Cyclic cohomology for one-parameter smooth crossed products

by

G. A. ELLIOTT — *University of Copenhagen, Copenhagen, Denmark*

T. NATSUME — *University of Copenhagen, Copenhagen, Denmark* and *Saitama University, Shimo-okubo, Urawa, Japan*

and

R. NEST — *University of Copenhagen, Copenhagen, Denmark*

Introduction

In [1] A. Connes proved that, for an arbitrary C^*-dynamical system $(A, \alpha, \mathbf{R})$, there is a natural isomorphism

$$\phi_\alpha\colon K_i(A) \to K_{i+1}(A \times_\alpha \mathbf{R}), \quad i \in \mathbf{Z}/2\mathbf{Z}.$$

He showed also that, given an α-invariant trace τ on A, with dual trace $\hat{\tau}$ on $A \times_\alpha \mathbf{R}$, the equality

$$\hat{\tau}(\phi_\alpha([u])) = \frac{1}{2\pi i}\tau(u^*\delta(u)) \qquad (*)$$

holds for any unitary u in the domain of the infinitesimal generator δ of α.

In the terminology of [2], the right hand side of the above equality is just the pairing of a unitary and a cyclic one-cocycle. Of course, $\hat{\tau}$ is a zero-cocycle. Therefore the above equality reveals a certain relation between cyclic cocyles on an algebra and those on its crossed product.

The purpose of the present paper is to construct a machine which makes precise the relation between the cyclic theory of an algebra and that of a one-parameter crossed product.

Given a Fréchet algebra $\mathscr{A}$ and a one-parameter group α of automorphisms of $\mathscr{A}$ satisfying certain smoothness conditions (see Section 2.1), a Fréchet algebra that we shall call the smooth crossed product $\mathscr{A} \times_\alpha \mathbf{R}$ can be defined. Our main result is as follows.

THEOREM 1. *There exists a natural map*

$$\#_\alpha\colon H_\lambda^*(\mathscr{A}) \to H_\lambda^{*+1}(\mathscr{A}\times_\alpha \mathbf{R})$$

which commutes with the operator S and defines isomorphisms

$$H^{\mathrm{ev}}(\mathscr{A}) \simeq H^{\mathrm{odd}}(\mathscr{A}\times_\alpha \mathbf{R}),$$

$$H^{\mathrm{odd}}(\mathscr{A}) \simeq H^{\mathrm{ev}}(\mathscr{A}\times_\alpha \mathbf{R}).$$

In the course of the proof we prove the following result.

THEOREM 2 (Stability). *Let $\mathscr{K}^\infty$ denote the algebra of smooth compact operators (defined in Section* 2.6). *Then there exists an isomorphism*

$$H^*(\mathscr{A}) \simeq H^*(\mathscr{A}\otimes \mathscr{K}^\infty).$$

As a corollary to Theorem 1 we also have

THEOREM 3 (Bott periodicity). *There exists an isomorphism*

$$H^*(\mathscr{A}\otimes \mathscr{S}(\mathbf{R})) \simeq H^{*+1}(\mathscr{A}),$$

and hence an isomorphism

$$H^*(\mathscr{A}\otimes \mathscr{S}(\mathbf{R}^2)) \simeq H^*(\mathscr{A}).$$

The contents of this paper are as follows. In the second section we construct the smooth crossed product $\mathscr{A}\times_\alpha\mathbf{R}$ of $\mathscr{A}$ by a smooth action α of $\mathbf{R}$. We also define the algebra $\mathscr{K}^\infty$, and prove a smooth version of the Takesaki-Takai duality theorem. In the third section we construct the map $\#_\alpha$ and derive its basic properties. In the fourth section we prove the stability theorem. This is used in the proof of the main theorem, which is given in Section 5. The sixth section is devoted to the comparison of the map $\#_\alpha$ with Connes's map ϕ_α in K-theory. We obtain the following generalisation of $(*)$:

THEOREM 4. *The equality $\langle \varphi, x\rangle = \langle \#_\alpha\varphi, \phi_\alpha(x)\rangle$ holds for any cyclic cocycle φ on $\mathscr{A}$ and K-class x of $\mathscr{A}$.*

Finally, the last section is devoted to a variant of the main theorem in the case that the action α is not smooth in the sense of Section 2.1.

All the cochains considered in this paper will be understood (proved if necessary) to be continuous. By the tensor product of locally convex spaces (that are not C^*-algebras) we will mean the complete projective tensor product.

2. Smooth one-parameter crossed products

2.1. Let $\mathcal{A}$ be a Fréchet algebra with topology given by an increasing sequence of seminorms $\|\cdot\|_n$, $n\in\mathbf{N}$.

Definition. A homomorphism

$$\alpha\colon \mathbf{R}\to \operatorname{Aut}\mathcal{A}$$

is called a *smooth action* if the following two conditions are satisfied.

(1) For each $a\in\mathcal{A}$ the function

$$t\mapsto \alpha_t(a)$$

is strongly infinitely differentiable.

(2) For arbitrary $m, k\in\mathbf{N}$ there exist $n, j\in\mathbf{N}$ and a positive constant C such that, for all $a\in\mathcal{A}$,

$$\left\|\frac{d^k}{dt^k}\alpha_t(a)\right\|_m \leqslant C(1+t^2)^{j/2}\|a\|_n.$$

2.2. A typical example of a smooth action is given by a smooth flow on a closed C^∞-manifold. An especially pertinent example is translation on $\mathcal{S}(\mathbf{R})$.

2.3. *Notation.* $\mathcal{S}(\mathbf{R})$ denotes the Fréchet algebra of rapidly decreasing functions with pointwise multiplication. $\mathcal{S}^*(\mathbf{R})$ denotes the Fréchet algebra of rapidly decreasing smooth functions with convolution.

2.4. *Remark.* Since $\mathcal{S}(\mathbf{R})$ is nuclear, the tensor product $\mathcal{S}(\mathbf{R})\otimes\mathcal{A}$ can be considered as a function space, $\mathcal{S}(\mathbf{R},\mathcal{A})$, for any complete Hausdorff locally convex space $\mathcal{A}$. If, in particular, $\mathcal{A}$ is as in Section 2.1, the topology of $\mathcal{S}(\mathbf{R})\otimes\mathcal{A}$ is given by the seminorms

$$\|f\|_{k,m} = \sup_{t\in\mathbf{R}}(1+t^2)^{k/2}\left\|\frac{d^m}{dt^m}f(t)\right\|_k.$$

2.5. Suppose that α is a smooth action of $\mathbf{R}$ on $\mathcal{A}$. Then it is easy to see that the formula

$$(f\ast g)(t) = \int_{\mathbf{R}} f(s)\,\alpha_s(g(t-s))\,ds$$

defines a jointly continuous product on $\mathcal{S}(\mathbf{R},\mathcal{A})$.

Definition. $\mathscr{A}\times_\alpha\mathbf{R}$ denotes the Fréchet algebra $\mathscr{S}(\mathbf{R},\mathscr{A})$ with the product defined above and is called the *smooth one-parameter crossed product* of $\mathscr{A}$ by α.

2.6. *Example.* Let us consider the action γ of $\mathbf{R}$ on $\mathscr{S}(\mathbf{R})$ by translations. This extends to an action of $\mathbf{R}$ on the C^*-algebra $C_0(\mathbf{R})$, and it is easily seen that $\mathscr{S}(\mathbf{R})\times_\gamma\mathbf{R}$ is embedded as a dense subalgebra of the C^*-algebra crossed product $C_0(\mathbf{R})\times_\gamma\mathbf{R}$. Via the canonical isomorphism of $C_0(\mathbf{R})\times_\gamma\mathbf{R}$ onto the algebra $K(L^2(\mathbf{R}))$ of all compact operators on $L^2(\mathbf{R})$, the subalgebra $\mathscr{S}(\mathbf{R})\times_\gamma\mathbf{R}$ is identified with the subalgebra $\mathscr{K}^\infty$ of those Hilbert-Schmidt operators whose integral kernels belong to $\mathscr{S}(\mathbf{R}^2)$. Furthermore, this subalgebra consists of trace class operators ([3], Proof of Theorem 3).

2.7. Let $(\mathscr{A},\alpha)$ be as in Section 2.1. The dual action $\hat{\alpha}$ of $\mathbf{R}$ on $\mathscr{A}\times_\alpha\mathbf{R}$ is given by

$$\hat{\alpha}_\omega(f)(t)=e^{2\pi i\omega t}f(t).$$

It is easy to see that $\hat{\alpha}$ is a smooth action, and so we may form the iterated smooth crossed product

$$\mathscr{A}\times_\alpha\mathbf{R}\times_{\hat{\alpha}}\mathbf{R}.$$

The following lemma will play a crucial role in the proof of our main result.

2.8. LEMMA (Takesaki-Takai duality). *The two Fréchet algebras*

$$\mathscr{A}\times_\alpha\mathbf{R}\times_{\hat{\alpha}}\mathbf{R}\quad \text{and}\quad \mathscr{A}\otimes\mathscr{K}^\infty$$

are isomorphic.

Proof. The proof follows the usual proof in the C^*-algebra case, with the simplifications due to the fact that we are dealing with function spaces instead of operator algebras. For later use we shall give some of the details.

Let γ be the action of $\mathbf{R}$ on $\mathscr{A}\otimes\mathscr{S}(\mathbf{R})$ given by

$$(\gamma_t f)(s)=\alpha_s(f(s-t)),$$

and let β be the action of $\mathbf{R}$ on $\mathscr{A}\times_\iota\mathbf{R}$, where $\iota_t=\mathrm{Id}$, $t\in\mathbf{R}$, given by

$$(\beta_t f)(\tau)=e^{-2\pi i t\tau}\alpha_t(f(\tau)).$$

We define maps

$$\pi\colon \mathcal{A}\otimes\mathcal{K}^\infty \to (\mathcal{A}\otimes\mathcal{S}(\mathbf{R}))\times_\gamma \mathbf{R},$$
$$\varrho\colon (\mathcal{A}\otimes\mathcal{S}(\mathbf{R}))\times_\gamma \mathbf{R}\to (\mathcal{A}\times_\iota \mathbf{R})\times_\beta \mathbf{R},$$
$$\psi\colon (\mathcal{A}\times_\iota \mathbf{R})\times_\beta \mathbf{R}\to \mathcal{A}\times_\alpha \mathbf{R}\times_{\hat{\alpha}} \mathbf{R}$$

as follows:

$$\pi(f)(s,t) = \alpha_s(f(s,s-t)),$$
$$\varrho(f)(\tau,t) = \int_{\mathbf{R}} f(s,t)\exp(-2\pi i s\tau)\,ds,$$
$$\psi(f)(t,\tau) = e^{2\pi i t\tau} f(\tau,t).$$

Since we are dealing with function spaces, it is straightforward to see that π, ϱ and ψ are isomorphisms of topological algebras, and hence the composed map

$$T = \psi\varrho\pi$$

gives the required isomorphism.

2.9. *Notation.* We shall denote by T_α the isomorphism

$$T_\alpha\colon \mathcal{A}\otimes\mathcal{K}^\infty \to \mathcal{A}\times_\alpha \mathbf{R}\times_{\hat{\alpha}} \mathbf{R}$$

constructed in the proof of Lemma 2.8.

3. Construction of the map $\#_\alpha$

3.1. Let $\mathcal{A}$ be a locally convex topological algebra. The construction of the universal differential graded algebra $(\Omega(\mathcal{A}), d)$ given in Section II.1 of [2] extends to the topological case if we set

$$\Omega(\mathcal{A}) = \bigoplus_{n\geq 0} \Omega_n(\mathcal{A}),$$
$$\Omega_n(\mathcal{A}) = \mathcal{A}^{\sim}\otimes\otimes^n\mathcal{A}, \quad n>0,$$
$$\Omega_0(\mathcal{A}) = \mathcal{A},$$

where $\mathcal{A}^{\sim}$ denotes the algebra $\mathcal{A}$ with unit adjoined, and the graded multiplication and differential are extended by continuity to the projective completions of the algebraic tensor products. Furthermore, any automorphism α of $\mathcal{A}$ has a natural extension to an automorphism of $\Omega(\mathscr{A})$ commuting with d. In the particular case when $\mathscr{A}$ is a Fréchet algebra and α acts smoothly, it is immediate to see that for each $n \geq 0$, the extension acts smoothly on $\Omega_n(\mathscr{A})$.

3.2. Set

$$E = \Omega(\mathscr{S}^*(\mathbf{R}))/\bigoplus_{n\geq 2} \Omega_n(\mathscr{S}^*(\mathbf{R})).$$

E carries a differential graded algebra structure induced by the quotient map

$$\Omega(\mathscr{S}^*(\mathbf{R})) \to E.$$

In what follows we shall assume that $\mathscr{A}$ is a Fréchet algebra and that α is a smooth action of $\mathbf{R}$ on $\mathscr{A}$. We will endow the space

$$\Omega(\mathscr{A}) \otimes E$$

with a structure of locally convex differential graded algebra as follows.

(1) Define

$$d\colon \Omega(\mathscr{A}) \otimes E \to \Omega(\mathscr{A}) \otimes E$$

by

$$d(\omega \otimes x) = d\omega \otimes x + (-1)^{\deg \omega} \omega \otimes dx.$$

(2) Define a left E_0-module structure on $\Omega(\mathscr{A}) \otimes E_0$ as the one induced from the product structure of

$$\Omega(\mathscr{A})^\sim \times_\alpha \mathbf{R}$$

and the inclusion of $1 \otimes E_0$ into the algebra $\Omega(\mathscr{A})^\sim \times_\alpha \mathbf{R}$.

(3) Define a left E-module structure on $\Omega(\mathscr{A}) \otimes E$ by the formulas

$$f(\omega \otimes g\,dh) = (f\omega \otimes g)\,dh,$$
$$df(\omega \otimes g) = d(f\omega \otimes g)) - fd(\omega \otimes g),$$
$$df(\omega \otimes g\,dh) = 0.$$

(4) Define the product in $\Omega(\mathscr{A}) \otimes E$ by

$$(\omega \otimes x)(\omega^1 \otimes x^1) = \omega(x(w^1 \otimes x^1)).$$

It is a straightforward computation to check that the above formulas extend by continuity to all of $\Omega(\mathscr{A})\otimes E$ and together with the differential d define the required structure.

Definition. $\Omega(\mathscr{A})\otimes_\alpha E$ denotes the differential graded algebra constructed above.

3.3. Suppose that φ is a closed graded trace of degree n on $\Omega(\mathscr{A})$. Set, for $f\in\Omega(\mathscr{A})\otimes E$,

$$\#_\alpha\varphi(f)=\begin{cases}2\pi i\int_{-\infty}^{\infty}dt\int_0^t d\lambda\,\varphi(\alpha_\lambda(f(-t,t))) & \text{if } f\in\Omega_n\otimes\mathscr{S}^*(\mathbf{R})\otimes\mathscr{S}^*(\mathbf{R})\subset\Omega_n\otimes E_1\\ 0 & \text{otherwise.}\end{cases}$$

LEMMA. *$\#_\alpha\varphi$ is a continuous closed graded trace of degree $n+1$ on $\Omega(\mathscr{A})\otimes_\alpha E$.*

Proof. This is seen by a routine computation.

Using Proposition II.1 of [2], we may reformulate this lemma as stating the existence of a linear map

$$\#_\alpha\colon Z_\lambda^n(\mathscr{A})\to Z_\lambda^{n+1}(\mathscr{A}\times_\alpha\mathbf{R}).$$

3.4. *Remark.* Note that the map $\#_\alpha$ is natural with respect to smooth actions, i.e., given two smooth actions $(\mathscr{A},\alpha)$ and $(\mathscr{B},\beta)$ and an equivariant homomorphism $\varrho\colon\mathscr{A}\to\mathscr{B}$, then

$$\hat{\varrho}^*\#_\beta=\#_\alpha\varrho^*,$$

where $\hat{\varrho}$ is the induced homomorphism of crossed products.

3.5. LEMMA. $\#_\alpha(B_\lambda^n(\mathscr{A}))\subseteq B_\lambda^{n+1}(\mathscr{A}\times_\alpha\mathbf{R})$.

Proof. Let C denote the Banach subalgebra of $B(l^2(\mathrm{N}))$ generated by all infinite matrices (a_{ij}), $i,j\in\mathbf{N}$, $a_{ij}\in\mathbf{C}$ such that

(I) the set of complex numbers $\{a_{ij}\}$ is finite,

(II) the number of non zero a_{ij}'s per row or column is bounded.

Let $\psi\in B_\lambda^n(\mathscr{A})$. We can extend ψ to an element $\tilde{\psi}\in B_\lambda^n((\mathscr{A}^\sim)$. Using the argument of Proposition II.8 of [2] we get a cyclic n-cocycle T on $\mathscr{A}^\sim\otimes C$. Following the construction of Section 3.2 applied to $\mathscr{A}^\sim\otimes C$ and the action $\alpha\otimes\iota$, we get a differential graded algebra

$$\Omega(\mathscr{A}^\sim\otimes C)\otimes_{\alpha\otimes\iota}E^\sim$$

and a closed graded trace $\#_{\alpha\otimes\iota} T$ of degree $n+1$. Since we have a natural homomorphism

$$(\mathscr{A}\times_\alpha \mathbf{R})^\sim \otimes C \to (\Omega(\mathscr{A}^\sim \otimes C) \otimes_{\alpha\otimes\iota} E^\sim)^0$$

this leads to a cyclic $(n+1)$-cocycle $\tilde{T}$ on $(\mathscr{A}\times_\alpha \mathbf{R})^\sim \otimes C$ such that

$$\varrho^* \tilde{T} = \#_\alpha \psi,$$

where ϱ is the map $x \mapsto x \otimes e_{11}$ from $\mathscr{A}\times_\alpha \mathbf{R}$ into $(\mathscr{A}\times_\alpha \mathbf{R})^\sim \otimes C$, and an application of Corollary II.6 of [2] finishes the proof.

3.6. According to Lemmas 3.3 and 3.5 the linear map $\#_\alpha$ descends to cyclic cohomology, i.e.

$$\#_\alpha \colon H_\lambda^n(\mathscr{A}) \to H_\lambda^{n+1}(\mathscr{A}\times_\alpha \mathbf{R})$$

is well defined.

3.7. LEMMA. $\#_\alpha S = S \#_\alpha$.

Proof. This is obtained by an argument analogous to that of Proposition 8.2 of [4].

3.8. Let us denote by ι the trivial action of $\mathbf{R}$ on $\mathbf{C}$. We have

$$\mathbf{C}\times_\iota \mathbf{R} = \mathscr{S}^*(\mathbf{R})$$

and

$$\mathbf{C}\times_\iota \mathbf{R} \times_{\hat{\iota}} \mathbf{R} \simeq \mathscr{K}^\infty.$$

Let τ be the normalised trace on $\mathbf{C}$ and set

$$\varepsilon = \#_\iota \tau,$$
$$\omega = \#_\iota \varepsilon.$$

PROPOSITION. *The following equalities hold*:

(1) $\varepsilon(f, g) = 2\pi i \int_{\mathbf{R}} t f(-t)\, g(t)\, dt$ *for* $f, g \in \mathscr{S}^*(\mathbf{R})$,

(2) $\omega(f, g, h) = -2\pi\{\mathrm{Tr}(f[D, g][M, h]) - \mathrm{Tr}(f[M, g][D, h]\}$ *for* $f, g, h \in \mathscr{K}^\infty$,

where D and M are the unbounded operators on $L^2(\mathbf{R})$ given by $(D\xi)(x) = -i\, d\xi(x)/dt$ and $(M\xi)(x) = x\xi(x)$, respectively.

Proof. (1) By definition,

$$\begin{aligned}\varepsilon(f,g) &= (\#_\iota \tau)(fdg)\\ &= 2\pi i \int_{\mathbf{R}} dt \int_0^t d\lambda\, \tau(f(-t)\, g(t))\\ &= 2\pi i \int_{\mathbf{R}} t f(-t)\, g(t)\, dt.\end{aligned}$$

(2) Using the Fourier transform, $\mathscr{S}^*(\mathbf{R})$ becomes identified with $\mathscr{S}(\mathbf{R})$, ε becomes the cyclic cocycle given by

$$\varepsilon(f,g) = \int fdg,$$

and the dual action $\hat{\iota}$ becomes the action of $\mathbf{R}$ on $\mathscr{S}(\mathbf{R})$ by translations γ. The rest of the computation consists of a straightforward chasing of the definition of $\#_\gamma \varepsilon$, and an application of the fact that $\mathscr{S}(\mathbf{R})\times_\gamma \mathbf{R}$ acts on $L^2(\mathbf{R})$ as integral operators with kernels

$$f(s, s-t), \quad f \in \mathscr{S}(\mathbf{R})\times_\gamma \mathbf{R} \simeq \mathscr{S}(\mathbf{R}^2).$$

3.9. Let p be a rank one projection inside $\mathscr{K}^\infty$. Then, using the equality $[D, M] = 1/i$, we get

$$\omega(p,p,p) = -2\pi i.$$

3.10. Note that in the case of trivial action,

$$\mathscr{A}\times_\iota \mathbf{R} \simeq \mathscr{A} \otimes \mathscr{S}(\mathbf{R})$$

and

$$\#_\iota \varrho = \varphi \# \varepsilon, \quad \varphi \in H^n_\lambda(\mathscr{A}).$$

3.11. Suppose that α is a smooth action of $\mathbf{R}$ on $\mathscr{A}$ and that τ is an α-invariant trace on $\mathscr{A}$. Then the formula

$$\hat{\tau}(f) = \tau(f(0)), \quad f \in \mathscr{A}\times_\alpha \mathbf{R},$$

defines a trace on $\mathscr{A}\times_\alpha \mathbf{R}$. Let e be the projection of $L^2(\mathbf{R})$ onto the one-dimensional subspace spanned by h_0.

Then $e \in \mathscr{K}^\infty$, and we can define a homomorphism

$$r\colon \mathscr{A} \to \mathscr{A} \otimes \mathscr{K}^\infty$$
$$a \mapsto a \otimes e.$$

PROPOSITION. *If δ denotes the derivation of $\mathscr{A}$ associated to α and T_α is the isomorphism constructed in Section 2.9, then*

$$(r^* T_\alpha^* \#_{\hat{\alpha}} \hat{\tau})(a^0, a^1) = \tau(a^0 \delta(a^1)).$$

Proof. First, recall that

$$(T_\alpha r(a))(s, \lambda) = \int_{\mathbf{R}} e^{2\pi i s \lambda} e^{-2\pi i u \lambda} \alpha_u(a)\, h_0(u-s)\, h_0(u)\, du. \tag{$*$}$$

Given $y^0, y^1 \in (\mathscr{A} \times_\alpha \mathbf{R}) \times_{\hat{\alpha}} \mathbf{R}$ we have, by definition,

$$\#_{\hat{\alpha}} \hat{\tau}(y^0 y^1) = 2\pi i \int_{\mathbf{R}} \lambda \hat{\tau}(y^0(-\lambda)\, \hat{\alpha}_{-\lambda}(y^1(\lambda)))\, d\lambda. \tag{$**$}$$

Inserting ($*$) in ($**$) we get, by a routine computation,

$$\#_{\hat{\alpha}} \hat{\tau}(T_\alpha r(a^0), T_\alpha r(a^1)) = -\tau(\delta(a^0)\, a^1) = \tau(a^0 \delta(a^1)).$$

3.12. One illustration of Proposition 3.11 comes from differential topology. Let X be a closed C^∞-manifold and α a one-parameter automorphism group of $C^\infty(X)$ generated by a smooth vector field ξ. For any α-invariant measure μ on X, a one-dimensional current C, called the Ruelle-Sullivan current, is defined by

$$C(\omega) = \int \omega(\xi)\, d\mu.$$

This current is, in a natural way, a cyclic one-cocycle on $C^\infty(X)$. Proposition 3.11 says that

$$\#_{\hat{\alpha}} \hat{\mu} = C.$$

4. The stability theorem

4.1. Given an n-cycle (Ω, ϱ, T) over $\mathscr{A}$, the canonical extension $(\Omega^\sim, \varrho^\sim, T^\sim)$ is a n-cycle over $\mathscr{A}^\sim$. Moreover, if (Ω, ϱ, T) and $(\mathscr{A}', \varrho', T')$ are cobordant over $\mathscr{A}$, then $(\Omega^\sim, \varrho^\sim, T^\sim)$ and $(\Omega'^\sim, \varrho'^\sim, T'^\sim)$ are cobordant over $\mathscr{A}^\sim$. By Lemma II.28 of [2], the characters τ_1 and τ_2 of two cobordant cycles over $\mathscr{A}$ satisfy

$$\tilde{\tau}_1 - \tilde{\tau}_2 = B\varphi$$

for some Hochschild cocycle $\varphi \in Z^{n+1}(\mathscr{A}^\sim, (\mathscr{A}^\sim)^*)$; hence

$$[S\tilde{\tau}_1] = [S\tilde{\tau}_2] \quad \text{in} \quad H_\lambda^{n+3}(\mathscr{A}^\sim).$$

Since the operator S commutes with the restriction map

$$H_\lambda^*(\mathscr{A}^\sim) \to H_\lambda^*(\mathscr{A}),$$

it follows that $[S\tau_1] = [S\tau_2]$ in $H_\lambda^{n+3}(\mathscr{A})$.

This observation in particular extends the homotopy invariance of $H^*(\mathscr{A})$ ([2], p. 341) to the case of nonunital $\mathscr{A}$.

4.2. THEOREM. *Let $\mathscr{K}$ be a locally convex topological algebra. Suppose that*

(1) *there exists an idempotent $e \in \mathscr{K}$ and a cyclic cocycle $\omega \in Z_\lambda^{2k}(\mathscr{A})$ such that $\omega(e, \ldots, e) = k!(2\pi i)^k$,*

(2) *the flip $\sigma \in \operatorname{Aut}(\mathscr{K} \otimes \mathscr{K})$ defined by $\sigma(a \otimes b) = b \otimes a$ is connected to the identity by a C^1-path of endomorphisms of $\mathscr{K} \otimes \mathscr{K}$.*

Then, for any locally convex topological algebra $\mathscr{A}$, the map

$$H^*(\mathscr{A}) \to H^*(\mathscr{A} \otimes \mathscr{K})$$
$$[\varphi] \mapsto [\varphi \# \omega]$$

is an isomorphism.

Proof. Denote by r and r' the homomorphisms $\mathscr{A} \to \mathscr{A} \otimes \mathscr{K}$ and $\mathscr{A} \otimes \mathscr{K} \to \mathscr{A} \otimes \mathscr{K} \otimes \mathscr{K}$ produced by tensoring with e. We have immediately

$$(r')^*(\varphi \# \omega) = S^k \varphi,$$
$$(r')^*(\mathrm{Id}_{\mathscr{A}} \otimes \sigma)^*(\varphi \# \omega) = r^*\varphi \# \varphi,$$

for any $\varphi \in Z_\lambda^*(\mathscr{A} \otimes \mathscr{K})$.

According to Section 4.1 and our assumptions,

$$(\mathrm{Id}_{\mathscr{A}} \otimes \sigma)^* = \mathrm{Id} \quad \text{on} \quad H^*(\mathscr{A} \otimes \mathscr{K} \otimes \mathscr{K})$$

and hence

$$[\varphi] = [r^*\varphi \# \omega] \in H^*(\mathscr{A} \otimes \mathscr{K}).$$

If we denote by $\#\omega$ the map $\varphi\mapsto\varphi\#\omega$, the last equality says that

$$(\#\omega)r^* = \mathrm{Id} \quad \text{on} \quad H^*(\mathscr{A}\otimes\mathscr{K}).$$

For any cyclic cocycle φ on $\mathscr{A}$,

$$(r^*(\#\omega))\varphi = S^k\varphi.$$

It follows that $\#\omega$ and r^* are inverses of each other at the level of periodic cyclic cohomology; in particular, they are isomorphisms.

4.3. THEOREM (Stability). *The Fréchet algebra $\mathscr{K}^\infty$ constructed in Section* 2.6 *satisfies the conditions of Theorem* 4.2. *Consequently*

$$H^*(\mathscr{K}^\infty)\simeq H^*(\mathbf{C})$$

and

$$H^*(\mathscr{A}\otimes\mathscr{K}^\infty)\simeq H^*(\mathscr{A})$$

for any locally convex topological algebra $\mathscr{A}$.

Proof. The existence of an idempotent and an even cocycle as required in Theorem 4.2(1) was shown in Sections 3.8 and 3.9. To deal with the flip automorphism σ, note that $\mathscr{K}^\infty$ consists of the integral operators with smooth rapidly decreasing kernels, and that one has isomorphisms of topological vector spaces

$$\mathscr{K}^\infty\otimes\mathscr{K}^\infty\simeq\mathscr{S}(\mathbf{R}^2)\otimes\mathscr{S}(\mathbf{R}^2)\simeq\mathscr{S}(\mathbf{R}^4).$$

Since any one-parameter subgroup of rotations in the coordinate space acts smoothly on $\mathscr{S}(\mathbf{R}^4)$ and gives automorphisms (unitarily implemented in $L^2(\mathbf{R}^2)$) of the algebra $\mathscr{K}^\infty\otimes\mathscr{K}^\infty$, the result follows.

5. Main theorem

5.1. Assume for the rest of this section that $\mathscr{A}$ is a Fréchet algebra and α is a smooth action of $\mathbf{R}$ on $\mathscr{A}$.

PROPOSITION. *Given $\varphi\in Z^n_\lambda(\mathscr{A})$, the classes of the cocycles $S(\varphi\#\omega)$ and $S(T^*_\lambda\#_{\hat\alpha}\#_\alpha)\varphi$ coincide in $H^{n+4}_\lambda(\mathscr{A}\otimes\mathscr{K}^\infty)$.*

Proof. We will use the notation introduced in the proof of Lemma 2.8. Extending the homomorphism $\psi\varrho$ we get a homomorphism

$$\psi\varrho\colon (\Omega(\mathscr{A})\otimes E)\otimes_\alpha E \to (\Omega(\mathscr{A})\otimes_\alpha E)\otimes_{\hat{\alpha}} E$$

and hence an equality

$$(\psi\varrho)^* \#_{\hat{\alpha}} \#_\alpha \varphi = \#_\gamma(\varphi \# \varepsilon),$$

where ε is the canonical one-cocycle on $\mathscr{S}(\mathbf{R})$ (see the proof of II.3.2 of [2]).

Let γ^t be the action of $\mathbf{R}$ on

$$\mathscr{A}\otimes\mathscr{S}(\mathbf{R}) \simeq \mathscr{S}(\mathbf{R},\mathscr{A}),$$

given by

$$((\gamma^t)_u f)(s) = \alpha_{tu}(f(s-u)),$$

$t\in[0,1]$. Define an action β on

$$\mathscr{S}(\mathbf{R},\mathscr{A})\otimes C^1([0,1])$$

by

$$\beta_u(f_t(s)) = ((\gamma^t)_u f)(s), \quad t\in[0,1].$$

Notice that β is smooth and therefore we can form the smooth crossed product

$$(\mathscr{S}(\mathbf{R},\mathscr{A})\otimes C^\infty([0,1]))\times_\beta \mathbf{R}.$$

The evaluation maps $(g_0 f)(s)=f_0(s)$, and $(g_1 f)(s)=f_1(s)$ are equivariant homomorphisms

$$g_i\colon \mathscr{S}(\mathbf{R},\mathscr{A})\otimes C^\infty([0,1]) \to \mathscr{S}(\mathbf{R},\mathscr{A}), \quad i=0,1,$$

and hence give rise to homomorphisms

$$\hat{g}_0\colon (\mathscr{S}(\mathbf{R},\mathscr{A})\otimes C^\infty([0,1]))\times_\beta \mathbf{R} \to \mathscr{S}(\mathbf{R},\mathscr{A})\times_{\gamma^0}\mathbf{R},$$

$$\hat{g}_1\colon (\mathscr{S}(\mathbf{R},\mathscr{A})\otimes C^\infty([0,1]))\times_\beta \mathbf{R} \to \mathscr{S}(\mathbf{R},\mathscr{A})\times_{\gamma^1}\mathbf{R}.$$

We define a homomorphism

$$\varrho\colon \mathscr{A}\otimes\mathscr{K}^\infty \to (\mathscr{S}(\mathbf{R},\mathscr{A})\otimes C^\infty([0,1]))\times_\beta \mathbf{R}$$

by

$$\varrho(f)_t(s,r) = \alpha_{st}(f(s,r)),$$

where $t \in [0,1]$ and r is the variable introduced by the crossed product with β. It is an easy observation that

$$\varrho^* \hat{g}_0^*(\#_{\gamma^0}(\varphi \# \hat{\varepsilon})) = T_\iota^* \#_\iota \#_\iota \varphi,$$

$$\varrho^* \hat{g}_1^*(\#_{\gamma^1}(\varphi \# \hat{\varepsilon})) = T_\alpha^* \#_{\hat{\alpha}} \#_\alpha \varphi.$$

Let us consider the two $(n+2)$-cycles over $\mathscr{A} \otimes \mathscr{K}^\infty$ given by

$$((\Omega(\mathscr{A}) \otimes E) \otimes_{\gamma^0} E,\ \hat{g}_0 \varrho,\ \#_{\gamma^0}(\varphi \# \hat{\varepsilon})),$$

$$((\Omega(\mathscr{A}) \otimes E) \otimes_{\gamma^1} E,\ \hat{g}_1 \varrho,\ \#_{\gamma^1}(\varphi \# \hat{\varepsilon})).$$

We claim that these two cycles are cobordant. In fact, let $\hat{\psi}$ be the canonical graded trace of degree one on the differential graded algebra $\Omega^*([0,1])$ of smooth differential forms on $[0,1]$ (see [2], p. 341). The graded trace

$$\#_\beta((\varphi \# \varepsilon) \# \hat{\psi})$$

over

$$(\Omega(\mathscr{A}) \otimes E \otimes \Omega^*([0,1])) \otimes_\beta E$$

gives us the required cobordism. As a result, applying Section 4.1, we get

$$ST_\iota^* \#_\iota \#_\iota \varphi = ST_\alpha^* \#_{\hat{\alpha}} \#_\alpha \varphi \quad \text{in} \quad H_\lambda^{n+4}(\mathscr{A} \otimes \mathscr{K}^\infty).$$

5.2. THEOREM. *For a smooth action α of* **R** *on a Fréchet algebra $\mathscr{A}$ the map $\#_\alpha$ induces isomorphisms*

$$H^{\mathrm{ev}}(\mathscr{A}) \xrightarrow{\sim} H^{\mathrm{odd}}(\mathscr{A} \times_\alpha \mathbf{R}),$$

$$H^{\mathrm{odd}}(\mathscr{A}) \xrightarrow{\sim} H^{\mathrm{ev}}(\mathscr{A} \times_\alpha \mathbf{R}).$$

Proof. Consider the diagram

$$\begin{array}{ccccc}
H^*(\mathscr{A}) & \xrightarrow{\#\omega} & H^*(\mathscr{A} \otimes \mathscr{K}^\infty) & \xrightarrow{r^*} & H^*(\mathscr{A}) \\
\Big\downarrow \#_\alpha & & \Big\uparrow T_\alpha^* & & \\
H^*(\mathscr{A} \times_\alpha \mathbf{R}) & \xrightarrow{\#_{\hat{\alpha}}} & H^*(\mathscr{A} \times_\alpha \mathbf{R} \times_{\hat{\alpha}} \mathbf{R}). & &
\end{array}$$

According to Proposition 5.1 and the stability theorem, this diagram is commutative and $\#\omega$ is an isomorphism. In particular, $\#_\alpha$ is injective and $\#_{\hat{\alpha}}$ is surjective (since $r^*T^*_\alpha$ is an isomorphism). Applying the same argument to the dual action $\hat{\alpha}$ we conclude that $\#_{\hat{\alpha}}$ is injective, and hence that $\#_\alpha$ is surjective.

5.3. Corollary. (1) $H^*(\mathscr{S}(\mathbf{R}^n)) \simeq \mathbf{C}$, *with a generator given by the n-cocycle*

$$(f^0, \ldots, f^n) \mapsto \int f^0 df^1 \ldots df^n.$$

(2) $H^*(\mathscr{A} \otimes \mathscr{S}(\mathbf{R})) \simeq H^{*+1}(\mathscr{A})$.
(3) $H^*(\mathscr{A} \otimes \mathscr{S}(\mathbf{R}^2)) \simeq H^*(\mathscr{A})$.

5.4. Let G be any connected, simply connected nilpotent Lie group. Since G can be written as an iterated semidirect product

$$G \simeq \mathbf{R} \rtimes \mathbf{R} \rtimes \ldots \rtimes \mathbf{R},$$

and since nilpotency of G implies smoothness of the successive actions, we can apply Theorem 5.2 and get

$$H^*(\mathscr{S}^*(G)) \simeq H^{*+m}(\mathbf{C}),$$

where $\mathscr{S}^*(G)$ is $\mathscr{S}(\mathbf{R}^m)$, as a topological vector space, with convolution over G as product.

5.5. In the case of the Heisenberg group the corresponding generator of $H^*(\mathscr{S}^*(H))$ is given by the cyclic 3-cocycle

$$\tau(f^0, f^1, f^2, f^3) = (2\pi i)^3 \int\int\int_{g_0 g_1 g_2 g_3 = 1} f^0(g_0) f^1(g_1) f^2(g_2) f^3(g_3) c(g_1, g_2, g_3)\, dg_1\, dg_2\, dg_3$$

where c is a continuous normalised group 3-cocycle generating $H^3_c(H; \mathbf{R}) \simeq \mathbf{R}$.

6. Comparison with the Connes isomorphism in K-theory

6.1. Let α be an action of $\mathbf{R}$ on a C^*-algebra A. In [1], Connes constructed a map

$$\phi^i_\alpha \colon K_i(A) \to K_{i+1}(A \times_\alpha \mathbf{R}),$$

satisfying certain natural axioms, and proved that it is unique up to a choice of orientation, and is an isomorphism. We review briefly the construction of ϕ^0_α.

The subalgebra $\mathscr{A}$ of smooth elements of A with respect to α has, in a natural way, a structure of Fréchet algebra, and the inclusion of $\mathscr{A}$ into A induces an isomorphism

$$K_0(\mathscr{A}) \to K_0(A).$$

Working, if necessary, in a matrix algebra over $\mathscr{A}^{\sim}$, we may assume that we are given a projection $e \in \mathscr{A}$.

With $h = \delta(e)e - e\delta(e)$ we get

$$(\delta - \operatorname{ad} h)(e) = 0,$$

and hence $\delta - \operatorname{ad} h$ generates an $\mathbf{R}$-action α' on $\mathscr{A}$ such that $\alpha'_t(e) = e$. Note that both actions α and α' are smooth on $\mathscr{A}$. By Lemma 1.3 of [1], there exists an isomorphism

$$i\colon A \times_\alpha \mathbf{R} \to A \times_{\alpha'} \mathbf{R},$$

and it is easy to see that i (as constructed in [1]) also defines an isomorphism

$$i\colon \mathscr{A} \times_\alpha \mathbf{R} \to \mathscr{A} \times_{\alpha'} \mathbf{R}.$$

Let U be a unitary in $C^*(\mathbf{R})^{\sim}$ such that the class of U is the positive generator of $K_1(C^*(\mathbf{R}))$, and $U - 1 \in \mathscr{S}^*(\mathbf{R})$. Then

$$i^{-1}(1 - e + eU)$$

is a unitary element of $(\mathscr{A} \times_\alpha \mathbf{R})^{\sim}$, and represents $\phi_\alpha^0([e])$ in $K_1(A \times_\alpha \mathbf{R})$. For the definition of K_1 for a Fréchet algebra see Section 12 of [3]. According to Lemma 12.1 of [4], $\phi_\alpha^0([e])$ can be evaluated on any odd-dimensional cyclic cocycle on $\mathscr{A} \times_\alpha \mathbf{R}$.

6.2. THEOREM. *Given* $\varphi \in H_\lambda^{2n}(\mathscr{A})$,

$$\langle \varphi, [e] \rangle = \langle \#_\alpha \varphi, \phi_\alpha^0([e]) \rangle.$$

Proof. Assume first that e is actually α-invariant. Then the homomorphism

$$\varrho\colon \mathbf{C} \to \mathscr{A}$$
$$\lambda \mapsto \lambda e,$$

induces a map

$$\hat{\varrho}\colon \mathbf{C} \times_\iota \mathbf{R} \to \mathscr{A} \times_\alpha \mathbf{R}.$$

By naturality of both ϕ_α^0 and $\#_\alpha$, we can pull everything back to $\mathbf{C}\times_\iota\mathbf{R}$ and thus reduce the computation to the case $\mathscr{A}=\mathbf{C}$ and $\alpha=\iota$. We have thus to show that

$$\langle \omega, 1\rangle = \langle \#_\iota\, \omega, \phi_\iota^0(1)\rangle.$$

By construction, $\phi_\iota^0(1)$ is the positively oriented generator of $K_1(C^*(\mathbf{R}))\simeq\mathbf{Z}$. Using the Fourier transform, we identify $\mathscr{S}^*(\mathbf{R})$ with $\mathscr{S}(\mathbf{R})$, and choose

$$\phi_\iota^0(1) = \exp 2\pi i h$$

where h is any C^∞ real-valued function on $\mathbf{R}$ such that

$$h(t) = 0 \quad \text{for} \quad t\leqslant 0,$$

$$h(t) = 1 \quad \text{for} \quad t\geqslant 1.$$

Since $\#_\iota$ commutes with S, we may assume that ω is equal to τ, the normalised trace on $\mathbf{C}$. Then

$$\langle \tau, 1\rangle = 1,$$

and

$$\begin{aligned}\langle \#_\iota\, \tau, \phi_\iota^0(1)\rangle &= \langle \varepsilon, \exp 2\pi i h\rangle \\ &= \frac{1}{2\pi i}\,\varepsilon(\exp(-2\pi i h)-1, \exp(2\pi i h)-1) \\ &= \frac{1}{2\pi i}\int e^{-2\pi i h} d(e^{2\pi i h}) = 1.\end{aligned}$$

The following lemma finishes the proof of Theorem 6.2.

6.3. LEMMA. *The following diagram is commutative.*

$$\begin{array}{ccc} H^*(\mathscr{A}) & \xrightarrow{\ \#_{\alpha'}\ } & H^{*+1}(\mathscr{A}\times_{\alpha'}\mathbf{R}) \\ & {\scriptstyle \#_\alpha}\searrow \quad \swarrow{\scriptstyle i^*} & \\ & H^{*+1}(\mathscr{A}\times_\alpha \mathbf{R}). & \end{array}$$

Proof. Let us recall first the construction of the isomorphism i. Given an action α and a smooth α-cocycle V_t, one defines an action γ of $\mathbf{R}$ on

$$M_2(\mathscr{A})$$

such that

$$\gamma(a \otimes e_{11}) = \alpha(a) \otimes e_{11}$$
$$\gamma(a \otimes e_{22}) = \alpha'(a) \otimes e_{22}.$$

From the smoothness of α and V it follows that γ is a smooth action and so we can construct the crossed product

$$M_2(\mathscr{A}) \times_\gamma \mathbf{R}$$

and the two imbeddings

$$\hat{\varrho}_1 : \mathscr{A} \times_\alpha \mathbf{R} \to M_2(\mathscr{A}) \times_\gamma \mathbf{R}$$
$$\hat{\varrho}_2 : \mathscr{A} \times_{\alpha'} \mathbf{R} \to M_2(\mathscr{A}) \times_\gamma \mathbf{R},$$

where $\varrho_i(a) = a \otimes e_{ii}$. The isomorphism i is now given by

$$\mathrm{Ad}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Since

$$\#_\alpha \varphi = \hat{\varrho}_1^*(\#_\gamma \varphi),$$
$$\#_{\alpha'} \varphi = \hat{\varrho}_2^*(\#_\gamma \varphi),$$

and since

$$\mathrm{Ad}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

is connected to the identity by a smooth path in $\mathrm{Aut}(M_2(\mathscr{A}) \times_\gamma \mathbf{R})$, an application of Section 4.1 (homotopy invariance of H^*) gives the required formula

$$i^* \#_{\alpha'} = \#_\alpha.$$

6.4. Let us consider $\phi_\alpha^1 : K_1(A) \to K_0(A \times_\alpha \mathbf{R})$. Note that ϕ_α^1 is the inverse of $\phi_{\hat{\alpha}}^0$. Since $\#_\alpha$ is the inverse of $\#_{\hat{\alpha}}$, we have by the above the following result.

PROPOSITION. *Let (A, α) satisfy the conditions of* 6.1. *Let u be a unitary in $\mathscr{A}^\sim$,* *and suppose that $\phi_\alpha^1([u])$ is represented by an element of $K_0(\mathscr{A}\times_\alpha \mathbf{R})$. Then, for any odd-dimensional cyclic cocycle ψ on $\mathscr{A}$,*

$$\langle \psi, [u]\rangle = \langle \#_\alpha \psi, \phi_\alpha^1([u])\rangle.$$

7. Further remarks

7.1. Let us consider the action of $\mathbf{R}$ on $\mathbf{R}$ given by

$$\alpha_t(s) = e^{2t}s.$$

The corresponding semidirect product group is a minimal parabolic subgroup of $PSL_2(\mathbf{R})$. If we try to use the procedure of Section 2.1, we get the action α of $\mathbf{R}$ on $\mathscr{S}^*(\mathbf{R})$ given by

$$(\alpha_t f)(s) = e^{-2t} f(e^{-2t}s),$$

which is not smooth in our terminology. We will sketch below how to extend our results to such a situation.

7.2. Suppose we are given a Fréchet algebra $\mathscr{A}$ and a strongly infinitely differentiable action of $\mathbf{R}$ on $\mathscr{A}$. Suppose, moreover, that there exists an increasing sequence of functions

$$\varrho_n\colon \mathbf{R}\to\mathbf{R}_+,$$

satisfying the following conditions:

(1) $(1+t^2)^{1/2}\varrho_{n-1}(t)\le\varrho_n(t), \quad t\in\mathbf{R}.$

(2) $\varrho_n(t)\le\varrho_n(s)\,\varrho_n(t-s), \quad s,t\in\mathbf{R}.$

(3) $\|D^k\alpha_t(a)\|_n\le\varrho_{n'}(t)\|a\|_{n'}$ uniformly in $t\in\mathbf{R}$ and $a\in\mathscr{A}$, $n'=n'(n,k)$.

Denote by $\mathscr{S}_\varrho^*(\mathbf{R})$ the convolution algebra of functions satisfying

$$\|f\|_{k,n} = \sup_t \varrho_n(t)|D^{k-1}f(t)| < \infty.$$

One checks that $\mathscr{S}_\varrho^*(\mathbf{R})$ with topology defined by $\|\ \|_{k,n}$, $k,n\in\mathbf{N}$, becomes a Fréchet algebra, nuclear as a locally convex space.

Define

$$\mathscr{A} \times_\alpha^\varrho \mathbf{R} = \mathscr{S}_\varrho^*(\mathbf{R}, \mathscr{A}) \quad (= \mathscr{S}_\varrho^*(\mathbf{R}) \otimes \mathscr{A})$$

with the multiplication defined by the formula in Section 2.5.

It is straightforward to see that the construction of $\#_\alpha$ goes through and defines a map

$$\#_\alpha \colon H^*(\mathscr{A}) \to H^{*+1}(\mathscr{A} \times_\alpha^\varrho \mathbf{R}).$$

The all-important, though completely trivial, fact is that the dual action $\hat{\alpha}$ on $\mathscr{A} \times_\alpha^\varrho \mathbf{R}$ is smooth, and so we can form $(\mathscr{A} \times_\alpha^\varrho \mathbf{R}) \times_{\hat{\alpha}} \mathbf{R}$. It is easy to see that the proof of Takesaki-Takai duality still goes through and gives

$$(\mathscr{A} \times_\alpha^\varrho \mathbf{R}) \times_{\hat{\alpha}} \mathbf{R} \simeq \mathscr{A} \otimes \mathscr{K}^\varrho,$$

where $\mathscr{K}^\varrho = \mathscr{S}_\varrho^*(\mathbf{R}) \times_t \mathbf{R}$. Furthermore, $\mathscr{K}^\varrho$ satisfies the conditions of Theorem 4.2.

Using Theorems 4.2 and 5.2 we get

$$\begin{aligned} H^*(\mathscr{A} \times_\alpha^\varrho \mathbf{R}) &\simeq H^{*+1}(\mathscr{A} \times_\alpha^\varrho \mathbf{R} \times_{\hat{\alpha}} \mathbf{R}) \\ &\simeq H^{*+1}(\mathscr{A} \otimes \mathscr{K}^\varrho) \simeq H^{*+1}(\mathscr{A}). \end{aligned}$$

We have thus sketched the proof of the following result.

7.3. PROPOSITION. *Suppose that $\mathscr{A}$, α, ϱ satisfy the conditions of Section 7.2. Then $\#_\alpha$ gives an isomorphism*

$$\#_\alpha \colon H^*(\mathscr{A}) \to H^{*+1}(\mathscr{A} \times_\alpha^\varrho \mathbf{R}).$$

7.4. The preceding proposition applies to the example of Section 7.1 if we set

$$\varrho_n(t) = e^{2n|t|}.$$

In general, given an infinitely differentiable action α of $\mathbf{R}$ on a Fréchet algebra $\mathscr{A}$ such that α_t is continuous with respect to each of the seminorms defining the topology of $\mathscr{A}$, then we can use the weight functions ϱ_n given by

$$\varrho_n(s) = \sum_{k=0}^{n} \sum_{i=0}^{n} \left(\int_{-s}^{s} \left\| \frac{d^k}{dt^k} \alpha_t \right\|_i dt \right)^n$$

(cf. Section 1 of [4]).

Incidentally, it is only in this last construction that $\mathscr{A}$ need be a Fréchet algebra—everywhere else in this paper $\mathscr{A}$ may be any locally convex algebra.

Acknowledgements. The first author was partially supported by the Carlsberg Foundation. The second author was partially supported by the Danish Natural Science Research Council.

References

[1] Connes, A., An analogue of the Thom isomorphism for crossed products of a C^*-algebra by an action of **R**. *Adv. in Math.*, 39 (1981), 31–55.

[2] — Non-commutative differential geometry, Parts I and II. *Publ. I.H.E.S.*, 62 (1985), 257–360.

[3] Dixmier, J., Sur les représentations unitaires des groupes de Lie nilpotents, IV. *Canad. J. of Math.*, 11 (1959), 321–344.

[4] Nest, R., Cyclic cohomology of crossed products with **Z**. To appear in *J. Funct. Anal.*

Received June 1, 1987