The fundamental conjecture for homogeneous Kähler manifolds

by

JOSEF DORFMEISTER and KAZUFUMI NAKAJIMA

The University of Georgia
Athens, GA, U.S.A.

Kyoto University
Kyoto, Japan

The purpose of this paper is to prove the

FUNDAMENTAL CONJECTURE. *Every homogeneous Kähler manifold is a holomorphic fiber bundle over a homogeneous bounded domain in which the fiber is (with the induced Kähler metric) the product of a flat homogeneous Kähler manifold and a compact simply connected homogeneous Kähler manifold.*

This conjecture has been stated first by Gindikin and Vinberg [25] in 1967. At that time it was known from results of Borel and Matsushima [1], [14] that the Fundamental Conjecture holds if the manifold admits a transitive reductive group of automorphisms (≡ biholomorphic isometries).

Gindikin and Vinberg proved the Fundamental Conjecture for the case that the manifold admits a split solvable transitive group of automorphisms [25].

What was known about the Fundamental Conjecture at that time (in 1967) is contained in the very readible survey article [9].

In the following years only few results concerning general homogeneous Kähler manifolds were published. There are clearly three basic types of homogeneous Kähler manifolds occuring in the Fundamental Conjecture. Here the flat type is trivial and the compact type was known by the work of Wang [27]. Between 1970 and 1980 the structure of bounded homogeneous domains and their infinitesimal automorphisms has been classified by various authors. Knowledge of the "fine structure" of homogeneous bounded domains is used in several places of our proof of the Fundamental Conjecture.

In the last five years or so several papers have been published discussing the Fundamental Conjecture under various additional assumptions. A survey on these results can be found in [7]. We would like to mention two developments in more detail, since their combination eventually led to a proof of the Fundamental Conjecture.

On one hand, a few authors investigated homogeneous Kähler manifolds without flat homogeneous Kähler submanifolds. The main result, proven by Nakajima [17] is: Fundamental Conjecture holds for homogeneous Kähler manifolds without flat homogeneous Kähler submanifolds and for homogeneous Kähler manifolds associated with effective j-algebras. He obtained the above result by using techniques developed in [26] and [16].

The second string of investigations allows flat homogeneous Kähler submanifolds but uses "solvability conditions". The following results were proven by Dorfmeister [6], [8]: (a) If a homogeneous Kähler manifold admits a solvable transitive group of automorphisms, then the Fundamental Conjecture holds, and (b) (Radical Conjecture) If an effective Kähler algebra $(\mathfrak{g}, \mathfrak{k}, j, \varrho)$ contains a solvable ideal $\mathfrak{r}$ such that $\mathfrak{g}=\mathfrak{r}+j\mathfrak{r}+\mathfrak{k}$, then there exists a solvable Kähler subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ satisfying $\mathfrak{g}=\mathfrak{s}+\mathfrak{k}$ and $\mathfrak{s}\cap\mathfrak{k}=0$.

This latter result is particularly important for our proof of the Fundamental Conjecture and is used frequently in this paper. Especially, it plays an essential role at the starting point of our investigation of Kähler algebras.

We now explain our method more precisely. After some preparations in § 1, we introduce in § 2 the notion of a quasi-normal Kähler algebra and prove that for every homogeneous Kähler manifold M, one can find a quasi-normal Kähler algebra $(\mathfrak{g}, \mathfrak{k}, j, \varrho)$ which generates a transitive subgroup of $\mathrm{Aut}(M)$ (Theorem 2.1). This can be done by using the Radical Conjecture and modifications [6]. The major part of our paper is devoted to proving: Every quasi-normal Kähler algebra $(\mathfrak{g}, \mathfrak{k}, j, \varrho)$ is decomposed as $\mathfrak{g}=\mathfrak{a}+\mathfrak{h}$, where $\mathfrak{a}$ is an abelian Kähler ideal and $\mathfrak{h}$ is a Kähler subalgebra such that the homogeneous Kähler manifold corresponding to $\mathfrak{h}$ is a holomorphic fiber space over a homogeneous bounded domain and the fiber is a compact simply connected homogeneous Kähler manifold (Theorem 2.5).

Let $(\mathfrak{g}, \mathfrak{k}, j, \varrho)$ be an effective Kähler algebra and $\mathfrak{n}$ the nilpotent radical of $\mathfrak{g}$. Consider the subalgebra $\mathfrak{g}'=\mathfrak{n}+j\mathfrak{n}+\mathfrak{k}$. By the Radical Conjecture together with [6], we can decompose $\mathfrak{g}'$ as $\mathfrak{g}'=\mathfrak{a}+\mathfrak{t}+\mathfrak{k}$, where $\mathfrak{a}$ (resp. $\mathfrak{t}$) is a Kähler subalgebra corresponding to a flat homogeneous Kähler manifold (resp. to a homogeneous bounded domain). We consider the two cases where $\mathfrak{t}=0$ and where $\mathfrak{t}\neq 0$ separately.

In § 3 we study the first possibility. Using arguments like in Case 1 of [8] we show that if $\mathfrak{t}=0$ then $j\,\mathrm{rad}(\mathfrak{g})\subset \mathrm{rad}(\mathfrak{g})+\mathfrak{k}$ where $\mathrm{rad}(\mathfrak{g})$ denotes the radical of $\mathfrak{g}$ (Theorem 3.2). Then the orthogonal complement $\mathfrak{h}$ of $\mathrm{rad}(\mathfrak{g})$ relative to ϱ is a j-invariant subalgebra. If we further assume that $\mathfrak{g}$ is quasi-normal, then $\mathrm{rad}(\mathfrak{g})$ becomes abelian and $\mathfrak{h}$ is semi-simple. Now the decomposition $\mathfrak{g}=\mathrm{rad}(\mathfrak{g})+\mathfrak{h}$ satisfies the desired properties of Theorem 2.5.

§§4, 5 and 6 are devoted to the study of the case $\mathfrak{t}\neq 0$. In this case, we denote by e the maximal idempotent of $\mathfrak{t}$ and let $\mathfrak{g}_\lambda$ denote the weight spaces of the real part of $\mathrm{ad}\,je$. We then have $\mathfrak{g}=\mathfrak{g}_{-1/2}+\mathfrak{g}_0+\mathfrak{g}_{1/2}+\mathfrak{g}_1$ and $\mathfrak{g}_0=j\mathfrak{g}_1+\mathfrak{s}$, where $\mathfrak{s}$ is a j-invariant subalgebra containing $\mathfrak{k}$ (Theorem 4.4). Proofs and techniques are similar to those of [7]. In the rest of §4, we follow [16] in our setting and prove that $\mathrm{ad}\,\mathfrak{g}_0|\mathfrak{g}_1$ is the Lie algebra of a transitive group of automorphisms of a homogeneous convex cone in $\mathfrak{g}_1$ (Proposition 4.8).

Next, in §5, we show that $\mathfrak{g}_{-1/2}+\mathfrak{g}_{1/2}\subset\mathrm{rad}(\mathfrak{g})$ (Theorem 5.1). To do so, we first reduce to the case $\dim\mathfrak{g}_1=1$. This is an improvement over a similar reduction used in [8]. To prove that (under our assumptions) a maximal semisimple subalgebra of $\mathfrak{g}$ has to be contained in $\mathfrak{g}_0$, we use a second weight space decomposition. It is determined by an element $E\in\mathfrak{g}_0$ which is more convenient in the present situation than the element $f_0\in\mathfrak{g}_0$ used in [8].

In §6, we first study the structure of $\mathfrak{g}_0$ in great detail under the additional assumption that $\mathfrak{g}$ is quasi-normal. To obtain the description of $\mathfrak{g}_0$ that we need, we use the results obtained in the previous sections and in addition the knowledge of the fine structure of homogeneous cones (Theorem 6.2) which will be proven in Appendix 1.

Finally, set $\tilde{\mathfrak{a}}=\mathfrak{g}_{-1/2}+\mathrm{rad}(\mathfrak{s}_0)+[e,\mathfrak{g}_{1/2}]$, where $\mathfrak{s}_0=\{x\in\mathfrak{s};\ [x,\mathfrak{g}_1]=0\}$. We show that $\tilde{\mathfrak{a}}$ is an abelian Kähler ideal of $\mathfrak{g}$ (Theorem 6.5). Let $\tilde{\mathfrak{h}}$ be the orthogonal complement of $\tilde{\mathfrak{a}}$ relative to ϱ. Then from the arguments of [17] the decomposition $\mathfrak{g}=\tilde{\mathfrak{a}}+\tilde{\mathfrak{h}}$ satisfies the properties of Theorem 2.5.

In §7, we construct a fibering of the homogeneous Kähler manifold M and prove the Fundamental Conjecture. Let G be as in Theorem 2.1. Then $M=G/K$ for some subgroup K. Taking the universal covering group instead of G, we may assume G is simply connected. Let $\mathfrak{a}$ and $\mathfrak{n}$ be as in Theorem 2.5. Denote by L the connected subgroup of G corresponding to $\mathfrak{a}+\mathfrak{u}$. One can show that L is a closed subgroup of G containing K and obtain a fibering: $M=G/K\to G/L$. We show that this fibering has the desired properties of the Fundamental Conjecture. Here we use properties of the decomposition of a quasi-normal Kähler algebra and construct a G-equivariant holomorphic imbedding of M onto an open set of a complex homogeneous space of a complex Lie group. This last part of our proof follows an idea of [25].

In Appendix 1, as is mentioned before, we give a proof of Theorem 6.2 which describes a decomposition of a homogeneous convex cone C according to an arbitrary transitive algebraic subalgebra $\mathfrak{F}$ of Lie Aut C. Here we use the results of [26] on j-algebras, regarding $\mathfrak{F}$ as a subalgebra of Lie Aut $D(C)$, where $D(C)$ denotes the Siegel domain associated with the cone C.

In Appendix 2, we state and prove a result involving Levi decompositions of a Lie algebra, which is used frequently in this paper. We would expect that this result is well known; but we were unable to find a reference for it.

§ 1. Preliminaries

1.1. Let G be a connected real Lie group and K a closed subgroup of G. Then the homogeneous manifold $M=G/K$ is called a *homogeneous Kähler manifold* if it is endowed with a G-invariant Kähler structure.

Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K respectively. Then the G-invariant Kähler structure induces an endomorphism j of $\mathfrak{g}$ and a skew-symmetric bilinear from ϱ on $\mathfrak{g}$ such that for all $x, y, z \in \mathfrak{g}$ and $k \in \mathfrak{k}$ the following conditions hold [9].

$$j\mathfrak{k} \subset \mathfrak{k}; \quad j^2 x \equiv -x \pmod{\mathfrak{k}} \tag{1.1.1}$$

$$[k, jx] \equiv j[k, x] \pmod{\mathfrak{k}} \tag{1.1.2}$$

$$[jx, jy] \equiv [x, y] + j[jx, y] + j[x, jy] \pmod{\mathfrak{k}} \tag{1.1.3}$$

$$\varrho(jx, jy) = \varrho(x, y) \tag{1.1.4}$$

$$\varrho(k, x) = 0 \tag{1.1.5}$$

$$\varrho([x, y], z) + \varrho([y, z], x) + \varrho([z, x], y) = 0 \tag{1.1.6}$$

$$\varrho(jx, x) > 0 \quad \text{if} \quad x \notin \mathfrak{k}. \tag{1.1.7}$$

Conversely, let $\mathfrak{g}$ be a Lie algebra equipped with an endomorphism j and a skew-symmetric bilinear from ϱ and let $\mathfrak{k}$ be a subalgebra of $\mathfrak{g}$. Then the system $(\mathfrak{g}, \mathfrak{k}, j, \varrho)$ or simply $\mathfrak{g}$ is called *a Kähler algebra* if the above conditions are satisfied.

PROPOSITION. *Let $(\mathfrak{g}, \mathfrak{k}, j, \varrho)$ be a Kähler algebra. Let G be the connected and simply connected Lie group with Lie algebra $\mathfrak{g}$ and let K be the connected subgroup of G corresponding to $\mathfrak{k}$. Then K is closed in G and the homogeneous space $M(\mathfrak{g})=G/K$ admits a G-invariant Kähler structure corresponding to j and ϱ.*

Proof. If K is closed in G, then $M(\mathfrak{g})$ is a manifold and it is straightforward to define on $M(\mathfrak{g})$ a G-invariant Kähler structure which corresponds to j and ϱ. Therefore it suffices to show that K is closed. Set $\hat{\mathfrak{g}}=\mathfrak{g}\oplus\mathbf{R}$ and $\hat{\mathfrak{k}}=\mathfrak{k}\oplus\mathbf{R}$ and define an algebra structure on $\hat{\mathfrak{g}}$ by

$$[[x\oplus a, y\oplus b]] = [x,y] \oplus \varrho(x,y) \quad \text{for} \quad x,y\in\mathfrak{g},\ a,b\in\mathbf{R}.$$

Then $\hat{\mathfrak{g}}$ is a Lie algebra relative to $[[\ ,\ ]]$ and $\mathbf{R}\cong 0\oplus\mathbf{R}$ is an ideal of $\hat{\mathfrak{g}}$. Extend j to an endomorphism $\hat{j}$ of $\hat{\mathfrak{g}}$ by putting $\hat{j}(\mathbf{R})=0$ and define a linear form ω on $\hat{\mathfrak{g}}$ by setting $\omega(x\oplus a)=a$ for $x\in\mathfrak{g}$, $a\in\mathbf{R}$. Then $\omega([[x\oplus a, y\oplus b]])=\varrho(x,y)$ for $x,y\in\mathfrak{g}$ and $a,b\in\mathbf{R}$. Therefore we know that $(\hat{\mathfrak{g}},\hat{\mathfrak{k}},j,-d\omega)$ is a Kähler algebra. Let $\hat{G}$ be the connected and simply connected Lie group with Lie algebra $\hat{\mathfrak{g}}$ and consider the subgroup $\hat{K}'=\{g\in\hat{G}; \omega(\operatorname{Ad} gx)=\omega(x)$ for all $x\in\hat{\mathfrak{g}}\}$ of $\hat{G}$. Clearly, $\hat{K}'$ is closed in $\hat{G}$ and the Lie algebra $\hat{\mathfrak{k}}'$ of $\hat{K}'$ consists of all $y\in\hat{\mathfrak{g}}$ satisfying $\omega([y,x])=-d\omega(y,x)=0$ for any $x\in\hat{\mathfrak{g}}$. Therefore $\hat{\mathfrak{k}}'=\hat{\mathfrak{k}}$, whence the connected subgroup $\hat{K}$ of $\hat{G}$ corresponding to $\hat{\mathfrak{k}}$ is the identity component of $\hat{K}'$. In particular, $\hat{K}$ is a closed subgroup of $\hat{G}$. Since $0\oplus\mathbf{R}$ is an ideal of $\hat{\mathfrak{g}}$, $\mathfrak{g}\cong\hat{\mathfrak{g}} \bmod (0\oplus\mathbf{R})$ and $\hat{G}$ is simply connected, the canonical homomorphism from $\hat{\mathfrak{g}}$ onto $\mathfrak{g}$ induces a homomorphism π of $\hat{G}$ onto G. The kernel $\hat{C}$ of π is the closed and connected subgroup of $\hat{G}$ corresponding to $0\oplus\mathbf{R}$. Since $0\oplus\mathbf{R}$ is contained in the center of $\hat{\mathfrak{g}}$, $\hat{C}$ acts trivially on $M(\hat{\mathfrak{g}})=\hat{G}/\hat{K}$. Hence, from $G=\hat{G}/\hat{C}$ and $K=\hat{K}/\hat{C}$ it follows that G acts on $M(\hat{\mathfrak{g}})$ in a natural manner and that the isotropy subgroup of G at the origin $\hat{K}$ of $M(\hat{\mathfrak{g}})$ is the group K. Therefore K is a closed subgroup of G proving the proposition.

From now on, we denote by $M(\mathfrak{g})$ the *homogeneous Kähler manifold associated with the Kähler algebra* $\mathfrak{g}$ by the proposition above.

1.2. A Kähler algebra $(\mathfrak{g},\mathfrak{k},j,\varrho)$ is called *of flat type, of domain type*, or *of compact type* if $M(\mathfrak{g})$ is a flat Kähler manifold, or a homogeneous bounded domain with a G-invariant Kähler structure where G is the Lie group associated with $\mathfrak{g}$, or a compact simply connected homogeneous Kähler manifold, respectively.

Let j' be another endomorphism of $\mathfrak{g}$ satisfying $jx\equiv j'x \pmod{\mathfrak{k}}$ for all $x\in\mathfrak{g}$. Then $(\mathfrak{g},\mathfrak{k},j',\varrho)$ is also a Kähler algebra and its associated homogeneous Kähler manifold is the same as one associated to $(\mathfrak{g},\mathfrak{k},j,\varrho)$. Such a change of j will be called an *inessential change of j*.

Let $\mathfrak{g}'$ be a subalgebra of $\mathfrak{g}$ satisfying $j\mathfrak{g}'\subset\mathfrak{g}'+\mathfrak{k}$. Hence, after an inessential change of j, we can assume $j\mathfrak{g}'\subset\mathfrak{g}'$. Then $(\mathfrak{g}',\mathfrak{g}'\cap\mathfrak{k},j,\varrho)$ is also a Kähler algebra. We call such a subalgebra $\mathfrak{g}'$ of $\mathfrak{g}$ a *Kähler subalgebra*. The notion of a *Kähler ideal* is defined similarly.

It is easy to see that for any ideal $\mathfrak{r}$ of $\mathfrak{g}$, $\mathfrak{g}'=\mathfrak{r}+j\mathfrak{r}+\mathfrak{k}$ is a Kähler subalgebra of $\mathfrak{g}$.

1.3. *A Kähler derivation* of a Kähler algebra $(\mathfrak{g}, \mathfrak{k}, j, \varrho)$ is a derivation D of the Lie algebra $\mathfrak{g}$ satisfying the following conditions: $D\mathfrak{k} \subset \mathfrak{k}$, $Djx \equiv jDx \pmod{\mathfrak{k}}$ for all $x \in \mathfrak{g}$ and $\varrho(Dx, y) + \varrho(x, Dy) = 0$ for all $x, y \in \mathfrak{g}$. Clearly, the set of all Kähler derivations of the Kähler algebra $\mathfrak{g}$ is a Lie algebra. We will denote this Lie algebra by $\mathrm{Der}_K(\mathfrak{g})$.

Let $\mathfrak{A}$ be a subalgebra of $\mathrm{Der}_K(\mathfrak{g})$ and consider the sum of vector spaces $\mathfrak{g}(\mathfrak{A}) = \mathfrak{g} \oplus \mathfrak{A}$. We introduce an algebra structure on $\mathfrak{g}(\mathfrak{A})$ by

$$[x \oplus D_1, y \oplus D_2] = [x, y] + D_1 y - D_2 x \oplus [D_1, D_2]$$

where $x, y \in \mathfrak{g}$ and $D_1, D_2 \in \mathfrak{A}$.

It is easy to see that $\mathfrak{g}(\mathfrak{A})$ is a Lie algebra. Set $\mathfrak{k}(\mathfrak{A}) = \mathfrak{k} \oplus \mathfrak{A}$ and extend j and ϱ to $\mathfrak{g}(\mathfrak{A})$ by putting $j\mathfrak{A} = 0$ and $\varrho(\mathfrak{g}(\mathfrak{A}), \mathfrak{A}) = 0$. Then $(\mathfrak{g}(\mathfrak{A}), \mathfrak{k}(\mathfrak{A}), j, \varrho)$ is also a Kähler algebra. Clearly, $M(\mathfrak{g}(\mathfrak{A})) = M(\mathfrak{g})$.

1.4. For a Lie algebra $\mathfrak{g}$ we denote by $\mathrm{rad}(\mathfrak{g})$ the *radical* of $\mathfrak{g}$ and by $\mathrm{nil}(\mathfrak{g})$ the *nilpotent radical* of $\mathfrak{g}$, i.e. $\mathrm{nil}(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}] \cap \mathrm{rad}(\mathfrak{g}) = \mathrm{rad}([\mathfrak{g}, \mathfrak{g}])$. We note that for any representation τ of $\mathfrak{g}$ on a finite dimensional vector space V, $\tau(x)$ is a nilpotent endomorphism of V for all $x \in \mathrm{nil}(\mathfrak{g})$ ([2]).

A Kähler algebra $(\mathfrak{g}, \mathfrak{k}, j, \varrho)$ is called *effective* if $\mathfrak{k}$ contains no non-trivial ideal of $\mathfrak{g}$.

LEMMA. *Let $\mathfrak{g}'$ be a Kähler subalgebra of the effective Kähler algebra $\mathfrak{g}$ and let $\mathfrak{k}_0$ be the largest ideal of $\mathfrak{g}'$ contained in $\mathfrak{k}' = \mathfrak{g}' \cap \mathfrak{k}$. Then there exists an effective Kähler ideal, $\check{\mathfrak{g}}'$ of $\mathfrak{g}'$ such that $\mathfrak{g}' = \check{\mathfrak{g}}' \oplus \mathfrak{k}_0$.*

Proof. Let $\mathfrak{h}$ be a maximal semi-simple subalgebra of $\mathfrak{g}'$. Then

$$\mathfrak{k}_0 = \mathrm{rad}(\mathfrak{g}') \cap \mathfrak{k}_0 \oplus \mathfrak{h} \cap \mathfrak{k}_0.$$

We set

$$\mathfrak{p} = \{x \in \mathrm{rad}(\mathfrak{g}'); \mathrm{ad}\, x \text{ has only real eigenvalues on } \mathfrak{g}\}. \tag{1.4.1}$$

Then $\mathfrak{p}$ is an ideal of $\mathfrak{g}'$ containing $\mathrm{nil}(\mathfrak{g}')$. From the effectiveness of $\mathfrak{g}$ we derive $\mathfrak{p} \cap \mathfrak{k} = 0$. Thus we can find a subspace $\mathfrak{c}$ of $\mathrm{rad}(\mathfrak{g}')$ so that $\mathrm{rad}(\mathfrak{g}') = \mathfrak{p} \oplus \mathrm{rad}(\mathfrak{g}') \cap \mathfrak{k}_0 \oplus \mathfrak{c}$. Since $\mathfrak{h}$ is semi-simple, there exists a semi-simple ideal $\mathfrak{h}'$ of $\mathfrak{h}$ satisfying $\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{h} \cap \mathfrak{k}_0$. We set $\check{\mathfrak{g}}' = \mathfrak{p} \oplus \mathfrak{c} \oplus \mathfrak{h}'$. Then $\mathfrak{g}' = \check{\mathfrak{g}}' \oplus \mathfrak{k}_0$ and $\check{\mathfrak{g}}'$ is an effective Kähler ideal of $\mathfrak{g}'$.

Remark. By the result above we can treat $\mathfrak{g}'$ as an "effective" Kähler algebra.

1.5. In this subsection and the following ones we recall some facts on symplectic representations.

Let W be a real vector space together with a complex structure J and skew-symmetric bilinear form Ω. We say W is a *symplectic space* if the following conditions are satisfied

$$\Omega(Jw, Jw') = \Omega(w, w') \quad \text{for} \quad w, w' \in W$$

$$\Omega(Jw, w) > 0 \quad \text{for} \quad w \in W, w \neq 0.$$

An endomorphism p of W is called *symplectic* if p satisfies

$$\Omega(pw, w') + \Omega(w, pw') = 0 \quad \text{for} \quad w, w' \in W.$$

The following fact is used frequently in this paper.

LEMMA ([19]). *Let p and q be symplectic endomorphisms satisfying*

$$[p, q] = q \quad \textit{and} \quad p \circ J - J \circ p = q + J \circ q \circ J.$$

Then p is semi-simple and W is decomposed into the sum of subspaces $W = W_{-1/2} + W_0 + W_{1/2}$ such that for $\lambda = 0, \pm 1/2$

(a) *p leaves W_λ invariant and every eigenvalue of p on W_λ has real part λ,*

(b) *$JW_\lambda = W_{-\lambda}$,*

(c) *$q|W_0 + W_{1/2} = 0$ and $qw = jw$ for $w \in W_{-1/2}$.*

1.6. For a symplectic space W with complex structure J let $\mathfrak{sp}(W)$ denote the Lie algebra of all symplectic endomorphisms of W. We also set $\mathfrak{k}(W) = \{f \in \mathfrak{sp}(W); f \circ J = J \circ f\}$.

LEMMA. *Let $(\mathfrak{g}, \mathfrak{k}, j, \varrho)$ be a Kähler algebra of flat or compact type and let τ be a homomorphism of $\mathfrak{g}$ to $\mathfrak{sp}(W)$. Assume that $\tau(\mathfrak{k}) \subset \mathfrak{k}(W)$ holds and assume that $\tau(jx) \circ J - J \circ \tau(jx) = \tau(x) + J \circ \tau(x) \circ J$ for all $x \in \mathfrak{g}$. Then $\tau(x) \circ J = J \circ \tau(x)$ for all $x \in \mathfrak{g}$,* i.e. $\tau(\mathfrak{g}) \subset \mathfrak{k}(W)$.

Proof. Let $Sp(W)$ and $K(W)$ be the connected subgroups of $GL(W)$ corresponding to $\mathfrak{sp}(W)$ and $\mathfrak{k}(W)$ respectively. The homogeneous space $Sp(W)/K(W)$ is well known as "Siegel's upper half plane". Here the endomorphism I of $\mathfrak{sp}(W)$ corresponding to the invariant complex structure is given by $I(g) = \frac{1}{2}[J, g]$ for $g \in \mathfrak{sp}(W)$ (cf. [18]). Let $M(\mathfrak{g})$ be the homogeneous Kähler manifold associated with $\mathfrak{g}$. From the assumptions, we obtain

$\tau(\mathfrak{k}) \subset \mathfrak{k}(W)$ and $\tau(jx) \equiv I \cdot \tau(x) \pmod{\mathfrak{k}(W)}$ for any $x \in \mathfrak{g}$. This means that τ induces a holomorphic mapping of $M(\mathfrak{g})$ to $Sp(W)/K(W)$. Since $M(\mathfrak{g})$ is biholomorphically equivalent to $\mathbf{C}^N$ or compact, the image of $M(\mathfrak{g})$ must be a single point. This implies that $\tau(\mathfrak{g}) \subset \mathfrak{k}(W)$ holds, proving the lemma.

1.7. Let x be an element of a real Lie algebra $\mathfrak{g}$. Consider the endomorphism $\mathrm{ad}\,x$. There exist pairwise commuting derivations R, I and N of $\mathfrak{g}$ such that R has only real eigenvalues, I has only imaginary eigenvalues, N is nilpotent, R and I are semi-simple and $\mathrm{ad}\,x = R + I + N$. We note that R, I and N are polynomials in $\mathrm{ad}\,x$ without constant term [3]. We call R, I and N the *real part, imaginary part* and *nilpotent part* of $\mathrm{ad}\,x$ respectively and write $\mathrm{Re}(\mathrm{ad}\,x)$ for R and also $\mathrm{Im}(\mathrm{ad}\,x)$ for I.

§ 2. Quasi-normal Kähler algebras

2.1. Let M be a homogeneous Kähler manifold and let $\mathrm{Aut}(M)$ be the group of all biholomorphic isometries of M. For the study of M, we have to find a transitive subgroup of $\mathrm{Aut}(M)$ which has nice properties.

We say a Kähler algebra $\mathfrak{g}$ is *quasi-normal* if $\mathrm{ad}\,x$ has only real eigenvalues for all $x \in \mathrm{rad}(\mathfrak{g})$.

We want to prove

THEOREM. *For every homogeneous Kähler manifold M, there exists a connected subgroup G of* $\mathrm{Aut}(M)$ *satisfying*

(a) *G acts transitively on M,*

(b) *The Lie algebra $\mathfrak{g}$ of G is quasi-normal.*

This theorem will be proven in section 2.4.

2.2. We show the following

LEMMA. *Let $(\mathfrak{g}, \mathfrak{k}, j, \varrho)$ be a Kähler algebra and let $x \in \mathrm{rad}(\mathfrak{g})$. Assume that* $\mathrm{Im}(\mathrm{ad}\,x)$ *is a Kähler derivation of the subalgebra* $\mathfrak{g}' = \mathrm{rad}(\mathfrak{g}) + j\,\mathrm{rad}(\mathfrak{g}) + \mathfrak{k}$ *of* $\mathfrak{g}$. Then $\mathrm{Im}(\mathrm{ad}\,x)$ *is also a Kähler derivation of* $\mathfrak{g}$.

Proof. Let R, I and N be the real, imaginary and nilpotent part of $\mathrm{ad}\,x$ respectively. We can decompose $\mathfrak{g}$ as $\mathfrak{g} = \oplus_{\alpha \geq 0} \mathfrak{g}_\alpha$ so that $I^2|\mathfrak{g}_\alpha = \alpha^2$. Since $I\mathfrak{g} \subset \mathfrak{g}'$, $\mathfrak{g}_\alpha$ is contained in $\mathfrak{g}'$ if $\alpha \neq 0$. We want to show

$$\varrho(\mathfrak{g}_0, \mathfrak{g}_\alpha) = 0 \quad \text{for} \quad \alpha \neq 0. \tag{2.2.1}$$

Let $u_0 \in \mathfrak{g}_0$ and $v_\alpha \in \mathfrak{g}_\alpha$, $\alpha \neq 0$. Then we have

$$\frac{d}{dt}\varrho(e^{t\,\mathrm{ad}\,x}u_0, e^{t\,\mathrm{ad}\,x}v_\alpha) = \varrho(x, e^{t\,\mathrm{ad}\,x}[u_0, v_\alpha]). \tag{2.2.2}$$

Since $[u_0, v_\alpha] \in \mathfrak{g}'$ and since I is a Kähler derivation of $\mathfrak{g}'$, we have

$$\varrho(x, e^{t\,\mathrm{ad}\,x}[u_0, v_\alpha]) = \varrho(x, e^{tR}\cdot e^{tN}[u_0, v_\alpha]).$$

For the proof of (2.2.1), it is sufficient to show $\varrho(u_0, v_\alpha)=0$ under the additional assumption that u_0 and v_α are eigenvectors for R. This implies that (2.2.2) is of type

$$\frac{d}{dt}e^{st}(X(t)\cos\alpha t + Y(t)\sin\alpha t) = e^{st}Z(t), \tag{2.2.3}$$

where $X(t)$, $Y(t)$ and $Z(t)$ are polynomial functions of t. An integration of (2.2.3) yields

$$e^{st}(X(t)\cos\alpha t + Y(t)\sin\alpha t) = e^{st}W(t)+\text{const.}, \tag{2.2.4}$$

where $W(t)$ is also a polynomial. Since $\alpha \neq 0$, the equation (2.2.4) implies $X(t)= Y(t)=W(t)=0$. This proves (2.2.1).

From our assumption we know $I\mathfrak{k}\subset\mathfrak{k}$ and $\varrho(Ix, y)+\varrho(x, Iy)=0$ for all $x, y \in \oplus_{\alpha\neq 0}\,\mathfrak{g}_\alpha$. Using (2.2.1) we obtain $\varrho(Ix, y)+\varrho(x, Iy)=0$ for all $x, y \in \mathfrak{g}$. Hence it remains to show that I commutes with j modulo $\mathfrak{k}$. Consider the set $\mathfrak{u}=\{x \in \mathfrak{g}; \varrho(x, \mathfrak{g}')=0\}$. Then $I\mathfrak{u}\subset\mathfrak{u}\cap\mathfrak{g}'=\mathfrak{k}$. Since $\mathfrak{u}$ is j-invariant, we also have $Ij\mathfrak{u}\equiv 0 \pmod{\mathfrak{k}}$. Therefore $I\circ j\equiv j\circ I \pmod{\mathfrak{k}}$ on $\mathfrak{u}$. On $\mathfrak{g}'$ this identity holds by assumption; hence it also holds on $\mathfrak{g}'+\mathfrak{u}=\mathfrak{g}$. This finishes the proof of the lemma.

2.3. Let $(\mathfrak{g}, \mathfrak{k}, j, \varrho)$ be an effective Kähler algebra. We set

$$\mathfrak{p}_0 = \{x \in \mathrm{rad}(\mathfrak{g}); \mathrm{ad}\,x \text{ has only real eigenvalues}\},$$

$$\mathfrak{p}_1 = \{x \in \mathrm{rad}(\mathfrak{g}); \varrho(jx, y)=0 \text{ for all } y \in \mathfrak{p}_0\}.$$

Since $[\mathfrak{g}, \mathrm{rad}(\mathfrak{g})]\subset\mathrm{nil}(\mathfrak{g})\subset\mathfrak{p}_0$, the space $\mathfrak{p}_0$ is an ideal of $\mathfrak{g}$. Moreover, $\mathfrak{k}$ leaves $\mathfrak{p}_0$ and $\mathfrak{p}_1$ invariant, whence $[\mathfrak{k}, \mathfrak{p}_1]=0$.

LEMMA. *Let* $x \in \mathfrak{p}_1$, *then* $\mathrm{Im}(\mathrm{ad}\,x) \in \mathrm{Der}_K(\mathfrak{g})$.

Proof. By Lemma 2.2, it is sufficient to show that $\mathrm{Im}(\mathrm{ad}\,x)$ is a Kähler derivation of $\mathfrak{g}'=\mathrm{rad}(\mathfrak{g})+j\,\mathrm{rad}(\mathfrak{g})+\mathfrak{k}$. From the Radical Conjecture (see Introduction) together with the consideration in section 1.4, we can find a solvable Kähler subalgebra $\mathfrak{s}$ such that $\mathfrak{s}\supset\mathrm{rad}(\mathfrak{g})$ and $\mathfrak{g}'=\mathfrak{s}+\mathfrak{k}$. Here $\mathfrak{s}\cap\mathfrak{k}$ may not be zero but we know that $\mathfrak{s}\cap\mathfrak{k}$ is abelian. Let us set $\mathfrak{s}'=\{x\in\mathfrak{s};\ \mathrm{ad}\,x$ has only real eigenvalues in $\mathfrak{g}\}$. Then $[\mathfrak{s},\mathfrak{s}]\subset\mathfrak{s}'$ and $\mathfrak{s}'\cap\mathfrak{k}=0$. Let $\mathfrak{c}$ be a complementary subspace of $\mathfrak{s}'+\mathfrak{s}\cap\mathfrak{k}$ in $\mathfrak{s}$ and put $\mathfrak{s}''=\mathfrak{s}'+\mathfrak{c}$. Then $\mathfrak{s}''$ is an ideal of $\mathfrak{s}$ satisfying $\mathfrak{s}=\mathfrak{s}''\oplus\mathfrak{s}\cap\mathfrak{k}$. By construction we have $\mathfrak{g}'=\mathfrak{s}''\oplus\mathfrak{k}$ hence $\mathfrak{s}''$ is a Kähler subalgebra of $\mathfrak{s}$. Therefore we may assume that $\mathfrak{s}''$ is j-invariant. By [6], every solvable Kähler algebra with vanishing isotropy subalgebra is a modification of a split solvable Kähler algebra. Therefore there exists a linear map D (called a modification map) of $\mathfrak{s}''$ to $\mathrm{Der}_K(\mathfrak{s}'')$ satisfying the following properties (1) and (2):

(1) $[D(x),D(y)]=0,\ D([x,y])=0,\ D(D(x)y)=0$ for $x,y\in\mathfrak{s}''$.

Define a product $(\ ,\ )$: $\mathfrak{s}''\times\mathfrak{s}''\to\mathfrak{s}''$ by

$$(x,y)=[x,y]+D(x)y-D(y)x \quad \text{for} \quad x,y\in\mathfrak{s}''.$$

From the properties (1) it follows that the product $(\ ,\ )$ also defines a Lie algebra structure on $\mathfrak{s}''$ and that $(\mathfrak{s}'',0,j,\varrho)$ together with the product $(\ ,\ )$ is a Kähler algebra. Moreover, this new Lie algebra has the additional property

(2) the adjoint representation relative to $(\ ,\ )$ has only real eigenvalues.

We define the set $\mathfrak{s}''_0=\{x\in\mathfrak{s}'';D(x)=0\}$ and $\mathfrak{s}''_1=\{x\in\mathfrak{s}'';\ \varrho(jx,y)=0$ for all $y\in\mathfrak{s}''_0\}$. Then $\mathfrak{s}''=\mathfrak{s}''_0\oplus\mathfrak{s}''_1$. Using the properties (1) and (2) we can easily show

(3) $D(\mathfrak{s}'')\mathfrak{s}''_1=0$.

(4) If $x\in\mathfrak{s}''$ and $D(\mathfrak{s}'')x=0$, then $D(x)=-\mathrm{Im}\,(\mathrm{ad}\,x)|\mathfrak{s}''$.

Therefore, for any $x\in\mathfrak{s}''$, $D(x)$ can be expressed as a polynomial without constant term of $\mathrm{ad}\,z$ for a suitable $z\in\mathfrak{s}''$. Next we show for $x\in\mathfrak{s}''$,

(5) $D(x)=0$ if and only if $\mathrm{Im}(\mathrm{ad}\,x)|\mathfrak{s}''=0$.

First we note that both endomorphisms are semi-simple and map $\mathfrak{s}''_1$ into $\mathfrak{s}''_0$ and leave $\mathfrak{s}''_0$ invariant. Therefore it suffices to prove (5) for the restrictions to $\mathfrak{s}''_0$. We note that on $\mathfrak{s}''_0$ we have $D(x)y=(x,y)-[x,y]$. Assume now $D(x)=0$, then $\mathrm{ad}\,x|\mathfrak{s}''_0$ has only real eigenvalues, whence $\mathrm{Im}(\mathrm{ad}\,x)|\mathfrak{s}''_0=0$. Assume on the other hand that $\mathrm{Im}(\mathrm{ad}\,x)|\mathfrak{s}''_0=0$ holds. Then $\mathrm{ad}\,x$ has only real eigenvalues on $\mathfrak{s}''_0$. Since $\mathfrak{s}''$ is solvable, it is easy to see that the map $y\to(x,y)-[x,y], y\in\mathfrak{s}''_0$, has only real eigenvalues. Hence $D(x)|\mathfrak{s}''_0=0$, proving (5). From (5) we obtain $\mathfrak{s}'\subset\mathfrak{s}''_0$. Hence $[\mathfrak{s},\mathfrak{s}]\subset\mathrm{nil}(\mathfrak{s})\subset\mathfrak{s}'\subset\mathfrak{s}''_0$. Therefore $\mathfrak{s}''_0$ is an ideal of $\mathfrak{s}$. Noting that $\mathfrak{s}''$ is an ideal of $\mathfrak{s}$, and that $\mathfrak{s}\cap\mathfrak{k}$ leaves invariant $\mathfrak{s}''_0$ and $\mathfrak{s}''_1$, we then have $[\mathfrak{s}''_1,\mathfrak{s}\cap\mathfrak{k}]=0$. In particular, $\mathrm{Im}(\mathrm{ad}\,x)|\mathfrak{s}\cap\mathfrak{k}=0$ for all $x\in\mathfrak{s}''_1$. Now let $x\in\mathfrak{p}_1$ and $z\in\mathrm{rad}(\mathfrak{g})$. Then $x=x_1+x_2$ and $z=z_1+z_2$, where $x_1,z_1\in\mathfrak{s}''$ and $x_2,z_2\in\mathfrak{s}\cap\mathfrak{k}$. For $y\in\mathfrak{s}''_1$ we thus obtain $\varrho(\mathrm{Im}(\mathrm{ad}\,y)x, jz)=-\varrho(D(y)x_1, jz_1)=\varrho(x_1, jD(y)z_1)=\varrho(x, j\,\mathrm{Im}(\mathrm{ad}\,y)z)=0$, because $\mathrm{Im}(\mathrm{ad}\,y)z\in\mathfrak{p}_0$. Hence we have $D(y)x_1=\mathrm{Im}(\mathrm{ad}\,y)x=0$ for all $y\in\mathfrak{s}_1''$. Then, by (4), $\mathrm{Im}(\mathrm{ad}\,x_1)|\mathfrak{s}''\in\mathrm{Der}_K(\mathfrak{s}'')$. Since $[\mathfrak{k},\mathfrak{p}_1]=0$ and $\mathfrak{s}\cap\mathfrak{k}$ is abelian, we have $[x_1,\mathfrak{s}\cap\mathfrak{k}]=0$. This implies $\mathrm{Im}(\mathrm{ad}\,x_1)\in\mathrm{Der}_K(\mathfrak{s})$ and $\mathrm{Im}(\mathrm{ad}\,x)=\mathrm{Im}(\mathrm{ad}\,x_1)+\mathrm{Im}(\mathrm{ad}\,x_2)$. Hence we also have $\mathrm{Im}(\mathrm{ad}\,x)\in\mathrm{Der}_K(\mathfrak{s})$. Consequently, $\mathrm{Im}(\mathrm{ad}\,x)\in\mathrm{Der}_K(\mathfrak{g}')$ because $\mathfrak{g}'=\mathfrak{s}+\mathfrak{k}$ and $\mathrm{Im}(\mathrm{ad}\,x)|\mathfrak{k}=0$. Thus we have proved the lemma.

2.4. In this section we prove Theorem 2.1. Take a subgroup G satisfying (a). Let K be the isotropy subgroup of G and $\mathfrak{k}$ the Lie algebra of K. Assume that $\mathrm{rad}(\mathfrak{g})\cap\mathfrak{k}\neq 0$. We can find a subspace $\mathfrak{c}$ of $\mathrm{rad}(\mathfrak{g})$ so that $\mathrm{rad}(\mathfrak{g})=\mathrm{nil}(\mathfrak{g})\oplus(\mathrm{rad}(\mathfrak{g})\cap\mathfrak{k})\oplus\mathfrak{c}$. We set $\mathfrak{g}'=\mathrm{nil}(\mathfrak{g})\oplus\mathfrak{c}\oplus\mathfrak{h}$, where $\mathfrak{h}$ is a maximal semi-simple subalgebra of $\mathfrak{g}$. Clearly $\mathfrak{g}=\mathfrak{g}'+\mathfrak{k}$ and $\mathrm{rad}(\mathfrak{g}')\cap\mathfrak{k}=0$. Therefore, by taking the subgroup corresponding to $\mathfrak{g}'$ instead of G, we may assume $\mathrm{rad}(\mathfrak{g})\cap\mathfrak{k}=0$.

Let $\mathfrak{p}_0$ and $\mathfrak{p}_1$ be as in section 2.3. Take $a\in\mathfrak{p}_1$ and set $I=\mathrm{Im}(\mathrm{ad}\,a)$. Then, by Lemma 2.3, $I\in\mathrm{Der}_K(\mathfrak{g})$. Consider the Kähler algebra $\mathfrak{g}(\mathfrak{A})$ constructed in section 1.3, where $\mathfrak{A}=\mathbf{R}I$. We also write $\mathfrak{g}=\mathfrak{p}_0\oplus\mathbf{R}a\oplus\mathfrak{p}'\oplus\mathfrak{h}$, where $\mathfrak{h}$ is a maximal semi-simple subalgebra of $\mathfrak{g}$ and $\mathfrak{p}'\subset\mathfrak{p}_1$. Define a linear map $\xi\colon\mathfrak{g}\to\mathfrak{g}(\mathfrak{A})$ by $\xi(x)=x$ if $x\in\mathfrak{p}_0\oplus\mathfrak{p}'\oplus\mathfrak{h}$ and $\xi(a)=a-I$. Set $\hat{\mathfrak{g}}=\xi(\mathfrak{g})$. Then $\hat{\mathfrak{g}}$ is an ideal of $\mathfrak{g}(\mathfrak{A})$ since $[\mathfrak{g}(\mathfrak{A}),\mathfrak{g}(\mathfrak{A})]=[\mathfrak{g},\mathfrak{g}]=\mathrm{nil}(\mathfrak{g})+\mathfrak{h}\subset\mathfrak{p}_0+\mathfrak{h}\subset\hat{\mathfrak{g}}$ holds. Moreover, $\mathrm{rad}(\mathfrak{g}(\mathfrak{A}))=\mathrm{rad}(\mathfrak{g})\oplus\mathbf{R}I$ because $[I,\mathfrak{g}(\mathfrak{A})]\subset\mathrm{nil}(\mathfrak{g})$. This implies $\mathrm{rad}(\hat{\mathfrak{g}})=\mathrm{rad}(\mathfrak{g}(\mathfrak{A}))\cap\hat{\mathfrak{g}}=\xi(\mathrm{rad}(\mathfrak{g}))$ and we also obtain $\mathrm{rad}(\hat{\mathfrak{g}})\cap\mathfrak{k}(\mathfrak{A})=\xi(\mathrm{rad}(\mathfrak{g})\cap\mathfrak{k})=0$. It is important to note that $\mathrm{Im}(\mathrm{ad}\,x)=0$ in $\hat{\mathfrak{g}}$ for all $x\in\xi(\mathfrak{p}_0+\mathbf{R}a)$. We thus have $\dim\hat{\mathfrak{p}}_0>\dim\mathfrak{p}_0$ where $\hat{\mathfrak{p}}_0$ is defined for $\hat{\mathfrak{g}}$ as $\mathfrak{p}_0$ is for $\mathfrak{g}$ in section 2.3. A repetition of this procedure will therefore yield the assertion if we are able to find a subgroup $\hat{G}\subset\mathrm{Aut}(M)$ which acts transitively on M and has $\hat{\mathfrak{g}}$ as its Lie algebra.

Assume first that M is simply connected. In this case it is straightforward to see that our assertion holds. With this all assertions in the rest of this paper hold under the additional assumption that M is simply connected. In particular the Fundamental Conjecture holds in this case. Let now M be arbitrary. Let G^* and $G_{\mathfrak{A}}$ be the simply connected Lie groups corresponding to $\mathfrak{g}$ and $\mathfrak{g}(\mathfrak{A})$ respectively. We denote by W the 1-dimensional connected subgroup of $G_{\mathfrak{A}}$ generated by $\mathfrak{A}=\mathbf{R}I$. We then have $G_{\mathfrak{A}}=G^*W$ (semi-direct product) and both G^* and W are closed subgroups of $G_{\mathfrak{A}}$. Let π denote the projection $G^*\to G$ and $K^*=\pi^{-1}(K)$. Then $M=G^*/K^*$ and $M^*=G^*/K_0^*=G_{\mathfrak{A}}/K_0^*W$ is the universal covering space of M, where K_0^* is the identity component of K^*. As is mentioned above the Fundamental Conjecture holds for M^*. Therefore M^* is a holomorphic fiber bundle over a homogenous bounded domain D and the fiber is $\mathbf{C}^n\times N$, N denoting a compact simply connected homogenous Kähler manifold. Then $D=G^*/L^*$, $\mathbf{C}^n\times N=L^*/K_0^*$, $N=L^*/F^*$ and $\mathbf{C}^n=F^*/K_0^*$ for some connected closed subgroups $L^*\supset F^*\supset K_0^*$. We can see $K^*\subset L^*$ in a similar way as in the proof of Lemma 7.6. Then using the arguments of [28; Appendix] we have $K^*\subset F^*$. We also have $D=G_{\mathfrak{A}}/L^*W$ and $N=L^*W/F^*W$. Therefore $\mathfrak{f}+\mathbf{R}I$ is a Kähler subalgebra of $\mathfrak{g}(\mathfrak{A})$, where $\mathfrak{f}$ is the Lie algebra of F^*. We can find a Kähler ideal $\mathfrak{a}$ of $\mathfrak{f}$ such that $\mathfrak{f}=\mathfrak{a}+\mathfrak{k}$ (cf. section 3.3) and $\mathfrak{a}$ is decomposed as $\mathfrak{a}=\mathfrak{a}_0+\mathfrak{a}_1$, where $\mathfrak{a}_0=[\mathfrak{a},\mathfrak{a}]$ and $\mathfrak{a}_1$ is the orthogonal complement of $\mathfrak{a}_0$ in $\mathfrak{a}$ ([6]). Here we can assume that $\mathfrak{a}\supset\mathrm{nil}(\mathfrak{f})$. But then $\mathfrak{a}$, $\mathfrak{a}_0$ and $\mathfrak{a}_1$ are invariant under I.

We want to show that $\mathrm{Ad}\,kI=I$ for all $k\in K^*$. Since $\mathrm{Ad}\,ga\equiv a\ (\mathrm{mod\,nil}(\mathfrak{g}))$ for all $g\in G^*$, $\mathrm{Ad}\,K^*a=a$ follows. Therefore we have $\mathrm{Ad}\,k\circ I\circ\mathrm{Ad}\,k^{-1}=I$ for all $k\in K^*$. This means that $\mathrm{Ad}\,kI-I$ is in the center of $\mathfrak{g}$. Recall that both $\mathfrak{a}_0$ and $\mathfrak{a}_1$ are abelian ([6]). We have

$$K^*=(K^*\cap\exp\mathfrak{a}_0\cdot\exp\,\mathfrak{a}_1)\cdot K_0^*.$$

Therefore we may assume that $k=\exp x_0\cdot\exp x_1$, where $x_i\in\mathfrak{a}_i$. Then $\mathrm{Ad}\,kI-I=-Ix_0-Ix_1-[x_0,Ix_1]$. Since $Ix=0$ for any x contained in the center of $\mathfrak{g}$, we have $Ix_0=Ix_1=0$. Hence $\mathrm{Ad}\,kI=I$, proving our assertion. But then $K_{\mathfrak{A}}=K^*W$ is a closed subgroup of $G_{\mathfrak{A}}$. Therefore $G_{\mathfrak{A}}$ acts on $M=G^*/K^*=G_{\mathfrak{A}}/K_{\mathfrak{A}}$. Clearly this action is holomorphic and isometric.

Let $\hat{G}$ be the connected subgroup of $G_{\mathfrak{A}}$ generated by $\hat{\mathfrak{g}}$. From the above arguments it follows that $\hat{G}$ acts transitively on M. Since $\mathrm{rad}(\mathfrak{g})\cap\mathfrak{k}=0$ we obtain $\hat{\mathfrak{g}}\cap\mathfrak{k}(\mathfrak{A})\subset\mathfrak{k}$, whence $\hat{G}$ acts almost effectively on M. Therefore, after dividing by a discrete subgroup (if necessary), we can assume $\hat{G}\subset\mathrm{Aut}(M)$. As mentioned before, from this the theorem follows.

2.5. From Theorem 2.1 it follows that for the study of homogeneous Kähler manifolds we only have to know the structure of quasi-normal Kähler algebras.

The following theorem is a fundamental result of our paper.

THEOREM. *Let $(\mathfrak{g},\mathfrak{k},j,\varrho)$ be an effective quasi-normal Kähler algebra. Then $\mathfrak{g}$ is decomposed as*

$$\mathfrak{g}=\mathfrak{a}+\mathfrak{h},\ \textit{where}\ \mathfrak{a}\cap\mathfrak{h}=0\ \textit{and}\ \varrho(\mathfrak{a},\mathfrak{h})=0,$$

and where $\mathfrak{a}$ is an abelian Kähler ideal of $\mathfrak{g}$ and $\mathfrak{h}$ is a quasi-normal Kähler subalgebra *containing* $\mathfrak{k}$. *Moreover, there exists a reductive Kähler subalgebra* $\mathfrak{u}$ *of* $\mathfrak{h}$ *which satisfies the following properties:*

(a) $\mathfrak{u}$ *contains* $\mathfrak{k}$, *the semi-simle part of* $\mathfrak{u}$ *is compact, the center of* $\mathfrak{u}$ *is contained in* $\mathfrak{k}$, *and* $[u, jx] \equiv j[u, x] \pmod{\mathfrak{u}}$ *holds for all* $u \in \mathfrak{u}$, $x \in \mathfrak{h}$.

(b) *Let* H *be the connected simply connected Lie group with Lie algebra* $\mathfrak{h}$ *and* U *the connected subgroup of* H *corresponding to* $\mathfrak{u}$. *Then* U *is closed in* H *and the homogeneous space* H/U, *equipped with the* H*-invariant complex structure induced from the operator* j, *is biholomorphically equivalent to a homogeneous bounded domain.*

Remark. Let K be the connected subgroup of U corresponding to $\mathfrak{k}$ and consider the homogeneous Kähler manifold U/K. From (a) we also know that a connected compact semi-simple subgroup of U acts on U/K transitively. Therefore, by [1], U/K is compact and simply connected. But then from the proof of [17; Theorem B], we know that there exists a linear form ω on $\mathfrak{h}$ so that $(\mathfrak{h}, \mathfrak{k}, j, \omega)$ becomes a j-algebra in the sense of [18]. Therefore our decomposition $\mathfrak{g}=\mathfrak{a}+\mathfrak{h}$ is a generalization of a result of [25] and [9] which is obtained under the additional assumptions that $\mathfrak{g}$ is solvable and $\mathfrak{k}=0$.

2.6. In order to prove Theorem 2.5, we divide effective Kähler algebras $(\mathfrak{g}, \mathfrak{k}, j, \varrho)$ into two classes as follows. Consider the Kähler subalgebra $\mathfrak{g}'=\mathfrak{n}+j\mathfrak{n}+\mathfrak{k}$, where $\mathfrak{n}=\mathrm{nil}(\mathfrak{g})$. By the Radical Conjecture [8], there exists a solvable Kähler subalgebra $\mathfrak{m}$ of $\mathfrak{g}'$ such that $\mathfrak{g}'=\mathfrak{m}+\mathfrak{k}$ and $\mathfrak{m} \cap \mathfrak{k}=0$. In view of [7; 4.7], we can assume that $\mathfrak{m}$ contains $\mathfrak{n}$. Hence after an inessential change of j we can assume $\mathfrak{m}=\mathfrak{n}+j\mathfrak{n}$. By [6] every solvable Kähler algebra with vanishing isotropy subalgebra is decomposed into the sum of Kähler algebras of flat type and of domain type which are orthogonal to each other. Hence we can write $\mathfrak{m}=\mathfrak{a}+\mathfrak{t}$ where $\mathfrak{a} \cap \mathfrak{t}=0$ and $\varrho(\mathfrak{a}, \mathfrak{t})=0$ and where $\mathfrak{a}$ (resp. $\mathfrak{t}$) is a Kähler algebra of flat type (resp. of domain type). We will consider the following two cases:

Case I: $\mathfrak{t}=0$ (containing the case where $\mathfrak{n}=0$).

Case II: $\mathfrak{t} \neq 0$.

Clearly, in the first case, $\mathfrak{g}'$ is of flat type and in the second case $\mathfrak{g}'$ is not of flat type.

From the next section on, we will investigate Kähler algebras of type Case I and of type Case II separately and prove Theorem 2.5 in both cases.

§ 3. Kähler algebras of type case I

3.1. In this section, we shall study the structure of a Kähler algebra $\mathfrak{g}$ for which the subalgebra $\mathrm{nil}(\mathfrak{g})+j\,\mathrm{nil}(\mathfrak{g})+\mathfrak{k}$ is of flat type. We first show

PROPOSITION. *Let* $(\mathfrak{g}, \mathfrak{k}, j, \varrho)$ *be an effective Kähler algebra. Then* $\mathrm{nil}(\mathfrak{g})+j\,\mathrm{nil}(\mathfrak{g})+\mathfrak{k}$ *is of flat type if and only* if $\mathrm{rad}(\mathfrak{g})+j\,\mathrm{rad}(\mathfrak{g})+\mathfrak{k}$ *is of flat type.*

Proof. Set $\tilde{\mathfrak{g}}=\mathrm{rad}(\mathfrak{g})+j\,\mathrm{rad}(\mathfrak{g})+\mathfrak{k}$ and $\mathfrak{g}'=\mathrm{nil}(\mathfrak{g})+j\,\mathrm{nil}(\mathfrak{g})+\mathfrak{k}$. It is easy to see that $\mathfrak{g}'$ is an ideal of $\tilde{\mathfrak{g}}$ and that $\tilde{\mathfrak{g}}/\mathfrak{g}'$ is abelian. Consider the Kähler subalgebra $\mathfrak{b}$ given by $\mathfrak{b}=\{x\in\tilde{\mathfrak{g}};\ \varrho(x,y)=0 \text{ for all } y\in\mathfrak{g}'\}$. Then $\mathfrak{k}$ is an ideal of $\mathfrak{b}$. Therefore, by Lemma 1.4 we can find a Kähler ideal $\check{\mathfrak{b}}$ of $\mathfrak{b}$ such that $\mathfrak{b}=\check{\mathfrak{b}}\oplus\mathfrak{k}$. But then $\tilde{\mathfrak{g}}=\mathfrak{g}'\oplus\check{\mathfrak{b}}$, $[\check{\mathfrak{b}},\mathfrak{k}]=0$ and $\varrho(\mathfrak{g}',\check{\mathfrak{b}})=0$. Since $\tilde{\mathfrak{g}}/\mathfrak{g}'$ is abelian, we know that $\check{\mathfrak{b}}$ is abelian. We may assume that $\check{\mathfrak{b}}$ is j-invariant. Then $\check{\mathfrak{b}}$ is a Kähler algebra of flat type.

Let $x\in\check{\mathfrak{b}}$. Consider the action $\tau(x)$ on $V=\mathfrak{g}'/\mathfrak{k}$ induced from the adjoint representation. The vector space V equipped with the skew-symmetric bilinear from Ω induced by ϱ and with the complex structure J induced by j is a symplectic space. Then $\tau(x)$ is a symplectic endomorphism. From (1.1.3), we have $\tau(jx)\circ J-J\circ\tau(jx)=\tau(x)+J\circ\tau(x)\circ J$. Therefore, by Lemma 1.6, we have $\tau(x)\circ J=J\circ\tau(x)$ for any $x\in\check{\mathfrak{b}}$. This means $[x,jy]\equiv j[x,y] \pmod{\mathfrak{k}}$ for any $x\in\check{\mathfrak{b}}$ and $y\in\mathfrak{g}'$. Since also $[\check{\mathfrak{b}},\mathfrak{k}]=0$ and $\varrho([x,a],b)+\varrho(a,[x,b])=\varrho(x,[a,b])=0$ for all $a,b\in\tilde{\mathfrak{g}}$ we have $\mathrm{ad}\,x|\tilde{\mathfrak{g}}\in\mathrm{Der}_K(\tilde{\mathfrak{g}})$ for any $x\in\check{\mathfrak{b}}$. Set $\mathfrak{B}=\{\mathrm{ad}\,x|\tilde{\mathfrak{g}};x\in\check{\mathfrak{b}}\}$. By section 1.3, $\tilde{\mathfrak{g}}(\mathfrak{B})=\tilde{\mathfrak{g}}\oplus\mathfrak{B}$ is a Kähler algebra and $M(\tilde{\mathfrak{g}}(\mathfrak{B}))=M(\tilde{\mathfrak{g}})$. Let $\mathfrak{b}'=\{x-\mathrm{ad}\,x|\mathfrak{g};\ x\in\check{\mathfrak{b}}\}$. Then $\mathfrak{b}'$ is abelian and both $\mathfrak{b}'$ and $\mathfrak{g}'$ are Kähler ideals of $\tilde{\mathfrak{g}}(\mathfrak{B})$. Moreover, $\tilde{\mathfrak{g}}(\mathfrak{B})=\mathfrak{g}'\oplus\mathfrak{b}'\oplus\mathfrak{B}$ holds. Therefore, $M(\tilde{\mathfrak{g}})=M(\mathfrak{g}')\times M(\mathfrak{b}')$. Since $M(\mathfrak{b}')$ is flat, $M(\tilde{\mathfrak{g}})$ is flat if and only if $M(\mathfrak{g}')$ is flat. This finishes the proof of the proposition.

3.2. In the following five sections we will prove the theorem below. Our arguments are similar to the ones used in [8] for the corresponding (flat) case.

THEOREM. *Let* $(\mathfrak{g}, \mathfrak{k}, j, \varrho)$ *be an effective Kähler algebra of type Case* I. *Then* $\mathrm{rad}(\mathfrak{g})$ *is a Kähler ideal of* $\mathfrak{g}$ *of flat type.*

3.3. Let $(\mathfrak{g}, \mathfrak{k}, j, \varrho)$ be an effective Kähler algebra of type Case I and let $\mathfrak{a}=\mathfrak{n}+j\mathfrak{n}$ where $\mathfrak{n}=\mathrm{nil}(\mathfrak{g})$. As in [8], we can choose j so that $\mathfrak{a}$ is a Kähler ideal of $\mathfrak{g}'=\mathfrak{n}+j\mathfrak{n}+\mathfrak{k}$. In fact, since $\mathfrak{g}'$ is of flat type, every semi-simple subalgebra of $\mathfrak{g}'$ is compact. Therefore we can assume that a maximal semi-simple subalgebra of $\mathfrak{g}'$ is a subalgebra of $\mathfrak{k}$. But then $\mathfrak{g}'=\mathrm{rad}(\mathfrak{g}')+\mathfrak{k}$. Let $\mathfrak{p}$ be defined as in (1.4.1). Then we can find an $\mathrm{ad}\,\mathfrak{k}$-invariant subspace $\mathfrak{c}$ so that $\mathrm{rad}(\mathfrak{g}')=\mathfrak{p}\oplus\mathrm{rad}(\mathfrak{g}')\cap\mathfrak{k}\oplus\mathfrak{c}$. Then $\mathfrak{g}'=\mathfrak{p}\oplus\mathfrak{c}\oplus\mathfrak{k}$ and $\mathfrak{p}\oplus\mathfrak{c}$ is an ideal of $\mathfrak{g}'$ containing $\mathfrak{n}$. We can assume that $\mathfrak{p}\oplus\mathfrak{c}$ is j-invariant. But then $\mathfrak{n}+j\mathfrak{n}=\mathfrak{p}\oplus\mathfrak{c}$, whence $\mathfrak{n}+j\mathfrak{n}$ is an ideal of $\mathfrak{g}'$. We then have $[k,jx]=j[k,x]$ for $k\in\mathfrak{k}$ and $x\in\mathfrak{a}$. Set $\mathfrak{a}_0=[\mathfrak{a},\mathfrak{a}]$. By [6; 3.3], $\mathfrak{a}_0$ is an abelian Kähler ideal of $\mathfrak{a}$ and its orthogonal complement $\mathfrak{a}_1$ in $\mathfrak{a}$ relative to ϱ is an abelian Kähler subalgebra. We also know from [8; 1.2] that $\mathfrak{n}=\mathfrak{n}_0+\mathfrak{n}_1$ holds, where $\mathfrak{n}_\lambda=\mathfrak{n}\cap\mathfrak{a}_\lambda$, $\lambda=0,1$. We would like to point out that $\mathfrak{a}$ is regarded as the Lie algebra of a transitive subgroup of euclidian transformations of some $\mathbf{C}^r$. Therefore, if $\operatorname{ad}x\,|\,\mathfrak{a}$ is nilpotent, then x generates a translation. As a result, $[x,y]=0$ holds for $x,y\in\mathfrak{a}$ if both $\operatorname{ad}x\,|\,\mathfrak{a}$ and $\operatorname{ad}y\,|\,\mathfrak{a}$ are nilpotent. Using this we have

LEMMA ([8;1.3]). (1) $\mathfrak{n}+j\mathfrak{n}_0$ *is abelian and for any* $x\in\mathfrak{n}+j\mathfrak{n}_0$, $\operatorname{ad}x$ *is a nilpotent endomorphism of* $\mathfrak{g}$.

(2) $[\mathfrak{n}_1,\mathfrak{a}]=0$.

(3) $j\mathfrak{n}_1$ *is an abelian ideal of* $j\mathfrak{n}_1+\mathfrak{k}$ *and we have*

$$\operatorname{Im}(\operatorname{ad}jx)\,|\,j\mathfrak{n}_1+\mathfrak{k}=0 \quad \textit{for any} \quad x\in\mathfrak{n}_1.$$

3.4. In this section we show that sections 1.5 and 1.7 of [8] are still valid in our context. The first part of the lemma below can be proven as in [8; 1.5] and will therefore be omitted. The proof of the second part is a simplification of the proof of [8; 1.7].

LEMMA. *For every* $x\in\mathfrak{n}_1$, $\operatorname{ad}x$ *has only imaginary eigenvalues and*

$$\operatorname{Im}(\operatorname{ad}jx)\circ j\equiv j\circ\operatorname{Im}(\operatorname{ad}jx) \pmod{\mathfrak{k}}.$$

Proof. Since $[\mathfrak{k},\mathfrak{n}_1]$ is $\operatorname{ad}\mathfrak{k}$-invariant, we can find a subspace $\mathfrak{n}_1'$ such that $\mathfrak{n}_1=[\mathfrak{k},\mathfrak{n}_1]\oplus\mathfrak{n}_1'$ and $[\mathfrak{k},\mathfrak{n}_1']=0$. Then $j[\mathfrak{k},\mathfrak{n}_1]=[\mathfrak{k},j\mathfrak{n}_1]\subset\operatorname{nil}(\mathfrak{a})$, whence $\operatorname{Im}(\operatorname{ad}j[\mathfrak{k},x])=0$ for $x\in\mathfrak{n}_1$. Therefore it is sufficient to show the assertion for every element x satisfying $[x,\mathfrak{k}]=0$. But then $[jx,\mathfrak{k}]=0$. Hence both $\operatorname{ad}x$ and $\operatorname{ad}jx$, induce an endomorphism of $\mathfrak{g}/\mathfrak{k}$, which will be denoted by q and p respectively. We also denote by J the complex structure of $\mathfrak{g}/\mathfrak{k}$ induced by j. We then have

$$p\circ J-J\circ p=q+J\circ q\circ J \tag{3.4.1}$$

because of (1.1.3). We also know from Lemma 3.3,

$$[p,q]=0, \quad q^2=0 \quad \text{and} \quad q\circ J\circ q=0. \tag{3.4.2}$$

Let p_1 denote the J-linear part of p. From (3.4.1) and (3.4.2) we can easily see that the semi-simple part of p coincides with the semi-simple part of p_1 (see [6; p. 173]). This implies the assertion, because p has only imaginary eigenvalues.

3.5. Set $\mathfrak{U}=\{\mathrm{Im}(\mathrm{ad}\,jx); x\in\mathfrak{n}_1\}$. Then $\mathfrak{U}$ is an abelian space of semi-simple derivations of $\mathfrak{g}$. Let U denote the closure of the connected subgroup of $GL(\mathfrak{g})$ generated by $\mathfrak{U}$. Then U is a compact group. Define a skew-symmetric bilinear from $\bar{\varrho}$ on $\mathfrak{g}$ by

$$\bar{\varrho}(x,y)=\int_U \varrho(ux,uy)\,du \quad \text{for} \quad x,y\in\mathfrak{g},$$

where du denotes the normalized Haar measure of U. By Lemma 3.4, $(\mathfrak{g},\mathfrak{k},j,\bar{\varrho})$ is a Kähler algebra and $\mathfrak{U}\subset\mathrm{Der}_K(\mathfrak{g};\bar{\varrho})$. Consider the Kähler algebra $(\mathfrak{g}(\mathfrak{U}),\mathfrak{k}(\mathfrak{U}),j,\bar{\varrho})$ where $\mathfrak{g}(\mathfrak{U})$ and $\mathfrak{k}(\mathfrak{U})$ is defined in section 1.3. We can perform an inessential change j' of j so that $j'x=jx$ for $x\in\mathfrak{n}_0$ and $j'x=jx-\mathrm{Im}(\mathrm{ad}\,x)$ for $x\in\mathfrak{n}_1$. We set $\mathfrak{a}'=\mathfrak{n}+j'\mathfrak{n}$. From the construction, it is clear that $\mathfrak{a}'$ is a solvable subalgebra of $\mathfrak{g}(\mathfrak{U})$ and that $\mathrm{ad}\,x$ is a nilpotent endomorphism of $\mathfrak{g}(\mathfrak{U})$ for all $x\in\mathfrak{a}'$. Moreover $\mathfrak{a}'\cap\mathfrak{k}(\mathfrak{U})=0$ and $[\mathfrak{a}',\mathfrak{k}(\mathfrak{U})]\subset\mathfrak{a}'$. Since $\mathfrak{a}'$ is of flat type and $\mathrm{ad}\,\mathfrak{a}'$ consists of nilpotent endomorphisms, we can conclude that $\mathfrak{a}'$ is abelian. Then the arguments in sections 1.11 to 1.16 of [8] are still valid for our $\mathfrak{g}(\mathfrak{U})$ and $\mathfrak{a}'$. In particular, we have

$$j'\mathfrak{n}\subset\mathrm{rad}(\mathfrak{g}(\mathfrak{U})). \tag{3.5.1}$$

3.6. Since $[\mathfrak{g}(\mathfrak{U}),\mathfrak{g}(\mathfrak{U})]=[\mathfrak{g},\mathfrak{g}]$, $\mathfrak{n}$ coincides with $\mathrm{nil}(\mathfrak{g}(\mathfrak{U}))$. But then, $\mathfrak{a}'$ is an abelian Kähler ideal by (3.5.1). Let $\mathfrak{h}$ be the orthogonal complement of $\mathfrak{a}'$ in $\mathfrak{g}(\mathfrak{U})$ with respect to $\bar{\varrho}$. Then $\mathfrak{h}$ is a Kähler subalgebra containing $\mathfrak{k}(\mathfrak{U})$. Since $\mathfrak{a}'\cap\mathfrak{k}(\mathfrak{U})=0$, we have $\mathfrak{g}(\mathfrak{U})=\mathfrak{a}'\oplus\mathfrak{h}$. Clearly $\mathrm{nil}(\mathfrak{h})=0$ and hence $\mathfrak{h}$ is reductive. Let $\mathfrak{c}$ and $\mathfrak{h}'$ be the center of $\mathfrak{h}$ and the semi-simple part of $\mathfrak{h}$ respectively. By [14], $\bar{\varrho}(\mathfrak{c},\mathfrak{h}')=0$ and both $\mathfrak{c}$ and $\mathfrak{h}'$ are Kähler ideals of $\mathfrak{h}$. Clearly $\mathfrak{h}'\subset\mathfrak{g}$ and $\mathrm{rad}(\mathfrak{g}(\mathfrak{U}))=\mathfrak{a}'\oplus\mathfrak{c}$. This implies that $\mathfrak{h}'$ is maximal semi-simple in $\mathfrak{g}$, whence $\mathfrak{g}=\mathrm{rad}(\mathfrak{g})\oplus\mathfrak{h}'$. Moreover, since $\bar{\varrho}(\mathrm{rad}(\mathfrak{g}(\mathfrak{U})),\mathfrak{h}')=0$ and $\mathrm{rad}(\mathfrak{g})=\mathrm{rad}(\mathfrak{g}(\mathfrak{U}))\cap\mathfrak{g}$ we also have $\bar{\varrho}(\mathrm{rad}(\mathfrak{g}),\mathfrak{h}')=0$. Since $j'\mathfrak{h}'\subset\mathfrak{h}'+\mathfrak{k}(\mathfrak{U})$ and $j\mathfrak{h}'\equiv j'\mathfrak{h}'$ $(\mathrm{mod}\,\mathfrak{k}(\mathfrak{U}))$ we obtain $j\mathfrak{h}'\subset\mathfrak{h}'+\mathfrak{k}$. Therefore the orthogonal complement of $\mathfrak{h}'$ in $\mathfrak{g}$ relative to $\bar{\varrho}$ is j-invariant and it coincides with $\mathrm{rad}(\mathfrak{g})+\mathfrak{k}$. Thus we have $j\,\mathrm{rad}(\mathfrak{g})\subset\mathrm{rad}(\mathfrak{g})+\mathfrak{k}$. Therefore $\mathrm{rad}(\mathfrak{g})$ is a Kähler ideal. Now Theorem 3.2 follows from Proposition 3.1.

3.7. As a final preparation for the proof of Theorem 2.5 we remark

LEMMA. *Let* $(\mathfrak{g},\mathfrak{k},j,\varrho)$ *be a Kähler algebra. Assume that* $\mathrm{rad}(\mathfrak{g})$ *is a Kähler ideal of* $\mathfrak{g}$. *Then there exists a semi-simple Kähler subalgebra* $\mathfrak{h}$ *satisfying*

(a) $\mathfrak{g}=\mathrm{rad}(\mathfrak{g})+\mathfrak{h}$, $\varrho(\mathrm{rad}(\mathfrak{g}),\mathfrak{h})=0$,

(b) $\mathfrak{k}=\mathfrak{k}\cap\mathrm{rad}(\mathfrak{g})+\mathfrak{k}\cap\mathfrak{h}$.

Proof. Let $\mathfrak{s}$ be the orthogonal complement of $\mathrm{rad}(\mathfrak{g})$ in $\mathfrak{g}$ relative to ϱ. From the assumptions it follows that $\mathfrak{s}$ is a Kähler subalgebra containing $\mathfrak{k}$. Since $\mathfrak{s}+\mathrm{rad}(\mathfrak{g})\ (\mathrm{mod}\,\mathrm{rad}(\mathfrak{g}))\cong\mathfrak{s}\ (\mathrm{mod}\,\mathfrak{s}\cap\mathrm{rad}(\mathfrak{g}))$ is semi-simple we obtain $\mathrm{rad}(\mathfrak{s})=\mathrm{rad}(\mathfrak{g})\cap\mathfrak{s}\subset\mathfrak{k}$. Let $\mathfrak{h}$ be a maximal semi-simple subalgebra of $\mathfrak{s}$. Then $\mathfrak{s}=\mathrm{rad}(\mathfrak{s})\oplus\mathfrak{h}$. Since $\mathrm{rad}(\mathfrak{s})\subset\mathfrak{k}$, $\mathfrak{h}$ is a Kähler subalgebra satisfying (a). Let $k=k_1+k_2$ be an element of $\mathfrak{k}$, where $k_1\in\mathrm{rad}(\mathfrak{g})$ and $k_2\in\mathfrak{h}$. Then $\varrho(k_1,\mathrm{rad}(\mathfrak{g}))=\varrho(k,\mathrm{rad}(\mathfrak{g}))=0$. Therefore $\varrho(k_1,\mathfrak{g})=0$, whence $k_1\in\mathfrak{k}$, proving (b).

3.8. In this section we prove Theorem 2.5 for Case I. Let $\mathfrak{g}$ be an effective quasi-normal Kähler algebra of type Case I. By Theorem 3.2, $\mathrm{rad}(\mathfrak{g})$ is a Kähler ideal of flat type. Let $\mathfrak{g}=\mathrm{rad}(\mathfrak{g})+\mathfrak{h}$, be a decomposition of $\mathfrak{h}$ as in Lemma 3.7. Since $\mathfrak{g}$ is quasi-normal, $\mathrm{rad}(\mathfrak{g})\cap\mathfrak{k}=0$. Therefore $\mathrm{rad}(\mathfrak{g})$ is the Lie algebra of a transitive group of eluclidian transformations on $M(\mathrm{rad}(\mathfrak{g}))$. Hence $\mathrm{ad}\,x$ has only imaginary eigenvalues for any $x\in\mathrm{rad}(\mathfrak{g})$. But $\mathfrak{g}$ is quasi-normal, whence $\mathrm{ad}\,x$ is nilpotent and $\mathrm{rad}(\mathfrak{g})$ is abelian. From $\mathrm{rad}(\mathfrak{g})\cap\mathfrak{k}=0$ we also know $\mathfrak{h}\supset\mathfrak{k}$ by Lemma 3.7. Since $\mathfrak{h}$ is a semi-simple Kähler algebra, we know from [1] (see also [21]) that a maximal compact subalgebra $\mathfrak{u}$ of $\mathfrak{h}$ containing $\mathfrak{k}$ satisfies the properties (a) and (b) of Theorem 2.5. This completes the proof of Theorem 2.5.

§ 4. The canonical decomposition of Kähler algebras in Case II

4.1. Let $(\mathfrak{g},\mathfrak{k},j,\varrho)$ be an effective Kähler algebra of type Case II and let $\mathfrak{n}$, $\mathfrak{g}'$, $\mathfrak{m}$, $\mathfrak{a}$, and $\mathfrak{t}$ be as in section 2.6.

In the sections 4.1 to 4.4, we essentially assert that the statements 4.8 to 4.32 of [7] still hold with minor changes in our setting. (Recall that we use $\mathfrak{n}=\mathrm{nil}(\mathfrak{g})$ here.)

Note that $\mathfrak{t}$ is the Lie algebra of a Lie group which acts simply transitively on a homogeneous bounded domain. Let e be the principal idempotent of the maximal abelian ideal of $\mathfrak{t}$ of the first kind. (For the definitions of an abelian ideal of the first kind and its principal idempotent, see [26].) We call e the maximal idempotent of $\mathfrak{t}$. By [26] we have

$$\mathfrak{t}=\mathfrak{t}_0+\mathfrak{t}_{1/2}+\mathfrak{t}_1, \tag{4.1.1}$$

where $\mathfrak{t}_\lambda$ is the eigenspace of $R=\mathrm{Re}(\mathrm{ad}\,je)$ for the eigenvalue λ.

$$j\mathfrak{t}_{1/2}=\mathfrak{t}_{1/2},\quad j\mathfrak{t}_0=\mathfrak{t}_1\quad\text{and}\quad e\in\mathfrak{t}_1. \tag{4.1.2}$$

$$[jx,e]=x\quad\text{for}\quad x\in\mathfrak{t}_1. \tag{4.1.3}$$

It is not hard to see that e is the unique element of $\mathfrak{t}$ having the properties (4.1.1) to (4.1.3). We decompose the Lie algebra $\mathfrak{g}$ into the sum of eigenspaces $\mathfrak{g}_\lambda$ of R. From the properties of modifications [6; 3.1] it follows immediately that $\mathrm{ad}\, je$ and $\mathrm{ad}\, e$ leave $\mathfrak{a}$ invariant. Clearly $\mathfrak{m}$ and $\mathfrak{n}$ are $\mathrm{ad}\, je$-invariant. Hence $\mathfrak{a}$, $\mathfrak{m}$ and $\mathfrak{n}$ are also invariant under R and decompose into eigenspaces $\mathfrak{a}_\lambda$, $\mathfrak{m}_\lambda$ and $\mathfrak{n}_\lambda$ respectively. Using Lemma 1.5 we have ([6])

$$\mathfrak{a} = \mathfrak{a}_{-1/2} + \mathfrak{a}_0 + \mathfrak{a}_{1/2}, \quad j\mathfrak{a}_\lambda = \mathfrak{a}_{-\lambda}, \tag{4.1.4}$$

$$[e, x] = jx \text{ for } x \in \mathfrak{a}_{-1/2} \quad \text{and} \quad [e, \mathfrak{a}_0 + \mathfrak{a}_{1/2}] = 0. \tag{4.1.5}$$

We can use the proofs of 4.10 and 4.11 of [7] without change in our setting and obtain

$$\mathfrak{n}_1 = \mathfrak{t}_1 \tag{4.1.6}$$

$$\mathfrak{n}_{1/2} = \mathfrak{a}_{1/2} + (\mathfrak{t}_{1/2} \cap \mathfrak{n}_{1/2}) \tag{4.1.7}$$

$$\mathfrak{n}_{1/2} + j\mathfrak{n}_{1/2} = \mathfrak{m}_{1/2} + \mathfrak{m}_{-1/2}. \tag{4.1.8}$$

4.2. The following result has been stated in [7] without proof.

LEMMA. $[\mathfrak{k}, e] = 0$.

Proof. Let $M(\mathfrak{g}')$ denote the homogeneous Kähler manifold associated with $\mathfrak{g}'$. By [6], the Fundamental Conjecture holds for $M(\mathfrak{g}')$. Thus $M(\mathfrak{g}')$ is a holomorphic fiber bundle over a homogeneous bounded domain whose fiber is a complex euclidian space. Then $\mathfrak{a}+\mathfrak{k}$ corresponds to the group that leaves the fiber invariant. In particular $\mathfrak{a}+\mathfrak{k}$ is a subalgebra. Clearly $\mathfrak{t}+\mathfrak{k} = \{x \in \mathfrak{g}'; \varrho(x, \mathfrak{a}+\mathfrak{k}) = 0\}$ by [6]. Then we obtain $\varrho([\mathfrak{t}, \mathfrak{k}], \mathfrak{a}) = \varrho(\mathfrak{t}, [\mathfrak{k}, \mathfrak{a}]) = 0$. Hence $\mathfrak{t}+\mathfrak{k}$ is also a subalgebra. Since $\mathrm{ad}\, jx - j \circ \mathrm{ad}\, x$ induces an endomorphism of $(\mathfrak{t}+\mathfrak{k})/\mathfrak{k}$, we can define a linear form ψ on $\mathfrak{t}+\mathfrak{k}$ by $\psi(x) = \mathrm{Trace}(\mathrm{ad}\, jx - j \circ x)|(\mathfrak{t}+\mathfrak{k})/\mathfrak{k}$ for $x \in \mathfrak{t}+\mathfrak{k}$. From [13] we know $\psi([jx, jy]) = \psi([x, y])$ and $\psi([x, \mathfrak{k}]) = 0$ for any $x, y \in \mathfrak{t}+\mathfrak{k}$. Moreover, the form $\psi([jx, y])$ corresponds to the Bergman metric of the homogeneous bounded domain $M(\mathfrak{t}+\mathfrak{k})$. Therefore $\psi([jx, x]) \geqslant 0$ for all $x \in \mathfrak{t}+\mathfrak{k}$ and equality holds if and only if $x \in \mathfrak{k}$. Since $(\mathfrak{t}+\mathfrak{k}) \cap \mathfrak{n} = \mathfrak{t} \cap \mathfrak{n}$, we see that $\mathfrak{t} \cap \mathfrak{n}$ is an ideal of $\mathfrak{t}+\mathfrak{k}$ which contains e. It is straightforward to show $\psi(\mathfrak{t}_{1/2}) = 0$ and $\psi(\mathfrak{t}_0 \cap \mathfrak{n}) = 0$. This together with (4.1.3) yields

$$\psi(x) = \psi([je, x]) \quad \text{for any} \quad x \in \mathfrak{t} \cap \mathfrak{n}.$$

For any $k\in\mathfrak{k}$ and $x\in\mathfrak{t}\cap\mathfrak{n}$ we thus obtain

$$\psi([j[e,k],x]) = \psi([je,k],x]) = \psi([je,[k,x]]) = \psi([k,x]) = 0.$$

This implies $[e,k]=0$, because $[e,k]\in\mathfrak{t}\cap\mathfrak{n}$, and finishes the proof of the lemma.

4.3. From Lemma 4.1 we obtain $\mathfrak{k}\subset\mathfrak{g}'_0$. Therefore, by (4.1.1) and (4.1.4) we have

$$\mathfrak{g}' = \mathfrak{g}'_{-1/2}+\mathfrak{g}'_0+\mathfrak{g}'_{1/2}+\mathfrak{g}'_1, \quad \mathfrak{g}'_\lambda = \mathfrak{g}'\cap\mathfrak{g}_\lambda.$$

Then the results of 4.13 to 4.25 of [7] still hold in our setting. In particular for all $u, v\in\mathfrak{g}$ there exist $a, b, c\in\mathbf{R}$ such that

$$\varrho(e^{t\,\mathrm{ad}\,je}u, e^{t\,\mathrm{ad}\,je}v) = e^t a+e^{-t}b+c \tag{4.3.1}$$

holds. Moreover, we have

$$\varrho(\mathfrak{g}_\lambda, \mathfrak{g}_\mu) = 0 \quad \text{if} \quad \lambda+\mu \neq 0, \pm 1 \tag{4.3.2}$$

$$\mathfrak{g}_\lambda = 0 \quad \text{if} \quad \lambda\notin\mathbf{Z}/2 \tag{4.3.3}$$

$$j\mathfrak{g}_n\subset\mathfrak{g}_n+\mathfrak{g}'_1+\mathfrak{g}'_0 \quad \text{for all} \quad n\in\mathbf{Z}. \tag{4.3.4}$$

$$\bigoplus_{n\in\mathbf{Z}}\mathfrak{g}_n \quad \text{is a Kähler subalgebra.} \tag{4.3.5}$$

The following result is [7; 4.28]. We give here a somewhat shorter proof.

LEMMA. $\mathfrak{g}_\lambda=0$ *if* $\lambda\notin\{0, \pm 1/2, \pm 1, \pm 3/2, -2\}$.

Proof. First note that by (4.3.2) we have $\varrho(\mathfrak{g}_\lambda, \mathfrak{g}')=0$ if $\lambda+\{0, \pm 1/2, 1\}$ does not contain any of the numbers $0, \pm 1$, i.e., if $\lambda\notin\mathcal{M}$, where $\mathcal{M}=\{0, \pm 1/2, \pm 1, \pm 3/2, -2\}$. Since $j\mathfrak{g}'\subset\mathfrak{g}'$ this also implies $\varrho(j\mathfrak{g}_\lambda, \mathfrak{g}')=0$. But $j\mathfrak{g}_\lambda\subset\mathfrak{g}_\lambda+\mathfrak{g}'$ by 4.15 and 4.18 of [7]. Therefore even $j\mathfrak{g}_\lambda\subset\mathfrak{g}_\lambda+\mathfrak{k}$ holds for $\lambda\notin\mathcal{M}$. Hence, from (4.3.2) we obtain $\varrho(j\mathfrak{g}_\lambda, \mathfrak{g}_\lambda)=0$ since $2\lambda\neq 0, \pm 1$ if $\lambda\notin\mathcal{M}$ and $\mathfrak{g}_\lambda=0$ follows.

4.4. We prove in this section that the eigenvalues $\lambda=-1, \pm 3/2, -2$ do not occur. First we note that $\mathrm{rad}(\mathfrak{g})\cap\mathfrak{g}_\lambda=\mathfrak{n}_\lambda$ if $\lambda\neq 0$. Therefore

$$\mathrm{rad}(\mathfrak{g})\cap\mathfrak{g}_\lambda = 0 \quad \text{if} \quad \lambda\notin\{0, \pm 1/2, 1\}. \tag{4.4.1}$$

Since $R=\mathrm{Re}(\mathrm{ad}\,je)$ is a semi-simple derivation of $\mathfrak{g}$, by Appendix 2, we can find a maximal semi-simple subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ which is invariant under R. Then all eigenvalues of R in $\mathfrak{h}$ occur together with their negatives. This implies that -2 does not occur in $\mathfrak{h}$, whence $\mathfrak{g}_{-2}\subset\mathrm{rad}(\mathfrak{g})$. This means $\mathfrak{g}_{-2}=0$ by (4.4.1).

By (4.3.4), $\hat{\mathfrak{g}}=\mathfrak{g}_0+\mathfrak{g}_1$ is a Kähler subalgebra. Consider the subspace

$$\mathfrak{s}=\{x\in\hat{\mathfrak{g}};[e,x]=[e,jx]=0\}.$$

Then $\mathfrak{s}$ is a j-invariant subspace containing $\mathfrak{k}$. Moreover $\mathfrak{s}$ is $\mathrm{ad}\,je$-invariant, whence $\mathfrak{s}=\mathfrak{s}_0+\mathfrak{s}_1$, where $\mathfrak{s}_\lambda=\mathfrak{s}\cap\mathfrak{g}_\lambda$. Using (4.1.3) and (4.1.6), we have $x-j[x,e]-[jx,e]\in\mathfrak{s}$ for any $x\in\hat{\mathfrak{g}}$. Therefore $\hat{\mathfrak{g}}=\mathfrak{s}+\mathfrak{n}_1+j\mathfrak{n}_1$. Clearly $\mathfrak{s}\cap(\mathfrak{n}_1+j\mathfrak{n}_1)=0$. From (1.1.3) we have $[je,jx]\equiv j[je,x] \pmod{\mathfrak{k}}$ for any $x\in\mathfrak{s}$. This implies $j\mathfrak{s}_1\subset\mathfrak{s}_1+\mathfrak{k}$. Then by (4.3.2), $\varrho(j\mathfrak{s}_1,\mathfrak{s}_1)=0$. Therefore $\mathfrak{s}_1=0$ and hence $\mathfrak{g}_1=\mathfrak{n}_1$. But this shows that R has not the eigenvalue -1 in $\mathfrak{h}$. Hence we get $\mathfrak{g}_{-1}\subset\mathrm{rad}(\mathfrak{g})$ and $\mathfrak{g}_{-1}=0$ follows. We have also proved $\mathfrak{g}_0=j\mathfrak{g}_1+\mathfrak{s}$ and $\mathfrak{s}=\{x\in\mathfrak{g}_0;\ [x,e]=0\}$. In particular, $\mathfrak{s}$ is a j-invariant subalgebra of $\mathfrak{g}_0$. Since $\mathfrak{g}_{-1}=0$, we derive from [7; 4.19] that the term in (4.3.1) involving e^{-t} does not occur. Thus (4.3.2) holds if $\lambda+\mu\neq 0,1$. Therefore $\varrho(\mathfrak{g}_{-3/2},\mathfrak{g}')=0$. But then we obtain $\mathfrak{g}_{-3/2}=0$ as in the proof of Lemma 4.3. It follows $\mathfrak{g}_{3/2}\subset\mathrm{rad}(\mathfrak{g})$, whence $\mathfrak{g}_{3/2}=0$.

Since $\varrho(\mathfrak{g}_{-1/2}+\mathfrak{g}_{1/2},\mathfrak{g}_0+\mathfrak{g}_1)=0$ and since $\mathfrak{g}_0+\mathfrak{g}_1$ is a Kähler subalgebra, we know $j(\mathfrak{g}_{-1/2}+\mathfrak{g}_{1/2})\subset\mathfrak{g}_{-1/2}+\mathfrak{g}_{1/2}+\mathfrak{k}$. Thus we have proved

THEOREM. *Let $(\mathfrak{g},\mathfrak{k},j,\varrho)$ be an effective Kähler algebra of type Case* II. *Let $\mathfrak{g}_\lambda$ be the weight space of* $\mathrm{Re}(\mathrm{ad}\,je)$ *in $\mathfrak{g}$, where e is the idempotent given in section* 4.1. *Then*

(1) $\mathfrak{g}=\mathfrak{g}_{-1/2}+\mathfrak{g}_0+\mathfrak{g}_{1/2}+\mathfrak{g}_1$,

(2) $j(\mathfrak{g}_{-1/2}+\mathfrak{g}_{1/2})\subset\mathfrak{g}_{-1/2}+\mathfrak{g}_{1/2}+k$,

(3) $\mathfrak{g}_0+\mathfrak{g}_1$ *is a Kähler subalgebra and* $\mathfrak{g}_0=j\mathfrak{g}_1+\mathfrak{s}$, *where* $\mathfrak{s}=\{x\in\mathfrak{g}_0;[x,e]=0\}$ *is also a Kähler algebra containing* $\mathfrak{k}$.

Remark. The choice of $\mathfrak{t}$ in $\mathfrak{g}'$ is not unique. But the subalgebra $\mathfrak{t}+\mathfrak{k}$ is uniquely determined in $\mathfrak{g}'$. Lemma 4.2 shows that $\mathfrak{t}_1$ is the maximal abelian ideal of the first kind in $\mathfrak{t}+\mathfrak{k}$ and e is its principal idempotent. As is stated in [26], the maximal abelian ideal of the first kind is unique. Consequently, e is obtained uniquely from $\mathfrak{g}$. We can easily see that $\mathrm{Re}(\mathrm{ad}\,je)=\mathrm{Re}(\mathrm{ad}\,j'e)$ for any inessential change j' of j. Indeed, as in [26], from $[e,\mathfrak{k}]=0$ and (1.1.2) we derive $[je,\mathfrak{k}]\subset\mathfrak{k}$. Hence $\mathfrak{k}$ is an ideal of the Lie algebra $\mathbf{R}je+\mathfrak{k}$.

Since $\operatorname{ad}\mathfrak{k}$ acts completely reducibly, there exists a one-dimensional subspace $\mathfrak{l}$ of $\mathbf{R}je+\mathfrak{k}$ satisfying $\mathbf{R}je+\mathfrak{k}=\mathfrak{l}+\mathfrak{k}$ and $[\mathfrak{l},\mathfrak{k}]=0$. It is easy to see that the $\mathfrak{l}$-component x_0 of je coincides with the $\mathfrak{l}$-component of $j'e$. Clearly $\operatorname{Re}(\operatorname{ad} je)=\operatorname{Re}(\operatorname{ad} x_0)=\operatorname{Re}(\operatorname{ad} j'e)$. Therefore the decomposition $\mathfrak{g}=\oplus\mathfrak{g}_\lambda$ is obtained in a unique way from $\mathfrak{g}$ and it is even independent of the choice of j.

4.5. In the following sections we will study the structure of $\mathfrak{g}_0+\mathfrak{g}_1$ somewhat closer. It turns out that also in the present setting one can proceed along the lines of [26] and [16]. Recall that we can choose j so that $j\mathfrak{g}_1=\mathfrak{t}_0$ holds. The following lemma is well-known.

Lemma ([26], [5], [16]). *There exist $c_1,\dots,c_m\in\mathfrak{g}_1$ and a decomposition*

$$\mathfrak{g}_1=\oplus_{1\le i\le k\le m}\mathfrak{r}_{ik}$$

satisfying

(a) $\mathfrak{r}_{ii}=\mathbf{R}c_i$ *and* $c_1+\dots+c_m=e$,

(b) $[jc_i,jc_k]=0$ *and* $[jc_i,c_k]=\delta_{ik}$,

(c) *Let* $R_i=\operatorname{Re}(\operatorname{ad} jc_i)$, *then* $R_i=(\delta_{is}+\delta_{it})/2$ *on* $\mathfrak{r}_{st}$ *and* $R_i=(\delta_{is}-\delta_{it})/2$ *on* $j\mathfrak{r}_{st}$ *for all* $s\le t$.

By [5], we also have

$$D(c_i)=0 \quad \text{for all} \quad D\in\operatorname{Der}_K(j\mathfrak{g}_1+\mathfrak{g}_1). \tag{4.5.1}$$

Since the family $\{R_i; i=1,\dots,m\}$ is abelian, we can consider the corresponding "root space" decomposition $\mathfrak{g}=\oplus\mathfrak{g}^{[\Gamma]}$. Clearly $\mathfrak{g}_\lambda=\oplus\mathfrak{g}_\lambda^{[\Gamma]}$, where $\mathfrak{g}_\lambda^{[\Gamma]}=\mathfrak{g}^{[\Gamma]}\cap\mathfrak{g}_\lambda$. Let us denote by Δ_i the root defined by $\Delta_i(R_k)=\delta_{ik}$. Then we have

$$\mathfrak{g}_1=\bigoplus_{i\le k}\mathfrak{g}_1^{[(\Delta_i+\Delta_k)/2]} \quad\text{and}\quad \mathfrak{g}_1^{[(\Delta_i+\Delta_k)/2]}=\mathfrak{r}_{ik}. \tag{4.5.2}$$

Remark. An element c of $\mathfrak{g}_1$ is called an *idempotent* if it satisfies $[jc,c]=c$. An idempotent c is called *minimal* if c can not be written as the sum of two idempotents. One can prove that the set $\{c_1,\dots,c_m\}$ is nothing else but the set of all minimal idempotents of $\mathfrak{g}_1$. Note that this definition of an idempotent depends on the choice of the operator j.

4.6. From now on, we restrict our investigation to the subalgebra $\hat{\mathfrak{g}}=\mathfrak{g}_0+\mathfrak{g}_1$. First we consider the eigenspace decomposition $\hat{\mathfrak{g}}=\oplus_{a\in\mathbf{R}}\hat{\mathfrak{g}}^{(a)}$ for fixed R_i. Note that (1.1.3) implies $j\hat{\mathfrak{g}}^{(a)}\subset\hat{\mathfrak{g}}^{(a)}+\mathfrak{g}''$, where $\mathfrak{g}''=\mathfrak{g}_1+j\mathfrak{g}_1+\mathfrak{k}$. We will use Lemma 4.2 of [16] in the following section. For the convenience of the reader we state this result here.

LEMMA ([16]). *Let* $u\in\hat{\mathfrak{g}}^{(a)}$ *and put* $\mathfrak{g}_1^{(b)}=\hat{\mathfrak{g}}^{(b)}\cap\mathfrak{g}_1$. *Then* $ju=v+x+jy+k$, *where* $v\in\hat{\mathfrak{g}}^{(a)}$, $x, y\in\mathfrak{g}_1^{(a)}+\mathfrak{g}_1^{(a+1)}$ *and* $k\in\mathfrak{k}$.

4.7. We need to know which eigenvalues of R_i can occur in $\mathfrak{g}_0$. The following result is [16; Lemma 4.3]. Our proof follows the spirit of the proof there but is adjusted to the present setting which is somewhat different from the one in [16].

LEMMA ([16]). $\mathfrak{g}_0^{(a)}\neq 0$ *only if* $-1/2\leq a\leq 1/2$, *where* $\mathfrak{g}_0^{(a)}=\hat{\mathfrak{g}}^{(a)}\cap\mathfrak{g}_0$.

Proof. Assume $a>1/2$ is the maximal eigenvalue of R_i in $\mathfrak{g}_0$. Let $u\in\mathfrak{g}_0^{(a)}$. Then $ju=v+x+jy+k$ as in Lemma 4.6. Note $\mathfrak{g}_1^{(a+1)}=0$, whence $x, y\in\mathfrak{g}_1^{(a)}$. Then $j(u-y)-k=v+x\in\hat{\mathfrak{g}}^{(a)}$ and $[j(u-y)-k, u-y]\in\hat{\mathfrak{g}}^{(2a)}$. Since $2a>1$, we have $\hat{\mathfrak{g}}^{(2a)}=\mathfrak{g}_0^{(2a)}$, whence $\hat{\mathfrak{g}}^{(2a)}=0$ by the maximality of a. Note that for

$$A(t)=\varrho(e^{t\,\mathrm{ad}\,jc_i}(j(u-y)-k),\ e^{t\,\mathrm{ad}\,jc_i}(u-y)),$$

we have

$$dA(t)/dt=\varrho(jc_i, e^{t\,\mathrm{ad}\,jc_i}[j(u-y)-k, u-y])=0.$$

Hence $A(t)=A(0)=\varrho(j(u-y), u-y)$. But $A(t)$ grows like e^{2ta}, whence $A(0)=0$ follows. This implies $u-y\in\mathfrak{k}$. Since $u\in\mathfrak{g}_0$, $\mathfrak{k}\subset\mathfrak{g}_0$ and $y\in\mathfrak{g}_1$, we obtain $y=0$ and $u\in\mathfrak{k}$. But $\mathrm{ad}\,u$ is nilpotent, because $u\in\mathfrak{g}_0^{(a)}$. Thus we get $u=0$.

Assume now $b<-1/2$ is minimal among all eigenvalues of R_i in $\mathfrak{g}_0$. Then for $u\in\mathfrak{g}_0^{(b)}$, we have $ju=v+x+jy+k$, where $v\in\hat{\mathfrak{g}}^{(b)}$, $x, y\in\mathfrak{g}_1^{(b+1)}$ and $k\in\mathfrak{k}$ by Lemma 4.6. Then

$$j(u-y)-k=v+x\in\hat{\mathfrak{g}}^{(b)}+\hat{\mathfrak{g}}^{(b+1)}$$

and

$$[j(u-y)-k, u-y]=[v, u]-[v, y]+[x, u]\in\hat{\mathfrak{g}}^{(2b)}+\mathfrak{g}_1^{(2b+1)}.$$

We have $\mathfrak{g}_1^{(2b+1)}=0$, since $2b+1<0$. Moreover, $2b<0$ implies $\hat{\mathfrak{g}}^{(2b)}=\mathfrak{g}_0^{(2b)}$. But the minimality of b now implies $\mathfrak{g}_0^{(2b)}=0$, whence $\hat{\mathfrak{g}}^{(2b)}=0$. Using the same argument as above we arrive at $\mathfrak{g}_0^{(b)}=0$. Hence the lemma follows.

4.8. By virtue of Lemma 4.7, we can carry out the proofs of Lemma 4.4 to 4.6 and the proof of Proposition 4.1 of [16] without changes. In particular, we have

$$\mathfrak{g}_0^{[\Gamma]}=(j\mathfrak{g}_1)\cap\mathfrak{g}_0^{[\Gamma]}+\mathfrak{s}\cap\mathfrak{g}_0^{[\Gamma]} \quad \text{for} \quad \Gamma=(\Delta_i-\Delta_j)/2, \quad i\leqslant j. \tag{4.8.1}$$

$$\mathrm{Trace}(\mathrm{ad}\,s|\mathfrak{g}_1)=0 \quad \text{for all} \quad s\in\mathfrak{s}. \tag{4.8.2}$$

It follows from [26; § 1] that $j\mathfrak{g}_1+\mathfrak{g}_1$ is a Lie algebra of affine transformations of the tube domain over some homogeneous convex cone C in $\mathfrak{g}_1$. In such a realization we have $e\in C$ and $\mathrm{ad}\,j\mathfrak{g}_1|\mathfrak{g}_1$ is a subalgebra of Lie Aut C, the Lie algebra of the group of all linear transformations of C. By (4.8.2) we can apply [24; IV, Proposition 4] and obtain

PROPOSITION. $\mathrm{ad}\,\mathfrak{g}_0|\mathfrak{g}_1$ *is a subalgebra of* Lie Aut C *and its isotropy subalgebra at* e *is* $\mathrm{ad}\,\mathfrak{s}|\mathfrak{g}_1$. *Therefore,* $\mathrm{ad}\,\mathfrak{s}|\mathfrak{g}_1$ *acts by skew-adjoint endomorphisms (relative to some inner product on the vector space* $\mathfrak{g}_1$*).*

§ 5. The subspace $\mathfrak{g}_{-1/2}+\mathfrak{g}_{1/2}$.

5.1. Let $\mathfrak{g}=\mathfrak{g}_{-1/2}+\mathfrak{g}_0+\mathfrak{g}_{1/2}+\mathfrak{g}_1$ be the decomposition of the effective Kähler algebra of type Case II given by Theorem 4.4. We keep the notations used in § 4. The purpose of this section is to prove

THEOREM. $\mathfrak{g}_{-1/2}+\mathfrak{g}_{1/2}\subset\mathrm{rad}(\mathfrak{g})$.

5.2. By Theorem 4.4, after an inessential change of j, we can (and will) assume $j(\mathfrak{g}_{-1/2}+\mathfrak{g}_{1/2})=\mathfrak{g}_{-1/2}+\mathfrak{g}_{1/2}$. Then it is easy to see $[j\mathfrak{g}_1, j\mathfrak{g}_{1/2}]\subset j\mathfrak{g}_{1/2}$. We set

$$\mathfrak{w}_{1/2}=\{x\in\mathfrak{g}_{1/2};\, jx\in\mathfrak{g}_{1/2}\} \quad \text{and} \quad \mathfrak{u}_{1/2}=\{x\in\mathfrak{g}_{1/2};\, jx\in\mathfrak{g}_{-1/2}\}, \tag{5.2.1}$$

and show as in [8; 3.7]

$$\mathfrak{g}_{1/2}=\mathfrak{w}_{1/2}+\mathfrak{u}_{1/2}, \quad \mathfrak{u}_{1/2}=\mathfrak{a}_{1/2}\subset\mathfrak{n}_{1/2} \tag{5.2.2}$$

$$[j\mathfrak{g}_1, \mathfrak{w}_{1/2}]\subset\mathfrak{w}_{1/2}. \tag{5.2.3}$$

By (5.2.3), $\mathfrak{g}_1+j\mathfrak{g}_1+\mathfrak{w}_{1/2}$ is a Kähler subalgebra of $\mathfrak{g}$.

LEMMA. *The homogeneous Kähler manifold associated with the Kähler algebra* $\mathfrak{g}_1+j\mathfrak{g}_1+\mathfrak{w}_{1/2}$ *is biholomorphically equivalent to a homogeneous bounded domain.*

Proof. Since $\mathfrak{g}_1+j\mathfrak{g}_1$ is a solvable Kähler algebra it is easy to see that $\mathfrak{g}_1+j\mathfrak{g}_1+\mathfrak{w}_{1/2}$ is a solvable Kähler algebra with vanishing isotropy subalgebra. Therefore by [6], it is the sum of a Kähler algebra of flat type and of a Kähler algebra of domain type. Suppose that the flat summand is not zero. Then there exists $x\neq 0$ such that $[jx,x]=0$ ([6; Lemma 3.5.1]). We write $x=x_1+jx_2+w$, where $x_1, x_2\in\mathfrak{g}_1$ and $w\in\mathfrak{w}_{1/2}$. Consider the function $A(t)=\varrho(e^{t\,\mathrm{ad}\,je}jx, e^{t\,\mathrm{ad}\,je}x)$. By (4.3.2) and (1.1.4) we have $0=\varrho(\mathfrak{g}_1,\mathfrak{g}_1)=\varrho(j\mathfrak{g}_1, j\mathfrak{g}_1)$ and $\varrho(\mathfrak{w}_{1/2}, j\mathfrak{g}_1)=\varrho(\mathfrak{w}_{1/2},\mathfrak{g}_1)=0$. Therefore

$$A(t)=\varrho(e^{t\,\mathrm{ad}\,je}jx_1, e^{t\,\mathrm{ad}\,je}x_1)+\varrho(e^{t\,\mathrm{ad}\,je}jw, e^{t\,\mathrm{ad}\,je}w)+\varrho(e^{t\,\mathrm{ad}\,je}jx_2, e^{t\,\mathrm{ad}\,je}x_2).$$

Since $[je,jw]=j[je,w]$, we get $e^{t\,\mathrm{ad}\,je}jw=je^{t\,\mathrm{ad}\,je}w$. We also have for $y\in\mathfrak{g}_1$, $[je,jy]=j[je,y]+j[e,jy]=j[je,y]-jy$, whence $(\mathrm{ad}\,je+\mathrm{id})jy=j\circ\mathrm{ad}\,je\,y$. Therefore

$$e^{t\,\mathrm{ad}\,je}jy=e^{-t}e^{t(\mathrm{ad}\,je+\mathrm{id})}jy=e^{-t}je^{t\,\mathrm{ad}\,je}y \quad\text{for}\quad y\in\mathfrak{g}_1.$$

We then have

$$A(t)=e^{-t}\varrho(je^{t\,\mathrm{ad}\,je}x_1, e^{t\,\mathrm{ad}\,je}x_1)+\varrho(je^{t\,\mathrm{ad}\,je}w, e^{t\,\mathrm{ad}\,je}w)+e^{-t}\varrho(je^{t\,\mathrm{ad}\,je}x_2, e^{t\,\mathrm{ad}\,je}x_2).$$

From this it is clear that $A(t)$ grows like e^t if it does not vanish identically. But $dA(t)/dt=\varrho(je, e^{t\,\mathrm{ad}\,je}[jx,x])=0$, whence $A(t)$ is constant. This is a contradiction, proving the lemma.

5.3. Let $c_1,\dots,c_m$ be as in Lemma 4.5. Consider the decomposition $\mathfrak{g}_{-1/2}=\oplus\,\mathfrak{g}^{[\Gamma]}_{-1/2}$, $\mathfrak{g}_{1/2}=\oplus\,\mathfrak{g}^{[\Gamma]}_{1/2}$, introduced in section 4.5. Since $\mathfrak{w}_{1/2}$ is invariant by $\mathrm{ad}\,jc_i$ for all i, we also have $\mathfrak{w}_{1/2}=\oplus\,\mathfrak{w}^{[\Gamma]}_{1/2}$. From Lemma 5.2, one can derive the usual root space decomposition of $\mathfrak{w}_{1/2}$ (see e.g. [16; p. 280]),

$$\mathfrak{w}_{1/2}=\bigoplus_i \mathfrak{w}^{[\Delta_i/2]}_{1/2},\quad j\mathfrak{w}^{[\Delta_i/2]}_{1/2}=\mathfrak{w}^{[\Delta_i/2]}_{1/2}. \tag{5.3.1}$$

A similar decomposition holds for the subalgebra $\mathfrak{a}$ of $\mathfrak{g}'$ defined in section 2.6.

By (4.5.1), we know from [6; 3.3] that $\mathfrak{a}$ is invariant under $\operatorname{ad} jc_i$ and $\operatorname{ad} c_i$ for all i. In particular, we also have the decompositions $\mathfrak{a}_{-1/2}=\oplus\mathfrak{a}_{-1/2}^{[\Gamma]}$ and $\mathfrak{a}_{1/2}=\oplus\mathfrak{a}_{1/2}^{[\Gamma]}$.

LEMMA. $\mathfrak{a}_{1/2}=\bigoplus_i\mathfrak{a}_{1/2}^{[\Delta_i/2]}$, $\mathfrak{a}_{-1/2}=\bigoplus_i\mathfrak{a}_{-1/2}^{[-\Delta_i/2]}$ *and* $j\mathfrak{a}_{-1/2}^{[-\Delta_i/2]}=\mathfrak{a}_{1/2}^{[\Delta_i/2]}$.

Proof. Since $\varrho(\mathfrak{a},\mathfrak{t})=0$, $\operatorname{ad} jc_i$ and $\operatorname{ad} c_i$ are symplectic endomorphisms of $\mathfrak{a}_{-1/2}+\mathfrak{a}_{1/2}$ (relative to ϱ and j) satisfying the conditions of Lemma 1.5. Therefore if $\mathfrak{a}_{1/2}^{[\Gamma]}\neq 0$, then $\Gamma(R_i)\in\{0,\pm 1/2\}$. Suppose $\Gamma(R_i)=-1/2$. Then $j\mathfrak{a}_{1/2}^{[\Gamma]}=[e_i,\mathfrak{a}_{1/2}^{[\Gamma]}]=0$, a contradiction. Therefore $\Gamma(R_i)=0$ or $1/2$. Since $\Sigma_{i=1}^m\Gamma(R_i)=1/2$, there exists exactly one i such that $\Gamma(R_i)=1/2$ and $\Gamma(R_k)=0$ if $k\neq i$. Thus we have $\mathfrak{a}_{1/2}=\oplus\mathfrak{a}_{1/2}^{[\Delta_i/2]}$. Since $j\mathfrak{a}_{1/2}^{[\Delta_i/2]}\subset\mathfrak{a}_{-1/2}^{[-\Delta_i/2]}$, the remaining assertions also hold and the lemma is proven.

5.4. Since $\{R_1,\dots,R_m\}$ is a commutative family of semi-simple derivations of $\mathfrak{g}$, Appendix 2 assures the existence of a maximal semi-simple subalgebra $\mathfrak{h}$ invariant by R_i for all i. Hence we have the decomposition $\mathfrak{h}=\oplus\mathfrak{h}_\lambda^{[\Gamma]}$, where $\mathfrak{h}_\lambda^{[\Gamma]}\subset\mathfrak{g}_\lambda^{[\Gamma]}$. From (5.2.2), (5.3.1) and Lemma 5.3, we already know

$$\mathfrak{g}_{1/2}=\oplus\,\mathfrak{g}_{1/2}^{[\Delta_i/2]}. \tag{5.4.1}$$

Therefore, if $\mathfrak{h}_{1/2}^{[\Gamma]}\neq 0$, then $\Gamma=\Delta_i/2$ for some i. Since $\mathfrak{h}$ is semi-simple, $\mathfrak{h}_\lambda^{[\Gamma]}\neq 0$ if and only if $\mathfrak{h}_{-\lambda}^{[-\Gamma]}\neq 0$. Hence we have

$$\mathfrak{h}_{1/2}=\bigoplus_i\mathfrak{h}_{1/2}^{[\Delta_i/2]},\quad \mathfrak{h}_{-1/2}=\bigoplus_i\mathfrak{h}_{-1/2}^{[-\Delta_i/2]}. \tag{5.4.2}$$

Recall that $\operatorname{rad}(\mathfrak{g})\cap\mathfrak{g}_{-1/2}\subset\mathfrak{n}_{-1/2}\subset\mathfrak{a}_{-1/2}$. Thus by Lemma 5.3 and (5.4.2), we have

$$\mathfrak{g}_{-1/2}=\oplus\,\mathfrak{g}_{-1/2}^{[-\Delta_i/2]}. \tag{5.4.3}$$

Next we show

LEMMA. $\mathfrak{g}_{-1/2}^{[-\Delta_i/2]}+\mathfrak{g}_{1/2}^{[\Delta_i/2]}$ *is j-invariant.*

Proof. We already know $j\mathfrak{g}_{1/2}^{[\Delta_i/2]}\subset\mathfrak{g}_{-1/2}^{[-\Delta_i/2]}+\mathfrak{g}_{1/2}^{[\Delta_i/2]}$. For an element x of $\mathfrak{g}_{-1/2}+\mathfrak{g}_{1/2}$, we denote by $x^{[\Gamma]}$ the $\mathfrak{g}^{[\Gamma]}$-component of x. Let $x\in\mathfrak{g}_{-1/2}^{[-\Delta_i/2]}$ and decompose jx as $jx=\Sigma(jx)^{[\Gamma]}$.

Since

$$[jc_i, jx] = [c_i, x] + j[jc_i, x] + j[c_i, jx],$$
$$[c_i, x] \in \mathfrak{g}_{1/2}^{[\Delta_i/2]}$$

and

$$j[c_i, jx] = j[c_i, (jx)^{[-\Delta_i/2]}] \in \mathfrak{g}_{1/2}^{[\Delta_i/2]} + \mathfrak{g}_{-1/2}^{[-\Delta_i/2]},$$

we have $[jc_i, (jx)^{[\Gamma]}] = (j[jc_i, x])^{[\Gamma]}$ for $\Gamma \neq \pm\Delta_i/2$. Denote by α_Γ the mapping of $\mathfrak{g}_{-1/2}^{[-\Delta_i/2]}$ to $\mathfrak{g}^{[\Gamma]}$ defined by $\alpha_\Gamma(x) = (jx)^{[\Gamma]}$. Then $\mathrm{ad}\, jc_i \circ \alpha_\Gamma = \alpha_\Gamma \circ \mathrm{ad}\, jc_i$ for $\Gamma \neq \pm\Delta_i/2$. This implies $R_i \circ \alpha_\Gamma = \alpha_\Gamma \circ R_i$ for $\Gamma \neq \pm\Delta_i/2$. As a consequence we obtain $\alpha_\Gamma = 0$, because $R_i = -1/2$ on $\mathfrak{g}_{-1/2}^{[-\Delta_i/2]}$ and $R_i = 0$ on $\mathfrak{g}_{-1/2}^{[\Gamma]} + \mathfrak{g}_{1/2}^{[\Gamma]}$ if $\Gamma \neq \pm\Delta_i/2$. Hence the lemma follows.

5.5. Recall the decomposition $\mathfrak{g}_0 = j\mathfrak{g}_1 + \mathfrak{s}$ given in Theorem 4.4. From (4.8.1) we derive

$$\mathfrak{g}_0^{[0]} = \bigoplus_{i=1}^{m} \mathbf{R} jc_i \oplus \mathfrak{s} \cap \mathfrak{g}_0^{[0]}.$$

Set

$$\mathfrak{s}' = \mathfrak{s} \cap \mathfrak{g}_0^{[0]}. \tag{5.5.1}$$

In the following sections we investigate $\mathfrak{s}'$. We start by showing

LEMMA. $j\mathfrak{s}' \subset \mathfrak{s}' + \mathfrak{k}$.

Proof. By (4.5.2), (5.4.1) and (5.4.3), the equality $\mathfrak{g}^{[0]} = \mathfrak{g}_0^{[0]}$ holds. Since

$$[jc_i, jx] \equiv j[jc_i, x] \pmod{\mathfrak{g}_1 + j\mathfrak{g}_1 + \mathfrak{k}}$$

for any i and for any $x \in \mathfrak{g}_0 + \mathfrak{g}_1$, we have $j\mathfrak{s}' \subset \mathfrak{g}_0^{[0]} + \mathfrak{g}_1 + j\mathfrak{g}_1 + \mathfrak{k}$. We also know $j\mathfrak{s}' \subset \mathfrak{s}$. Let $x \in \mathfrak{s}'$. Then $jx = x' + y + jz + k$ for some $x' \in \mathfrak{g}_0^{[0]}$, $y, z \in \mathfrak{g}_1$ and $k \in \mathfrak{k}$. Further $x' = x'' + jr$ for some $x'' \in \mathfrak{s}'$ and $r \in \mathfrak{g}_1$. Since $jx \in \mathfrak{s}$ we obtain $jx = x'' + k$, proving the lemma.

5.6. In this section we show

LEMMA. (1) $\mathfrak{s}' = \{x \in \mathfrak{g}_0^{[0]};\ [x, c_i] = 0$ *for all* $i\}$.

(2) $\mathfrak{s}' + \mathbf{R} jc_i = \{x \in \mathfrak{g}_0^{[0]};\ [x, c_k] = 0$ *for all* $k \neq i\}$.

(3) *Both,* $\mathfrak{s}'$ *and* $\mathfrak{s}' + \mathbf{R} jc_i$, *are ideals of* $\mathfrak{g}_0^{[0]}$.

Proof. Clearly, $[\mathfrak{s}', c_i]\subset \mathbf{R}c_i$. By Proposition 4.8, $\operatorname{ad} x|\mathfrak{g}_1$ has only imaginary eigenvalues for any $x\in\mathfrak{s}$. Therefore, $[\mathfrak{s}', c_i]=0$ for all i. Now the lemma follows immediately from the equations $\mathfrak{g}_0^{[0]}=\oplus_{i=1}^m \mathbf{R}jc_i\oplus\mathfrak{s}'$ and $[jc_i, c_k]=\delta_{ik}c_k$.

5.7. The following result provides an important piece of information on $\mathfrak{s}'$.

LEMMA. $\operatorname{rad}(\mathfrak{s}')$ *is a Kähler ideal of* $\mathfrak{s}'$ *of flat type.*

Proof. Let $\mathfrak{h}$ be as in section 5.4. It is easy to see that $\mathfrak{h}_0^{[0]}$ is reductive. From this we obtain $\operatorname{rad}(\mathfrak{g}_0^{[0]})=\mathfrak{g}_0^{[0]}\cap\operatorname{rad}(\mathfrak{g})+\mathfrak{c}_0$, where $\mathfrak{c}_0$ is the center of $\mathfrak{h}_0^{[0]}$. Therefore $\operatorname{nil}(\mathfrak{g}_0^{[0]})\subset\operatorname{nil}(\mathfrak{g})$. Since $\mathfrak{s}'$ is an ideal of $\mathfrak{g}_0^{[0]}$, $\operatorname{nil}(\mathfrak{s}')\subset\operatorname{nil}(\mathfrak{g}_0^{[0]})\subset\operatorname{nil}(\mathfrak{g})$. By Lemma 5.5, we can assume that $\mathfrak{s}'$ is j-invariant. Hence $\mathfrak{s}'$ is a Kähler algebra, whence by the Radical Conjecture we may assume that $\operatorname{nil}(\mathfrak{s}')+j\operatorname{nil}(\mathfrak{s}')$ is a solvable subalgebra of $\mathfrak{s}'$. Recall that $\mathfrak{g}_0'=\mathfrak{t}_0+\mathfrak{a}_0+\mathfrak{k}$, $\mathfrak{t}_0=j\mathfrak{g}_1$ and $[\mathfrak{a}_0, e]=0$. Therefore $\mathfrak{g}_0'\cap\mathfrak{s}=\mathfrak{a}_0+\mathfrak{k}$. Consequently, $\operatorname{nil}(\mathfrak{s}')+j\operatorname{nil}(\mathfrak{s}')\subset\mathfrak{a}+\mathfrak{k}$. This shows that $\operatorname{nil}(\mathfrak{s}')+j\operatorname{nil}(\mathfrak{s}')$ is of flat type. An application of Proposition 3.1 and Theorem 3.2 to $\mathfrak{s}'$ yields the assertion.

5.8. For every i we consider the subspace

$$\mathfrak{g}(i)=\mathfrak{g}_{-1/2}^{[-\Delta_i/2]}+\mathfrak{s}'+\mathbf{R}jc_i+\mathfrak{g}_{1/2}^{[\Delta_i/2]}+\mathbf{R}c_i. \tag{5.8.1}$$

Lemma 5.4 and Lemma 5.5 show that, after an inessential change of j, we can assume that $\mathfrak{g}(i)$ is j-invariant. Since $[\mathfrak{g}_{-1/2}^{[-\Delta_i/2]}, c_k]=0$ if $i\neq k$, we have

$$[\mathfrak{g}_{-1/2}^{[-\Delta_i/2]}, \mathfrak{g}_{1/2}^{[\Delta_i/2]}]\subset\mathfrak{s}'+\mathbf{R}jc_i$$

by Lemma 5.6. Therefore $\mathfrak{g}(i)$ is a subalgebra of $\mathfrak{g}$. It is easy to see that $\mathfrak{h}_{-1/2}^{[-\Delta_i/2]}+[\mathfrak{h}_{-1/2}^{[-\Delta_i/2]}, \mathfrak{h}_{1/2}^{[\Delta_i/2]}]+\mathfrak{h}_{1/2}^{[\Delta_i/2]}$ is a semi-simple subalgebra of $\mathfrak{g}(i)$. Therefore in order to prove Theorem 5.1, it is enough to show for all i

$$\mathfrak{g}_{1/2}^{[\Delta_i/2]}+\mathfrak{g}_{-1/2}^{[-\Delta_i/2]}\subset\operatorname{rad}(\mathfrak{g}(i)). \tag{5.8.2}$$

In what follows, we only consider the Kähler algebra $\mathfrak{g}(i)$ for fixed i and prove (5.8.2). To simplify the notation, we use $\mathfrak{g}, \mathfrak{k}, c, \mathfrak{g}_{1/2}, \mathfrak{g}_0, \mathfrak{s}$ and $\mathfrak{g}_{-1/2}$ instead of $\mathfrak{g}(i), \mathfrak{k}\cap\mathfrak{g}(i), c_i, \mathfrak{g}_{1/2}^{[\Delta_i/2]}, \mathfrak{s}'+\mathbf{R}jc_i, \mathfrak{s}'$, and $\mathfrak{g}_{-1/2}^{[-\Delta_i/2]}$ respectively. We also denote by $\mathfrak{r}$ the radical of $\mathfrak{g}(i)$. Then $\mathfrak{r}=\oplus\mathfrak{r}_\lambda$, where $\mathfrak{r}_\lambda=\mathfrak{r}\cap\mathfrak{g}_\lambda$, $\lambda=0, \pm 1/2, 1$.

5.9. The purpose of this section is to prove

LEMMA. *After an inessential change of j we obtain* $\mathrm{rad}(\mathfrak{g}_0)=\mathrm{rad}(\mathfrak{s})+\mathbf{R}jc$.

Proof. By Lemma 5.7 and Lemma 3.7, we can find a semi-simple Kähler subalgebra $\mathfrak{h}'$ of $\mathfrak{s}$ satisfying $\mathfrak{s}=\mathrm{rad}(\mathfrak{s})+\mathfrak{h}'$, $\varrho(\mathrm{rad}(\mathfrak{s}),\mathfrak{h}')=0$ and $\mathfrak{k}=\mathfrak{k}\cap\mathrm{rad}(\mathfrak{s})+\mathfrak{k}\cap\mathfrak{h}'$. Since $\mathfrak{s}$ is an ideal of $\mathfrak{g}_0$, $\mathrm{ad}\,jc$ leaves $\mathfrak{s}$ invariant. Hence there exists a unique element h of $\mathfrak{h}'$ such that $\mathrm{ad}(jc-h)$ maps $\mathfrak{h}'$ into $\mathrm{rad}(\mathfrak{s})$. Then for any $k\in\mathfrak{k}\cap\mathfrak{h}'$, we obtain $[h,k]\equiv[jc,k]\equiv j[c,k]\equiv 0 \ (\mathrm{mod}\,\mathrm{rad}(\mathfrak{s})+\mathfrak{k})$. This implies that $\mathrm{ad}\,h$ leaves $\mathfrak{k}\cap\mathfrak{h}'$ invariant. It is well-known that the normalizer of the isotropy subalgebra of a semi-simple Kähler algebra coincides with the isotropy subalgebra (see, e.g. [14]). Therefore $h\in\mathfrak{k}$. Consider the inessential change of j given by $j'c=jc-h$. Note that $\mathrm{Re}(\mathrm{ad}\,j'c)=\mathrm{Re}(\mathrm{ad}\,jc)$ by Remark 4.4 and $[j'c,\mathfrak{h}']\subset\mathrm{rad}(\mathfrak{s})$ by construction of h. From this it is easy to derive $\mathrm{rad}(\mathfrak{g}_0)=\mathrm{rad}(\mathfrak{s})\oplus\mathbf{R}j'c$, proving the lemma.

5.10. Let $\mathfrak{h}$ be a maximal semi-simple subalgebra of $\mathfrak{g}$ invariant under $\mathrm{Re}(\mathrm{ad}\,jc)$. Then $\mathfrak{g}=\mathfrak{r}+\mathfrak{h}$ and $\mathfrak{h}=\mathfrak{h}_{-1/2}+\mathfrak{h}_0+\mathfrak{h}_{1/2}$, where $\mathfrak{h}_\lambda\subset\mathfrak{g}_\lambda$. There exists a unique element E of $\mathfrak{h}$ such that $\mathrm{ad}\,E|\mathfrak{h}_\lambda=\lambda$. Clearly E is in the center of $\mathfrak{h}_0$, whence $E\in\mathrm{rad}(\mathfrak{g}_0)$. From Lemma 5.9, we obtain

$$E=\alpha jc+s_0,\quad \alpha\in\mathbf{R}\quad\text{and}\quad s_0\in\mathrm{rad}(\mathfrak{s}). \tag{5.10.1}$$

Since E is a real diagonal element of the semi-simple Lie algebra $\mathfrak{h}$, $\mathrm{ad}\,E$ is a semi-simple endomorphism of $\mathfrak{g}$ with only real eigenvalues (see, e.g. [15]).

An important property of the element $E\in\mathfrak{h}$ is proven in

LEMMA. $\mathrm{ad}\,E=0$ *on* $\mathfrak{g}_0$.

Proof. By Lemma 5.7, $\mathrm{ad}\,s_0|\mathfrak{g}_0$ has only imaginary eigenvalues. The same is true for $\mathrm{ad}\,jc|\mathfrak{g}_0$. Noting that jc and s_0 are contained in the solvable subalgebra $\mathrm{rad}(\mathfrak{g}_0)$, we obtain that also $\mathrm{ad}\,E|\mathfrak{g}_0$ has only imaginary eigenvalues. But as mentioned above, $\mathrm{ad}\,E$ has only real eigenvalues on $\mathfrak{g}$ and the lemma follows.

5.11. Denote by P^a (resp. Q^a) the eigenspace of $\mathrm{ad}\,E$ in $\mathfrak{r}_{-1/2}$ (resp. $\mathfrak{r}_{1/2}$) corresponding to the eigenvalue a.

LEMMA. $\mathfrak{r}_{-1/2}=P^0+P^{-1/2}$.

Proof. If $[P^a,\mathfrak{h}_{1/2}]\neq 0$, then $a+1/2=0$ by Lemma 5.10. In this case $a=-1/2$. Assume $[P^a,\mathfrak{h}_{1/2}]=0$. Then P^a is $\mathrm{ad}\,\mathfrak{h}$-invariant. Therefore $\mathrm{Trace}(\mathrm{ad}\,E|P^a)=0$, whence $a=0$.

5.12. In the remaining sections of this § 5 we discuss the possibilities $\alpha=1$, $\alpha=0$ and $\alpha\neq 0,1$ for the coefficient α in (5.10.1).

LEMMA. *The case $\alpha=1$ does not occur.*

Proof. First we note that if $\alpha=1$, then $[E,c]=c$ holds, because $[c,s_0]=0$. Therefore $P^{-1/2}+\mathfrak{r}_0+Q^{1/2}+\mathfrak{g}_1$ is $\operatorname{ad}\mathfrak{h}$-invariant. Then the trace of $\operatorname{ad}E$ restricted to this space is equal to zero, whence $\dim P^{-1/2}>\dim Q^{1/2}$. On the other hand, $jP^{-1/2}=[c,P^{-1/2}]\subset Q^{1/2}$. Therefore $\dim P^{-1/2}\leqslant\dim Q^{1/2}$, a contradiction. Hence the case $\alpha=1$ does not occur.

5.13. Next we will discuss the case $\alpha=0$. To do this we need a generalization of a result of Matsushima. Our proof follows the corresponding arguments of [14] in our setting.

PROPOSITION. *Let $(\mathfrak{g},\mathfrak{k},j,\varrho)$ be a Kähler algebra and let $\tilde{\mathfrak{h}}$ be a semi-simple ideal of $\mathfrak{g}$. Denote by $\tilde{\mathfrak{r}}$ the centralizer of $\tilde{\mathfrak{h}}$ in $\mathfrak{g}$. Then $\tilde{\mathfrak{h}}$ and $\tilde{\mathfrak{r}}$ are Kähler ideals of $\mathfrak{g}$ and the following equations hold.*

$$\mathfrak{g}=\tilde{\mathfrak{r}}\oplus\tilde{\mathfrak{h}},\quad \varrho(\tilde{\mathfrak{r}},\tilde{\mathfrak{h}})=0\quad \textit{and}\quad \mathfrak{k}=\mathfrak{k}\cap\tilde{\mathfrak{r}}\oplus\mathfrak{k}\cap\tilde{\mathfrak{h}}.$$

Proof. The first equation follows immediately from the complete reducibility of $\tilde{\mathfrak{h}}$. Further we have $\varrho(\tilde{\mathfrak{r}},\tilde{\mathfrak{h}})=\varrho(\tilde{\mathfrak{r}},[\tilde{\mathfrak{h}},\tilde{\mathfrak{h}}])=\varrho([\tilde{\mathfrak{r}},\tilde{\mathfrak{h}}],\tilde{\mathfrak{h}})=0$. Let $k=k_1+k_2$ be an element of $\mathfrak{k}$, where $k_1\in\tilde{\mathfrak{r}}$ and $k_2\in\tilde{\mathfrak{h}}$. Then $\varrho(k_1,\tilde{\mathfrak{r}})=\varrho(k,\tilde{\mathfrak{r}})=0$, whence $\varrho(k_1,\mathfrak{g})=0$. This means $k_1\in\mathfrak{k}$. Hence we obtain $\mathfrak{k}=\mathfrak{k}\cap\tilde{\mathfrak{r}}\oplus\mathfrak{k}\cap\tilde{\mathfrak{h}}$. Since $\tilde{\mathfrak{h}}$ is semi-simple, there exists a linear form ω on $\tilde{\mathfrak{h}}$ such that $d\omega=\varrho$ on $\tilde{\mathfrak{h}}$. Moreover, there exists a unique $w_0\in\tilde{\mathfrak{h}}$ so that $\omega(x)=B(w_0,x)$ for any $x\in\tilde{\mathfrak{h}}$, where B denotes the Killing form of $\tilde{\mathfrak{h}}$. It is easy to see that $\mathfrak{k}\cap\tilde{\mathfrak{h}}$ coincides with the centralizer of w_0 in $\tilde{\mathfrak{h}}$. In particular, $w_0\in\mathfrak{k}\cap\tilde{\mathfrak{h}}$. We can decompose $\tilde{\mathfrak{h}}$ as $\tilde{\mathfrak{h}}=\tilde{\mathfrak{h}}'\oplus\mathfrak{k}_0$, where $\mathfrak{k}_0$ is the largest ideal of $\tilde{\mathfrak{h}}$ contained in $\tilde{\mathfrak{h}}\cap\mathfrak{k}$ and $\tilde{\mathfrak{h}}'$ is an ideal of $\tilde{\mathfrak{h}}$. Clearly, $w_0\in\tilde{\mathfrak{h}}'\cap\mathfrak{k}$. The ideal $\tilde{\mathfrak{h}}'$ can be regarded as the Lie algebra of a Lie group which acts effectively and isometrically on a certain Riemannian manifold with $\tilde{\mathfrak{h}}'\cap\mathfrak{k}$ as its isotropy subalgebra. Therefore $\operatorname{ad}w_0$ is a semi-simple endomorphism of $\tilde{\mathfrak{h}}'$ and hence of $\tilde{\mathfrak{h}}$. Let $\mathfrak{n}$ denote the normalizer of $\tilde{\mathfrak{h}}\cap\mathfrak{k}$ in $\tilde{\mathfrak{h}}$. Since $\operatorname{ad}w_0$ is semi-simple, there exists an $\operatorname{ad}w_0$-invariant subspace $\mathfrak{n}'$ such that $\mathfrak{n}=\tilde{\mathfrak{h}}\cap\mathfrak{k}\oplus\mathfrak{n}'$. But then $[w_0,\mathfrak{n}']\subset\mathfrak{n}'\cap(\tilde{\mathfrak{h}}\cap\mathfrak{k})=0$. This means $\mathfrak{n}'\subset\tilde{\mathfrak{h}}\cap\mathfrak{k}$, whence $\mathfrak{n}=\tilde{\mathfrak{h}}\cap\mathfrak{k}$.

Let $x\in\tilde{\mathfrak{r}}$ and decompose $jx=x_1+x_2$, where $x_1\in\tilde{\mathfrak{r}}$ and $x_2\in\tilde{\mathfrak{h}}$. Let $k\in\mathfrak{k}\cap\tilde{\mathfrak{h}}$. Then $[x_2,k]=[jx,k]\equiv 0 \pmod{\mathfrak{k}}$. Therefore $[x_2,k]\in\mathfrak{k}\cap\tilde{\mathfrak{h}}$, whence $x_2\in\mathfrak{k}\cap\tilde{\mathfrak{h}}$. This shows that $\tilde{\mathfrak{r}}$ is a Kähler ideal of $\mathfrak{g}$. Finally $\tilde{\mathfrak{h}}+\mathfrak{k}=\{x\in\mathfrak{g};\varrho(x,\tilde{\mathfrak{r}})=0\}$ yields $j\tilde{\mathfrak{h}}\subset\tilde{\mathfrak{h}}+\mathfrak{k}$, proving the proposition.

5.14. In the following two sections we discuss the case $\alpha=0$. Here we have $[E,c]=0$. In this section we show

LEMMA. *If* $\alpha=0$, *then* $P^{-1/2}=Q^{1/2}=0$.

Proof. First we assert

$$\mathfrak{r}_{1/2}=Q^0+Q^{1/2}. \tag{5.14.1}$$

In fact, if $[Q^a,\mathfrak{h}_{-1/2}]\neq 0$, then by Lemma 5.10, we have $a=1/2$. Assume $[Q^a,\mathfrak{h}_{-1/2}]=0$. If also $[Q^a,\mathfrak{h}_{1/2}]=0$, then Q^a is ad$\mathfrak{h}$-invariant, whence $0=\mathrm{Trace}(\mathrm{ad}\,E|Q^a)$ and $a=0$ follows. If $[Q^a,\mathfrak{h}_{1/2}]\neq 0$, then $a+1/2=0$ since $\alpha=0$. Moreover, $Q^{-1/2}+\mathfrak{g}_1$ is ad$\,\mathfrak{h}$-invariant. Therefore $0=\mathrm{Trace}(\mathrm{ad}\,E|Q^{-1/2}+\mathfrak{g}_1)=-\dim Q^{-1/2}/2$, proving (5.14.1). We then have $jP^{-1/2}=[c,P^{-1/2}]\subset Q^{-1/2}=0$, whence $P^{-1/2}=0$. On the other hand, $P^{-1/2}+\mathfrak{r}_0+Q^{1/2}$ is ad$\,\mathfrak{h}$-invariant. Hence $\dim Q^{1/2}=\dim P^{-1/2}=0$, proving the lemma.

5.15. In this section we finish the case $\alpha=0$.

LEMMA. *If* $\alpha=0$, *then* $\mathfrak{g}_{-1/2}+\mathfrak{g}_{1/2}\subset\mathrm{rad}(\mathfrak{g})$.

Proof. Put $\tilde{\mathfrak{h}}=\{x\in\mathfrak{h};[x,\mathfrak{r}]=0\}$. Then $\tilde{\mathfrak{h}}$ is an ideal of $\mathfrak{g}$. Moreover, from $\alpha=0$ we obtain $\mathfrak{r}=P^0+\mathfrak{r}_0+Q^0+\mathfrak{g}_1$ by Lemma 5.11, (5.14.1) and Lemma 5.14. This implies that ad$E$ vanishes on $\mathfrak{r}$. Hence $[\mathfrak{h}_{-1/2},\mathfrak{r}]=0$, $[\mathfrak{h}_{1/2},\mathfrak{r}]=0$ and $\mathfrak{h}_{-1/2}+\mathfrak{h}_{1/2}\subset\tilde{\mathfrak{h}}$ follows. Let $\tilde{\mathfrak{r}}$ be the centralizer of $\tilde{\mathfrak{h}}$ in $\mathfrak{g}$. Then $c\in\mathfrak{r}\subset\tilde{\mathfrak{r}}$. By Proposition 5.13, $j\tilde{\mathfrak{r}}\subset\tilde{\mathfrak{r}}+\mathfrak{k}$. Therefore there exists $k\in\mathfrak{k}$ such that $jc-k\in\tilde{\mathfrak{r}}$. But then $[jc-k,\mathfrak{h}_{-1/2}+\mathfrak{h}_{1/2}]=0$, whence

$$\mathrm{Re}(\mathrm{ad}(jc-k))\,\mathfrak{h}_{-1/2}=\mathrm{Re}(\mathrm{ad}(jc-k))\,\mathfrak{h}_{1/2}=0.$$

As in Remark 4.4, $\mathrm{Re}(\mathrm{ad}(jc-k))=\mathrm{Re}(\mathrm{ad}\,jc)$ holds. Therefore $\mathfrak{h}_{-1/2}=\mathfrak{h}_{1/2}=0$, proving the assertion.

5.16. In this section we exclude the cases $\alpha\neq 0,1$. It is obvious that this finishes the proof of Theorem 5.1.

LEMMA. *The case* $\alpha\neq 0,1$ *does not occur.*

Proof. Under our assumptions we have $[\mathfrak{h}_{-1/2}, c] \neq 0$. In fact, if $[\mathfrak{h}_{-1/2}, c]=0$, then $\mathfrak{g}_1$ is invariant under $\mathrm{ad}\,\mathfrak{h}$, whence $\alpha=0$ follows. Since $\alpha \neq 1$, $[Q^{\alpha-1/2}, \mathfrak{h}_{-1/2}]=0$. Therefore $Q^{\alpha-1/2}+\mathfrak{g}_1$ is $\mathrm{ad}\,\mathfrak{h}$-invariant. It follows that $(\alpha-1/2)\dim Q^{\alpha-1/2}+\alpha=0$. In particular, $0<\alpha<1/2$. Let Q^a be any eigenspace of $\mathrm{ad}\,E$ in $\mathfrak{r}_{1/2}$. If $[Q^a, \mathfrak{h}_{-1/2}] \neq 0$ then $a=1/2$ and if $[Q^a, \mathfrak{h}_{-1/2}]=0$ then $a=0$ or $\alpha-1/2$ corresponding to $[Q^a, \mathfrak{h}_{1/2}]=0$ or $[Q^a, \mathfrak{h}_{1/2}] \neq 0$. Therefore $\mathfrak{r}_{1/2}=Q^0+Q^{1/2}+Q^{\alpha-1/2}$. It follows $jP^0=[c, P^0] \subset Q^\alpha=0$ because $0<\alpha<1/2$. Hence $P^0=0$. Since $jP^{-1/2}=[c, P^{-1/2}] \subset Q^{\alpha-1/2}$, we have $[jP^{-1/2}, P^{-1/2}]=0$. Therefore for any $x \in P^{-1/2}$, $\varrho(e^{\mathrm{ad}\,tE}jx, e^{\mathrm{ad}\,tE}x)$ is constant. But $\varrho(e^{\mathrm{ad}\,tE}jx, e^{\mathrm{ad}\,tE}x)$ grows as $e^{t(\alpha-1)}$ if it is not identically zero. Therefore it must vanish. Hence we get $\varrho(jx, x)=0$ and $P^{-1/2}=0$ follows. But then $\mathfrak{r}_{-1/2}=0$ and $\mathfrak{r}_0+Q^{1/2}$ is invariant under $\mathfrak{h}$. Therefore

$$0 = \mathrm{Trace}\,(\mathrm{ad}\,E|\mathfrak{r}_0+Q^{1/2}) = \dim Q^{1/2}/2$$

and $Q^{1/2}=0$ follows. Thus we have

$$\mathfrak{r}_{-1/2}=0 \quad \text{and} \quad \mathfrak{r}_{1/2}=Q^0+Q^{\alpha-1/2}. \tag{5.16.1}$$

Let $\mathfrak{w}_{1/2}$ and $\mathfrak{u}_{1/2}$ be the subspaces of $\mathfrak{g}_{1/2}$ defined by (5.2.1). Note that $\mathfrak{u}_{1/2}$ is contained in $\mathfrak{r}_{1/2}$. But then $[\mathfrak{g}_{-1/2}, \mathfrak{u}_{1/2}]=0$ because of Lemma 5.10 and (5.16.1). From this the usual argument shows that the function $A(t)=\varrho(e^{\mathrm{ad}\,tE}x, e^{\mathrm{ad}\,tE}y)$, $x \in \mathfrak{g}_{-1/2}, y \in \mathfrak{u}_{1/2}$, is constant. On the other hand, $A(t)$ grows exponentially if it does not vanish identically. Therefore $\varrho(\mathfrak{g}_{-1/2}, \mathfrak{u}_{1/2})=0$, whence $\mathfrak{u}_{1/2}=0$. Consequently $j\mathfrak{g}_{1/2}=\mathfrak{g}_{1/2}$. Then $\mathfrak{g}_{1/2}$ is a symplectic space relative to ϱ and j. Since we know $\varrho(\mathfrak{s}, \mathfrak{g}_1)=0$, $\mathrm{ad}\,x|\mathfrak{g}_{1/2}$ is a symplectic endomorphism for any $x \in \mathfrak{s}$. Therefore, from Lemma 1.6 and Lemma 5.7, it follows that $\mathrm{ad}\,s|\mathfrak{g}_{1/2}$ commutes with j for any $s \in \mathrm{rad}(\mathfrak{s})$. In particular, $\mathrm{ad}\,\mathfrak{s}_0$ has only imaginary eigenvalues in $\mathfrak{g}_{1/2}$. From this and $[s_0, \alpha jc]=[s_0, E]=0$ it follows that every eigenvalue of $\mathrm{ad}\,E$ in $\mathfrak{g}_{1/2}$ is equal to $\alpha/2$. This is a contradiciton, since $\alpha \neq 0, 1$, and finishes the proof of the lemma.

§ 6. Quasi-normal Kähler algebras of type Case II

6.1. We continue the investigation of the effective Kähler algebra $\mathfrak{g}$ of type Case II. We keep the notations used in § 4.

Recall the decomposition $\mathfrak{g}_0=j\mathfrak{g}_1+\mathfrak{s}$ (see Theorem 4.4). From Theorem 5.1 we derive

$$\mathrm{rad}(\mathfrak{g}_0) \subset \mathrm{rad}(\mathfrak{g}). \tag{6.1.1}$$

Set

$$\mathfrak{s}_0 = \{x \in \mathfrak{s}; [x, \mathfrak{g}_1] = 0\}. \tag{6.1.2}$$

Clearly $\mathfrak{s}_0$ is an ideal of $\mathfrak{g}_0$. Therefore $\mathrm{rad}(\mathfrak{s}_0) \subset \mathrm{rad}(\mathfrak{g})$.

LEMMA. $\mathrm{rad}(\mathfrak{s})$ *is a Kähler subalgebra of flat type.*

Proof. Let $x \in \mathrm{nil}(\mathfrak{s})$, then $\mathrm{ad}\,x$ is nilpotent. But by Proposition 4.8, $\mathrm{ad}\,x|\mathfrak{g}_1$ is a semi-simple endomorphism with only imaginary eigenvalues. Therefore $\mathrm{ad}\,x|\mathfrak{g}_1=0$. Hence $\mathrm{nil}(\mathfrak{s}) \subset \mathfrak{s}_0$. Consequently $\mathrm{nil}(\mathfrak{s}) \subset \mathrm{rad}(\mathfrak{s}_0) \subset \mathrm{rad}(\mathfrak{g})$, whence $\mathrm{nil}(\mathfrak{s}) \subset \mathrm{nil}(\mathfrak{g})$. Then $\mathrm{nil}(\mathfrak{s})+j\,\mathrm{nil}(\mathfrak{s})$ is a solvable subalgebra of $\mathfrak{m}=\mathrm{nil}(\mathfrak{g})+j\,\mathrm{nil}(\mathfrak{g})$. By [6], we can decompose $\mathrm{nil}(\mathfrak{s})+j\,\mathrm{nil}(\mathfrak{s})=\mathfrak{a}'+\mathfrak{t}'$, where $\mathfrak{a}'$ (resp. $\mathfrak{t}'$) is a Kähler subalgebra of flat type (resp. of domain type). If $\mathfrak{t}' \neq 0$, then $\mathfrak{t}'$ contains a maximal idempotent e'. As before, we can see $e' \in \mathrm{nil}(\mathfrak{s})$. Recall that $\mathfrak{m}_0=\mathfrak{t}_0+\mathfrak{a}_0$, $\mathfrak{t}_0=j\mathfrak{g}_1$ and that $[\mathfrak{a}_0, e]=0$. Therefore we get $e' \in \mathfrak{a}_0$, because $[e', e]=0$. But then $\mathrm{Re}(\mathrm{ad}\,je')|\mathfrak{a}=0$, because $\mathfrak{a}$ is of flat type. This is a contradiction since $[je', e']=e'$. Hence $\mathrm{nil}(\mathfrak{s})+j\,\mathrm{nil}(\mathfrak{s})$ is of flat type and the lemma follows from Proposition 3.1 and Theorem 3.2

6.2. In order to obtain more detailed information about the structure of the subalgebra $\mathfrak{s}$ we need some knowledge of the fine structure of homogeneous convex cones.

Let C be a (open) homogeneous convex cone in the real vector space V containing no entire line and let $\mathfrak{F} \subset \mathrm{Lie\,Aut}\,C$ be an algebraic subalgebra which generates a transitive subgroup of $\mathrm{Aut}\,C$. The following fact will be proved in Appendix 1.

THEOREM. *Let $e \in C$ and let $\mathfrak{F}_e$ be the isotropy subalgebra at e. Then there exist pairwise commuting elements $f_1, \ldots, f_q$ of $\mathfrak{F}$, decompositions*

$$V = \bigoplus_{1 \leq i \leq j \leq q} V_{ij}, \quad \mathfrak{F} = \bigoplus_{1 \leq i \leq j \leq q} \mathfrak{F}_{ij} \oplus \mathfrak{F}_0$$

and irreducible self dual cones $C_i \subset V_{ii}$ such that $f_i \in \mathfrak{F}_{ii}$ and

(1) $f_i=(\delta_{ij}+\delta_{ik})/2$ *on* V_{jk}, $\mathrm{ad}\,f_i=(\delta_{ij}-\delta_{ik})/2$ *on* $\mathfrak{F}_{jk}$.

(2) $\mathfrak{F}_0=\{g \in \mathfrak{F};\ gv=0$ *for any* $v \in \oplus_{i=1}^{q} V_{ii}\}$.

(3) $[\mathfrak{F}_0, \mathfrak{F}_{ii}]=0$ *for* $i=1, \ldots, q$, $[\mathfrak{F}_{ii}, \mathfrak{F}_{jj}]=0$ *if* $i \neq j$, $\mathfrak{F}_{ii} V_{jj}=0$ *if* $i \neq j$.

(4) *The spaces V_{jk} are invariant under $\mathfrak{F}_{ii}$ and the restriction of $\mathfrak{F}_{ii}$ to V_{ii} gives an isomorphism between $\mathfrak{F}_{ii}$ and* $\mathrm{Lie\,Aut}\,C_i$.

(5) $C_1 \times \ldots \times C_q = C \cap (\oplus_{i=1}^q V_{ii})$ *and* $e = \Sigma_{i=1}^q e_i$ *where* $e_i \in C_i$.

(6) $\mathfrak{F}_e = \oplus_{i=1}^q (\mathfrak{F}_e \cap \mathfrak{F}_{ii}) \oplus \mathfrak{F}_0$, $\mathfrak{F}_e \cap \mathfrak{F}_{ii} = \{f \in \mathfrak{F}_{ii}; fe_i = 0\}$.

This theorem is obtained in [4] and [24] for the case $\mathfrak{F} = \mathrm{Lie\,Aut}\, C$.

From (4) it follows that each $\mathfrak{F}_{ii}$ is reductive and its semi-simple $\mathfrak{H}_i$ is a simple Lie algebra. Set $\mathfrak{U}_i = \mathfrak{F}_e \cap \mathfrak{F}_{ii}$. Then

$$\mathfrak{F}_{ii} = \mathbf{R} f_i \oplus \mathfrak{H}_i \text{ and } \mathfrak{U}_i \text{ is a maximal compact subalgebra of } \mathfrak{H}_i. \tag{6.2.1}$$

We also have

$$\begin{aligned} \mathrm{rad}(\mathfrak{F}) &= \bigoplus_{i<j} \mathfrak{F}_{ij} + \bigoplus_{i=1}^{q} \mathbf{R} f_i \oplus \mathfrak{C}(\mathfrak{F}_0) \\ \mathrm{nil}(\mathfrak{F}) &= \bigoplus_{i<j} \mathfrak{F}_{ij}, \end{aligned} \tag{6.2.2}$$

where $\mathfrak{C}(\mathfrak{F}_0)$ denotes the center of $\mathfrak{F}_0$.

Recall that $\mathfrak{F}_{ii}$ is the Lie algebra of the automorphism group of the irreducible self dual cone C_i. The study of selfdual cones is mostly due to Koecher and his school. These cones and the corresponding Lie algebras have been completely classified and are listed in the table below (see e.g. [20; I, § 8]).

cone	Lie algebra	isotropy	center
$\mathrm{Pos}(n, \mathbf{R})$	$\mathfrak{sl}(n, \mathbf{R}) + \mathbf{R}\,\mathrm{id}.$	$\mathfrak{so}(n)$	$n=2$, $\mathfrak{so}(2)$
$\mathrm{Pos}(n, \mathbf{C})$	$\mathfrak{sl}(n, \mathbf{C}) + \mathbf{R}\,\mathrm{id}.$	$\mathfrak{su}(n)$	–
$\mathrm{Pos}(n, \mathbf{H})$	$\mathfrak{su}^*(2n) + \mathbf{R}\,\mathrm{id}.$	$\mathfrak{sp}(n)$	–
$\mathrm{Pos}(3, \mathbf{O})$	$\mathfrak{e}_{6(-26)} + \mathbf{R}\,\mathrm{id}.$	$\mathfrak{f}_4$	–
Light cone	$\mathfrak{so}(n, 1) + \mathbf{R}\,\mathrm{id}.$	$\mathfrak{so}(n)$	$n=2$, $\mathfrak{so}(2)$

Note that the two cases above with non-trivial center in the isotropy algebra are actually identical. From the table above we obtain immediately

$$\mathfrak{U}_i \text{ has non-trivial center if and only if } \mathfrak{F}_{ii} \simeq \mathfrak{gl}(2, \mathbf{R}). \text{ In this case } \dim \mathfrak{U}_i = 1. \tag{6.2.3}$$

6.3. In what follows, we assume that the Kähler algebra $\mathfrak{g}$ under consideration is quasi-normal (see section 2.1 for a definition).

We set $V = \mathfrak{g}_1$ and $\mathfrak{W} = \mathrm{ad}\, \mathfrak{g}_0 | \mathfrak{g}_1$. Denote by $\mathfrak{F}$ the algebraic hull of $\mathfrak{W}$. Let C be the cone associated with the Kähler algebra $\mathfrak{g}_1 + j\mathfrak{g}_1$. (See also section 4.8.) From Proposition 4.7 we obtain $\mathfrak{F}\subset\mathrm{Lie\,Aut}\,C$ and $\mathfrak{F}_e\cap\mathfrak{W}=\mathrm{ad}\,\mathfrak{s}|V$. Since $[\mathfrak{F},\mathfrak{F}]=[\mathfrak{W},\mathfrak{W}]$, we obtain in view of (6.2.1) and (6.2.2)

$$\mathfrak{W}=\left(\bigoplus_{i=1}^{q}\mathbf{R}f_i\oplus\mathfrak{C}(\mathfrak{F}_0)\right)\cap\mathfrak{W}+\bigoplus_{i<j}\mathfrak{F}_{ij}+\bigoplus_{i=1}^{q}\mathfrak{H}_i+[\mathfrak{F}_0,\mathfrak{F}_0]. \tag{6.3.1}$$

Here the first term is contained in $\mathrm{rad}(\mathfrak{W})=\mathrm{ad}(\mathrm{rad}(\mathfrak{g}_0))|V$. Therefore it is equal to $(\oplus_{i=1}^{q}\mathbf{R}f_i)\cap\mathfrak{W}$, because $\mathrm{ad}\,x$ has only real eigenvalues for $x\in\mathrm{rad}(\mathfrak{g}_0)\subset\mathrm{rad}(\mathfrak{g})$ by assumption. Hence we have

$$\mathrm{ad}\,\mathfrak{s}|V=\bigoplus_{i=1}^{q}\mathfrak{U}_i\oplus[\mathfrak{F}_0,\mathfrak{F}_0]. \tag{6.3.2}$$

Let I be the subset of $\{1,\dots,q\}$ consisting of all i such that $\mathfrak{F}_{ii}\simeq\mathfrak{gl}(2,\mathbf{R})$. From (6.2.3) we have

$$\mathrm{ad}\,\mathrm{rad}(\mathfrak{s})|V=\bigoplus_{i\in I}\mathfrak{U}_i. \tag{6.3.3}$$

For the subalgebra $\mathfrak{s}_0$ of $\mathfrak{s}$, defined by (6.1.2), we show

LEMMA. $\mathrm{rad}(\mathfrak{s}_0)$ *is an abelian Kähler subalgebra of* $\mathfrak{g}$.

Proof. Set $\mathfrak{g}^*=\oplus_{i=1}^{q}(V_{ii}+jV_{ii})+\mathfrak{s}$. Then $\mathfrak{g}^*$ is a j-invariant subalgebra of $\mathfrak{g}$ because

$$\mathrm{ad}\,(jV_{ii}+\mathfrak{s})\,|V\subset\mathfrak{F}_{ii}+\mathfrak{F}_e,\quad [\mathfrak{F}_{ii}+\mathfrak{F}_e,V_{ij}]\subset V_{ij}\quad\text{and}\quad [jV_{ii},\mathfrak{s}]\equiv j[V_{ii},\mathfrak{s}]\ (\mathrm{mod}\,\mathfrak{s}).$$

Clearly $\mathrm{rad}(\mathfrak{s}_0)+\oplus_{i=1}^{q}V_{ii}$ is a solvable ideal of $\mathfrak{g}^*$. Therefore, by the Radical Conjecture, there exists a solvable Kähler subalgebra $\mathfrak{b}$ of $\mathfrak{g}^*$ such that $\mathfrak{b}+\mathfrak{k}=\oplus_{i=1}^{q}(V_{ii}+jV_{ii})+\mathrm{rad}(\mathfrak{s}_0)+j\,\mathrm{rad}(\mathfrak{s}_0)+\mathfrak{k}$. We can assume that $\mathfrak{b}\supset\oplus_{i=1}^{q}V_{ii}+\mathrm{rad}(\mathfrak{s}_0)$ holds. Since $\mathfrak{b}$ is a Kähler subalgebra, after an inessential change of j, we can assume $j\mathfrak{b}\subset\mathfrak{b}$. For any $x\in\mathfrak{g}^*$, we set $\sigma_i(x)=\mathrm{ad}\,x|V_{ii}$. Then σ_i is a homomorphism of $\mathfrak{g}^*$ to $\mathrm{Lie\,Aut}\,C_i$. Let $i\in I$. Clearly $\mathfrak{F}_{ii}|V_{ii}=\sigma_i(\mathfrak{b})+\mathfrak{U}_i|V_{ii}$. Since $\mathfrak{b}$ is solvable, we have $\mathfrak{F}_{ii}|V_{ii}\neq\sigma_i(\mathfrak{b})$. Therefore $\sigma_i(\mathfrak{b})\cap(\mathfrak{U}_i|V_{ii})=0$, because $\dim\mathfrak{U}_i=1$. We also obtain from (6.3.2), $\sigma_i(j\,\mathrm{rad}(\mathfrak{s}_0))\subset\mathfrak{U}_i|V_{ii}$. Therefore $\sigma_i(j\,\mathrm{rad}(\mathfrak{s}_0))\subset\sigma_i(\mathfrak{b})\cap(\mathfrak{U}_i|V_{ii})=0$. By Lemma 6.1 and Lemma 3.7, there exists a semi-simple Kähler subalgebra $\mathfrak{h}$ of $\mathfrak{s}$ such that $\mathfrak{s}=\mathrm{rad}(\mathfrak{s})+\mathfrak{h}$ and $\mathfrak{k}=\mathrm{rad}(\mathfrak{s})\cap\mathfrak{k}\oplus\mathfrak{h}\cap\mathfrak{k}$. Clearly $\sigma_i(\mathfrak{h})=0$ for $i\in I$. Let $x\in\mathrm{rad}(\mathfrak{s}_0)\subset\mathrm{rad}(\mathfrak{s})$. Then $jx=y+z$, where $y\in\mathrm{rad}(\mathfrak{s})$ and $z\in\mathfrak{h}$. Since Lemma 6.1 implies $jx\in\mathrm{rad}(\mathfrak{s})+\mathfrak{k}\cap\mathfrak{h}$, we have $z\in\mathfrak{k}\cap\mathfrak{h}$. Then $\sigma_i(y)=0$ because $\sigma_i(jx)=\sigma_i(z)=0$ as shown above. It follows from (6.3.3) that even $\mathrm{ad}\,y|V=0$ holds, whence $y\in\mathrm{rad}(\mathfrak{s}_0)$. This shows that $\mathrm{rad}(\mathfrak{s}_0)$ is a Kähler subalgebra of $\mathfrak{g}$. Since

$\mathrm{rad}(\mathfrak{s}_0) \subset \mathrm{rad}(\mathfrak{s})$, we know that $\mathrm{rad}(\mathfrak{s}_0)$ is of flat type. However, since $\mathrm{ad}\, x$ has only real eigenvalues for $x \in \mathrm{rad}(\mathfrak{s}_0)$ by assumption, we conclude that $\mathrm{rad}(\mathfrak{s}_0)$ is abelian.

6.4. Let $\mathfrak{g}^{\#}$ be the orthogonal complement of $\mathrm{rad}(\mathfrak{s}_0)$ in $\mathfrak{g}_0+\mathfrak{g}_1$ relative to ϱ,

$$\mathfrak{g}^{\#} = \{x \in \mathfrak{g}_0+\mathfrak{g}_1; \varrho(x, \mathrm{rad}(\mathfrak{s}_0)) = 0\}.$$

By Lemma 6.3, $\mathfrak{g}^{\#}$ is a Kähler subalgebra of $\mathfrak{g}_0+\mathfrak{g}_1$ and $\mathfrak{g}_0+\mathfrak{g}_1=\mathfrak{g}^{\#}+\mathrm{rad}(\mathfrak{s}_0)$ holds. Since $\mathfrak{g}$ is quasi-normal and $\mathrm{rad}(\mathfrak{s}_0) \subset \mathrm{rad}(\mathfrak{g})$, we have $\mathfrak{k} \cap \mathrm{rad}(\mathfrak{s}_0)=0$. Therefore $\mathfrak{g}^{\#} \cap \mathrm{rad}(\mathfrak{s}_0)=\mathfrak{k} \cap \mathrm{rad}(\mathfrak{s}_0)=0$. It is easy to see that $\varrho(\mathfrak{s}_0, \mathfrak{g}_1)=0$ holds. Therefore $\mathfrak{g}^{\#} \supset \mathfrak{g}_1+j\mathfrak{g}_1+\mathfrak{k}$, whence, putting $\mathfrak{s}^{\#}=\mathfrak{g}^{\#} \cap \mathfrak{s}$, we have

$$\mathfrak{g}^{\#} = \mathfrak{g}_1+j\mathfrak{g}_1+\mathfrak{s}^{\#}, \quad j\mathfrak{s}^{\#} \subset \mathfrak{s}^{\#} \quad \text{and} \quad \mathfrak{s}=\mathfrak{s}^{\#}+\mathrm{rad}(\mathfrak{s}_0).$$

It follows $\mathrm{nil}(\mathfrak{s}^{\#}) \subset \mathrm{nil}(\mathfrak{s}) \cap \mathfrak{g}^{\#} \subset \mathrm{rad}(\mathfrak{s}_0) \cap \mathfrak{g}^{\#}=0$. This shows that $\mathfrak{s}^{\#}$ is a reductive Kähler subalgebra containing $\mathfrak{k}$. Let $\mathfrak{c}^{\#}=\mathrm{rad}(\mathfrak{s}^{\#})$ denote the center of $\mathfrak{s}^{\#}$ and let $\mathfrak{h}^{\#}=[\mathfrak{s}^{\#}, \mathfrak{s}^{\#}]$. By [14], both $\mathfrak{c}^{\#}$ and $\mathfrak{h}^{\#}$ can be assumed to be j-invariant and $\mathfrak{k}= \mathfrak{k} \cap \mathfrak{c}^{\#} \oplus \mathfrak{k} \cap \mathfrak{h}^{\#}$. We want to prove

LEMMA. *$\mathfrak{c}^{\#}$ is contained in $\mathfrak{k}$.*

Proof. Since $\mathfrak{s}=\mathfrak{s}^{\#}+\mathrm{rad}(\mathfrak{s}_0)$, we have $\mathrm{rad}(\mathfrak{s})=\mathfrak{c}^{\#}+\mathrm{rad}(\mathfrak{s}_0)$. Therefore $\mathrm{ad}\,\mathrm{rad}(\mathfrak{s})|V= \mathrm{ad}\,\mathfrak{c}^{\#}|V$ follows. Then from $\mathfrak{c}^{\#} \cap \mathrm{rad}(\mathfrak{s}_0)=0$ and (6.3.3) we have

$$\mathfrak{c}^{\#} \cong \bigoplus_{i \in I} \mathfrak{u}_i. \tag{6.4.1}$$

Consider the j-invariant subalgebra $\mathfrak{g}_I=\oplus_{i \in I}(V_{ii}+jV_{ii})+\mathfrak{s}^{\#}$. We will show that $\mathfrak{h}^{\#}$ is a Kähler ideal of $\mathfrak{g}_I$. More precisely we claim

$$\mathfrak{h}^{\#} = \left\{x \in \left(\bigoplus_{i \in I} jV_{ii}+\mathfrak{s}^{\#}\right); \left[x, \bigoplus_{i \in I} V_{ii}\right] = 0\right\}. \tag{6.4.2}$$

To verify this we denote by $\mathfrak{d}_I$ the right hand side of (6.4.2). Then $\mathfrak{d}_I$ is an ideal of $\mathfrak{g}_I$. Moreover, $[x, e_i]=0$ for all $i \in I$ implies $\mathfrak{d}_I \subset \mathfrak{s}^{\#}$. Since $\mathrm{ad}\, \mathfrak{h}^{\#}|V_{ii}=0$ for $i \in I$, we obtain $\mathfrak{h}^{\#} \subset \mathfrak{d}_I$. To prove the converse inclusion we consider $x \in \mathfrak{d}_I$ and write $x=h+u$ where $h \in \mathfrak{h}^{\#}$, $u \in \mathfrak{c}^{\#}$. Then $[x, v_i]=[u, v_i]$ for all $v_i \in V_{ii}$, $i \in I$. The assertion follows now from (6.4.1).

Let $\mathfrak{r}^\#$ be the centralizer of the j-invariant semi-simple ideal $\mathfrak{h}^\#$ in $\mathfrak{g}_I$. Then $\mathfrak{g}_I=\mathfrak{r}^\#\oplus\mathfrak{h}^\#$ and $\varrho(\mathfrak{r}^\#, \mathfrak{h}^\#)=0$ (cf. Proposition 5.13.). Moreover, $\mathfrak{r}^\#$ is a Kähler ideal of $\mathfrak{g}_I$. Therefore after an inessential change of j, we can assume that $j\mathfrak{r}^\#\subset\mathfrak{r}^\#$. We note also $\mathfrak{r}^\#=\oplus_{i\in I}(V_{ii}+jV_{ii})+\mathfrak{c}^\#$. Finally we consider the map π from $(\oplus_{i\in I}jV_{ii})+\mathfrak{c}^\#$ to $\oplus_{i\in I}\mathfrak{F}_{ii}$ which is given by the equation $\pi(x)=\mathrm{ad}\,x|\oplus_{i\in I}V_{ii}$. From (6.4.1) and the choice of I it follows that π is an isomorphism. Therefore, to the ideals $\mathfrak{H}_i\subset\mathfrak{F}_{ii}$, there correspond ideals $\mathfrak{h}_i$ of $(\oplus_{i\in I}jV_{ii})+\mathfrak{c}^\#$. Since $\mathfrak{U}_i\subset\mathfrak{H}_i$ we obtain $\mathfrak{c}^\#=\oplus_{i\in I}(\mathfrak{h}_i\cap\mathfrak{c}^\#)$ and $\dim(\mathfrak{h}_i\cap\mathfrak{c}^\#)=1$ for each $i\in I$. Since $\varrho(\mathfrak{h}_i, \mathfrak{h}_j)=\varrho(\mathfrak{h}_i, [\mathfrak{h}_j, \mathfrak{h}_j])=\varrho([\mathfrak{h}_i, \mathfrak{h}_j], \mathfrak{h}_i)=0$ if $i\neq j$, we have

$$\varrho(\mathfrak{c}^\#, \mathfrak{c}^\#)=\oplus\,\varrho(\mathfrak{h}_i\cap\mathfrak{c}^\#, \mathfrak{h}_i\cap\mathfrak{c}^\#)=0.$$

From this the lemma follows, because $j\mathfrak{c}^\#\subset\mathfrak{c}^\#$.

6.5. In this section, we show the following.

THEOREM. *Set* $\tilde{\mathfrak{a}}=\mathfrak{g}_{-1/2}+\mathrm{rad}(\mathfrak{s}_0)+[e, \mathfrak{g}_{-1/2}]$. *Then* $\tilde{\mathfrak{a}}$ *is an abelian Kähler ideal of* $\mathfrak{g}$.

Proof. From Theorem 5.1, we obtain $\mathfrak{g}_{-1/2}=\mathfrak{a}_{-1/2}$, $[e, \mathfrak{g}_{-1/2}]=\mathfrak{a}_{1/2}$ and $\mathfrak{w}_{1/2}=\mathfrak{t}_{1/2}$, where $\mathfrak{w}_{1/2}$ is the subspace given by (5.2.1). From the description of solvable Kähler algebras in [6], we have $[\mathfrak{a}_\lambda, \mathfrak{t}_\mu]\subset\mathfrak{a}_{\lambda+\mu}$ if $\lambda\neq0$. (Recall that $D(\mathfrak{a}_\lambda)=0$ if $\lambda\neq0$, where D is a modification map.) In particular, $[j\mathfrak{g}_1, \mathfrak{a}_{1/2}]=[\mathfrak{t}_0, \mathfrak{a}_{1/2}]\subset\mathfrak{a}_{1/2}$. We also have

$$[\mathfrak{s}, \mathfrak{a}_{1/2}]=[e, [\mathfrak{s}, \mathfrak{g}_{-1/2}]]\subset[e, \mathfrak{g}_{-1/2}]=\mathfrak{a}_{1/2}.$$

Therefore $[\mathfrak{g}_0, \tilde{\mathfrak{a}}]\subset\tilde{\mathfrak{a}}$. We also have $[\mathfrak{g}_{-1/2}, \mathfrak{g}_1]=[\mathfrak{a}_{-1/2}, \mathfrak{t}_1]\subset\mathfrak{a}_{1/2}$, whence $[\tilde{\mathfrak{a}}, \mathfrak{g}_1]\subset\tilde{\mathfrak{a}}$. Next we prove $[\mathfrak{g}_{1/2}, \tilde{\mathfrak{a}}]\subset\tilde{\mathfrak{a}}$. We already know that $[\mathfrak{g}_{1/2}, \mathfrak{a}_{1/2}]=[\mathfrak{t}_{1/2}, \mathfrak{a}_{1/2}]+[\mathfrak{a}_{1/2}, \mathfrak{a}_{1/2}]=0$. Since $\mathfrak{g}_{1/2}=\mathfrak{a}_{1/2}+\mathfrak{w}_{1/2}$ and since $[\mathfrak{g}_0, \mathfrak{a}_{1/2}]\subset\mathfrak{a}_{1/2}$, we know that for any $x\in\mathrm{rad}(\mathfrak{s}_0)$ there exists an endomorphism $\tau(x)$ of $\mathfrak{w}_{1/2}$ such that $[x, w]\equiv\tau(x)w\,(\mathrm{mod}\ \mathfrak{a}_{1/2})$ for any $w\in\mathfrak{w}_{1/2}$. The space $\mathfrak{w}_{1/2}$, equipped with ϱ and j, is a symplectic space. Hence, from $\varrho(\mathfrak{s}_0, \mathfrak{g}_1)=0$, $\varrho(\mathfrak{a}_{1/2}, \mathfrak{w}_{1/2})=0$ and from Lemma 6.3 it follows that τ is a symplectic representation satisfying the conditions of Lemma 1.6. Therefore $\tau(x)$ commutes with j and its eigenvalues are all imaginary. On the other hand, since $\mathrm{rad}(\mathfrak{s}_0)\subset\mathrm{rad}(\mathfrak{g})$, $\tau(x)$ has only real eigenvalues. This implies $\tau(x)=0$, whence $[\mathrm{rad}(\mathfrak{s}_0), \mathfrak{g}_{1/2}]\subset\mathfrak{a}_{1/2}$. Moreover,

$$[[\mathfrak{g}_{-1/2}, \mathfrak{g}_{1/2}], \mathfrak{g}_1]=[\mathfrak{a}_{1/2}, \mathfrak{g}_{1/2}]=0$$

shows that $[\mathfrak{g}_{-1/2}, \mathfrak{g}_{1/2}] \subset \mathfrak{s}_0 \cap \mathrm{nil}(\mathfrak{g}) \subset \mathrm{rad}(\mathfrak{s}_0)$ holds. Therefore $[\tilde{\mathfrak{a}}, \mathfrak{g}_{1/2}] \subset \tilde{\mathfrak{a}}$. From [6] we also know $[\mathfrak{g}_{-1/2}, \mathfrak{a}_{1/2}] = [\mathfrak{a}_{-1/2}, \mathfrak{a}_{1/2}] = 0$, whence $[\tilde{\mathfrak{a}}, \mathfrak{g}_{-1/2}] \subset \tilde{\mathfrak{a}}$ follows. Thus we have proved that $\tilde{\mathfrak{a}}$ is an ideal. Since $[\mathfrak{a}_{-1/2}, \mathfrak{a}_{1/2}] = 0$ the adjoint representation of $\mathrm{rad}(\mathfrak{s}_0)$ on $\mathfrak{a}_{-1/2} + \mathfrak{a}_{1/2}$ is also a symplectic representation satisfying the conditions of Lemma 1.6. Therefore, as before, $[\mathrm{rad}(\mathfrak{s}_0), \mathfrak{a}_{-1/2} + \mathfrak{a}_{1/2}] = 0$ follows. This together with Lemma 6.3, implies that $\tilde{\mathfrak{a}}$ is abelian. It is clear that $\tilde{\mathfrak{a}}$ is j-invariant. Hence we have shown that $\tilde{\mathfrak{a}}$ is an abelian Kähler ideal of $\mathfrak{g}$.

6.6. We are now in a position to prove Theorem 2.5 for the effective Kähler algebra $(\mathfrak{g}, \mathfrak{k}, j, \varrho)$ of type II.

We set $\tilde{\mathfrak{h}} = \mathfrak{g}_1 + j\mathfrak{g}_1 + \mathfrak{w}_{1/2} + \mathfrak{s}^{\#}$. It is clear that $\mathfrak{g} = \tilde{\mathfrak{a}} + \tilde{\mathfrak{h}}$, $\tilde{\mathfrak{a}} \cap \tilde{\mathfrak{h}} = 0$ and $\varrho(\tilde{\mathfrak{a}}, \tilde{\mathfrak{h}}) = 0$ holds. Therefore by the theorem above, $\tilde{\mathfrak{h}}$ is a j-invariant subalgebra. We show that the decomposition $\mathfrak{g} = \tilde{\mathfrak{a}} + \tilde{\mathfrak{h}}$ satisfies the desired properties of Theorem 2.5. It remains only to show that in $\tilde{\mathfrak{h}}$ there exists a Kähler subalgebra $\mathfrak{u}$ satisfying the properties (a) and (b) in Theorem 2.5. Recall that $\mathfrak{s}^{\#}$ is a reductive Kähler subalgebra containing $\mathfrak{k}$ and the center $\mathfrak{c}^{\#}$ of $\mathfrak{s}^{\#}$ is contained in $\mathfrak{k}$ by Lemma 6.4. At this point we can follow the arguments of § 4 to § 5 in [17]. Note that to this end we only have to change the letters $\tilde{\mathfrak{h}}$, $\mathfrak{g}_1$, $\mathfrak{w}_{1/2}$ and $\mathfrak{s}^{\#}$ to $\mathfrak{g}$, $\mathfrak{r}$, $\mathfrak{w}$, and $\mathfrak{s}$. Now putting $\mathfrak{u} = \mathfrak{c}^{\#} + \mathfrak{u}^{\#}$ where $\mathfrak{u}^{\#}$ is a maximal compact subalgebra of $\mathfrak{h}^{\#}$ $(= [\mathfrak{s}^{\#}, \mathfrak{s}^{\#}])$ containing $\mathfrak{h}^{\#} \cap \mathfrak{k}$, we know from the proof of [17; Theorem 11] that $\mathfrak{u}$ is a reductive Kähler subalgebra of $\tilde{\mathfrak{h}}$ satisfying (a) and (b). This completes the proof of Theorem 2.5 in Case II.

§ 7. Proof of the Fundamental Conjecture

7.1. Let $M = G/K$ be a homogeneous Kähler manifold and let $(\mathfrak{g}, \mathfrak{k}, j, \varrho)$ be the corresponding Kähler algebra. By Theorem 2.1 we can assume that $\mathfrak{g}$ is effective and quasi-normal. Moreover, by replacing G by its universal covering group we can assume that G is simply connected. Let $\mathfrak{g} = \mathfrak{a} + \mathfrak{h}$ be the decomposition of $\mathfrak{g}$ given by Theorem 2.5. Denote by A and H the connected subgroups of G corresponding to $\mathfrak{a}$ and $\mathfrak{h}$. These are closed simply connected subgroups of G and $G = AH$ is a semi-direct product. Let $\mathfrak{u}$ be the subalgebra of $\mathfrak{h}$ as in Theorem 2.5 and denote by U the connected subgroup of H corresponding to $\mathfrak{u}$. Then $U \supset K_0$, where K_0 denotes the identity component of K. We already know that

(i) H/U is a homogeneous bounded domain and the projection from $M(\mathfrak{h}) = H/K_0$ onto H/U is holomorphic.

(ii) U/K_0 is a compact simply connected homogeneous Kähler manifold.

We set $\mathfrak{l}=\mathfrak{a}+\mathfrak{u}$, which is a j-invariant subalgebra. Let L denote the connected subgroup of G corresponding to $\mathfrak{l}$. Clearly $L=AU$ and L is closed in G.

7.2. The homogeneous space G/K_0 is the universal covering space of M and it has the natural G-invariant Kähler structure.

LEMMA. *G/L admits a natural G-invariant complex structure, with respect to which G/L is a homogeneous bounded domain and the projection $\pi^*: G/K_0 \to G/L$ is holomorphic.*

Proof. Recall that $[u, jx] \equiv j[u, x] \pmod{\mathfrak{u}}$ for $u \in \mathfrak{u}$ and $x \in \mathfrak{h}$. Since $\mathfrak{a}$ is an ideal of $\mathfrak{g}$, we obtain the relation $[l, jx] \equiv j[l, x] \pmod{\mathfrak{l}}$ for $l \in \mathfrak{l}$ and $x \in \mathfrak{g}$. This combined with (1.1.3) and with $j\mathfrak{l} \subset \mathfrak{l}$ implies that G/L admits a G-invariant complex structure so that π^* is holomorphic. Since $G=AH$ and since $H \cap L=U$, we have $G/L=H/U$. It is clear that the invariant complex structure of G/L coincides with the one of H/U. This finishes the proof of the lemma.

7.3. The spaces L/K_0, U/K_0 and A have, as complex submanifolds of G/K_0, invariant Kähler structures. Since $L=AU$ and $A \cap U=\{e\}$, we have the natural decomposition $L/K_0=A \times U/K_0$ as real analytic manifolds. Here the action of $f \in L$ on $A \times U/K_0$ is expressed as $f(g_1, g_2K_0)=(f_1f_2g_1f_2^{-1}, f_2g_2K_0)$ for $(g_1, g_2K_0) \in A \times U/K_0$, where $f=f_1f_2$, $f_1 \in A$, $f_2 \in U$. Note that $(\mathfrak{u}, \mathfrak{k}, j, \varrho)$ is a Kähler algebra of compact type. Then the adjoint representation of $\mathfrak{u}$ on the abelian ideal $\mathfrak{a}$ is a symplectic representation satisfying the conditions of Lemma 1.6. This implies that the map $g_1 \to f_2g_1f_2^{-1}$ is an automorphism of the flat Kähler manifold A. Consequently, L acts on $A \times U/K_0$ as a holomorphic and isometric transformation group. It is now clear that the Kähler algebra structure of $\mathfrak{l}$ induced from the holomorphic isometric transformation group L of the Kähler manifold $A \times L/K_0$ coincides with the Kähler structure of $\mathfrak{l}$ as the Kähler subalgebra of $\mathfrak{g}$. Therefore we obtain

LEMMA. *$L/K_0=A \times U/K_0$ as Kähler manifolds.*

7.4. We introduce some notation which will be used in the rest of this paper. For a Lie algebra $\mathfrak{s}$, we denote by $\mathfrak{s}_C$ the complexification of $\mathfrak{s}$ and if $\mathfrak{t} \subset \mathfrak{s}_C$, we write $\mathfrak{n}(\mathfrak{s}_C, \mathfrak{t})$ for the normalizer of $\mathfrak{t}$ in $\mathfrak{s}_C$.

7.5. In this subsection, we collect some remarks on homogeneous bounded domains and compact simply connected homogeneous Kähler manifolds.

Let S/B be a homogeneous Kähler manifold of a connected Lie group S by a closed subgroup B. (We do not assume the effectiveness of the action of S here.) Let $(\mathfrak{s}, \mathfrak{b}, j, \varrho)$ be the corresponding Kähler algebra. Set

$$\mathfrak{s}_- = \{x+\sqrt{-1}\, jx; x \in \mathfrak{s}\} + \mathfrak{b}_{\mathrm{C}}.$$

It is easy to see that $\mathfrak{s}_{\mathrm{C}} = \mathfrak{s} + \mathfrak{s}_-$ and $\mathfrak{s}_- \cap \mathfrak{s} = \mathfrak{b}$ hold. Moreover, $\mathfrak{s}_-$ is a complex subalgebra of $\mathfrak{s}_{\mathrm{C}}$. We define a closed subgroup B' of S by

$$B' = \{g \in S; \operatorname{Ad} g\, \mathfrak{s}_- = \mathfrak{s}_-\}.$$

It is easy to see that an element $g \in S$ belongs to B' if and only if the following conditions are satisfied

$$\operatorname{Ad} g\, \mathfrak{b} = \mathfrak{b} \quad \text{and} \quad \operatorname{Ad} g \cdot jx \equiv j \operatorname{Ad} g\, x \pmod{\mathfrak{b}} \quad \text{for any } x \in \mathfrak{s}. \tag{7.5.1}$$

Clearly $B' \supset B$. Denote by $\mathfrak{b}'$ the Lie algebra of B'. From (7.5.1) we derive that $\mathfrak{b}'$ is j-invariant and that the following relations hold (cf. [11]).

$$\mathfrak{b}' = \mathfrak{n}(\mathfrak{s}_{\mathrm{C}}, \mathfrak{s}_-) \cap \mathfrak{s} \quad \text{and} \quad \mathfrak{n}(\mathfrak{s}_{\mathrm{C}}, \mathfrak{s}_-) = \mathfrak{s}_- + \mathfrak{b}'.$$

LEMMA. *Assume that S/B is a homogeneous bounded domain or a compact simply connected homogeneous Kähler manifold. Then $B' = B$ and $\mathfrak{n}(\mathfrak{s}_{\mathrm{C}}, \mathfrak{s}_-) = \mathfrak{s}_-$.*

Proof. From our assumption it follows that the Ricci tensor of S/B is negative definite or positive definite. As in the proof of Lemma 4.2, we define a form ψ on $\mathfrak{s}$ by

$$\psi(x) = \operatorname{Trace}(\operatorname{ad} jx - j \circ \operatorname{ad} x)|\mathfrak{s}/\mathfrak{k}.$$

By [13], the symmetric bilinear form $-\psi([jx, y])$, $x, y \in \mathfrak{s}$, corresponds to the Ricci tensor of S/B. As a result, $(\mathfrak{s}, \mathfrak{k}, j, \eta)$ is a Kähler algebra, where $\eta = -d\psi$ or $d\psi$. A simple computation shows

$$\psi(\operatorname{ad} g\, x) = \psi(x) \quad \text{for any } g \in B' \text{ and } x \in \mathfrak{s}. \tag{7.5.2}$$

This implies $\psi([\mathfrak{b}', \mathfrak{s}]) = 0$. Therefore $\mathfrak{b}' = \mathfrak{b}$ and hence $\mathfrak{n}(\mathfrak{s}_{\mathrm{C}}, \mathfrak{s}_-) = \mathfrak{s}_-$.

Since S/B is simply connected, B is connected. Therefore B is the identity component of B', whence S/B is a covering space of S/B'. By (7.5.1) and (7.5.2) we know that there exists an S-invariant Kähler structure on S/B' so that the projection S/B onto S/B' is holomorphic. In the case where S/B is a homogeneous bounded domain, the equation $B=B'$ follows from a result of [12]: Every homogeneous complex manifold which has a homogeneous bounded domain as universal covering space is simply connected. In the latter case, S/B' is a compact homogeneous Kähler manifold with positive definite Ricci tensor. Therefore a semi-simple group acts transitively on S/B' as an automorphism group. Hence by [1], S/B' is simply connected and we get $B'=B$.

Remark. In [11], the equation $\mathfrak{n}(\mathfrak{s}_C, \mathfrak{s}_-)=\mathfrak{s}_-$ is proved in a more general setting.

7.6. We return to the investigation of the homogeneous Kähler manifold M. Consider the homogeneous Kähler manifild H/K_0 and the homogeneous bounded domain H/U discussed in section 7.1. As in section 7.5, we set

$$\begin{aligned}\mathfrak{h}_- &= \{x+\sqrt{-1}\,jx; x\in\mathfrak{h}\}+\mathfrak{k}_C,\\ \mathfrak{h}_-^* &= \{x+\sqrt{-1}\,jx; x\in\mathfrak{h}\}+\mathfrak{u}_C,\\ K' &= \{g\in H; \operatorname{ad} g\,\mathfrak{h}_- = \mathfrak{h}_-\}.\end{aligned}$$

We already know from Lemma 7.5,

$$\mathfrak{n}(\mathfrak{h}_C, \mathfrak{h}_-^*) = \mathfrak{h}_-^*. \tag{7.6.1}$$

We want to show

LEMMA. (a) $K' = K_0$.

(b) $\mathfrak{n}(\mathfrak{h}_C, \mathfrak{h}_-) = \mathfrak{h}_-$.

Proof. Let k be an element of K'. From (7.5.1) we derive that k induces a holomorphic transformation γ_k of H/K_0 given by $\gamma_k(gK_0)=gkK_0$ for $gK_0\in H/K_0$. Consider the map $\pi^*\circ\gamma_k$, where π^* denotes the projection $H/K_0\to H/U$. Then $\pi^*\circ\gamma_k\,(U/K_0)$ consists of a single point, because U/K_0 is compact. This implies that $k\,U k^{-1}=U$ holds, whence $\operatorname{Ad} k\,\mathfrak{u}=\mathfrak{u}$. Clearly $\operatorname{Ad} kjx\equiv j\operatorname{Ad} kx \pmod{\mathfrak{u}}$ for any $x\in\mathfrak{h}$. Therefore an application of Lemma 7.5 to the homogeneous bounded domain H/U yields $k\in U$. Now we apply Lemma 7.5 to U/K_0 and obtain $k\in K_0$. We thus infer $K'=K$. Hence we also have $\mathfrak{n}(\mathfrak{h}_C, \mathfrak{h}_-)=\mathfrak{h}_-$ and the lemma is proved.

7.7. In this section we obtain the Fundamental Conjecture up to the fact that the fiber bundle is holomorphically locally trivial. This property will be established in the following section.

LEMMA. *$K=\Gamma K_0$, where Γ is a discrete subgroup of A contained in the center of G. In particular $K\subset L$.*

Proof. Let k be an element of K. We can write k in the form $k=k_1k_2$, where $k_1\in A$ and $k_2\in H$. Since $\mathfrak{a}$ is an abelian ideal of $\mathfrak{g}$ and since $\mathfrak{h}$ is the orthogonal complement of $\mathfrak{a}$, we have $\mathrm{Ad}\,k\,\mathfrak{a}=\mathfrak{a}$ and $\mathrm{Ad}\,k\,\mathfrak{h}=\mathfrak{h}$. Moreover $\mathrm{Ad}\,k_1 x\equiv x \pmod{\mathfrak{a}}$ for any $x\in\mathfrak{g}$. This implies $\mathrm{Ad}\,kx\equiv\mathrm{Ad}\,k_2x \pmod{\mathfrak{a}}$ for all $x\in\mathfrak{h}$. It follows that $\mathrm{Ad}\,k_2x=\mathrm{Ad}\,kx=\mathrm{Ad}\,k_1\,\mathrm{Ad}\,k_2\,x$ holds for all $x\in\mathfrak{h}$. Consequently, $\mathrm{Ad}\,k_1|\mathfrak{h}$ is the identity map. From this we conclude that k_1 is in the center of G. It is clear now that k_2 has the following properties: $\mathrm{Ad}\,k_2\mathfrak{k}=\mathfrak{k}$ and $\mathrm{Ad}\,k_2jx\equiv j\,\mathrm{Ad}\,k_2x \pmod{\mathfrak{k}}$ for any $x\in\mathfrak{h}$. Therefore by Lemma 7.6, k_2 is an element of K_0. Putting $\Gamma=K\cap A$, the assertion of the lemma follows.

Combining Lemmata 7.2, 7.3 and 7.7, we have shown

PROPOSITION. *The homogeneous Kähler manifold $M=G/K$ is a real analytic fiber bundle over the homogeneous bounded domain $D=G/L$ with a holomorphic projection $\pi: M\to D$ and the typical fiber $F=L/K$ is, with the induced Kähler structure, the direct product of the flat homogeneous Kähler manifold A/Γ and the compact simply connected homogeneous Kähler manifold U/K_0.*

7.8. The final step of the proof of the Fundamental Conjecture is to show that the fiber bundle obtained in Proposition 7.7 is holomorphically locally trivial.

Let G_C be the simply connected Lie group with $\mathfrak{g}_C$ as its Lie algebra. We denote by A_C and H_C the connected subgroups of G_C corresponding to $\mathfrak{a}_C$ and $\mathfrak{h}_C$ respectively. We also denote by σ the natural homomorphism of G into G_C. It is clear that σ is injective on A and that $G_C=A_CH_C$ is a semi-direct product. We denote by H_- and H^*_- the connected subgroups of H_C corresponding to $\mathfrak{h}_-$ and $\mathfrak{h}^*_-$ defined in section 7.6. By (7.6.1) (resp. Lemma 7.6), the group H^*_- (resp. H_-) is the identity component of the normalizer of $\mathfrak{h}^*_-$ (resp. $\mathfrak{h}_-$) in H_C. Therefore both H^*_- and H_- are closed complex subgroups of H_C.

Next we define subspaces $\mathfrak{a}_\pm$ and subgroups $A_\pm$ by

$$\mathfrak{a}_\pm=\{a\in\mathfrak{a}_C;\, ja=\pm\sqrt{-1}\,a\},$$
$$A_\pm=\{\exp a;\, a\in\mathfrak{a}_\pm\},$$

where exp denotes the exponential mapping from $\mathfrak{a}_C$ to A_C. We put

$$G_- = \sigma(\Gamma)A_-H_- \quad \text{and} \quad G_-^* = A_C H_-^*.$$

Note that $[\mathfrak{a}_-, \mathfrak{h}_-] \subset \mathfrak{a}_-$. Therefore both, G_- and G_-^*, are closed complex subgroups of G_C and $G_-^* \supset G_-$. From the definitions of G_- and G_-^* it follows that $\sigma(K) \subset G_-$ and $\sigma(L) \subset G_-^*$ hold (recall $\mathfrak{l} = \mathfrak{a} + \mathfrak{u}$). Therefore σ induces G-equivariant mappings $\Phi: G/K \to G_C/G_-$ and $\varphi: G/L \to G_C/G_-^*$. Clearly $\pi_C \circ \Phi = \varphi \circ \pi$, where π_C denotes the projection of G_C/G_- onto G_C/G_-^* and $\pi: M \to D$ was defined in Proposition 7.7. The Fundamental Conjecture will be a direct consequence of the

PROPOSITION. *The mappings Φ and φ are holomorphic imbeddings of G/K and of G/L onto open sets of G_C/G_- and of G_C/G_-^* respectively. Moreover Φ maps L/K onto G_-^*/G_-.*

Proof. Clearly $jx \equiv \sqrt{-1}\, x (\mathrm{mod}\, \mathfrak{a}_- + \mathfrak{h}_-)$ for any $x \in \mathfrak{g}$. This implies that Φ and φ are holomorphic. Let g be an element of G. We decompose g as $g = g_1 g_2$, where $g_1 \in A$ and $g_2 \in H$. Assume that $\sigma(g) \in G_-$. Then $\sigma(g_1) \in \sigma(\Gamma) A_-$ and $\sigma(g_2) \in H_-$. Since σ is injective on A and since $\sigma(A) \cap A_- = \{e\}$, we have $g_1 \in \Gamma$. From $\sigma(g_2) \in H_-$, we obtain $\mathrm{Ad}\, g_2\, \mathfrak{h}_- = \mathfrak{h}_-$. Therefore, by Lemma 7.6, we know $g_2 \in K_0$, whence $g \in K$ by Lemma 7.7. This implies that Φ is an imbedding.

Next we assume that $\sigma(g) \in G_-^*$. The Lie algebra $\mathfrak{g}_-^*$ of G_-^* is given by

$$\mathfrak{g}_-^* = \{x + \sqrt{-1}\, jx; x \in \mathfrak{g}\} + \mathfrak{l}_C.$$

Therefore an application of Lemma 7.5 to the homogeneous bounded domain G/L yields $g \in L$. This shows that also φ is an imbedding.

Clearly, $\dim G/K = \dim G_C/G_-$ and $\dim G/L = \dim G_C/G_-^*$. From this it follows that $\Phi(G/K)$ and $\varphi(G/L)$ are open sets of G_C/G_- and of G_C/G_-^* respectively. Hence $\Phi(L/K)$ is also an open set of $G_-^*/G_- = A_C/\sigma(\Gamma)A_- \times H_C/G_-^*$. Since $A_C = A_+ A_-$, we obtain $\Phi(A/\Gamma) = A_C/\sigma(\Gamma)A_-$. We also have $\Phi(U/K_0) = H_C/H_-^*$, because U/K_0 is compact. Therefore $\Phi(L/K) = G_-^*/G_-$, completing the proof of the proposition.

Remarks: (1) From the proposition above it follows that the fiber bundle (M, π, D, F) is the restriction of the holomorphic fiber bundle $(G_C/G_-, \pi_C, G_C/G_-^*, G_-^*/G_-)$. Therefore it is also a holomorphic fiber bundle. This finishes the proof of the Fundamental Conjecture.

(2) Moreover, in the above fibering the structure group can be taken to be the complex Lie group G^*_-. Therefore, as is mentioned in [25], this bundle is holomorphically trivial by a result of [10] because the base space is topologically trivial. Thus we get.

THEOREM. *Every homogeneous Kähler manifold is as a complex manifold, the product of a homogeneous bounded domain and* $\mathbf{C}^n/\Gamma$ *and a compact simply connected homogeneous complex manifold, where* Γ *denotes a discrete subgroup of translations of* $\mathbf{C}^n$.

Appendix 1: Description of algebraic transitive Lie algebras on homogeneous convex cones (Proof of Theorem 6.2)

A1.1. We will give here the proof of Theorem 6.2.

Recall that C is a homogeneous convex cone in a real vector space V, $\mathfrak{F}$ is an algebraic Lie algebra which generates a transitive subgroup of $\operatorname{Aut} C$, and that $\mathfrak{F}_e$ denotes the isotropy subalgebra of F at a point $e\in C$. Consider the sum $\mathfrak{G}=V\oplus\mathfrak{F}$. We can introduce in $\mathfrak{G}$ a Lie algebra structure as follows.

$$[v_1\oplus f_1, v_2\oplus f_2]=f_1v_1-f_2v_2\oplus [f_1,f_2] \quad \text{for} \quad v_1, v_2\in V, f_1, f_2\in\mathfrak{F}.$$

Then $\mathfrak{G}$ can be regarded as the Lie algebra of a transitive subgroup of $\operatorname{Aut} D(C)$, where $D(C)=\{x+\sqrt{-1}\,y; x\in V, y\in C\}$ denotes the Siegel domain of the first kind associated with the cone C and $\mathfrak{F}_e$ is the isotropy subalgebra at the point $\sqrt{-1}\,e\in D(C)$. Let j denote an endomorphism of $\mathfrak{G}$ corresponding to the complex structure of $D(C)$ and satisfying $j\mathfrak{F}_e\subset\mathfrak{F}_e$ Then $(\mathfrak{G}, \mathfrak{F}_e, j)$ is a j-algebra. It is clear that V is an abelian ideal of the first kind [26] and e is its principal idempotent. Since $\mathfrak{F}$ is algebraic, we can decompose $\mathfrak{F}$ as $\mathfrak{F}=\mathfrak{T}+\mathfrak{F}_e$, where $\mathfrak{T}$ is a split solvable subalgebra [23]. Then for any $f\in\mathfrak{T}$, $\operatorname{ad} f$ has only real eigenvalues in $\mathfrak{G}$. By a suitable change of j, we can assume that $jV=\mathfrak{T}$ holds.

We note that if $v\in V$ satisfies $[v, \mathfrak{F}_e]=0$, then $[jv, \mathfrak{F}_e]=0$. Indeed, as in Remark 4.4, we can write $jv=x+y$, where $y\in\mathfrak{F}_e$ and $[x, \mathfrak{F}_e]=0$. Then $\operatorname{ad} jv|\mathfrak{F}_e=\operatorname{ad} y|\mathfrak{F}_e$. Since $\operatorname{ad} jv$ has only real eigenvalues, we have $\operatorname{ad} jv|\mathfrak{F}_e=0$. In particular, we infer $[je, \mathfrak{F}_e]=0$.

A1.2. Let $c_1, \ldots, c_m$ be the elements of V used in Lemma 4.5 and consider the decomposition $V=\oplus_{i\leqslant k}\mathfrak{r}_{ik}$. Since $\mathfrak{T}$ is split solvable, $V+\mathfrak{T}$ is a normal j-algebra.

Therefore we have the following

LEMMA (Takeuchi [22]). *Let c be an element of V satisfying $[jc, c]=0$. Then $c=\Sigma_{i\in I}c_i$, where I is a subset of $\{1,\dots,m\}$.*

We also note that from the description of normal j-algebras in [19] we know $\mathrm{ad}jc_i=\mathrm{Re}(\mathrm{ad}jc_i)$ for all i. Therefore we have $\mathrm{ad}jc=\mathrm{Re}(\mathrm{ad}jc)$ for every element $c\in V$ satisfying $[jc, c]=c$.

A1.3. Let V_{11} be an $\mathfrak{F}$-invariant subspace of V of minimal dimension. Then V_{11} is an abelian ideal of the first kind in $\mathfrak{G}$. Let e_1 be its principal idempotent. Then $[e_1, \mathfrak{F}_e]=0$ ([26]) and hence $[je_1, \mathfrak{F}_e]=0$. By Lemma Al.2, there exists a subset I_1 of $\{1,\dots,m\}$ such that $e_1=\Sigma_{i\in I_1}c_i$. Recall that by [26] the equality $V_{11}=\{x\in\mathfrak{G};\ \mathrm{ad}je_1x=x\}$ holds. Therefore we have $V_{11}=\oplus_{i,k\in I_1}\mathfrak{r}_{ik}$. Set $V_1^{1/2}=\oplus_{i\in I,k\in I_1'}\mathfrak{r}_{ik}$ and $V_1^0=\oplus_{i,k\in I_1'}\mathfrak{r}_{ik}$, where $I_1'=\{1,\dots,m\}\setminus I_1$. We note that $\mathfrak{r}_{ik}=0$ if $i\in I_1'$ and $k\in I_1$. Indeed, for such i,k we have $[c_k, j\mathfrak{r}_{ik}]\subset V_{11}$, because V_{11} is $\mathfrak{F}$-invariant. But $[c_k, j\mathfrak{r}_{ik}]=-\mathfrak{r}_{ik}$, whence $\mathfrak{r}_{ik}=0$. Therefore we have $V=V_{11}\oplus V_1^{1/2}\oplus V_1^0$. Clearly, V_{11}, $V_1^{1/2}$ and V_1^0 are $\mathfrak{F}_e$-invariant and $[jV_{11}+V_{11}, jV_1^0+V_1^0]=0$ and $[jV_1^{1/2}, V_{11}]=0$ holds. In particular, the group generated by $jV_{11}+\mathfrak{F}_e$ acts irreducibly on V_{11}.

Set $e_1'=e-e_1$, $C_1=\exp(jV_{11}+\mathfrak{F}_e)\,e_1$ and $C_1'=\exp(jV_1^0+\mathfrak{F}_e)\,e_1'$. Then C_1 and C_1' are homogeneous cones in V_{11} and V_1^0. Moreover, $C_1\times C_1'\subset C$.

Consider the j-invariant subalgebra $\mathfrak{G}'=V_0^1+jV_1^0+\mathfrak{F}_e$. If there exists a $(jV_1^0+\mathfrak{F}_e)$-invariant subspace of V_1^0, then by the same arguments as before we find subsets $I_2, I_2'\subset I_1'$, a decomposition $V_1^0=V_{22}+V_2^{1/2}+V_2^0$ and cones C_2, C_2'.

Repeating the procedure above, we obtain $I_\alpha'=I_{\alpha+1}\cup I_{\alpha+1}'$ and subspaces

$$V_{\alpha\alpha}=\oplus_{i,k\in I_\alpha}\mathfrak{r}_{ik},\quad V_\alpha^{1/2}=\oplus_{i\in I_\alpha,k\in I_\alpha'}\mathfrak{r}_{ik}\quad\text{and}\quad V_\alpha^0=\oplus_{i,k\in I_\alpha'}\mathfrak{r}_{ik}.$$

Thus we get

$$\{1,\dots,m\}=\bigcup_{\alpha=1}^{q}I_\alpha\quad\text{(disjoint union)}\tag{Al.1}$$

$$e_\alpha=\sum_{i\in I_\alpha}e_i$$

and a decomposition

$$V=\oplus_{\alpha\leq\beta}V_{\alpha\beta},\quad\text{where}\quad V_{\alpha\beta}=\oplus_{i\in I_\alpha,k\in I_\beta}\mathfrak{r}_{ik}.$$

Note that $V_\alpha^{1/2}=\oplus_{\alpha<\beta}V_{\alpha\beta}$ and $V_\alpha^0=\oplus_{\alpha<\beta\leq\gamma}V_{\beta\gamma}$ holds. Set $f_\alpha=je_\alpha$. By construction, we have

$$\operatorname{ad}f_\alpha|V_{\beta\gamma}=(\delta_{\alpha\beta}+\delta_{\alpha\gamma})/2 \quad\text{and}\quad \operatorname{ad}f_\alpha|jV_{\beta\gamma}=(\delta_{\alpha\beta}-\delta_{\alpha\gamma})/2. \tag{A1.2}$$

We also have

$$[jV_{\alpha\alpha}+V_{\alpha\alpha},jV_{\beta\beta}+V_{\beta\beta}]=0 \quad\text{if}\quad \alpha\neq\beta. \tag{A1.3}$$

A1.4. We have chosen e_α so that $[e_\alpha,\mathfrak{F}_e]=0$ holds. Therefore $[je_\alpha,\mathfrak{F}_e]=0$. Hence all $V_{\alpha\beta}$ are invariant under $\mathfrak{F}_e$. Set

$$C_\alpha=\exp(jV_{\alpha\alpha}+\mathfrak{F}_e)e_\alpha.$$

We already know that C_α is a homogeneous convex cone in $V_{\alpha\alpha}$ and

$$C\supset C_1\times\ldots\times C_q. \tag{A1.4}$$

Consider the subalgebra $\tilde{\mathfrak{F}}=\oplus_{\alpha=1}^q jV_{\alpha\alpha}+\mathfrak{F}_e$. The correspondence $\varrho_\alpha:f\to f|V_{ii}$ gives a homomorphism of $\tilde{\mathfrak{F}}$ to $\operatorname{Lie}\operatorname{Aut}(C_\alpha)$. Then ϱ_α is an irreducible representation. Therefore $\operatorname{Aut}(C_\alpha)$ acts irreducibly on $V_{\alpha\alpha}$. But then the following facts are well-known:

(i) C_α is an irreducible self dual cone.

(ii) $\operatorname{Lie}\operatorname{Aut}(C_\alpha)$ is reductive, its semi-simple part is simple, its center is 1-dimensional and generated by $\varrho(je_\alpha)$.

We decompose $\tilde{\mathfrak{F}}$ into the sum of root spaces of $\{jc_1,\ldots,jc_m\}$, $\tilde{\mathfrak{F}}=\oplus\tilde{\mathfrak{F}}^\Gamma$. Let $\mathfrak{F}_0=\{f\in\tilde{\mathfrak{F}};f|V_{\alpha\alpha}=0$ for all $\alpha\}$. Then $\mathfrak{F}_0\subset\mathfrak{F}_e$, because $e\in\oplus V_{\alpha\alpha}$. Moreover, $\mathfrak{F}_0=\bigcap_{\alpha=1}^q\operatorname{Kernel}\varrho_\alpha$ and it follows that $\mathfrak{F}_0$ is an ideal of $\tilde{\mathfrak{F}}$. It is easy to see that $\mathfrak{F}_0\subset\tilde{\mathfrak{F}}^0$. Since $\tilde{\mathfrak{F}}/\mathfrak{F}_0$ is identified with a transitive subalgebra of $\oplus_{\alpha=1}^q\operatorname{Lie}\operatorname{Aut}(C_\alpha)$, we know from [26] that every root Γ is of the form $(\Delta_i-\Delta_j)/2$ and for $i<k$, $\tilde{\mathfrak{F}}^{(\Delta_i-\Delta_k)/2}=j\mathfrak{r}_{ik}$ holds. For each α we set

$$\mathfrak{H}_\alpha=\bigoplus_{\substack{i,k\in I_\alpha\\ i\neq k}}\tilde{\mathfrak{F}}^{(\Delta_i-\Delta_k)/2}+\sum_{\substack{i,k\in I_\alpha\\ i\neq k}}[\tilde{\mathfrak{F}}^{(\Delta_i-\Delta_k)/2},\tilde{\mathfrak{F}}^{(\Delta_k-\Delta_i)/2}]$$

$$\mathfrak{F}_{\alpha\alpha}=\mathfrak{H}_\alpha+\mathbf{R}f_\alpha.$$

It is obvious that $\varrho_\beta(\mathfrak{F}_{\alpha\alpha})=0$ if $\alpha\neq\beta$ and $[\mathfrak{F}_0,\mathfrak{F}_{\alpha\alpha}]=0$.

LEMMA. *ϱ_α is an isomorphism of $\mathfrak{F}_{\alpha\alpha}$ onto* Lie Aut C_α.

Proof. Let $\tilde{\mathfrak{F}}$=Lie AutC_α and let $f_i'=\varrho_\alpha(jc_i)$ for $i\in I_\alpha$. We decompose $\tilde{\mathfrak{F}}$ into the sum of root spaces of $\{f_i'; i\in I_\alpha\}$, $\tilde{\mathfrak{F}}=\oplus\tilde{\mathfrak{F}}^\Gamma$. Clearly, for $i, k\in I_\alpha$ we have

$$\varrho_\alpha(\mathfrak{F}^{(\Delta_i-\Delta_k)/2})\subset\tilde{\mathfrak{F}}^{(\Delta_i-\Delta_k)/2}.$$

We also know from [26] that if $\tilde{\mathfrak{F}}^\Gamma\neq 0$, then Γ is of the form $(\Delta_i-\Delta_k)/2$ for some $i, k\in I_\alpha$ and that $\tilde{\mathfrak{F}}^{(\Delta_i-\Delta_k)/2}=\varrho_\alpha(j\mathfrak{r}_{ik})$ for $i, k\in I_\alpha$ satisfying $i<k$. From the irreducibility, it follows that $\varrho_\alpha(\mathfrak{F})$ is reductive. Therefore $\varrho_\alpha(\mathfrak{F}^\Gamma)\neq 0$ if and only if $\varrho_\alpha(\mathfrak{F}^{-\Gamma})\neq 0$ and $\dim\varrho_\alpha(\mathfrak{F}^\Gamma)=\dim\varrho_\alpha(\mathfrak{F}^{-\Gamma})$. The same assertions also hold for $\tilde{\mathfrak{F}}^\Gamma$ and $\tilde{\mathfrak{F}}^{-\Gamma}$. We thus obtain that $\varrho_\alpha(\mathfrak{H}_\alpha)$ is an ideal of $\tilde{\mathfrak{F}}$ contained in the semi-simple part $[\tilde{\mathfrak{F}},\tilde{\mathfrak{F}}]$ of $\tilde{\mathfrak{F}}$, whence $\varrho_\alpha(\mathfrak{H}_\alpha)=[\tilde{\mathfrak{F}},\tilde{\mathfrak{F}}]$. It is now clear that ϱ_α is injective on $\mathfrak{F}_{\alpha\alpha}$. This implies the lemma.

A1.5. By Lemma Al.4, we have

$$\mathfrak{F}=\bigoplus_{\alpha=1}^{q}\mathfrak{F}_{\alpha\alpha}\oplus\mathfrak{F}_0\quad\text{(direct sum of ideals)}.\tag{Al.5}$$

It is clear that $\mathfrak{F}_e=\oplus_{\alpha=1}^q(\mathfrak{F}_e\cap\mathfrak{F}_{\alpha\alpha})\oplus\mathfrak{F}_0$ and that $\mathfrak{F}_e\cap\mathfrak{F}_{\alpha\alpha}=\{f\in\mathfrak{F}_{\alpha\alpha}; fe_\alpha=0\}$. Now we set $\mathfrak{F}_{\alpha\beta}=jV_{\alpha\beta}$ for $\alpha<\beta$ and get the decomposition $\mathfrak{F}=\oplus_{\alpha\leq\beta}\mathfrak{F}_{\alpha\beta}\oplus\mathfrak{F}_0$. Theorem 6.2 now follows from (Al.1) to (Al.5).

Appendix 2. On maximal semi-simple subalgebras of Lie algebras

Let $\mathfrak{g}$ be a Lie algebra over **R** or **C**. It is well known that there exists a maximal semi-simple subalgebra $\mathfrak{h}$ such that $\mathfrak{g}=\mathrm{rad}(\mathfrak{g})+\mathfrak{h}$. In this Appendix 2, we shall prove the theorem below by using ideas similar to the ones used in [15]. We remark that this theorem can also be proved by using [8; Appendix].

THEOREM. *Let D be an abelian family of semi-simple derivations of a Lie algebra $\mathfrak{g}$. Then there exist a semi-simple subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ such that* $\mathfrak{g}=\mathrm{rad}(\mathfrak{g})+\mathfrak{h}$ *and* $D\mathfrak{h}\subset\mathfrak{h}$.

Proof. We prove this theorem by induction on $\dim\mathfrak{g}$. Let $\mathfrak{r}=\mathrm{rad}(\mathfrak{g})$ and $\mathfrak{r}'=[\mathfrak{r},\mathfrak{r}]$. Assume that $\mathfrak{r}'\neq 0$. Consider $\check{\mathfrak{g}}=\mathfrak{g}/\mathfrak{r}'$. Clearly $\mathrm{rad}(\check{\mathfrak{g}})=\mathfrak{r}/\mathfrak{r}'$. Since every element $f\in D$ induces a derivation $\check{f}$ of $\mathfrak{g}/\mathfrak{r}'$, we can apply the induction hypothesis and obtain a semi-simple subalgebra $\check{\mathfrak{h}}$ of $\check{\mathfrak{g}}$ such that $\check{\mathfrak{g}}=\mathrm{rad}(\check{\mathfrak{g}})+\check{\mathfrak{h}}$ and $\check{f}\check{\mathfrak{h}}\subset\check{\mathfrak{h}}$ for any $f\in D$. Set $\mathfrak{g}'=\pi^{-1}(\check{\mathfrak{h}})$, where π denotes the projection of $\mathfrak{g}$ onto $\check{\mathfrak{g}}$. Then $\dim\mathfrak{g}'<\dim\mathfrak{g}$ and $\mathrm{rad}(\mathfrak{g}')=\mathfrak{r}'$. By construction $D\mathfrak{g}'\subset\mathfrak{g}'$. Therefore, applying again the induction hypothesis, we can find a semi-simple subalgebra $\mathfrak{h}$ which is invariant under D such that $\mathfrak{g}'=\mathfrak{r}'+\mathfrak{h}$ holds. But then $\mathfrak{g}=\mathfrak{r}+\mathfrak{h}$ and we obtain the assertion in this case.

Now assume $\mathfrak{r}'=0$. We consider the semi-simple Lie algebra $\tilde{\mathfrak{g}}=\mathfrak{g}/\mathfrak{r}$. For every $f\in D$, we denote by $\tilde{f}$ the induced derivations of $\tilde{\mathfrak{g}}$. First we consider the complex case. Then $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ are decomposed into the sum of "root spaces" $\mathfrak{g}=\oplus_\Gamma\mathfrak{g}^\Gamma$ and $\tilde{\mathfrak{g}}=\oplus_\Gamma\mathfrak{g}^\Gamma$, where $\mathfrak{g}^\Gamma=\{x\in\mathfrak{g};fx=\Gamma(f)x$ for any $f\in D\}$ and $\tilde{\mathfrak{g}}^\Gamma=\{x\in\tilde{\mathfrak{g}};\tilde{f}x=\Gamma(f)x$ for any $f\in D\}$. Take a semi-simple subalgebra $\mathfrak{h}$ so that $\mathfrak{g}=\mathfrak{r}+\mathfrak{h}$. Since $\mathfrak{h}$ is isomorphic to $\tilde{\mathfrak{g}}$, $\mathfrak{h}$ is decomposed as $\mathfrak{h}=\oplus_\Gamma\mathfrak{h}^\Gamma$, corresponding to the decomposition of $\tilde{\mathfrak{g}}$. Let $x\in\mathfrak{h}^\Gamma$. We write $x=\Sigma x^\Lambda$, where $x^\Lambda\in\mathfrak{g}^\Lambda$. Then $x^\Lambda\in\mathfrak{r}$ if $\Lambda\neq\Gamma$. The correspondence $x\to x^\Gamma$ gives an injective map ϱ^Γ of $\mathfrak{h}^\Gamma$ to $\mathfrak{g}$. Let $x\in\mathfrak{h}^\Gamma$ and $y\in\mathfrak{h}^{\Gamma'}$. Then $[x,y]\in\mathfrak{h}^{\Gamma+\Gamma'}$. We write $x=\Sigma x^\Gamma$ and $y\in\Sigma y^\Gamma$, where x^Γ, $y^\Gamma\in\mathfrak{g}^\Gamma$. Since $\mathfrak{r}$ is abelian, the $\mathfrak{g}^{\Gamma+\Gamma'}$-component of $[x,y]$ is equal to $[x^\Gamma,y^{\Gamma'}]$. Therefore $\varrho^{\Gamma+\Gamma'}([x,y])=[\varrho^\Gamma(x),\varrho^{\Gamma'}(y)]$. This shows that $\varrho=\oplus\varrho^\Gamma$ is an injective homomorphism of $\mathfrak{h}$ to $\mathfrak{g}$. Then $\varrho(\mathfrak{h})$ has the desired properties.

It remains to consider the case where $\mathfrak{g}$ is a real Lie algebra. In this case we take a semi-simple subalgebra $\mathfrak{h}$ so that $\mathfrak{g}=\mathfrak{r}+\mathfrak{h}$ holds and consider the complexification $\mathfrak{g}_C$, $\mathfrak{h}_C$ and $(\mathfrak{g}/\mathfrak{r})_C$. As before, we have $\mathfrak{g}_C=\oplus\mathfrak{g}_C^\Gamma$, $(\mathfrak{g}/\mathfrak{r})_C=\oplus(\mathfrak{g}/\mathfrak{r})_C^\Gamma$, $\mathfrak{h}_C=\oplus\mathfrak{h}_C^\Gamma$ and define a map $\varrho=\oplus\varrho^\Gamma$. We must prove that $\varrho(\mathfrak{h})\subset\mathfrak{g}$ holds. It is clear that $\overline{\mathfrak{g}_C^\Gamma}=\mathfrak{g}_C^{\bar{\Gamma}}$ and $\overline{\mathfrak{h}_C^\Gamma}=\mathfrak{h}_C^{\bar{\Gamma}}$ hold. Hence every element of $\mathfrak{h}$ is a sum of the form $x+\bar{x}$, where $x\in\mathfrak{h}_C^\Gamma$ and $\bar{x}\in\mathfrak{h}_C^{\bar{\Gamma}}$. Let x^Γ be the $\mathfrak{g}_C^\Gamma$-component of x. Then the $\mathfrak{g}_C^{\bar{\Gamma}}$-component of $\bar{x}$ is equal to $\overline{x^\Gamma}$. Therefore $\varrho(x+\bar{x})=x^\Gamma+\overline{x^\Gamma}\in\mathfrak{g}$ and the assertion follows.

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