An extended Euler-Poincaré theorem

by

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Dedicated to the memory of Alex Zabrodsky

1. Introduction

Suppose that Γ is a finite simplicial complex and $\mathbf{k}$ a field. Let $f=(f_0,f_1,\dots)$ and $b=(b_0,b_1,\dots)$ be the *f-vector* and *Betti sequence* of Γ, i.e., $f_i=\mathrm{card}\{F\in\Gamma \mid \dim F=i\}$ and $b_i=\dim_{\mathbf{k}} \tilde{H}_i(\Gamma,\mathbf{k})$, $i\geqslant 0$. The well-known 1899 theorem of Poincaré [P1, P2] (usually called the *Euler-Poincaré formula*) states that

$$\sum_{i\geqslant 0}(-1)^i f_i=\sum_{i\geqslant 0}(-1)^i b_i. \tag{1.1}$$

It was later shown by Mayer [M] that no other linear relation holds between f and b.

In this paper we introduce d (where $d=\dim\Gamma$) non-linear relations which f and b are shown to satisfy. Also, we prove that (1.1) together with these new relations completely characterize the pairs (f, b) of numerical sequences which arise as f-vectors and Betti sequences of finite simplicial complexes. Several related results are discussed concerning such (f, b)-pairs for simplicial complexes, and a characterization is given of those integer sequences which can arise as Betti sequences of simplicial complexes on at most n vertices.

In recent years f-vectors of various classes of simplicial and polyhedral complexes have been intensively studied. We refer the reader to the surveys [Bj, BK, St]. A basic result is the Kruskal-Katona theorem, which characterizes the f-vectors of arbitrary

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simplicial complexes. This theorem and the Sperner theorem—two cornerstones of set-theoretic combinatorics—are reviewed in Section 2. While the Kruskal-Katona theorem can be said to characterize the projection onto the *first* coordinate of all (f, b)-pairs, it follows from our work that the Sperner theorem and its relatives characterize the projection onto the *second* coordinate of all (f, b)-pairs with $f_0 \leq n$. The characterization of (f, b)-pairs given here was previously known in one special case, namely for acyclic complexes [K2].

We now proceed to give precise statements of the main results. Proofs and further details will be found in later sections.

All simplicial complexes Γ in this paper are supposed to be finite. For the usual definitions of complexes and their homology, see [LW] or [Sp]. Some special combinatorial properties are briefly reviewed in Section 2 to fix terminology and notation.

Here $\tilde{H}_i(\Gamma, \mathbf{k})$ will denote i-dimensional reduced simplicial homology of Γ over a field $\mathbf{k}$. Also $\beta_i = \dim_{\mathbf{k}} \tilde{H}_i(\Gamma, \mathbf{k})$ is the ith *(reduced) Betti number* of Γ over $\mathbf{k}$. The reduced Betti numbers β_i differ from the ordinary Betti numbers b_i only in dimension zero: $b_0 = \beta_0 + 1$ (assuming that Γ is nonempty, in which case b_0 counts the number of connected components). For the rest of this paper we will consider only reduced Betti numbers, and therefore prefer this reformulation of the Euler-Poincaré formula (1.1):

$$\sum_{i \geq 0} (-1)^i f_i = \sum_{i \geq 0} (-1)^i \beta_i + 1. \tag{1.2}$$

The following number-theoretic function will be of importance. If $n, k \geq 1$ there is a unique expansion

$$n = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \ldots + \binom{a_i}{i},$$

such that $a_k > a_{k-1} > \ldots > a_i \geq i \geq 1$. This given, define

$$\partial_{k-1}(n) = \binom{a_k}{k-1} + \binom{a_{k-1}}{k-2} + \ldots + \binom{a_i}{i-1}. \tag{1.3}$$

Also, let $\partial_{k-1}(0) = 0$.

Let $\mathbf{N}_0^{(\infty)}$ denote the set of ultimately vanishing sequences of nonnegative integers. For $\alpha = (\alpha_0, \alpha_1, \ldots) \in \mathbf{N}_0^{(\infty)}$ we call *α-wedge of spheres* the topological space obtained by wedging together α_i copies of the i-dimensional sphere for $i \geq 0$. The 0-wedge of spheres is just a point.

THEOREM 1.1. *Suppose that* $f=(f_0,f_1,\ldots)$, $\beta=(\beta_0,\beta_1,\ldots)\in\mathbf{N}_0^{(\infty)}$ *are two given sequences and* $\mathbf{k}$ *is a field. Then the following conditions are equivalent:*

(a) *f is the f-vector and β the Betti sequence over* $\mathbf{k}$ *of some simplicial complex,*

(b) *let* $\chi_{k-1}=\Sigma_{j\geq k}(-1)^{j-k}(f_j-\beta_j)$, *for* $k\geq 0$; *then*

(i) $\chi_{-1}=1$, (1.4)

(ii) $\partial_k(\chi_k+\beta_k)\leq\chi_{k-1}$, *for all* $k\geq 1$, (1.5)

(c) *f is the f-vector of some simplicial complex which is homotopy equivalent to the β-wedge of spheres.*

One observes that in the numerical characterization (b) of (f,β)-pairs of d-dimensional complexes condition (i) is the Euler-Poincaré formula (1.2) and condition (ii) gives d additional non-trivial relations. Also, just as the original Euler-Poincaré formula the new relations and therefore the entire characterization is independent of field characteristic.

The relations (1.5) have a homological interpretation which is discussed in Remark 4.4. The formulation of condition (c) can be sharpened from homotopy class to a combinatorially defined class of "near-cones", see Section 4. For $\beta=0$ these near-cones are ordinary cones, and Theorem 1.1 specializes to the characterization of f-vectors of acyclic complexes given in [K2].

Let us call a pair $f,\beta\in\mathbf{N}_0^{(\infty)}$ *compatible* if f is the f-vector and β the Betti sequence of some simplicial complex. This is, by Theorem 1.1, a purely combinatorial relation, independent of field characteristic. Every subset of $\mathbf{N}_0^{(\infty)}$ will be considered partially ordered by the componentwise ordering: $(n_0,n_1,\ldots)\leq(n_0',n_1',\ldots)$ if $n_i\leq n_i'$ for all $i\geq 0$.

THEOREM 1.2. (a) *Suppose that f is the f-vector of some simplicial complex. Then the set B_f of all compatible Betti sequences has a unique maximal element. Define* $\psi(f)=\max B_f$.

(b) *Suppose that* $\beta\in\mathbf{N}_0^{(\infty)}$. *The set F_β of all compatible f-vectors has a unique minimal element. Define* $\varphi(\beta)=\min F_\beta$.

(c) *Suppose that (f,β) is a compatible pair. Then $\varphi(\beta)=f$ if and only if conditions* (1.5) *hold with equality everywhere.*

(d) $\psi(\varphi(\beta))=\beta$, *for all* $\beta\in\mathbf{N}_0^{(\infty)}$.

(e) $\varphi(\psi(f))\leq f$, *for all* f-vectors f.

More detailed observations can be made about the combinatorics of compatible (f,β)-pairs. For this see Section 5.

Our next result gives a characterization of Betti sequences of simplicial complexes on a bounded number of vertices. Sperner families and their f-vectors are defined in Section 2.

THEOREM 1.3. *Let $n \geq 0$ be a fixed integer, $\beta=(\beta_0, \beta_1, \ldots) \in \mathbf{N}_0^{(\infty)}$, and let $\mathbf{k}$ be a field. Then β is the Betti sequence over $\mathbf{k}$ of some simplicial complex with at most $n+1$ vertices if and only if β is the f-vector of some Sperner family of subsets of $\{1, 2, \ldots, n\}$.*

The f-vectors of Sperner families have a known characterization (see Theorem 2.5), which in conjunction with Theorem 1.3 gives a complete numerical characterization of the possible Betti sequences of complexes on at most $n+1$ vertices.

For a simplicial complex Γ and field $\mathbf{k}$, let $\tilde{\beta}(\Gamma) = \Sigma_{i \geq 0} \beta_i(\Gamma)$. Clearly, $|\chi(\Gamma)| \leq \tilde{\beta}(\Gamma)$, where $\chi(\Gamma) = \Sigma_{i \geq 0} (-1)^i \beta_i(\Gamma)$ is the reduced Euler characteristic (1.2). The following is a direct consequence of Theorem 1.3 and the well-known theorem of Sperner (see Theorem 2.4).

THEOREM 1.4. *Let Γ be a simplicial complex with at most $n+1$ vertices, and let $\tilde{\beta}(\Gamma)$ be the sum of its Betti numbers over $\mathbf{k}$. Then*

$$|\chi(\Gamma)| \leq \tilde{\beta}(\Gamma) \leq \binom{n}{[n/2]}.$$

Furthermore, the following conditions are equivalent:

(i) $|\chi(\Gamma)| = \binom{n}{[n/2]}$,

(ii) $\tilde{\beta}(\Gamma) = \binom{n}{[n/2]}$,

(iii) *Γ is the k-skeleton of an n-simplex, where $k=n/2-1$ if n is even and $k=(n-1)/2$ or $k=(n-3)/2$ if n is odd.*

The Euler-Poincaré formula (1.2) is true for all finite topological cell complexes. In an appendix we discuss the pairs of f-vectors and Betti sequences which arise from cell complexes and regular cell complexes. For such pairs some rather obvious necessary conditions turn out to also be sufficient. This way a characterization result similar to Theorem 1.1 is obtained, see Theorem 6.1. In view of its simplicity this result may well be known, however we have failed to find any reference to it in the literature.

Viewed in the larger setting of cell complexes, it is tempting to speculate about the exact domain of validity for the stringent "simplicial" relations (1.5). We conjecture that relations (1.5) are satisfied by the (f,β)-pairs of all regular cell complexes such that any nonempty intersection of two closed cells is a closed cell, and hence in particular are valid for all polyhedral complexes, cf. Remark 6.3 and the Note added in proof.

2. Shifting and compression

Let $\mathbf{N}=\{1,2,\ldots\}$, $\mathbf{N}_0=\{0,1,2,\ldots\}$, and $[n]=\{1,2,\ldots,n\}$. The cardinality of a set A is written $|A|$ or $\operatorname{card} A$. For a set A, let $\binom{A}{k}$ denote the family of all k-element subsets of A. In the sequel, $S=\{i_1,i_2,\ldots,i_k\}_<$ will denote an ordered set $S=\{i_1,i_2,\ldots,i_k\}$ such that $i_1<i_2<\ldots<i_k$.

We will consider three orderings of the k-subsets of $\mathbf{N}$. They are defined as follows for $S,T\in\binom{\mathbf{N}}{k}$, $S=\{i_1,i_2,\ldots,i_k\}_<$ and $T=\{j_1,j_2,\ldots,j_k\}_<$:

partial order: $S\leqslant_{\mathrm{P}} T$ if $i_e\leqslant j_e$ for all $1\leqslant e\leqslant k$,
lexicographic order: $S\leqslant_{\mathrm{L}} T$ if $S=T$ or $\min(S\Delta T)\in S$,
antilexicographic order: $S\leqslant_{\mathrm{AL}} T$ if $S=T$ or $\max(S\Delta T)\in T$,

where $S\Delta T=(S\setminus T)\cup(T\setminus S)$. Thus for instance, $\{1,2\}<_{\mathrm{L}}\{1,3\}<_{\mathrm{L}}\{1,4\}<_{\mathrm{L}}\{2,3\}$ and $\{1,2\}<_{\mathrm{AL}}\{1,3\}<_{\mathrm{AL}}\{2,3\}<_{\mathrm{AL}}\{1,4\}$. Both the lexicographic and the antilexicographic orderings are extensions of the partial ordering to a total order.

A family $A\subseteq\binom{\mathbf{N}}{k}$ is *compressed* if it is an initial set with respect to the antilexicographic order, meaning that $S<_{\mathrm{AL}} T\in A$ implies $S\in A$. A is *shifted* if it is an initial set with respect to the partial order $<_{\mathrm{P}}$. Note that a compressed family is shifted, but not conversely.

Let Γ be a finite simplicial complex whose vertex set is contained in $\mathbf{N}$. Denote $\Gamma_k=\{S\in\Gamma:\dim S=k\}$, where $\dim S=\operatorname{card} S-1$. The complex Γ is *compressed* if Γ_k is compressed for every $k\geqslant 0$, and Γ is *shifted* if Γ_k is shifted for all $k\geqslant 0$.

Let $A\subseteq\binom{\mathbf{N}}{k}$. The *shadow* ∂A of A is defined by

$$\partial A=\left\{S\in\binom{\mathbf{N}}{k-1}:\ S\subseteq T \text{ for some } T\in A\right\}.$$

For $n\in\mathbf{N}_0$, denote by I_k^n the compressed family in $\binom{\mathbf{N}}{k}$ of size n. It is easy to show (see [GK]) that ∂I_k^n is a compressed subset of $\binom{\mathbf{N}}{k-1}$, and that $|\partial I_k^n|=\partial_{k-1}(n)$. (The function $\partial_{k-1}(n)$ was defined by (1.3) in the previous section.)

We are ready to formulate the Kruskal-Katona theorem [Kr, Ka], which gives a complete characterization of f-vectors of simplicial complexes:

THEOREM 2.1. *A sequence* $f=(f_0,f_1,\ldots)\in\mathbf{N}_0^{(\infty)}$ *is the f-vector of some simplicial complex if and only if*

$$\partial_k(f_k)\leqslant f_{k-1}, \quad \textit{for every} \quad k\geqslant 1. \tag{2.1}$$

Note that the Kruskal-Katona theorem implies (and is in fact equivalent to) that for every simplicial complex Γ there is a (necessarily unique) compressed complex with the same f-vector. For an f-vector f, let K_f be this compressed complex:

$$K_f=\bigcup_{k\geqslant 0} I_{k+1}^{f_k}. \tag{2.2}$$

We will need the following combinatorial lemma in Section 5.

LEMMA 2.2. Let $A\subseteq\binom{\mathbf{N}}{k}$ be a shifted family, $|A|=n$. Then

$$\operatorname{card}\{S\in A\colon 1\in S\}\geqslant\operatorname{card}\{S\in I_k^n\colon 1\in S\}.$$

Proof. Put $X=\{S\in\binom{\mathbf{N}}{k}\colon 1\in S\}$. Let $B\subseteq X$ be a shifted family, and define

$$a_p(B)=\operatorname{card}\{S\in B\colon \min(S\setminus\{1\})=p\},$$
$$b_p(B)=\operatorname{card}\{S\in B\colon \min(S\setminus\{1\})\geqslant p\}=a_p(B)+a_{p+1}(B)+\ldots, \quad \text{for } p\geqslant 2.$$

Claim. There is a unique maximal (with respect to inclusion) shifted family $D=D(B)\subseteq\binom{\mathbf{N}}{k}$, such that $D\cap X=B$. Moreover, $|D|=\Sigma_{p\geqslant 2}(p-1)\,a_p(B)=\Sigma_{p\geqslant 2}\,b_p(B)$.

To verify this claim, just take

$$D=\{\{i,j_2,j_3,\ldots,j_k\}_<\colon \{1,j_2,j_3,\ldots,j_k\}_<\in B\}.$$

We now prove Lemma 2.2 by induction on k. It is clearly true for $k=1$.

Let $C_1,C_2\subseteq X$ be shifted families such that $|C_1|\leqslant|C_2|$ and C_2 is initial with respect to $<_{\mathrm{AL}}$ restricted to X. Observe that C_1' is shifted and C_2' compressed as families of $(k-1)$-subsets of $\{2,3,\ldots\}$, where $C_i'=\{S\setminus\{1\}\colon S\in C_i\}$. Hence, from the induction hypothesis it follows that $b_p(C_1)\leqslant b_p(C_2)$ for all $p\geqslant 2$, and therefore $|D(C_1)|\leqslant|D(C_2)|$.

Now, let $A\subseteq\binom{\mathbb{N}}{k}$ be shifted and $|A|=n$. Put

$$B_1=\{S\in A: 1\in S\},\quad B_2=\{S\in I_k^n: 1\in S\}$$

and assume to the contrary that $|B_1|<|B_2|$. Let B_3 be obtained from B_2 by deleting the last element with respect to $<_{\mathrm{AL}}$. It is easy to see that $D(B_3)\subsetneq I_k^n$. Since $A\subseteq D(B_1)$ and $|B_1|\leqslant|B_3|$, we have

$$|A|\leqslant|D(B_1)|\leqslant|D(B_3)|<n;$$

a contradiction. □

Remark 2.3. Lemma 2.2 is implicit in Frankl [F1]. In fact, from the last page of [F1] one can extract a proof of the equivalence of Lemma 2.2 and the Kruskal-Katona theorem 2.1. A simple inductive proof of the Kruskal-Katona theorem is obtained from Lemma 2.2 as follows:

First note that if $A\subseteq\binom{\mathbb{N}}{k}$ is shifted and $A_1=\{S\setminus\{1\}: 1\in S\in A\}$, then

$$\partial A=A_1\cup\{R\cup\{1\}: R\in\partial A_1\}.$$

Thus,

$$|\partial A|=|A_1|+|\partial A_1|. \tag{2.3}$$

Now, let $A\subseteq\binom{\mathbb{N}}{k}$ and $|A|=n$. By a standard combinatorial shifting method (see [EKR, F2]) there exists a shifted k-family B, such that $|B|=n$ and $|\partial B|\leqslant|\partial A|$. By Lemma 2.2, formula (2.3) and an induction hypothesis we get that $|\partial I_k^n|\leqslant|\partial B|$. □

A *Sperner family* S is a family of nonempty subsets of a finite set with no proper containment relations (i.e., if $T_1, T_2\in S, T_1\subseteq T_2$, then $T_1=T_2$). The fundamental result of Sperner [Spe] is:

THEOREM 2.4. *Let S be a Sperner family of subsets of $[n]$. Then $|S|\leqslant\binom{n}{[n/2]}$. Furthermore, equality holds if and only if $S=\binom{[n]}{n/2}$ for n even and $S=\binom{[n]}{(n+1)/2}$ or $S=\binom{[n]}{(n-1)/2}$ for n odd.*

The definition of f-vector $f=(f_0,f_1,\ldots)$ can be directly extended to arbitrary families S of nonempty subsets of a finite set: $f_i=\mathrm{card}\{T\in S: |T|=i+1\}$, $i\geqslant 0$. The following characterization of f-vectors of Sperner families follows from the Kruskal-Katona theorem. It was found by Clements [C] and independently by Daykin, Godfrey and Hilton [DGH].

THEOREM 2.5. *A sequence* $f=(f_0,f_1,\ldots)\in \mathbf{N}_0^{(\infty)}$ *is the f-vector of some Sperner family of subsets of* $[n]$ *if and only if either* $f=0$ *or else*

$$f_k+\partial_{k+1}(f_{k+1}+\partial_{k+2}(f_{k+2}+\ldots+\partial_{l-1}(f_{l-1}+\partial_l(f_l))\ldots))\leq \binom{n}{k+1}, \tag{2.4}$$

where k *and* l *are the smallest and largest indices* i *for which* $f_i\neq 0$.

3. Algebraic shifting and homology

A *shifting operation* is a map which assigns to every simplicial complex Γ a shifted simplicial complex $\Delta(\Gamma)$ with the same f-vector. A well-known combinatorial shifting operation, introduced by Erdös, Ko and Rado [EKR], has been of great use in extremal set theory. See the survey [F2]. We will need a shifting operation which does not change the Betti numbers of Γ. Such an operation is introduced in [K1, K3] and will be described in this section.

THEOREM 3.1 (Kalai [K3]). *Given a simplicial complex* Γ *on* n *vertices and a field* $\mathbf{k}$, *there exists a canonically defined simplicial complex* $\Delta=\Delta(\Gamma,\mathbf{k})$ *on vertices* $\{1,2,\ldots,n\}$ *such that*

(i) $f_i(\Delta)=f_i(\Gamma)$, *for* $i\geq 0$,

(ii) $\beta_i(\Delta)=\beta_i(\Gamma)$, *for* $i\geq 0$, (*Betti numbers with coefficients in* $\mathbf{k}$)

(iii) Δ *is shifted.*

We will give here a selfcontained proof of Theorem 3.1 which differs from the one in [K3]. First a few algebraic preliminaries are needed.

Let E be an n-dimensional vector space over the field $\mathbf{k}$ with a distinguished basis $\mathbf{e}=\{e_1,e_2,\ldots,e_n\}$. Let ΛE be the exterior algebra over E, cf. [Bo]. Thus $\Lambda E=\oplus_{k=0}^{n}\Lambda^k E$ is a 2^n-dimensional graded algebra over $\mathbf{k}$ with exterior product "$\wedge$". As a vector space, $\Lambda^k E$ is spanned by the basis $\{e_S: S\in\binom{[n]}{k}\}$, where $e_\varnothing=1$ and $e_S=e_{i_1}\wedge e_{i_2}\wedge\ldots\wedge e_{i_k}$ for $S=\{i_1,i_2,\ldots,i_k\}_<$.

Suppose that $\mathbf{f}=\{f_1,f_2,\ldots,f_n\}$ is another basis of E, $f_i=\Sigma_{j=1}^{n}\alpha_{ij}e_j$, $1\leq i\leq n$. We use analogous notation $f_S = f_{i_1} \wedge f_{i_2} \wedge \ldots \wedge f_{i_k} \in \Lambda^k E$ for $S = \{i_1, i_2, \ldots, i_k\}_<$, etc. If $S, T \in \binom{[n]}{k}$, let $\alpha_{S,T} = \det A_{S,T}$, where $A_{S,T}$ is the submatrix of $A = (\alpha_{ij})$ with rows S and columns T. Then clearly $f_S = \Sigma\, \alpha_{S,T} e_T$, sum over all $T \in \binom{[n]}{k}$.

Proof of Theorem 3.1. By an arbitrary labeling we may assume that the vertex set of Γ is $[n]$. Let

$$\bar{C}_k(\Gamma) = \operatorname{span}\left\{ e_S : S \in \binom{[n]}{k+1}, S \notin \Gamma \right\}.$$

Then $\bar{C}(\Gamma) = \oplus_k \bar{C}_k(\Gamma)$ is a homogeneous graded ideal in ΛE. Define the *face algebra* $\Lambda(\Gamma)$ of the simplicial complex Γ by

$$\Lambda[\Gamma] = \Lambda E / \bar{C}(\Gamma).$$

For $x \in \Lambda E$, denote $\tilde{x} = x + \bar{C}(\Gamma) \in \Lambda[\Gamma]$. Let

$$\Lambda_k[\Gamma] = \{\tilde{x} : x \in \Lambda^{k+1} E\} \cong \Lambda^{k+1} E / \bar{C}_k(\Gamma).$$

Note that $\Lambda[\Gamma] = \oplus_{k=-1}^{\dim \Gamma} \Lambda_k[\Gamma]$ is a graded algebra whose Hilbert function is the f-vector of Γ, i.e., $\dim \Lambda_k[\Gamma] = \operatorname{card} \Gamma_k = f_k(\Gamma)$.

Let $\mathbf{f} = \{f_1, f_2, \ldots, f_n\}$ be a basis of E. Define

$$\Delta_{\mathbf{f}}(\Gamma) = \{S \subseteq [n] : \tilde{f}_S \notin \operatorname{span}\{\tilde{f}_R : R <_{\mathrm{L}} S\}\}. \tag{3.1}$$

The set family $\Delta_{\mathbf{f}}(\Gamma)$ can be assembled by inspecting the k-subsets of $[n]$ in lexicographic order and deleting those whose $\mathbf{f}$-representative in $\Lambda[\Gamma]$ is in the span of earlier ones. (The different cardinalities may be taken in arbitrary order.) It follows that $\{\tilde{f}_S : S \in \Delta_{\mathbf{f}}(\Gamma)\}$ is a basis for $\Lambda[\Gamma]$, so $\Delta_{\mathbf{f}}(\Gamma)$ and Γ have the same f-vector.

In the following we need to consider a basis $\mathbf{f}$ which is "generic" with respect to $\mathbf{e}$. For this, let α_{ij}, $1 \leq i,j \leq n$, be n^2 transcendentals, algebraically independent over $\mathbf{k}$, and replace $\mathbf{k}$ by its extension $\hat{\mathbf{k}} = \mathbf{k}(\alpha_{11}, \ldots, \alpha_{nn})$. Let $f_i = \alpha_{i1} e_1 + \alpha_{i2} e_2 + \ldots + \alpha_{in} e_n$, $1 \leq i \leq n$, and consider the "generic" basis $\mathbf{f} = (f_1, f_2, \ldots, f_n)$. Define $\Delta = \Delta_{\mathbf{f}}(\Gamma)$.

Claim 1. The set family Δ is a shifted simplicial complex. It is independent of the choice of generic basis $\mathbf{f}$ and independent of the labeling of the vertices of Γ.

Claim 2. $\beta_k(\Gamma) = \beta_k(\Delta) = \operatorname{card}\{S \in \Delta_k : S \cup \{1\} \notin \Delta\}$, for $k \geq 0$.

We first show that Δ is a simplicial complex. Suppose that $S \notin \Delta$ and $T \supset S$. Then

$$\tilde{f}_S = \sum \{\gamma_R \tilde{f}_R : R <_{\mathrm{L}} S\},$$

and

$$\tilde{f}_T = \pm \tilde{f}_{T \setminus S} \wedge \tilde{f}_S = \sum \{\pm \gamma_R \tilde{f}_{R \cup (T \setminus S)} : R <_{\mathrm{L}} S \text{ and } R \cap (T \setminus S) = \varnothing\}.$$

As is easily seen, if $R <_{\mathrm{L}} S$ and $R \cap (T \setminus S) = \varnothing$, then $R \cup (T \setminus S) <_{\mathrm{L}} T$. Hence, $T \notin \Delta$.

To prove that Δ is shifted we need the following.

PERMUTATION LEMMA. *Let $\mathscr{F}$ be a family of subsets of $[n]$ and suppose that for some $S \subseteq [n]$ we have*

$$\tilde{f}_S = \sum_{T \in \mathscr{F}} \gamma_T \tilde{f}_T, \quad \gamma_T \in \hat{\mathbf{k}}.$$

If π: $[n] \to [n]$ is a permutation, then

$$\tilde{f}_{\pi(S)} = \sum_{T \in \mathscr{F}} \gamma'_T \tilde{f}_{\pi(T)}, \tag{3.2}$$

for some coefficients $\gamma'_T \in \hat{\mathbf{k}}$.

Proof of lemma. Expand the given relation,

$$f_S = \sum_{T \in \mathscr{F}} \gamma_T f_T + \sum_{Y \notin \Gamma} \delta_Y e_Y,$$

in the $\mathbf{e}$ basis:

$$\sum_J \alpha_{S,J} e_J = \sum_{T \in \mathscr{F}} \gamma_T \sum_K \alpha_{T,K} e_K + \sum_{Y \notin \Gamma} \delta_Y e_Y.$$

This is equivalent to a collection of polynomial relations in the α_{ij}'s over the ground field $\mathbf{k}$ (recall that the coefficients γ_T and δ_Y are rational functions of the α_{ij}'s over $\mathbf{k}$). Being algebraically independent over $\mathbf{k}$, the transcendentals α_{ij} can be freely permuted in these relations. Let π permute the rows of the matrix (α_{ij}). Then π: $\alpha_{S,J} \to \alpha_{\pi S, J}$, etc.

We get

$$\sum_J \alpha_{\pi S,J}\, e_J = \sum_{T\in\mathcal{F}} \gamma'_T \sum_K \alpha_{\pi T,K}\, e_K + \sum_{Y\notin\Gamma} \delta'_Y e_Y,$$

which implies (3.2). □

We now show that Δ is shifted. Suppose that $S\notin\Delta$ and $S<_{\mathrm{P}} T$ for equicardinal subsets $S, T\subseteq[n]$. Define a permutation $\pi\colon[n]\to[n]$ such that $\pi(S)=T$ and the restriction of π to S and to $[n]\setminus S$ is order-preserving. Since $S\notin\Delta$, we have

$$\tilde{f}_S = \sum\{\gamma_R \tilde{f}_R : R<_{\mathrm{L}} S\}.$$

The permutation lemma shows that

$$\tilde{f}_T = \tilde{f}_{\pi(S)} = \sum\{\gamma'_R \tilde{f}_{\pi(R)} : R<_{\mathrm{L}} S\}.$$

It is easy to see that if $R<_{\mathrm{L}} S$ then $\pi(R)<_{\mathrm{L}} T$. Hence, $T\notin\Delta$.

We have now proven all parts of Claim 1. The independence of choice of generic basis is clear from the construction, and independence of labeling of vertices of Γ follows at once from the permutation lemma.

It is left to prove Claim 2, i.e., the fact that the generically shifted complex Δ has the same Betti numbers as Γ with respect to the field $\mathbf{k}$, or equivalently, over the field $\hat{\mathbf{k}}$ (since $\operatorname{char}\mathbf{k}=\operatorname{char}\hat{\mathbf{k}}$). We first show that

$$\beta_k(\Gamma) = \operatorname{card}\{S\in\Delta_k : S\cup\{1\}\notin\Delta\},\quad k\geq 0. \tag{3.3}$$

Since we are working over a field, we may regard $\beta_k(\Gamma)$ as the dimension (as a $\hat{\mathbf{k}}$-vector space) of the *(reduced) cohomology group* $\tilde{H}^k(\Gamma,\hat{\mathbf{k}})$.

Note that if $f=\alpha_1 e_1+\alpha_2 e_2+\ldots+\alpha_n e_n$ then $\tilde{f}\wedge\cdot$ is a "weighted" coboundary operator on $\Lambda[\Gamma]$: If $S\in\Gamma_k$ and $[n]\setminus S=\{i_1, i_2, \ldots, i_{n-k-1}\}$ then

$$\tilde{f}\wedge\tilde{e}_S = \sum \pm\alpha_{i_j}\tilde{e}_{S\cup\{i_j\}}, \tag{3.4}$$

with summation over all j such that $S\cup\{i_j\}\in\Gamma_{k+1}$ and with signs as in the usual "unweighted" cohomology (which is the $\alpha_1=\ldots=\alpha_n=1$ case). We will compute Betti numbers with respect to a weighted coboundary operator $\tilde{f}\wedge\cdot$ and need to first observe that if $\alpha_i \neq 0$ for all $1 \leq i \leq n$, then these weighted Betti numbers are the same as the usual unweighted Betti numbers. To see this, use that the Betti numbers of a simplicial coboundary operator (3.4) are determined by the ranks of the weighted incidence matrices M_k^α, where M_k^α has rows indexed by Γ_k and columns indexed by Γ_{k+1} and the row corresponding to a k-face S give the coefficients in the expansion of $\tilde{f} \wedge \tilde{e}_S$ as in (3.4). By multiplying the rows and dividing the columns by appropriate products of the α_i's, it is easy to transform M_k^α to the unweighted incidence matrix M_k without affecting the rank. Hence, the corresponding Betti numbers are also the same.

Now, define weighted k-cocycles and k-coboundaries:

$$Z^k = \{x \in \Lambda_k[\Gamma] : \tilde{f}_1 \wedge x = 0\} \quad \text{and} \quad B^k = \tilde{f}_1 \wedge \Lambda_{k-1}[\Gamma].$$

By the preceding discussion we have that $\beta_k(\Gamma) = \dim Z^k - \dim B^k$. We claim that

$$B^k = \operatorname{span}\{\tilde{f}_S : 1 \in S \in \Delta_k\}. \tag{3.5}$$

Let $\mathscr{A}_k^1 = \{S \in \binom{[n]}{k+1} : 1 \in S\}$. Since $\mathscr{A}_k^1$ is initial with respect to the lexicographic ordering of $\binom{[n]}{k+1}$ it follows from the construction (3.1) of Δ that $\{\tilde{f}_S : 1 \in S \in \Delta_k\}$ is a basis of $\operatorname{span}\{\tilde{f}_S : S \in \mathscr{A}_k^1\}$. If $1 \in S \in \Delta_k$, then $\tilde{f}_S = \tilde{f}_1 \wedge \tilde{f}_{S \setminus \{1\}} \in B^k$. Conversely, if $x \in \Lambda_{k-1}[\Gamma]$ and $x = \Sigma \gamma_R \tilde{f}_R$, then

$$\tilde{f}_1 \wedge x = \sum_{1 \notin R} \pm \gamma_R \tilde{f}_{R \cup \{1\}} \in \operatorname{span}\{\tilde{f}_S : S \in \mathscr{A}_k^1\} = \operatorname{span}\{\tilde{f}_S : 1 \in S \in \Delta_k\},$$

and (3.5) follows.

We have shown that $\dim B^k = \operatorname{card}\{S \in \Delta_k : 1 \in S\}$. Therefore,

$$\begin{aligned} \dim Z^k &= \dim \Lambda_k[\Gamma] - \dim B^{k+1} = \operatorname{card} \Delta_k - \operatorname{card}\{S \in \Delta_{k+1} : 1 \in S\} \\ &= \operatorname{card}\{S \in \Delta_k : 1 \in S \text{ or } S \cup \{1\} \notin \Delta_{k+1}\}. \end{aligned}$$

These facts together with $\beta_k(\Gamma) = \dim Z^k - \dim B^k$ imply (3.3).

To conclude the proof of Theorem 3.1 it remains to show that

$$\beta_k(\Delta) = \operatorname{card}\{S \in \Delta_k : S \cup \{1\} \notin \Delta\}.$$

Since Δ is shifted this will follow from Theorem 4.3, which proves a stronger fact for a somewhat larger class of complexes. $\square$

Remark 3.2. It can be shown that if $\mathbf{f}=(f_1,\ldots,f_n)$ is a basis of E such that f_1 is generic with respect to $\mathbf{e}$ (i.e., if $f_1=\Sigma\alpha_i e_i$ then $\alpha_1,\alpha_2,\ldots,\alpha_n$ are algebraically independent over $\mathbf{k}$), then $\Delta_{\mathbf{f}}(\Gamma)$ is a "near-cone" in the sense of Definition 4.1.

The construction given in the next section (proof of (b)$\Rightarrow$(c) in Theorem 1.1) will associate to each simplicial complex Γ and field $\mathbf{k}$ another simplicial complex D which satisfies assertions (i), (ii) and (iii) of Theorem 3.1. In fact, D depends *only* on the pair (f,β), where f is the f-vector and β the sequence of Betti numbers of Γ with respect to $\mathbf{k}$.

The algebraically shifted complex $\Delta(\Gamma,\mathbf{k})$ associated with a simplicial complex Γ, carries many topological and combinatorial properties of Γ (beyond those given by Theorem 3.1). Results and conjectures in this direction can be found in [BK, K1, K3]. It is worth noting that the operation $\Gamma\to\Delta(\Gamma,\mathbf{k})$ depends only on the characteristic of $\mathbf{k}$, and also that if Γ itself is shifted then $\Delta(\Gamma,\mathbf{k})=\Gamma$.

4. Proof of the main theorem

We begin by describing some properties of an auxiliary class of simplicial complexes called "near-cones".

Let Γ be a simplicial complex, and let v be a vertex not in Γ. Recall that the *cone over* Γ *with apex* v is the simplicial complex

$$v*\Gamma=\Gamma\cup\{S\cup\{v\}:S\in\Gamma\}.$$

If $f(v*\Gamma)=(f_0,f_1,f_2,\ldots)$, then clearly

$$f_{k-1}(\Gamma)=f_k-f_{k+1}+f_{k+2}-\ldots,\quad k\geq 0. \tag{4.1}$$

Definition 4.1. A simplicial complex Δ on $[n]$ is a *near-cone* if for every $S\in\Delta$, if $1\notin S$ and $j\in S$ then $S\setminus\{j\}\cup\{1\}\in\Delta$. For a near-cone Δ define

$$B(\Delta)=\{S\in\Delta: S\cup\{1\}\notin\Delta\}.$$

Every shifted complex is a near-cone.

LEMMA 4.2. *Let* Δ *be a near-cone. Then,*
(i) *every* $S\in B(\Delta)$ *is maximal in* Δ,
(ii) $B(\Delta)$ *is a Sperner family.*

Proof. Clearly (i) implies (ii). If $S \in B(\Delta)$ and $S \cup \{j\} \in \Delta$ for some $j \notin S$, then from the definition of a near-cone we get $S \cup \{1\} \in \Delta$, a contradiction. □

THEOREM 4.3. *Let Δ be a near-cone. Then Δ is homotopy equivalent to the $f(B(\Delta))$-wedge of spheres. In particular,*

$$\beta_k(\Delta) = f_k(B(\Delta)) = \operatorname{card}\{S \in \Delta_k : S \cup \{1\} \notin \Delta\}, \quad k \geq 0.$$

Proof. Represent Δ as a disjoint union $\Delta = B(\Delta) \cup (1 * C)$. Now, use the fact that if Δ' is a contractible subcomplex of a complex Δ then the quotient map $|\Delta| \to |\Delta|/|\Delta'|$ is a homotopy equivalence. This follows from the well-known homotopy extension property for simplicial pairs; a simple direct proof is given in [BW].

Since the cone $1 * C$ is contractible, we get that $|\Delta| \simeq |\Delta|/|1 * C|$. Now, $\Delta \setminus (1 * C) = B(\Delta)$ is, by Lemma 4.2, a family of maximal faces. So, contraction of the subcomplex $1 * C$ to a point turns each k-face in $B(\Delta)$ into a k-sphere, and these spheres of various dimensions are wedged together at the point corresponding to their identified boundaries. □

We are now ready to prove the main result.

Proof of Theorem 1.1. (a) $\Rightarrow$ (b). Let Γ be a simplicial complex. Let $\Delta = \Delta(\Gamma, \mathbf{k})$ be the associated shifted complex with respect to a field $\mathbf{k}$. By Theorem 3.1, for every $k \geq 0$,

$$f_k = f_k(\Gamma) = f_k(\Delta), \quad \text{and} \quad \beta_k = \beta_k(\Gamma) = \beta_k(\Delta) = f_k(B(\Delta)),$$

where $B(\Delta) = \{S \in \Delta : S \cup \{1\} \notin \Delta\}$. Represent Δ as a disjoint union $\Delta = B(\Delta) \cup (1 * C)$. Now, $f(1 * C) = (f_0 - \beta_0, f_1 - \beta_1, \ldots) = f - \beta$, hence by (4.1) $f(C) = (\chi_0, \chi_1, \ldots)$. For $k \geq 0$, $\chi_k + \beta_k$ is the number of k-faces in $E = C \cup B(\Delta)$, which is a simplicial complex obtained by restricting Δ to $\{2, \ldots, n\}$. By the Kruskal-Katona theorem $|\partial(E_k)| \geq \partial_k(\chi_k + \beta_k)$. By Lemma 4.2 every $S \in B(\Delta)$ is maximal in Δ, hence also in E. Thus we have $\partial(E_k) \subseteq C_{k-1}$. Therefore,

$$\partial_k(\chi_k + \beta_k) \leq |\partial(E_k)| \leq |C_{k-1}| = \chi_{k-1}, \quad \text{for every} \quad k \geq 1.$$

(b) $\Rightarrow$ (c). Condition (b) clearly implies that, for every $k \geq 0$, $\chi_k \geq 0$ and that for every $k \geq 1$, $\partial_k(\chi_k + \beta_k) \leq \chi_{k-1} + \beta_{k-1}$ and $\partial_k(\chi_k) \leq \chi_{k-1}$. Let E be the compressed complex on $\mathbf{N} \setminus \{1\}$, with $f_k(E) = \chi_k + \beta_k$, $k \geq -1$. Let C be the compressed complex on $\mathbf{N} \setminus \{1\}$

with $f_k(C)=\chi_k$, $k\geq -1$. C is a subcomplex of E. Define $D=(1*C)\cup E$. Since $\partial_k(\chi_k+\beta_k)\leq\chi_{k-1}$, we have that $\partial(E_k)\subseteq C_{k-1}$, for every $k\geq 0$. Thus, for every $S\in E$ and $i\in S$, $S\setminus\{i\}\in C$, hence $S\setminus\{i\}\cup\{1\}\in D$. D is therefore a near-cone. (It is easily seen that D is actually a shifted complex.) Now, $B(D)=E\setminus C$ and hence for every $k\geq 0$, $\beta_k(D)=|E_k|-|C_k|=\beta_k$. Since $f_k(C)=\chi_k$, by (4.1) we obtain that $f_k(1*C)=f_k-\beta_k$. Therefore $f_k(D)=f_k-\beta_k+f_k(B(D))=f_k-\beta_k+\beta_k=f_k$. The implication (b)$\Rightarrow$(c) now follows from Theorem 4.3.

The implication (c)$\Rightarrow$(a) is clear. This completes the proof of Theorem 1.1. □

Proof of Theorem 1.3. *Necessity*: Let Γ be a simplicial complex with vertices in $[n+1]$ and let Δ be the associated algebraically shifted complex. By Theorem 3.1, $\beta_i(\Gamma)=\beta_i(\Delta)=f_i(B(\Delta))$. By Lemma 4.2, $B(\Delta)$ is a Sperner family of subsets of $\{2,\ldots,n+1\}$.

Sufficiency: Let S be a Sperner family of subsets of $\{2,\ldots,n+1\}$. Let E be the simplicial complex generated by S and let $C=E\setminus S$. Define $D=(1*C)\cup S$. For $T\in S$ and $i\in T$, $T\setminus\{i\}\in C$ hence $T\setminus\{i\}\cup\{1\}\in D$. Therefore D is a near-cone and, as is easily seen, $B(D)=S$. Therefore for every $k\geq 0$, $\beta_k(D)=f_k(S)$. □

Remark 4.4. The homological interpretation of relation (1.5) is that the space of k-cycles and the space of $(k-1)$-boundaries of a simplicial complex must satisfy

$$\partial_k(\dim Z_k)\leq\dim B_{k-1} \tag{4.2}$$

for all $k\geq 1$. (See formula (6.5).)

Here is a brief sketch of a different proof of this relation. One needs the following two facts: (a) Let $A\subseteq\Lambda^{k+1}E$, $B\subseteq\Lambda^k E$ be subspaces and assume that for every $f\in E^*$ and $m\in A$, $f\llcorner m\in B$. (Here $f\llcorner m$, is the left interior product, cf. [Bo, K1]). Then $\partial_k(\dim A)\leq\dim B$. (This inequality implies that the Kruskal-Katona characterization of f-vectors of simplicial complexes applies to Hilbert functions of $\llcorner$-submodules of ΛE.) (b) For every $f\in E^*$ and $z\in Z_k$, $f\llcorner z\in B_{k-1}$.

To prove (a) one has to define algebraic shifting for arbitrary subspaces of $\Lambda^i E$, in a similar way to the definition in Section 3, and then the proof of Claim 1 in Section 3 extends directly.

To prove (b), consider a basis element $e_i\in E$ and notice that for every $z\in Z_k$, $e_i^*\llcorner z$ is a cycle supported in $\mathrm{lk}(i,\Gamma)$ (the link of vertex i in Γ). But $Z_{k-1}(\mathrm{lk}(i,\Gamma))\subseteq$

$B_{k-1}(\mathrm{st}(i,\Gamma))$, since $\mathrm{st}(i,\Gamma)$ is a cone and hence acyclic. ($\mathrm{st}(i,\Gamma)=i*\mathrm{lk}(i,\Gamma)$ denotes the star of i in Γ.) Thus $e_i^* \llcorner z \in B_{k-1}(\Gamma)$ for every i and every $z\in Z_k(\Gamma)$ and the claim follows.

The definition of cohomology and the characterization of (f,β)-pairs apply to any quotient algebra of ΛE.

5. Combinatorics of (f,β)-pairs

In this section we will prove Theorem 1.2 together with some additional results. Recall that we say an ordered pair (f,β) of sequences from $\mathbf{N}_0^{(\infty)}$ is *compatible* if f is the f-vector and β the Betti sequence of some simplicial complex. By Theorem 1.1 this is a purely combinatorial relation, independent of field characteristic. This relation will be denoted by $f\sim\beta$.

LEMMA 5.1. *Suppose that* $f\sim\beta$, $f=(f_0,f_1,\ldots,f_d)$, $\beta=(\beta_0,\beta_1,\ldots,\beta_d)$. *Fix* j, $1\leqslant j\leqslant d$, *and let* $\varepsilon=(\varepsilon_0,\varepsilon_1,\ldots,\varepsilon_d)$ *with* $\varepsilon_{j-1}=\varepsilon_j=1$ *and* $\varepsilon_i=0$ *for all* $i\neq j-1,j$.

(i) *If* $\partial_j(\chi_j+\beta_j)<\chi_{j-1}$, *then* $(f-\varepsilon)\sim\beta$.

(ii) *If* $\partial_j(\chi_j+\beta_j+1)<\chi_{j-1}$, *then* $f\sim(\beta+\varepsilon)$.

(iii) *If* $\beta_{j-1}>0$ *and* $\beta_j>0$, *then* $f\sim(\beta-\varepsilon)$.

Proof. The two first parts will be proven in tandem with the argument for (ii) in square brackets.

We pass from f to $f'=f-\varepsilon$, [from β to $\beta'=\beta+\varepsilon$], by subtracting 1 from, [by adding 1 to], the $(j-1)$- and j-coordinates. Hence, we pass from $\chi=(\chi_{-1},\chi_0,\ldots,\chi_{d-1})$ to $\chi'=(\chi'_{-1},\chi'_0,\ldots,\chi'_{d-1})$ by in both cases subtracting 1 from the $(j-1)$-coordinate only. Since (χ,β) satisfies equations (1.5) for all k, then so does automatically (χ',β), $[(\chi',\beta')]$, for all $k\neq j$. But also

$$\partial_j(\chi'_j+\beta_j)=\partial_j(\chi_j+\beta_j)\leqslant\chi_{j-1}-1=\chi'_{j-1},$$

$$[\partial_j(\chi'_j+\beta'_j)=\partial_j(\chi_j+\beta_j+1)\leqslant\chi_{j-1}-1=\chi'_{j-1}],$$

so (1.5) is satisfied also for $k=j$.

The reasoning for part (iii) is similar. We omit the details. □

A few definitions are needed in preparation for the next result. Fix a sequence $\beta=(\beta_0,\beta_1,\ldots,\beta_d)\in\mathbf{N}_0^{(\infty)}$, $\beta_d\neq 0$. Let $\alpha_d=0$, and define recursively

$$\alpha_i=\partial_{i+1}(\alpha_{i+1}+\beta_{i+1}),\quad d-1\geqslant i\geqslant -1. \tag{5.1}$$

Let $D=\bigcup_{i=0}^{d} \bar{I}_{i+1}^{\alpha_i}$, where $\bar{I}_{i+1}^{\alpha_i}$ is the initial segment of size α_i in the antilexicographic order of $(i+1)$-element subsets of $\{2,3,\ldots\}$. Equivalently, D is the compressed complex with f-vector $(\alpha_0,\alpha_1,\ldots,\alpha_d)$, as defined in Section 2, translated by the map $n\to n+1$. Similarly, let $A=\bigcup_{i=0}^{d} \bar{I}_{i+1}^{\alpha_i+\beta_i}$, and define the near-cone (cf. Definition 4.1)

$$\Delta_\beta = A\cup(1*D). \tag{5.2}$$

By construction, $B(\Delta_\beta)=A\setminus D$ is the compressed Sperner family on $\{2,3,\ldots\}$ with f-vector equal to β. In particular, Δ_β has Betti sequence β and χ-vector α.

THEOREM 5.2. *For $\beta\in\mathbf{N}_0^{(\infty)}$, let $F_\beta=\{f\in\mathbf{N}_0^{(\infty)}: f\sim\beta\}$.*

(a) *The set F_β has a unique componentwise minimal element $\varphi(\beta)=(\varphi_0,\varphi_1,\ldots)$; namely $\varphi_i=\beta_i+\alpha_i+\alpha_{i-1}$, with α_i defined by* (5.1).

(b) *For sequences f, $\beta\in\mathbf{N}_0^{(\infty)}$ the following conditions are equivalent:*

(i) $f=\varphi(\beta)$,

(ii) $\partial_k(\chi_k+\beta_k)=\chi_{k-1}$, *for all* $k\geq 1$, *and* $\chi_{-1}=1$, (cf. (1.5))

(iii) *f is the f-vector of the near-cone Δ_β (cf.* (5.2)).

(c) *The mapping φ from Betti sequences to f-vectors is injective and order-preserving.*

Proof. Suppose $f\in F_\beta$. If $\partial_j(\chi_j+\beta_j)<\chi_{j-1}$ for some j, then by Lemma 5.1 there exists $f^{(1)}\in F_\beta$ such that $f>f^{(1)}$. Repetition of this leads in a finite number of steps $f=f^{(0)}>f^{(1)}>\ldots>f^{(q)}$, to an f-vector $f^{(q)}\in F_\beta$ such that

$$\partial_k(\chi_k^{(q)}+\beta_k)=\chi_{k-1}^{(q)}, \quad \text{for all} \quad k\geq 1. \tag{5.3}$$

Since $\chi_k^{(q)}=\beta_k=0$ for k sufficiently large, equations (5.3) together with β determine the sequence $\chi^{(q)}=(\chi_0^{(q)},\chi_1^{(q)},\ldots)$ uniquely, and with β this determines $f^{(q)}$.

The preceding paragraph shows that there exists a unique f-vector $\varphi(\beta)=f^{(q)}\in F_\beta$, characterized by (5.3), such that $f\geq\varphi(\beta)$ for all $f\in F_\beta$.

Since $\chi(\Delta_\beta)=\alpha$ and $\beta(\Delta_\beta)=\beta$, equations (5.1) show that (5.3) is satisfied by the f-vector of the complex Δ_β. Hence, $\varphi(\beta)=(\varphi_0,\varphi_1,\ldots)=f(\Delta_\beta)$. From the construction of Δ_β one can read off its f-vector:

$$\varphi_i=\alpha_{i-1}+\alpha_i+\beta_i, \quad i\geq 0. \tag{5.4}$$

From the explicit description (5.1) and (5.4) of $\varphi(\beta)$ it is easy to see that $\beta\neq\beta'$ implies $\varphi(\beta)\neq\varphi(\beta')$ and $\beta\leq\beta'$ implies $\varphi(\beta)\leq\varphi(\beta')$. □

For $n, k \geq 1$, let

$$n = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \ldots + \binom{a_i}{i},$$

where $a_k > a_{k-1} > \ldots > a_i \geq i \geq 1$. Recall the definition (1.3) of $\partial_{k-1}(n)$. We will here need also the following function

$$\partial^{k-1}(n) = \binom{a_k - 1}{k} + \binom{a_{k-1} - 1}{k-1} + \ldots + \binom{a_i - 1}{i}, \tag{5.5}$$

and $\partial^{k-1}(0) = 0$.

THEOREM 5.3. *For an f-vector f, let* $B_f = \{\beta \in \mathbf{N}_0^{(\infty)} : f \sim \beta\}$.

(a) *The set* B_f *has a unique componentwise maximal element* $\psi(f) = (\psi_0, \psi_1, \ldots)$; *namely*

$$\psi_i = \partial^i(f_i) + \partial^{i+1}(f_{i+1}) - f_{i+1}, \quad i \geq 0. \tag{5.6}$$

(b) *For* $f, \beta \in \mathbf{N}_0^{(\infty)}$ *the following conditions are equivalent:*

(i) $\beta = \psi(f)$,

(ii) $\partial_k(\chi_k + \beta_k) \leq \chi_{k-1} \leq \partial_k(\chi_k + \beta_k + 1)$, *for all* $k \geq 1$, *and* $\chi_{-1} = 1$,

(iii) β *is the Betti sequence of the compressed complex* K_f (*cf.* (2.2)).

(c) *For an f-vector f, let* ψ_i *be as in* (5.6). *Define* $\gamma_{-1} = 0$ *and then recursively* $\gamma_k = \min(\psi_{k+1}, \psi_k - \gamma_{k-1})$, $k \geq 0$. *Let* $\delta_k = \psi_k - \gamma_k - \gamma_{k-1}$, *for* $k \geq 0$. *Then* $(\delta_0, \delta_1, \ldots) \in B_f$, *and*

$$\min_{\beta \in B_f} \sum_{i \geq 0} \beta_i = \sum_{k \geq 0} \delta_k = \sum_{k \geq 0} (\psi_k - 2\gamma_k). \tag{5.7}$$

Proof. Let $\beta \in B_f$. If $\chi_{j-1} > \partial_j(\chi_j + \beta_j + 1)$ for some j, then by Lemma 5.1 there exists $\beta^{(1)} \in B_f$ such that $\beta < \beta^{(1)}$. So in a finite number of steps (since the set B_f is obviously finite) $\beta = \beta^{(0)} < \beta^{(1)} < \ldots < \beta^{(q)}$ we reach a Betti sequence $\beta^{(q)} \in B_f$ such that

$$\partial_k(\chi_k^{(q)} + \beta_k^{(q)}) \leq \chi_{k-1}^{(q)} \leq \partial_k(\chi_k^{(q)} + \beta_k^{(q)} + 1), \quad \text{for all} \quad k \geq 1. \tag{5.8}$$

Knowledge of f and equations (5.8) uniquely determines the sequence $\beta^{(q)}$. To see this we use the following simple observation.

LEMMA. *Suppose that* g, h: $\mathbf{N} \to \mathbf{N}$ *are functions such that* $x < y$ *implies* $g(x) \leq g(y)$ *and* $h(x) > h(y)$, *for all* $x, y \in \mathbf{N}$. *Then the inequalities* $g(y) \leq h(y) \leq g(y+1)$ *have at most one solution.*

Now, assuming that $\beta_j^{(q)}$ is known for $j \geq k+1$, so that $\chi_k^{(q)}, \chi_{k+1}^{(q)}, \ldots$ are already determined numbers, we have that $\beta_k^{(q)}$ is a solution to the inequalities

$$\partial_k(\chi_k^{(q)}+y) \leq f_k - y - \chi_k^{(q)} \leq \partial_k(\chi_k^{(q)}+y+1).$$

By the lemma, $y=\beta_k^{(q)}$ is the only solution.

We have shown that there exists a unique Betti sequence $\psi(f)=\beta^{(q)} \in B_f$, characterized by (5.8), such that $\beta \leq \psi(f)$ for all $\beta \in B_f$.

Consider the class $\mathrm{Sh}(f)$ of all shifted complexes with a given f-vector f. By Theorem 3.1, $B_f = \{\beta(\Delta): \Delta \in \mathrm{Sh}(f)\}$, and by Theorem 4.3

$$\beta_k(\Delta) = \mathrm{card}\{S \in \Delta_k: 1 \notin S\} - \mathrm{card}\{S \in \Delta_{k+1}: 1 \in S\}, \tag{5.9}$$

for all $\Delta \in \mathrm{Sh}(f)$. Lemma 2.2 then shows that $\beta_k(\Delta)$ is maximized among all shifted complexes Δ, if Δ_k and Δ_{k+1} are the compressed families. Hence, the compressed complex K_f maximizes β_k for all $k \geq 0$, which means that $\psi(f)=\beta(K_f)$. The proof of parts (a) and (b) is then completed by observing that

$$\mathrm{card}\{S \in (K_f)_i: 1 \notin S\} = \partial^i(f_i), \quad i \geq 0, \tag{5.10}$$

and then using (5.9).

Equation (5.10) is immediately clear by inspection of the "cascade form" of compressed families: $(K_f)_i = I_{i+1}^{f_i}$ consists of all $(i+1)$-subsets of $[a_{i+1}]$, all i-subsets of $[a_i]$ augmented by $\{a_{i+1}+1\}$, all $(i-1)$-subsets of $[a_{i-1}]$ augmented by $\{a_i+1, a_{i+1}+1\}$, and so on, where

$$f_i = \binom{a_{i+1}}{i+1} + \binom{a_i}{i} + \binom{a_{i-1}}{i-1} + \ldots$$

as in (5.5).

It remains to prove part (c). This will be done following Lemma 5.5 below. $\square$

By other methods Sarkaria [Sa] has independently identified the compressed complex K_f as the complex which maximizes Betti numbers among all simplicial complexes with a given f-vector f.

In contrast to φ, the mapping ψ is not injective or order-preserving. E.g., $\psi(3,1)=\psi(4,2)=(1,0)$ and $\psi(3,3)=(0,1)$. About the compositions of these mappings the following can be said.

COROLLARY 5.4. (a) $\psi(\varphi(\beta))=\beta$, *for all* $\beta \in \mathbf{N}_0^{(\infty)}$.

(b) $\varphi(\psi(f))\leqslant f$, *for all f-vectors f.*

These relations follow from Theorems 5.2 and 5.3 most easily by comparing condition (ii) of part (b) in each theorem.

Whereas the minimal element of F_β and the maximal element of B_f are unique, the complexes which realize the corresponding (f,β)-pairs are in general not unique. For instance, there are two complexes with $f=(5,8)$ and $\beta=(0,4)$, and $f=\varphi(\beta)$, $\beta=\psi(f)$.

If $c=(c_0, c_1, \ldots)\in \mathbf{N}_0^{(\infty)}$ and $c_{j-1}, c_j>0$, we will say that the sequence

$$c' = (c_0, c_1, \ldots, c_{j-2}, c_{j-1}-1, c_j-1, c_{j+1}, \ldots)$$

is obtained from c by a *j-collapse*. Also, c is *collapsible to* c' if a sequence of such collapse steps (for different j) lead from c to c'. Using the notation of Theorem 5.3 we have that

$$B_f=\{\beta \in \mathbf{N}_0^{(\infty)}\colon \psi(f) \text{ is collapsible to } \beta\}. \tag{5.11}$$

The forward inclusion $\subseteq$ was shown in the proof of Theorem 5.3, the reverse inclusion follows from part (iii) of Lemma 5.1.

To minimize the *sum* of Betti numbers over the set B_f we are therefore led to the following combinatorial problem: Given $c\in \mathbf{N}_0^{(\infty)}$, find the longest collapsing sequence starting with c.

LEMMA 5.5. *An optimal collapsing sequence is obtained by making all possible j-collapses first for j=*1, *then for j=*2, *and so on in order of increasing j.*

Proof. Suppose that in a collapsing sequence of maximal length a j-collapse immediately precedes in i-collapse, and $j>i$. It is clearly legal to transpose this pair (first do the i-collapse, then the j-collapse), and this way one obtains a collapsing sequence of the same length but with fewer inversions. Hence a sequence of the same length with *no* inversions exists, as claimed. □

Starting with $c=(c_0, c_1, \ldots)$ the number of possible 1-collapses is $g_0=\min(c_0, c_1)$, and c collapses to $c^{(1)}=(c_0-g_0, c_1-g_0, c_2, c_3, \ldots)$. The number of possible 2-collapses in $c^{(1)}$ is $g_1=\min(c_1-g_0, c_2)$, and $c^{(1)}$ collapses to $c^{(2)}=(c_0-g_0, c_1-g_0-g_1, c_2-g_1, c_3, \ldots)$. Recursively define $g_k=\min(c_k-g_{k-1}, c_{k+1})$, for $k\geqslant 1$. Then the number of k-collapses in the optimal collapsing sequence of Lemma 5.5 is g_{k-1}. This together with (5.11) implies part (c) of Theorem 5.3.

6. Appendix: Cell complexes

By a *cell complex* X we shall mean a finite *CW* complex as defined in the literature, see e.g. [LW] or [Sp]. A cell complex is *regular* if the closure of each cell is homeomorphic to a ball. The f-vector $(f_0,f_1,\dots)$ and Betti sequence $(\beta_0,\beta_1,\dots)$ of a cell complex have the obvious meaning: f_i equals the number of i-dimensional cells and β_i equals the dimension over a field $\mathbf{k}$ of the i-dimensional cellular (or singular) homology.

The Euler-Poincaré formula (1.2) is true for all cell complexes. Some rather obvious additional necessary conditions on (f,β)-pairs of cell complexes turn out to also be sufficient. We shall state and prove here for the sake of completeness the description of such (f,β)-pairs.

THEOREM 6.1. *Suppose that* $f=(f_0,f_1,\dots,f_d)$, $\beta=(\beta_0,\beta_1,\dots,\beta_d)\in\mathbf{N}_0^{d+1}$ *are two given sequences and* $\mathbf{k}$ *is a field. Then the following conditions are equivalent:*

(a) *f is the f-vector and β the Betti sequence over* $\mathbf{k}$ *of some cell complex [or, regular cell complex],*

(b) *let* $\chi_{k-1}=\Sigma_{j\geq k}(-1)^{j-k}(f_j-\beta_j)$, *for* $0\leq k\leq d$; *then*

(i) $\chi_{-1}=1$

(ii) $\chi_k\geq 0$, *for* $0\leq k\leq d-1$ (6.1)

[or, $\chi_k\geq 1$, *for* $0\leq k\leq d-1$]. (6.2)

(c) *f is the f-vector of some cell complex [or, regular cell complex] which is homotopic to the β-wedge of spheres.*

Proof. Before entering any specific details let us record the following elementary observation: If

$$0\to C_p\xrightarrow{d_p}C_{p-1}\xrightarrow{d_{p-1}}\dots\to C_0\xrightarrow{d_0}C_{-1}\to 0 \tag{6.3}$$

is any sequence of finite-dimensional $\mathbf{k}$-vector spaces and linear maps such that $d_j\circ d_{j+1}=0$ for all j, and if $f_j=\dim C_j$ and $\beta_j=\dim(\operatorname{Ker} d_j/\operatorname{Im} d_{j+1})$, then

$$\dim(\operatorname{Im} d_k)=\sum_{j\geq k}(-1)^{j-k}(f_j-\beta_j) \tag{6.4}$$

for $-1\leq k\leq p$.

(a)⇒(b). Let X be a p-dimensional cell complex with f-vector f and Betti sequence β (over $\mathbf{k}$), and suppose that (6.3) is the augmented cellular chain complex of X over $\mathbf{k}$. So, d_0 is surjective and $\dim C_{-1}=1$. By (6.4),

$$\dim B_{k-1}=\chi_{k-1}, \tag{6.5}$$

so clearly $\chi_{-1}=1$ and $\chi_k\geq 0$ for $k\geq 0$. If X is regular, then the boundary of any $(k+1)$-cell is non-zero and hence $\dim B_k\geq 1$ for $0\leq k\leq p-1$, cf. [LW, p. 168].

(b)⇒(c). Suppose that we are given sequences $f,\beta\in\mathbf{N}_0^{(\infty)}$ such that $\chi_{-1}=1$ and $\chi_k\geq 0$ for $k\geq 0$. By definition,

$$f_i=\chi_{i-1}+\beta_i+\chi_i, \quad i\geq 0. \tag{6.6}$$

For each i, let $E_i=A_i\cup B_i\cup C_i$ be a partitioned set of f_i i-dimensional cells, such that $\operatorname{card} A_i=\chi_{i-1}$, $\operatorname{card} B_i=\beta_i$ and $\operatorname{card} C_i=\chi_i$.

We now construct a cell complex X as follows. E_0 is a set of vertices with distinguished base point e_0, where $A_0=\{e_0\}$. Suppose the $(i-1)$-skeleton X_{i-1} has been constructed. To obtain X_i, attach the i-cells in A_i so that they fill the interior of the $(i-1)$-spheres determined by the $(i-1)$-cells from C_{i-1}. Then attach the cells in $B_i\cup C_i$ with their whole boundary to e_0.

The complex X is homeomorphic to a wedge of spheres and balls, with β_i spheres and χ_{i-1} balls of dimension i, $i\geq 0$. Contraction of the balls shows that X is homotopically equivalent to the β-wedge of spheres.

Suppose next that

$$p=\max\{j: f_j\neq 0\}$$

and that $\chi_k\geq 1$ for $0\leq k\leq p-1$. Let $E_i=A_i\cup B_i\cup C_i$ be as before, $i\geq 0$. Then a *regular* cell complex Y is constructed in the following way.

Again start with the vertex set E_0. It has two distinguished vertices $e_0^1\in A_0$ and $e_0^2\in C_0$, where e_0^2 is arbitrarily chosen in C_0. Attach the 1-cells in A_1 with one endpoint to e_0^1 and the other to distinct vertices in C_0. Let e_1^1 by the A_1-cell connecting e_0^1 and e_0^2. Then attach the 1-cells in $B_1\cup C_1$ regularly to the 0-sphere $\{e_0^1, e_0^2\}$, and choose arbitrarily $e_1^2\in C_1$.

Suppose that the $(i-1)$-skeleton Y_{i-1} has been constructed and is a regular complex with two distinguished cells $e_j^1\in A_j$ and $e_j^2\in C_j$ in each dimension $j\leq i-1$ so that

$Z_k=\bigcup_{j=0}^{k}(e_j^1\cup e_j^2)$ is homeomorphic to the k-dimensional sphere, for $k\leq i-1$, and all $(i-1)$-cells in $B_{i-1}\cup C_{i-1}$ are regularly attached to Z_{i-2}. To obtain Y_i, attach the cells of A_i regularly to the $(i-1)$-spheres formed by e_{i-1}^1 and the distinct cells in C_{i-1}. Let e_i^1 be the unique A_i-cell thus attached to Z_{i-1}. Then regularly attach all the cells in $B_i\cup C_i$ to Z_{i-1}, and arbitrarily choose $e_i^2\in C_i$.

We omit the verification that the regular complex $Y=\bigcup Y_i$ is homotopic to the β-wedge of spheres. □

Remark 6.2. Suppose $\beta=(0,\ldots,0,1)$ is the Betti sequence of the d-dimensional sphere S^d. If $f=(f_0,f_1,\ldots,f_d)$ satisfies $\chi_{-1}=1$ and $\chi_k\geq 1$ for $0\leq k\leq d-1$, then by the above construction there exists a regular cell complex X with f-vector f which is homotopy equivalent to S^d. Bayer [Ba] has shown that such X can be found which is even homeomorphic to S^d.

Remark 6.3. As mentioned in the Introduction, we conjecture that the characterization of (f,β)-pairs of simplicial complexes given by Theorem 1.1 extends to all regular cell complexes whose face poset is a meet-semilattice (i.e., any nonempty intersection of two closed cells is a closed cell). In particular, this includes the polyhedral complexes.

The following can be cited as supporting evidence for this conjecture. Wegner [W] proved that the Kruskal-Katona conditions are satisfied by a large class of graded meet-semilattices, which includes all meet-semilattices arising as face posets from regular cell complexes. Also, by triangulating the faces of a polyhedral complex without introducing new vertices, one shows that Theorem 1.3 is valid for all polyhedral complexes. Thus, these characterization results for each component of simplicial (f,β)-pairs, *taken alone*, extend to all polyhedral complexes (and beyond).

In order to prove the above conjecture it would be desirable to find a way of extending the notion of shifting to classes of non-simplicial complexes.

Note added in proof (July 1988):

(1) The conjecture made in Remark 6.3 has now been verified. Details will appear in a subsequent paper.

(2) Theorem 1.3 can be given the following more detailed formulation (same proof): *$\beta\in\mathbf{N}_0^{(\infty)}$ is the Betti sequence of some simplicial complex Γ such that $|\Gamma_0|\leq n+1$ and* $\dim\Gamma\leq d$ *if and only if β is the f-vector of some Sperner family S of subsets of* $[n]$

such that $|A| \leqslant d+1$ *for all* $A \in S$. This leads to a corresponding sharpening of Theorem 1.4: *If* Γ *is a simplicial complex with* $|\Gamma_0| \leqslant n+1$ *and* $\dim \Gamma \leqslant d$, *then*

$$|\chi(\Gamma)| \leqslant \beta(\Gamma) \leqslant \begin{cases} \binom{n}{d+1}, & \text{if } d+1 < [n/2] \\ \binom{n}{[n/2]}, & \text{otherwise.} \end{cases}$$

The cases of equality occur only for the expected skeleta as in Theorem 1.4.

The information about the Euler characteristic $|\chi(\Gamma)|$ contained in this improved version of Theorem 1.4 was previously known and is due to J. Eckhoff (see Hilfsatz 2 and 3 in J. Eckhoff, "Die Euler-Characteristik von Vereinigungen konvexer Mengen im $\mathbf{R}^d$", Abhandl. Math. Sem. Univ. Hamburg, 50 (1980), 135–146).

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