$$\int_{x\in X} |F_i(h(x)+v)|\,dx \leq c|X| \int_U |F_i| \quad (1 \leq i \leq k) \tag{2}$$

on a set of $v \in B^m(\delta)$ *of positive measure. (Here* $B^m(\delta)$ *is the ball of radius* δ *centered at the origin in* $\mathbf{R}^m$.*)*

Proof.

$$\begin{aligned}\int_{|v|<\delta}\int_{x\in X} |F_i(h(x)+v)|\,dx\,dv &= \int_{x\in X}\int_{|v|<\delta} |F_i(h(x)+v)|\,dv\,dx \\ &\leq \int_{x\in X}\int_{z\in U} |F_i(z)|\,dz\,dx \\ &= |X| \int_U |F_i|.\end{aligned}$$

This immediately gives (1), and implies that for each i

$$\mathcal{L}^m\left\{ v\in B^m(\delta)\colon \int_X |F_i\circ h_v| > c|X| \int_U |F_i| \right\} \leq c^{-1}$$

(where $h_v(x)=h(x)+v$) so

$$\mathcal{L}^m \bigcup_{i=1}^{k} \left\{ v\in B^m(\delta)\colon \int_X |F_i\circ h_v| > c|X| \int_U |F_i| \right\} \leq kc^{-1}.$$

Thus we may let $c=2k(\mathcal{L}^m(B^m(\delta)))^{-1}$, for example. □

2. $H^{1,p}$ maps

THEOREM 2.1. *Let* M *and* N *be compact riemannian manifolds,* $p \geq 1$, *and* d *be the greatest integer strictly less than* p. *For every* $K<\infty$, *there is an* $\varepsilon>0$ *such that if* $f_1, f_2\colon M\to N$ *are lipschitz maps with*

$$\|f_i\|_{1,p} \leq K \quad (i=1,2)$$

$$\|f_1-f_2\|_p < \varepsilon$$

then $f_1|M^d$ *and* $f_2|M^d$ *are homotopic.*

Consequently each $f\in H^{1,p}(M,N)$ *has a well-defined d-homotopy type, and d-*

homotopy types are preserved by bounded weak convergence. Thus for each continuous map g from M to N, there is a map that minimizes $\int_M |Df|^p$ *among all maps* $f \in H^{1,p}$ *having the same d-homotopy type as g.*

Proof. Let U be a small tubular neighborhood of M and $R: U \to M$ be the nearest point retraction. Define $F_i: U \to N$ by

$$F_i(x) = f_i(R(x)).$$

Then clearly

$$\begin{aligned} \|F_1 - F_2\|_p &\leq c_1 \|f_1 - f_2\|_p \leq c_1 \varepsilon \\ \|F_i\|_{1,p} &\leq c_1 \|f_i\|_{1,p} \leq c_1 K \end{aligned} \tag{1}$$

where c_1 depends only on $R: U \to M$.

Let $h: M^d \to U$ be the inclusion map, and for $v \in \mathbf{R}^m$ write $h_v(x) = h(x) + v$. Then by (1) and the Fubini lemma 1.2, there exists a v with $|v| < \operatorname{dist}(M^d, \partial U)$ such that

$$\|F_1 \circ h_v - F_2 \circ h_v\|_p \leq c_2 c_1 \varepsilon$$

$$\|(DF_i) \circ h_v\|_p \leq c_2 c_1 K$$

and thus

$$\|D(F_i \circ h_v)\|_p \leq c_3 K$$

where $c_3 = c_2 c_1 \operatorname{Lip}(h_v)$.

Let $1 < \gamma < 1 - d/p$. By the Morrey theorem 1.1, we have for each $\eta > 0$,

$$\begin{aligned} |F_1 \circ h_v - F_2 \circ h_v|_{0,\gamma} &\leq \eta \|D(F_1 \circ h_v) - D(F_2 \circ h_v)\|_p \\ &\quad + C(\eta) \|F_1 \circ h_v - F_2 \circ h_v\|_p \\ &\leq 2\eta c_3 K + C(\eta) c_2 c_1 \varepsilon. \end{aligned} \tag{2}$$

Let W be a small tubular neighborhood of $N \subset \mathbf{R}^n$ that retracts onto N, and let $\delta = \operatorname{dist}(N, \partial W)$. Now we can choose η small enough that $2\eta c_3 K < \delta/3$, and, having chosen η, we can choose ε small enough that $C(\eta) c_2 c_1 \varepsilon < \delta/3$. Then by (2)

$$|F_1 \circ h_v - F_2 \circ h_v|_{0,\gamma} \leq \tfrac{2}{3}\delta$$

and so for every $t \in [0, 1]$,

$$(1-t)(F_1 \circ h_v)(x)+t(F_2 \circ h_v)(x) \in W.$$

Thus $F_1 \circ h_v$ and $F_2 \circ h_v$ are homotopic in W and, since W retracts onto N, therefore homotopic in N. Finally, $F_i \circ h$ is homotopic to $F_i \circ h_v$ in N ($t \mapsto F_i \circ h_{tv}$ is a homotopy), so $F_1 \circ h$ is homotopic to $F_2 \circ h$ in N.

This proves the first assertion. Now given $f \in H^{1,p}$, we can find a sequence $f_i: M \to N$ of lipschitz maps such that $\|f_i - f\|_p \to 0$ and such that $\|f_i\|_{1,p}$ is uniformly bounded. It follows from the first assertion that for i sufficiently large, the f_i all have the same d-homotopy type. If we define this common d-homotopy type to be the d-homotopy type of f, then the remaining conclusions follow immediately. □

3. $L^{1,p}$ maps

LEMMA 3.1. *Let X be a regular polyhedral complex, U be an open subset of $\mathbf{R}^m$, $h: X \to U$ be a lipschitz map, $f \in L^{1,p}(U, \mathbf{R}^n)$, and g be a distribution derivative of f. Let $\delta = \operatorname{dist}(h(X), \partial U)$. Define $h_v: X \to U$* by

$$h_v(x) = h(x) + v.$$

Then for almost every $v \in B^m(\delta)$, $f \circ h_v \in L^{1,p}$ and $g(h_v) \cdot Dh_v$ is a distribution derivative of $f \circ h_v$.

Proof. Let $f_i: U \to \mathbf{R}^n$ be a sequence of smooth maps such that

$$\|f_i - f\|_{1,p} \le 2^{-i}.$$

Then

$$\int_U F(z)\, dz < \infty$$

where

$$F(z) = \sum_i (|f_i(z) - f(z)|^p + |Df_i(z) - g(z)|^p).$$

By the Fubini lemma 1.2, we have that for almost all $|v| < \delta$

$$\int_{x\in X} F(h_v(x))\,dx<\infty$$

i.e.,

$$\sum \int_X (|f_i\circ h_v - f\circ h_v|^p + |Df_i\circ h_v - g\circ h_v|^p)<\infty.$$

Thus since h is lipschitz

$$\sum \int_X (|f_i\circ h_v - f\circ h_v|^p + |D(f_i\circ h_v) - g(h_v)\cdot Dh_v|^p)<\infty$$

so

$$\|f_i\circ h_v - f\circ h_v\|_p + \|D(f_i\circ h_v) - g(h_v)\cdot Dh_v\|_p \to 0.$$

Hence $f\circ h_v\in L^{1,p}$ and has distribution derivative $g(h_v)\cdot Dh_v$. □

PROPOSITION 3.2. *Let U be an open subset of $\mathbf{R}^m$ and let $f\in L^{1,p}(U,N)$. Let X be a regular polyhedral complex of dimension $d\leq[p-1]$ that is contained in the d-skeleton of a regular $(d+1)$-dimensional polyhedral complex Y. Let $h\colon X\to U$ be a lipschitz map that extends to a lipschitz map of Y into U. Then there is a homotopy class $f_\#[h]$ of continuous maps from X to N with the following properties. For almost every v with $|v|<\delta=\operatorname{dist}(h(X),\partial U)$, there is a continuous map $g^v\colon X\to N$ such that*

(1) *$f\circ h_v(x)=g^v(x)$ for $\mathcal{H}^k$-almost every $x\in X^k$ $(0\leq k\leq d)$.*

(2) *g^v extends to a continuous map from Y to N.*

(3) *$g^v\in f_\#[h]$.*

Furthermore, if $\psi\in \operatorname{Lip}(X,U)$ is homotopic to h, then $f_\#[\psi]=f_\#[h]$.

Proof. Define $\tilde X\subset X\times[0,1]^d$ and $\tilde h\colon \tilde X\to U$ by

$$\tilde X=\bigcup_{k=0}^{d}(X^k\times[0]^k\times[0,1]^{d-k})$$

$$\tilde h((x,t))=h(x).$$

Then $\tilde X$ is a regular polyhedral complex. By Lemma 3.1, for almost every $v\in B^m(\delta)$, $f\circ\tilde h_v\in L^{1,p}(\tilde X,N)$. By the Morrey theorem 1.2, for each such v, there is a continuous map $g^v\colon\tilde X\to N$ such that $f\circ\tilde h_v(x,t)=g^v(x,t)$ for almost every $(x,t)\in\tilde X$, i.e.,

$$f(h_v(x))=g^v(x,t)\quad\text{for } \mathcal{H}^d\text{-almost every } (x,t)\in\tilde X. \tag{4}$$

For each x, $\{t\colon (x,t)\in \tilde{X}\}$ is connected, so (4) implies that $g^v(x,t)$ is a function of x alone. Thus we may write

$$f\circ h_v(x) = g^v(x) \tag{5}$$

for $\mathscr{H}^d$-almost all $(x,t)\in\tilde{X}$. In particular, (5) holds for $\mathscr{H}^d$-almost every $(x,t)\in X^k\times[0]^k\times[0,1]^{d-k}$ and therefore $\mathscr{H}^k$-almost every $x\in X^k$. This proves (1).

To prove (2), let $\psi\colon Y\to U$ be a lipschitz map that extends h. Let $\hat{Y}=(X\times[0,1])\cup(Y\times[1])$ and define $\hat{\psi}\colon\hat{Y}\to U$ so that $\hat{\psi}(y,t)=\psi(y)$. Then for almost all v, $f\circ\hat{\psi}_v\in L^{1,p}(\hat{Y},N)$. Let $D(f\circ\hat{\psi}_v)$ be a distribution derivative. As in the proof of (1), $f\circ\hat{\psi}_v$ is essentially continuous on $X\times[0,1]$.

Let $\varepsilon>0$. For $r\in(0,1]$, let $H(y,r)$ be the point in $\mathbf{R}^n$ that minimizes

$$\int_{z\in\hat{Y},\,\mathrm{dist}(z,(y,r))<\varepsilon r}|f\circ\hat{\psi}_v(z)-H(y,r)|^p\,dz.$$

(If $p>1$, this point is unique since the L^p norm is strictly convex. For $p=1$, let $H(y,r)$ be the average of $f\circ\hat{\psi}_v$ over the set of $z\in\hat{Y}$ such that $\mathrm{dist}(z,(y,r))<\varepsilon r$.)

It is easy to see that H is continuous for $r>0$. And because $f\circ\hat{\psi}_v$ is essentially continuous on $X\times[0,1]$, we may extend H continuously to all of $\hat{Y}$ so that $H(x,0)=f(\hat{\psi}_v(x))$ for almost every $x\in X$. Now

$$\begin{aligned}\mathrm{dist}(H(x,r),N)^p &\leqslant C(\varepsilon r)^{-(d+1)}\int_{z\in\hat{Y},\,\mathrm{dist}(z,(y,r))<\varepsilon r}|f(\hat{\psi}_v(z))-H(y,r)|^p\,dz\\ &\leqslant C'\int_{z\in\hat{Y},\,\mathrm{dist}(z,(y,r))<3\varepsilon}|D(f\circ\hat{\psi}_v)(z)|^p\,dz\end{aligned} \tag{6}$$

by the Poincare inequality [W, 2]. (The proof in [W] for $p=d+1$ easily generalizes to $p\geqslant d+1$.)

By choosing ε small enough, we may can make (6) as small as we like. In particular, we can choose $\varepsilon>0$ small enough that the image of H lies in a tubular neighborhood W of N such that there exists a retraction $R\colon W\to N$. Then $R\circ H(\cdot,0)=g^v(\cdot)$, which is homotopic to $R\circ H(\cdot,1)|X$, which extends to $R\circ H(\cdot,1)\colon Y\to N$. This proves (2).

To prove (3), fix a small vector $u\in\mathbf{R}^m$. Consider the map $h\colon(X\times[0])\cup(X\times[1])\to U$ defined by $\bar{h}(x,t)=h(x)+tu$ and note that $\bar{h}$ extends to a lipschitz map of $Y=X\times[0,1]$ into U. Then by (1) and (2), for almost all (small) v, $f\circ\bar{h}_v$ is (essentially) continuous and extends to a continuous map of $X\times[0,1]$ into N. But that means $f\circ h_v$ and $f\circ h_{v+u}$ are essentially continuous and homotopic in N. Thus for almost every (small) u and v, g_u and g_{u+v} are homotopic. Now let $f_\#[h]$ be the common homotopy class. Then (3) is immediate.

To prove the last statement, let

$$\bar{h}: X \times \{0,1\} \to U$$
$$\bar{h}(x,0) = h(x)$$
$$\bar{h}(x,1) = \psi(x).$$

Then exactly as in the proof of (3), $f \circ h_v$ and $f \circ \psi_v$ are essentially continuous and homotopic in N for almost all small v. Thus $f_\#[h] = f_\#[\psi]$. □

PROPOSITION 3.3. *Let X be a regular polyhedral complex of dimension $d=[p-1]$, U be an open subset of $\mathbf{R}^m$, and $h \in \mathrm{Lip}(X, U)$. For every $K<\infty$, there is an $\varepsilon>0$ such that if*

$$f_1, f_2 \in L^{1,p}(U, N)$$
$$\|f_i\|_{1,p} < K \quad (i=1,2)$$
$$\|f_1 - f_2\|_p < \varepsilon$$

then $(f_1)_\#[h] = (f_2)_\#[h]$.

Proof. Let $\delta = \mathrm{dist}(h(X), \partial U)$. By the Fubini lemma 1.2, there is a set of $v \in B^m(\delta)$ of positive measure such that

$$\int_X |f_1 \circ h_v - f_2 \circ h_v|^p < (c_1 \varepsilon)^p \tag{1}$$

$$\int_X |(Df_i) \circ h_v|^p < (c_1 K)^p \quad (i=1,2) \tag{2}$$

where c_1 depends only on X and h.

By Lemma 3.1 and Proposition 3.2, for almost every $v \in B^m(\delta)$, $f_i \circ h_v$ is essentially continuous, has distribution derivative $Df_i(h_v) \cdot Dh_v$, and

$$f_i \circ h_v \in (f_i)_\#[h]. \tag{3}$$

Let $\eta>0$. By the Morrey theorem 2.1 there is a $C(\eta)$ such that

$$\begin{aligned}|f_1\circ h_v-f_2\circ h_v| &\leq \eta\|D(f_1\circ h_v)-D(f_2\circ h_v)\|_p+C(\eta)\,\|f_1\circ h_v-f_2\circ h_v\|_p\\ &\leq 2\eta\,(\mathrm{Lip}\,(h))\,c_1\,K+C(\eta)\,c_1\,\varepsilon\end{aligned}\tag{4}$$

(where $\mathrm{Lip}\,(h)$ is the lipschitz constant of h). Let $W\subset\mathbf{R}^n$ be a neighborhood of N that retracts onto N. Choose $\eta>0$ so that

$$2\eta\,(\mathrm{Lip}\,(h))\,c_1\,K<\mathrm{dist}\,(N,\partial W)/3$$

and then choose $\varepsilon>0$ so that

$$C(\eta)\,c_1\,\varepsilon<\mathrm{dist}\,(N,\partial W)/3.$$

It then follows from (4) that

$$tf_1(h_v(x))+(1-t)\,f_2(h_v(x))\in W$$

for $0\leq t\leq 1$ and $x\in X$. Thus $f_1\circ h_v$ and $f_2\circ h_v$ are homotopic in W and, since W retracts onto N, therefore homotopic in N. This with (3) implies that $(f_1)_\#\,[h]=(f_2)_\#\,[h]$. □

THEOREM 3.4. *Let $d=[p-1]$ be the greatest integer less than or equal to $p-1$. Then each $f\in L^{1,p}(M,N)$ has a d-homotopy type $f_\#[M^d]$. This d-homotopy type is a homotopy class of continuous mappings from M^d into N such that:*

(1) *If $f_i\in L^{1,p}(M,N)$, $\|f_i-f\|_p\to 0$, and $\|Df_i\|_p$ is uniformly bounded, then*

$$(f_i)_\#\,[M^d]=f_\#[M^d]$$

for sufficiently large i.

(2) *If $f\in L^{1,p}(M,N)$ is continuous at each $x\in M^d$, then*

$$f_\#[M^d]=[f|M^d].$$

(3) $\{f_\#[M^d]\colon f\in L^{1,p}(M,N)\}=\{[\varphi|M^d]\colon \varphi\in C^0(M^{d+1},N)\}.$

Proof. Let U be a small tubular neighborhood of $M\subset\mathbf{R}^m$ and $R\colon U\to M$ be the nearest point retraction. Note that if f is an $L^{1,p}$ function or map defined on M then $f\circ R$ is an $L^{1,p}$ function or map defined on U, and

$$\|f\circ R\|_p\leq c\|f\|_p$$
$$\|D(f\circ R)\|_p\leq c\|Df\|_p$$

(for some constant c). Now (using the notation of Proposition 3.2), we define $f_\#[M^d]$ to be $(f\circ R)_\#[\iota]$, where $\iota: M^d \to U$ is the inclusion map. Then (1) and (2) are immediate consequences of Propositions 3.2 and 3.3.

To prove (3), recall that $f_\#[M^d]=(f\circ R)_\#[\iota]$ is the homotopy class of a certain lipschitz map

$$g^v = (f\circ R)\circ \iota_v : M^d \to N$$

which (by conclusion (2) of Proposition 3.2), extends to a continuous map from M^{d+1} into N. Thus

$$\{f_\#[M^d]: f\in L^{1,p}(M,N)\} \subset \{[\varphi|M^d]: \varphi\in C^0(M^{d+1},N)\}.$$

To prove the reverse inclusion, recall that since $d+1\leq p$, there is a map $F\in L^{1,p}(M,M)$ that (continuously) retracts $M\setminus Y$ onto M^{d+1}, where $Y\subset M$ is an $(m-d-2)$-dimensional set disjoint from M^{d+1}. (See [W, p. 129], where F is written $F_{0,1}$.) Thus if $\varphi\in \mathrm{Lip}(M^{d+1},N)$, then $\varphi\circ F\in L^{1,p}(M,N)$. By (2),

$$(\varphi\circ F)_\#[M^d] = [\varphi|M^d].$$

Thus

$$\{[\varphi|M^d]: \varphi\in \mathrm{Lip}(M^{d+1},N)\} \subset \{f_\#[M^d]: f\in L^{1,p}(M,N)\}$$

and therefore (since continuous maps can be uniformly approximated by lipschitz maps)

$$\{[\varphi|M^d]: \varphi\in C^0(M^{d+1},N)\} \subset \{f_\#[M^d]: f\in L^{1,p}(M,N)\}. \qquad \square$$

The reader may wonder if it is possible for two maps in $L^{1,p}(M,N)$ to have the same $[p-1]$-homotopy type with respect to one triangulation of M but not with respect to another. More generally, one can ask if it is possible to have two maps $f_1,f_2\in L^{1,p}(M,N)$ with the same $[p-1]$-homotopy type (with respect to a triangulation of M) such that $(f_1)_\#[\psi]\neq(f_2)_\#[\psi]$ for some lipschitz map ψ of a polyhedral complex of dimension $\leq[p-1]$ into M. (Here $(f_i)_\#[\psi]$ is defined to be $(f_i\circ R)_\#[\psi]$ with R as in the proof of Theorem 3.4.) In fact it is not possible:

PROPOSITION 3.5. *Let $f_1,f_2\in L^{1,p}(M,N)$, X be a regular polyhedral complex of dimension $\leq[p-1]$, and ψ be a lipschitz map from X to M. If $(f_1)_\#[M^{[p-1]}]=(f_2)_\#[M^{[p-1]}]$, then $(f_1)_\#[\psi]=(f_2)_\#[\psi]$.*

Proof. By subdividing X, we may assume that ψ is homotopic to a map u such that $u(X^k)\subset M^k$ for $0\leq k\leq [p-1]$ and such that u is affine on each simplex of X.

Recall that for almost all $v\in\mathbf{R}^m$ with $|v|$ sufficiently small, there exist maps $g_i^v\in \mathrm{Lip}(M^{[p-1]}, N)$ such that for $0\leq k\leq [p-1]$

$$f_i(x+v)=g_i^v(x) \quad \text{for } \mathcal{H}^k\text{-almost every } x\in M^k \tag{1}$$

and such that

$$(f_i)_\#[M^{[p-1]}]=[g_i^v]. \tag{2}$$

By (1),

$$f_i(u(x)+v)=g_i^v(u(x)) \quad \text{for } \mathcal{H}^k\text{-almost every } x\in X^k.$$

Thus for almost every v with $|v|$ sufficiently small,

$$g_i^v\circ u\in (f_i)_\#[u]$$

and since u and ψ are homotopic:

$$g_i^v\circ u\in (f_i)_\#[\psi] \tag{3}$$

(by Proposition 3.2). Since $(f_1)_\#[M^{[p-1]}]=(f_2)_\#[M^{[p-1]}]$, it follows from (2) that g_1^v and g_2^v are homotopic. Thus $g_1^v\circ u$ and $g_2^v\circ u$ are homotopic. But then by (3), $(f_1)_\#[\psi]=(f_2)_\#[\psi]$. □

4. The Dirichlet problem for $L^{1,p}$ maps

THEOREM 4.1. *Let ψ be a lipschitz map from ∂M to N. Then there exists a map $f\in L^{1,p}(M,N)$ with boundary trace ψ if and only if ψ can be extended to a continuous map from $\partial M\cup M^{[p]}$ into N.*

Proof. Suppose first that $f\in L^{1,p}(M,N)$ has boundary trace $\psi\in\mathrm{Lip}(\partial M,N)$. Let $V\subset M$ be a small neighborhood of ∂M. Let $u: M\to M$ be a lipschitz map that retracts V onto ∂M and that maps $M\setminus V$ diffeomorphically onto M. Define $f': M\to N$ by

$$f'(x)=f(u(x)) \quad (x\in M\setminus V)$$
$$f'(x)=\psi(u(x)) \quad (x\in V).$$

Then $f' \in L^{1,p}(M,N)$ (because f has boundary trace ψ; indeed we could let this define what it means for $f \in L^{1,p}(M,N)$ to have boundary trace ψ).

Let $U \subset \mathbf{R}^m$ be a neighborhood of M and $R: U \to M$ be the nearest point retraction. Let $h: M^{[p-1]} \cup \partial M \to U$ be the inclusion map. Then exactly as in the proofs of Proposition 3.2 and Theorem 3.4, there exists a $v \in \mathbf{R}^m$ (with $|v|$ arbitrarily small) such that $(f' \circ R \circ h_v)|M^{[p-1]}$ is continuous and extends to a continuous map of $M^{[p]}$ into N. Moreover, since f' is continuous on V, $f' \circ R$ is continuous on a neighborhood of $\partial M \subset U$, and therefore $f' \circ R \circ h_v$ is continuous on ∂M.

We have shown that $(f' \circ R \circ h_v)|\partial M$ extends to a continuous map of $\partial M \cup M^{[p]}$ into N. Since $\psi = (f' \circ R \circ h_0)|\partial M$ is homotopic to $(f' \circ R \circ h_v)|\partial M$ (by the homotopy $t \mapsto (f' \circ R \circ h_{tv})|\partial M$), it follows that ψ extends to a continuous map of $\partial M \cup M^{[p]}$ into N.

Conversely, suppose $\psi \in \operatorname{Lip}(\partial M, N)$ extends to a continuous map Ψ from $\partial M \cup M^{[p]}$ into N. We may assume without loss of generality that Ψ is in fact lipschitz. We recall that there is a map $F \in L^{1,p}(M,M)$ that retracts M onto $\partial M \cup M^{[p]}$. (This F, which is discontinuous on a closed $(m-[p]-1)$-dimensional set disjoint from $\partial M \cup M^{[p]}$, is the $F_{0,1}$ of [W, p. 130], modified according to [W, Section 4].) Then since Ψ is lipschitz, $\Psi \circ F \in L^{1,p}(M,N)$, and we are done. □

One can also generalize Theorem 3.4 to manifolds M with boundary. Recall that two continuous maps $f, g: A \to N$ are said to be *homotopic relative to B* if there is a homotopy $H: [0,1] \times A \to N$ from f to g such that $H(\cdot, x) = f(x) = g(x)$ for all $x \in A \cap B$. The corresponding equivalence class of a continuous map f is called its *homotopy class* (rel$B$) and is noted by $[f(\mathrm{rel}\, B)]$.

THEOREM 4.2. *Let d be the greatest integer less than or equal to $p-1$, and suppose $\psi \in \operatorname{Lip}(\partial M, N)$. Then each $f \in L^{1,p}(M,N)$ with boundary trace ψ has a d-homotopy type $f_\#[M^d(\mathrm{rel}\,\partial M)]$. This d-homotopy type is a homotopy class* (rel ∂M) *of continuous mappings from M^d into N such that:*

(1) *If $f_i \in L^{1,p}(M,N)$ has boundary trace ψ, $\|f_i - f\|_p \to 0$, and $\|Df_i\|_p$ is uniformly bounded, then*

$$(f_i)_\#[M^d(\mathrm{rel}\,\partial M)] = f_\#[M^d(\mathrm{rel}\,\partial M)]$$

for sufficiently large i.

(2) *If $f \in L^{1,p}(M,N)$ has boundary trace ψ and is continuous at each $x \in M^d$, then*

$$f_\#[M^d(\mathrm{rel}\,\partial M)] = [(f|M^d)(\mathrm{rel}\,\partial M)].$$

(3) *The set*

$$\{f_{\#}[M^d\,(\text{rel}\,\partial M)]\colon f\in L^{1,p}(M,N) \textit{ has boundary trace } \psi\}$$

is equal to

$$\{[(\varphi|M^d)\,(\text{rel}\,\partial M)]\colon \varphi\in C^0(M^{d+1},N),\, \varphi(x)=\psi(x) \textit{ for } x\in M^d\cap\partial M\}$$

The proof is a fairly straightforward generalization of the proof of Theorem 3.4, using the map $u\colon M\to M$ as in the proof of Theorem 4.1 above.

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