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Uniqueness and related analytic properties for the Benjamin–Ono equation— a nonlinear Neumann problem in the plane

by

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1. Introduction

In a study of internal travelling solitary waves in a stable, two-layer perfect fluid of infinite depth contained above a rigid horizontal bottom, T. B. Benjamin [3] introduced a pseudo-differential equation of the form

$$u(x)^2-u(x)=\mathscr{F}(u)(x),\quad x\in\mathbf{R},\tag{1.1}$$

where

$$\mathscr{F}(\xi)(x)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}|k|\,e^{-ikx}\left\{\int_{-\infty}^{+\infty}\xi(z)\,e^{ikz}\,dz\right\}dk.\tag{1.2}$$

He then sought solutions u with $|u(x)|\to 0$ as $|x|\to\infty$. (In the present account dimensionless parameters have been normalised to equal unity.) The equation (1.1) was later discussed by Ono [5] in the context of soliton theory. In fact he derived a time-dependent equation

$$u_t+u_x+2uu_x-(\mathscr{L}(u))_{xx}=0.\tag{1.3}$$

Here $\mathscr{L}$ denotes the Hilbert transform (Ono uses the sign convention of Stein [7] for the Hilbert transform), and certain constants have been normalised to equal unity. If steady super-critical solitary wave solutions of (1.3) are sought in the form $u(x,t)=u(c_0x-ct)$ then equation (1.3) reduces to

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$$c_0^{-2}(c_0-c)\,u+c_0^{-1}u^2=(\mathscr{L}(u))_x,\quad 0<c_0<c, \tag{1.4}$$

(where a constant of integration has been taken to be zero). This coincides formally with (1.1) since for smooth functions u,

$$(\mathscr{L}(u))_x=\mathscr{F}(u),$$

and since a change of variables normalises the constants in (1.4) to be -1 and 1 respectively. (In 1967, equation (1.3) was written down by Benjamin as the time-dependent version of the travelling wave equation (1.1) upon which his attention was then focused. At the same time Davis and Acrivos derived equation (1.4) in an experimental and numerical study of similar travelling wave phenomena [4]. Equation (1.3), which was re-derived using formal nonlinear perturbation methods by Ono [5] in 1975, is generally known nowadays as the Benjamin–Ono equation.)

The present paper is concerned with the theory of equation (1.1) (or, equivalently, of equation (1.4)). In particular we present a proof that all its solutions are known in closed form, and we derive a rather surprising connection between the solubility of (1.1) and an initial-value problem for a complex analytic function in the upper half-plane ((1) and (2) below).

Our results arose from a separate investigation [2] in which the Benjamin–Ono equation is studied from the viewpoint of its role in the exact theory of the Euler equation governing such steady two fluid systems. There (1.1) is seen as giving, to leading order, the form of solitary waves where u tends to zero as $|x|\to\infty$. Such solitary waves were first found by Benjamin who wrote them down in closed form and who remarked upon their decay to zero at infinity being algebraic rather than exponential. For the study of periodic waves the operator $\mathscr{F}$ in (1.1) must be replaced by an appropriate analogue involving Fourier series rather than transforms of u. For that case too Benjamin wrote down solutions in closed form.

In the present paper we show, among other things, that (apart from translations) the solution found by Benjamin

$$u(x)=\frac{2}{1+x^2},\quad x\in\mathbf{R}, \tag{1.5}$$

is the only solution of (1.1) which converges to zero as $|x|\to\infty$. We will also show that Benjamin found the complete set of functions which satisfy the analogue of (1.1) which governs periodic travelling waves in those two fluid systems.

Our results are a consequence of a larger picture of the analytic structure of solitary and periodic waves in the present context. The key is an observation in Benjamin [1] that if u is a bounded solution of (1.1) and u is extended as a bounded harmonic function on the closure of the upper half-plane, then this harmonic function satisfies

$$\left.\begin{array}{c} \Delta u(x,y)=0, \quad x\in\mathbf{R},\ y>0, \\ u_y(x,0)=u(x,0)-u^2(x,0), \quad x\in\mathbf{R}, \\ u(x,0)\to 0 \quad as \quad |x|\to\infty, \\ \{u(x,y)\colon x\in\mathbf{R},\ y>0\} \quad is\ bounded. \end{array}\right\} (\mathscr{S})$$

The analogous problem for internal periodic waves is governed by the boundary-value problem

$$\left.\begin{array}{c} \Delta u(x,y)=0, \quad x\in\mathbf{R},\ y>0, \\ u_y(x,0)=u(x,0)-u^2(x,0), \quad x\in\mathbf{R}, \\ u(x,0)=u(x+p,0), \quad x\in\mathbf{R}, \\ \{u(x,y)\colon x\in\mathbf{R},\ y>0\} \quad is\ bounded \end{array}\right\} (\mathscr{P})$$

where $p\neq 0$ is a constant. In both cases $u\equiv 0$ is a solution and $u\equiv 1$ is a solution of $(\mathscr{P})$. In what follows we restrict attention to non-constant solutions. Note that the last condition in $(\mathscr{P})$ excludes the family of solutions

$$u(x,y)=\alpha+(\alpha-\alpha^2)y, \quad \alpha\in\mathbf{R}\setminus\{0,1\}.$$

More general than either $(\mathscr{S})$ or $(\mathscr{P})$ is the problem

$$\left.\begin{array}{c} \Delta u(x,y)=0, \quad x\in\mathbf{R},\ y>0, \\ u_y(x,0)=u(x,0)-u^2(x,0), \quad x\in\mathbf{R}, \\ \{u(x,y)\colon x\in\mathbf{R},\ y>0\} \quad is\ bounded \end{array}\right\} (\mathscr{B})$$

and our main results are the following:

(1) *every non-constant solution of* $(\mathscr{B})$ *is a solution of* $(\mathscr{S})$ *or* $(\mathscr{P})$;

(2) *if u is a non-zero solution of* $(\mathscr{S})$ *or* $(\mathscr{P})$, *then* $u(x,y)>0$, $x\in\mathbf{R}$, $y\geq 0$, *and u can be normalised with respect to translation so that*

$$u_x(0,0)=0. \tag{1.6}$$

Note that this normalisation is not necessarily unique. Let

$$c=u^2(0,0)-2u(0,0)\in\mathbf{R}. \tag{1.7}$$

Then there exists a unique complex function f, analytic in the closure of the upper half-plane, which satisfies the ordinary differential equation

$$\frac{df}{dz}(z) = \frac{i}{2}(f(z)^2+c), \quad z=x+iy, \ x\in\mathbf{R}, \ y\geqslant 0, \tag{1.8}$$

and such that

$$f(0) = u(0). \tag{1.9}$$

Moreover

$$u(x,y) = \mathrm{Real} f(x+iy), \quad x\in\mathbf{R}, \ y\geqslant 0. \tag{1.10}$$

Remark. We will write $u(x)$ instead of $u(x,0)$ whenever it is convenient. The fact that (1.6) does not uniquely determine u does not affect the validity of our conclusion. In particular $u_x(x_1,0)=u_x(x_2,0)$ implies that either $u(x_1)=u(x_2)$ or $u(x_1)+u(x_2)=2$. That is, for all solutions, $\frac{1}{2}(u_{\max}+u_{\min})=1$ in problem $(\mathscr{P})$, since the maxima and minima are, by the maximum principle, attained on the boundary.

The solutions of (1.7) and (1.8) are as follows: when $u(0,0)=2$, $c=0$, and the general solution of (1.8) is

$$f(z) = \frac{2i}{z+w}, \tag{1.11}$$

where $w=\alpha+i\beta$ is an arbitrary complex constant of integration. Then (1.9) implies that $w=i$ and (apart from translations)

$$u(x,0) = \frac{2}{1+x^2}.$$

This gives the solitary wave solution found by Benjamin.

If $u(0,0)\in(0,2)$, then $c\in[-1,0)$ and the general solution of (1.8) is

$$f(z) = -i\sqrt{|c|}\tan\left(\frac{1}{2}\sqrt{|c|}(z+w)\right), \quad w\in\mathbf{C}. \tag{1.12}$$

Hence (1.9) implies that w is determined by the requirement that

$$u(0) = -i\sqrt{2u(0)-u(0)^2}\tan\left(\frac{1}{2}w\sqrt{2u(0)-u(0)^2}\right). \tag{1.13}$$

If $u(0)\in(0,1)$ then $w=i\gamma+2n\pi/\sqrt{2u(0)-u(0)^2}$, $n\in\mathbf{Z}$, gives

$$\frac{u(0)}{\sqrt{2u(0)-u(0)^2}}=\tanh\left(\frac{\gamma}{2}\sqrt{2u(0)-u(0)^2}\right)$$

to determine γ, while if $u(0)\in(1,2)$ then

$$\frac{u(0)}{\sqrt{2u(0)-u(0)^2}}=\coth\left(\frac{\gamma}{2}\sqrt{2u(0)-u(0)^2}\right),$$

where $w=i\gamma+(2n+1)\pi/\sqrt{2u(0)-u(0)^2}$, $n\in\mathbf{Z}$. Then the periodic waves discovered by Benjamin can be recovered by taking the real part of (1.12). With c given by (1.7) and $d=\frac{1}{2}\sqrt{|c|}$ they are

$$u(x,y)=2d\sec^2 dx\left\{\frac{(1+(u(0)/2d)\tanh dy)((u(0)/2d)+\tanh dy)}{(1+(u(0)/2d)\tanh dy)^2+\tan^2 dx(\tanh dy+(u(0)/2d))^2}\right\}. \quad (1.14)$$

If $u(0)=1$, then (1.13) cannot be solved for w (except in the extended complex plane) and the only solution is $u(x,y)\equiv 1$.

Now if $u(0)\notin[0,2]$, then the general solution of (1.8) is

$$f(z)=i\sqrt{c}\tanh\left(\frac{1}{2}\sqrt{c}(z+w)\right),\quad w\in\mathbf{C}, \quad (1.15)$$

whose real part has infinitely many poles in the upper half plane, so there is no solution when $u(0)\notin[0,2]$. When $u(0,0)=0$ the solution is $u(x,y)\equiv 0$ by the boundary point lemma since $u_y(0,0)=u(0,0)=0$, and $u(x,y)\geq 0$, $x\in\mathbf{R}$, $y>0$.

Before proceeding to elaborate the proof of this result it is interesting to examine the implications of (1.8) for solutions of $(\mathscr{B})$.

Let H denote the upper half-plane $\{(x,y)\colon x=\mathbf{R},\ y>0\}$. Suppose that

$$f(x+iy)=u(x,y)+iv(x,y),\quad (x,y)\in\bar{H}, \quad (1.16)$$

is *any* non-constant solution of (1.8) with

$$u(x,y)>0. \quad (1.17)$$

Then

$$u_x+uv=0,\quad (x,y)\in H. \quad (1.18)$$

A differentiation with respect to x followed by substitution from (1.18) and a use of the Cauchy–Riemann equations gives

$$\frac{\partial}{\partial x}\left(\frac{u_x}{u}\right)=u_y. \quad (1.19)$$

Similarly, a differentiation of (1.18) with respect to y yields

$$\frac{\partial}{\partial x}\left(\frac{u_y}{u}+u\right)=0. \tag{1.20}$$

Hence there exists a function α, of y only, such that

$$u_y=\alpha(y)\,u-u^2 \quad \text{on } \bar{H}. \tag{1.21}$$

This means that on each line y=constant the boundary-condition is similar to that with $y=0$. If u satisfies ($\mathscr{B}$) we would conclude that

$$\alpha(0)=1. \tag{1.22}$$

A substitution from (1.19) into (1.21) then gives

$$\frac{\partial}{\partial x}\left(\frac{u_x}{u}\right)=\alpha(y)u-u^2, \quad (x,y)\in\bar{H}. \tag{1.23}$$

Now multiplying by (u_x/u) and an integration yields the existence of a function β, of y only, such that

$$u_x^2=2\alpha(y)\,u^3-u^4+\beta(y)\,u^2, \quad (x,y)\in\bar{H}. \tag{1.24}$$

Now differentiation with respect to x gives

$$u_x\{u_{xx}-3\alpha(y)\,u^2+2u^3-\beta(y)\,u\}=0, \quad x\in\mathbf{R},\ y\geqslant 0. \tag{1.25}$$

If u is not a constant function, the fact that u_x is real-analytic and so has isolated zeros then gives that for each fixed $y\geqslant 0$, u satisfies a second-order, autonomous ordinary differential equation

$$u_{xx}-3\alpha(y)\,u^2+2u^3-\beta(y)\,u=0, \quad x\in\mathbf{R},\ y\geqslant 0. \tag{1.26}$$

A differentiation of (1.21) with respect to y followed by a substitution from (1.21) gives

$$u_{yy}=\alpha'(y)\,u+(\alpha(y)-2u)(\alpha(y)-u)\,u, \quad x\in\mathbf{R},\ y\geqslant 0. \tag{1.27}$$

Since u is harmonic (1.26) and (1.27) together give

$$\alpha'(y)+\alpha^2(y)+\beta(y)=0. \tag{1.28}$$

The easiest case is when u corresponds to a solitary wave. Then it is one of the estimates proved in [1] that $|u(x,y)|+|u_x(x,y)/u(x,y)|\to 0$ as $x^2+y^2\to\infty$, and so, letting $x\to\infty$ in (1.24) for fixed y yields that $\beta(y)=0$ for all $y\geq 0$. Then (1.28) gives that $\alpha(y)=1/(y+1)$ in this case, since $\alpha(0)=1$. This then determines the solitary wave uniquely up to translation in the x-direction.

To study the periodic problem we proceed as follows. If (1.21) is differentiated with respect to x, then multiplied by u_x and a substitution from (1.24) is made we obtain

$$u_x u_{xy} = (\alpha(y)-2u)(2\alpha(y)\,u^3-u^4+\beta(y)\,u^2). \tag{1.29}$$

On the other hand, differentiation of (1.24) with respect to y gives

$$2u_x u_{xy} = 2\alpha'(y)u^3+\beta'(y)\,u^2+2\{3\alpha(y)\,u^2-2u^3+\beta(y)\,u\}(\alpha(y)\,u-u^2). \tag{1.30}$$

Together (1.28), (1.29) and (1.30) imply that

$$\beta'(y)\,u^2 = 0. \tag{1.31}$$

Hence β is constant in the case of periodic waves. Now it is clear from Section 2 that if u corresponds to a solution of $(\mathcal{P})$ then u is a periodic solution of (2.5) which is bounded below by a positive constant on the line $y=0$. Hence from (2.2) and (2.5) it follows that u is bounded below by a positive constant on the upper half plane and, in particular, $u(x,y)\not\to 0$ as $y\to\infty$ (for fixed x). An inspection of (1.24) on a line where x is constant and u_x is zero together with (1.28) then yields that $\beta\neq 0$ in the case of periodic waves. Hence, since (1.28) is to have a solution $\alpha(y)$, for all $y>0$, we conclude that $\beta<0$.

Let $\beta=-\sigma^2$, $\sigma>0$. Then if $\alpha(0)=1$ the solution of (1.28) is

$$\alpha(y) = \sigma\left\{\frac{\cosh\sigma y+\sigma\sinh\sigma y}{\sinh\sigma y+\sigma\cosh\sigma y}\right\},$$

and in particular

$$\alpha(y)\to\sigma=\sqrt{|\beta|} \quad\text{as}\quad y\to\infty.$$

If this is substituted into (1.24) on a line where x is constant and u_x is zero we find that $\lim_{y\to\infty} u(x,y)=\sqrt{|\beta|}$ along such a line. However, if u is bounded (as it is in the periodic case) elliptic estimates mean that $u_x(x,y)\to 0$ as $y\to\infty$ uniformly for $x\in\mathbf{R}$ and we conclude that

$$u(x,y)\to\sqrt{|\beta|} \text{ as } y\to\infty, \quad \text{uniformly in } x.$$

It is interesting to note that the equation

$$u_{xx}-3u^2+2u^3=0, \tag{1.32}$$

which is equation (1.26) in the case $y=0$, $\beta=0$, has periodic solutions as well as the solitary wave solution found by Benjamin

$$u(x)=\frac{2}{1+x^2}. \tag{1.33}$$

Since $\beta=0$, $\alpha(y)=1/(1+y)$ and hence (1.24) cannot hold if u is to be bounded below in the upper half-plane. Hence the periodic solutions of (1.32) *do not* correspond to periodic solutions of the Benjamin–Ono equation, or equivalently of problem $(\mathscr{P})$.

Likewise, when $\beta>0$ the equation

$$u_{xx}-3u^2+2u^3-\beta(0)\,u=0 \tag{1.34}$$

has non-trivial solitary wave solutions in the case $\beta(0)>0$. However, since (1.28) is not solvable for α, for all $y>0$, these solitary wave solutions are *not* solitary wave solutions of the Benjamin–Ono equation.

The layout of the paper is as follows. After a few preliminary observations are made in Section 2, the main result is proved in Section 3 as a consequence of the maximum principle and the Cauchy–Riemann equations. In Section 4 we show that, in the unit disc Ω in the plane, the problem

$$\Delta u=0 \quad \text{on } \Omega,$$

$$\frac{\partial u}{\partial r}=-u+u^2 \quad \text{on } \partial\Omega,$$

has no non-constant solutions. (Here r denotes the radial polar co-ordinate in the plane.)

It seems clear that his method has implications for other quadratic Neumann boundary conditions on planar domains other than half-spaces and discs.

One final remark: if the boundary condition in $(\mathscr{B})$ is replaced by

$$u_y=au^2+bu+c, \quad a\neq 0, \tag{1.35}$$

then an affine change of the dependent variable and a linear change of the independent variable leads to $(\mathscr{B})$ provided that $b^2>4ac$. Thus (1.35) is included in our treatment of $(\mathscr{B})$. In particular the case

$$u_y = -u - u^2$$

is included, and this corresponds to sub-critical travelling wave solutions of the Benjamin–Ono equation. We conclude that there do not exist sub-critical solitary wave solutions of the Benjamin–Ono equation since these would (after a change of variables) correspond to solutions of ($\mathscr{B}$) which satisfy the boundary condition

$$u(x) \to 1 \quad \text{as} \quad |x| \to \infty,$$

and no such solutions exist.

2. Preliminaries

Let H denote the open half-plane $\{(x, y) \in \mathbf{R}^2 \colon y > 0\}$ and $\bar{H}$ its closure. For any $(x, y) \in \bar{H}$ let

$$G(x, y) = \frac{1}{\pi} \int_0^\infty \frac{(y+s)\exp(-s)}{x^2 + (y+s)^2}\, ds. \tag{2.1}$$

Then $G > 0$ is harmonic in H,

$$\int_{\mathbf{R}} G(x, y)\, dx = 1, \quad y \geqslant 0, \tag{2.2}$$

$$G_y(x, y) = G(x, y) - \frac{1}{\pi}\left(\frac{y}{x^2 + y^2}\right), \tag{2.3}$$

for all $(x, y) \in H$. Hence if f is any bounded continuous function on $\mathbf{R}$ and w is defined by

$$w(x, y) = \int_{\mathbf{R}} G(x - t, y) f(t)\, dt, \tag{2.4}$$

then

$$\Delta w(x, y) = 0, \quad (x, y) \in H,$$

and

$$\lim_{\substack{(x, y) \to (\hat{x}, 0) \\ (x, y) \in H}} w(x, y) - w_y(x, y) = f(\hat{x}), \quad \hat{x} \in \mathbf{R}.$$

Here subscripts denote partial derivatives in the usual way. The function w is well-defined and bounded in H because f is bounded and (2.2) holds. Moreover, by (2.3),

$$|w_y(x,y)| \leqslant |w(x,y)| + \frac{1}{\pi}\int_{\mathbf{R}} \frac{|yf(t)|}{(x-t)^2+y^2}\,dt,$$

and so w_y is bounded on H as well. Hence by the Phragmen–Lindelöf principle (see, for example, [6], p. 94, Theorem 18) w is the unique solution of the boundary-value problem

$$\Delta u(x,y) = 0, \quad (x,y) \in H,$$

$$u(x,0) - u_y(x,0) = f(x), \quad x \in \mathbf{R},$$

$$(|u(x,y)| + |u_x(x,y)|)/(x^2+y^2)^{1/2} \to 0 \quad \text{as} \quad x^2+y^2 \to \infty.$$

In the context of the Benjamin–Ono equation we wish to consider all solutions of problem $(\mathscr{B})$ of Section 1. From the present discussion we conclude that a solution of $(\mathscr{B})$ exists if and only if

$$u(x,y) = \int_{\mathbf{R}} G(x-t,y)\,u(t)^2\,dt. \tag{2.5}$$

A simple bootstraping argument ensures that u and all its derivatives are bounded on $\bar{H}$. We need the following elementary observation.

Lemma 2.1. *Suppose that u $(\neq 0)$ satisfies* (2.5). *Then there exists a positive constant such that*

$$u(x,y) \geqslant \text{const.}\frac{1+y}{1+x^2+y^2}, \quad (x,y) \in H.$$

Proof. First note from (2.5) that

$$u > 0 \text{ on } \bar{H}, \tag{2.6}$$

since we are considering non-trivial solutions. Let $m = \inf\{u^2(x,0) \colon x \in [-1,1]\}$. Now an elementary calculation gives that

$$\begin{aligned} G(x,y) &\geqslant \frac{1}{2\pi}\int_0^\infty \frac{(y+w)\exp(-w)}{x^2+y^2+w^2}\,dw \\ &\geqslant \frac{1}{2\pi(x^2+y^2)}\int_0^\infty \frac{(y+w)\exp(-w)}{1+w^2}\,dw, \quad x^2+y^2 \geqslant 1, \\ &\geqslant \text{const.}\left\{\frac{1+y}{x^2+y^2}\right\}, \quad x^2+y^2 \geqslant 1. \end{aligned}$$

Hence

$$
\begin{aligned}
u(x,y) &\geqslant m\int_{-1}^{+1} G(x-t,y)\,dt \\
&\geqslant \text{const.}\int_{-1}^{+1}\frac{1+y}{(x-t)^2+y^2}\,dt, \quad x^2+y^2\geqslant 4, \\
&\geqslant \text{const.}\frac{(1+y)}{x^2+y^2}, \quad x^2+y^2\geqslant 4.
\end{aligned}
$$

Since u is bounded below on the set $\{(x,y)\colon x^2+y^2\leqslant 4\}$, the result follows. q.e.d.

LEMMA 2.2. *If u satisfies (2.5) then there exists a constant M such that for all $x>0$*

$$
\left|\int_{-x}^{0}(u(t)-u^2(t))\,dt\right|+\left|\int_0^x (u(t)-u^2(t))\,dt\right|\leqslant M\log(2+x).
$$

Proof. Let $x>0$. Then

$$
\begin{aligned}
\int_0^x (u(t)-u^2(t))\,dt &= \int_{-\infty}^{\infty} u^2(t)\int_0^x G(s-t,0)\,ds\,dt - \int_0^x u^2(t)\,dt \\
&= \int_0^x u^2(t)\left\{\int_0^x G(s-t,0)\,ds-1\right\}dt + \int_{-\infty}^{0} u^2(t)\int_0^x G(s-t,0)\,ds\,dt \\
&\quad + \int_x^{\infty} u^2(t)\int_0^x G(s-t,0)\,ds\,dt \\
&= I_1+I_2+I_3, \quad \text{say.}
\end{aligned}
$$

We will estimate the integrals on the right-hand side in turn after observing that

$$
\int_0^x G(s-t,0)\,ds = \frac{1}{\pi}\int_0^{\infty} e^{-w}\left\{\tan^{-1}\left(\frac{x-t}{w}\right)+\tan^{-1}\left(\frac{t}{w}\right)\right\}dw,
$$

and that

$$
r\left[1-\frac{2r}{\pi}\right]^{-1}\geqslant \tan r \geqslant \frac{2}{\pi}\left[\frac{r}{\frac{1}{2}\pi-r}\right], \quad r\in\left[0,\frac{1}{2}\pi\right).
$$

Hence

$$
|I_1|\leqslant \frac{1}{\pi}\|u\|_{\infty}^2\int_0^{\infty} e^{-w}\left\{\int_0^x\left(\frac{\pi}{2}-\tan^{-1}\left(\frac{x-t}{w}\right)\right)+\left(\frac{\pi}{2}-\tan^{-1}\left(\frac{t}{w}\right)\right)dt\right\}dw
$$

$$=\frac{2}{\pi}\|u\|_\infty^2\int_0^\infty e^{-w}\int_0^x\left(\frac{\pi}{2}-\tan^{-1}\left(\frac{t}{w}\right)\right)dt\,dw$$
$$\leqslant\frac{2}{\pi}\|u\|_\infty^2\int_0^\infty e^{-w}\int_0^x\frac{\pi^2}{4}\left[\frac{\pi}{2}+\frac{t}{w}\right]^{-1}dt\,dw,$$
$$=\frac{\pi}{2}\|u\|_\infty^2\int_0^\infty we^{-w}\log\left(1+\frac{2x}{\pi w}\right)dw\leqslant\text{const.}\log(2+x).$$

Also

$$|I_2|\leqslant\frac{1}{\pi}\|u\|_\infty^2\int_{-\infty}^0\int_0^\infty e^{-w}\left\{\tan^{-1}\left(\frac{x-t}{w}\right)+\tan^{-1}\left(\frac{t}{w}\right)\right\}dw\,dt$$
$$=\frac{1}{\pi}\|u\|_\infty^2\int_0^\infty\int_0^\infty e^{-w}\tan^{-1}\left(\frac{xw}{w^2+t(x+t)}\right)dw\,dt$$
$$\leqslant\frac{1}{\pi}\|u\|_\infty^2\int_0^\infty\int_0^\infty e^{-w}\left(\frac{\pi^2}{4}\right)\left(\frac{xw}{(\pi xw/2)+w^2+t(x+t))}\right)dt\,dw,$$
$$=\text{const.}\int_0^\infty e^{-w}\left\{\int_0^1+\int_1^\infty\right\}\left(\frac{xw}{(\pi xw/2)+w^2+t(t+x)}\right)dt\,dw$$
$$\leqslant\text{const.}\left\{1+\int_0^\infty we^{-w}\int_1^\infty\frac{x}{t(t+x)}dt\,dw\right\}\leqslant\text{const.}\log(2+x).$$

Finally,

$$|I_3|\leqslant\text{const.}\int_x^\infty\int_0^\infty e^{-w}\left\{\tan^{-1}\left(\frac{x-t}{w}\right)+\tan^{-1}\left(\frac{t}{w}\right)\right\}dw\,dt$$
$$=\text{const.}\int_0^\infty e^{-w}\int_x^\infty\tan^{-1}\left(\frac{xw}{w^2-t(x-t)}\right)dt\,dw$$
$$=\text{const.}\int_0^\infty e^{-w}\int_0^\infty\tan^{-1}\left(\frac{xw}{w^2+t(x+t)}\right)dt\,dw\leqslant\text{const.}\log(2+x),$$

by the argument which led to the bound for I_2. The proof for $x<0$ is identical. q.e.d.

3. The main result

Now we are in a position to prove the main result described in (1) and (2) of the Introduction. Let u be a solution of $(\mathscr{B})$ and let v denote any harmonic conjugate of u in H. Since $v_x(x,0)=-u_y(x,0)=u(x)^2-u(x)$, it follows from Lemma 2.2 that $|v(x,0)|\leqslant$

const. $\log(2+|x|)$, $x \in \mathbf{R}$. The Phragemen–Lindelöf principle then gives

$$|v(x, y)| \leq \text{const.} \log(2+x^2+y^2) \quad \text{on } \bar{H}.$$

Let

$$w(x, y) = u_x(x, y) + u(x, y)\, v(x, y), \quad (x, y) = \bar{H}.$$

Then w is a harmonic function on H which has the property that

$$|w(x, y)| \leq \text{const.} \log(2+x^2+y^2), \quad (x, y) \in \bar{H}, \tag{3.1}$$

since u and u_x are bounded on $\bar{H}$. A calculation using the Cauchy–Riemann equations yields that

$$w_y(x, 0) = (1-u(x, 0))\, w(x, 0). \tag{3.2}$$

Since

$$u_y(x, 0) = (1-u(x, 0))\, u(x, 0)$$

it follows that

$$W_y(x, 0) = 0, \quad x \in \mathbf{R}, \tag{3.3}$$

where

$$W(x, y) = \frac{w(x, y)}{u(x, y)}, \quad (x, y) \in \bar{H}. \tag{3.4}$$

Clearly W is well-defined because of (2.6), and a calculation based on the fact that u and w are both harmonic functions, yields that W satisfies the elliptic equation

$$\Delta W + \frac{2\nabla u}{u} \cdot \nabla W = 0 \quad \text{on } H \tag{3.5}$$

with the homogeneous Neumann boundary condition

$$W_y(x, 0) = 0, \quad x \in \mathbf{R}. \tag{3.6}$$

The following result is central to the argument which follows.

LEMMA 3.1. *Suppose that there exists an interval* $(a-\varepsilon, a+\varepsilon)$ *such that*

$$W(x, 0) < W(a, 0), \quad x \in (a, a+\varepsilon), \tag{3.7 a}$$

$$W(x,0) > W(a,0), \quad x \in (a-\varepsilon, a), \tag{3.7 b}$$

then

$$W(x,0) \leqslant W(a,0), \quad x \in (a, \infty). \tag{3.8}$$

Similarly if

$$W(x,0) > W(a,0), \quad x \in (a, a+\varepsilon)$$

$$W(x,0) < W(a,0), \quad x \in (a-\varepsilon, a)$$

then

$$W(x,0) \geqslant W(a,0), \quad x \in (a, \infty).$$

Proof. It will suffice to establish the first part, for the proof of the second part is identical. Let (3.7) hold.

Let $X \subset \bar{H}$ denote the maximal open connected subset of the set $\{(x,y) \in H: W(x,y) < W(a,0)\}$ which contains the interval $[a, a+\varepsilon] \times \{0\}$ in its closure. Then $[a, a+\varepsilon] \times \{0\} \subset \partial X$, the boundary of X in $\bar{H}$, and $X \subset H$. So, in particular,

$$Y = \partial X \cap \partial H \supset [a, a+\varepsilon] \times \{0\}. \tag{3.9}$$

To prove (3.8) we will show that $[a, \infty) \times \{0\} \subset Y$. Suppose for contradiction that this is not the case so that there exists $c \in (a, \infty)$ such that $[a, c] \times \{0\}$ denotes the maximal component of Y which contains $[a, a+\varepsilon] \times \{0\}$. (The left-hand end of this component is a because of (3.7 b).) We claim that if $(z, 0) \in Y \setminus ((a, c) \times \{0\})$, then

$$W(z,0) = W(a,0). \tag{3.10}$$

To see why this is so, suppose not. Then, since $W(a+\varepsilon/2, 0) < W(a,0)$, $W(z,0) < W(a,0)$, $(z,0) \in Y$, and X is an open connected set, there exists a continuous Jordan curve Γ in $\bar{X}$ such that $\Gamma = \{\varphi(t): t \in [0,1]\}$, and

$$\varphi((0,1)) \subset X, \quad \varphi(0) = (a+\varepsilon/2, 0) \quad \text{and} \quad \varphi(1) = (z, 0). \tag{3.11}$$

Let Ω denote the open bounded connected set enclosed by Γ and the x-axis between $\varphi(0)$ and $\varphi(1)$. (Without loss of generality we suppose that $z > c$; the case $z < a$ is similar.)

Now there exists $\hat{z} \in (c, z)$ with

$$W(\hat{z}, 0) > W(a, 0).$$

If this is false, then

$$W(x,y) \leq W(a,0) \quad \text{for all} \quad (x,y) \in \partial\Omega,$$

$$W(c,0) = W(a,0) \quad \text{and} \quad (c,0) \in \partial\Omega.$$

Indeed $(c,0)$ belongs to a flat portion of $\partial\Omega$ and so, by the boundary-point lemma, $W_y(c,0) \neq 0$. This contradicts (3.6). Thus $W(\hat{z},0) > W(a,0)$ for some $\hat{z} \in (c,z)$.

Now let Z denote the maximal component of $\{(x,y) \in H \colon W(x,y) > W(a,0)\}$ which contains $(\hat{z},0)$ in its closure. It follows that Z is a subset of Ω, and so it is bounded. Now $W(x,y) = W(a,0)$ if $(x,y) \in H \cap \partial Z$, which means that $W - W(a,0)$ takes its maximum on $\bar{Z}$ at a point of $\partial H \cap \partial Z$ where the boundary of ∂Z is a flat subset of ∂H. At that point $W_y \neq 0$ by (3.5) and the boundary-point lemma. This contradicts (3.6). Thus (3.10) has been established.

Now define $\delta < 0$ by

$$\delta = \min\{W(x,0) - W(a,0) \colon x \in [a,c]\} < 0$$

and let

$$W(d,0) = \delta + W(a,0),$$

for some $d \in (a,c)$. Now define the harmonic function h on X by

$$h = u_x + uv - W(d,0)\,u.$$

Since $W(x,y) = W(a,0)$ for all $(x,y) \in \partial X \cap H$ it follows that

$$h(x,y) = -\delta u(x,y) > 0, \quad (x,y) \in \partial X \cap H.$$

Moreover

$$h(d,0) = 0,$$

$(d,0)$ belongs to a flat portion of ∂X, and by (3.10) and the construction, $h(x,0) \geq 0$ on $\partial X \cap \partial H$. Since

$$|h(x,y)| \leq \text{const.} \log(2 + x^2 + y^2), \quad (x,y) \in X \subset H,$$

it follows by the Phragman–Lindelöf principle [6; p. 96, Remark (ii)] that $h \geq 0$ on X. Hence

$$W(x,y) \geq W(a,0) + \delta \quad \text{on } X$$

and

$$W(d,0)=W(a,0)+\delta.$$

Hence, once again the boundary-point lemma contradicts (3.6). q.e.d.

The significance of the result is in the following consequence.

COROLLARY 3.2. *The function W defined by* (3.4) *has the property that $W(x,0)$ is monotonic on* $\mathbf{R}$.

Proof. Suppose the result is false. Since $W(\cdot,0)$ is real-analytic, the zeros of $W_x(\cdot,0)$ are isolated, and $W(\cdot,0)$ must have a local maximum or minimum since it is not monotone. If x_0 is a local maximum then $W_x(x,0)\neq 0$ in a deleted neighbourhood of x_0. Then, by the lemma, $W(x,0)\leq W(x_0,0)\leq W(x,0)$ for all $x\geq x_0$ which contradicts the fact that $W_x(x,0)\neq 0$ in a deleted neighbourhood of x_0. Similarly for a local minimum. q.e.d.

The main result now follows.

THEOREM 3.3. *The function W defined by* (3.4) *is constant on H.*

Proof. Let C_R denote the contour in $\bar{H}$ given by a semi-circle with centre at the origin and radius R clockwise, the line segment joining $(R,0)$ to $(1,0)$, the anticlockwise semi-circle of radius 1 and the line segment from $(-1,0)$ to $(-R,0)$, $R>1$.

Let $f(z)=u(x,y)+iv(x,y)$, $z=x+iy$, $(x,y)\in\bar{H}$. Then by Cauchy's theorem

$$\begin{aligned}0&=\operatorname{Imag}\int_C\frac{f(z)}{z}\,dz\\&=\int_0^\pi u(e^{i\theta})\,d\theta-\int_0^\pi u(Re^{i\theta})\,d\theta-\int_1^R\frac{v(x)}{x}\,dx-\int_{-R}^{-1}\frac{v(x)}{x}\,dx.\end{aligned}$$

Hence, since u is bounded, there is a constant independent of R such that

$$\left|\int_1^R\frac{v(x)-v(-x)}{x}\,dx\right|\leq\text{const.},\quad R>1. \tag{3.12}$$

Also

$$\int_1^R\frac{u_x(x)}{xu(x)}\,dx=\frac{\log u(R)}{R}-\log u(1)+\int_1^R\frac{\log u(x)}{x^2}\,dx.$$

By Lemma 2.1 and the boundedness of u we find that

$$|\log u(x)| \leq \text{const.}(1+\log(1+x^2)), \quad x \in \mathbf{R},$$

and so

$$\left| \int_1^R \frac{u_x(x)}{xu(x)}\, dx \right| \leq \text{const.}, \quad R \geq 1. \tag{3.13}$$

Similarly

$$\left| \int_1^R \frac{u_x(-x)}{xu(-x)}\, dx \right| \leq \text{const.}, \quad R \geq 1. \tag{3.14}$$

Combining (3.12)–(3.14) we find that

$$\left| \int_1^R \frac{W(x,0)-W(-x,0)}{x}\, dx \right| = \left| \int_1^R \left\{ \left(\frac{u_x(x)}{xu(x)} - \frac{u_x(-x)}{xu(-x)} \right) + \left(\frac{v(x)-v(-x)}{x} \right) \right\} dx \right|$$
$$\leq \text{const.}, \quad R \geq 1.$$

However, since $W(\cdot,0)$ is monotonic, this implies that $W(\cdot,0)$ is constant on $\mathbf{R}$. Hence $u_x+uv-\alpha u=0$ on $y=0$ for some $\alpha \in \mathbf{R}$. This is a harmonic function on the upper half plane whose growth is at worst logarithmic. Hence for some $\alpha \in \mathbf{R}$,

$$u_x+uv-\alpha u=0 \quad \text{on } H.$$

This completes the proof. q.e.d.

This shows that if u is a solution of $(\mathscr{B})$ then, for some harmonic conjugate v of u (the harmonic conjugates differ by an additive constant)

$$u_x+uv=0 \quad \text{on } H.$$

The conjugate to u_x+uv is then constant on $\bar{H}$:

$$-u_y-\frac{1}{2}(u^2-v^2)=\frac{1}{2}c \quad \text{on } \bar{H}.$$

Thus if

$$f(z)=u(x,y)+iv(x,y), \quad (x,y)\in H, \ z=x+iy,$$

then f satisfies (1.8). The observations in (1) and (2) of the Introduction, and the remarks which follow, are then immediate. This completes the discussion of the travelling wave solutions of the Benjamin–Ono equation.

3. Related results

Let Ω denote the open unit disc in the plane, and suppose that

$$\Delta u(x,y)=0, \quad (x,y)\in\Omega, \tag{4.1 a}$$

$$\frac{\partial u}{\partial r}(x,y)=-u(x,y)+u^2(x,y), \quad (x,y)\in\partial\Omega. \tag{4.1 b}$$

Then, by the maximum principle and the boundary-point lemma, $u>0$ in $\bar{\Omega}$ if u is not identically zero. Let v be any harmonic conjugate of u, and let $w=u_\theta+uv$. Then w is harmonic in Ω. Moreover by the Cauchy–Riemann equations $u_r=v_\theta$ and $u_\theta=-v_r$ when $r^2=x^2+y^2=1$. (Here (r,θ) are polar co-ordinates in the plane.) Hence

$$\left.\frac{\partial w}{\partial r}\right|_{r=1}=(u-1)\,w,$$

and

$$\left.\frac{\partial u}{\partial r}\right|_{r=1}=(u-1)\,u,$$

together give

$$\left.\frac{\partial W}{\partial r}\right|_{r=1}=0,$$

where $W=w/u$. It is immediate, since

$$\Delta W+\frac{2\nabla u}{u}\cdot\nabla W=0 \quad \text{on } \Omega,$$

that W is constant on $\bar{\Omega}$. Hence $u_\theta+uv=0$ on $\bar{\Omega}$ for some harmonic conjugate v of u in $\bar{\Omega}$. As in preceding sections, therefore, $u=\text{Real}(f)$, where

$$\frac{df}{dz}(z)=\frac{i}{2}(f(z)^2+c), \quad \text{for some } c\in\mathbf{R}.$$

Since $u_\theta+uv=0$ on $\bar{\Omega}$, a differentiation with respect to r yields that

$$\frac{\partial}{\partial\theta}\left\{\frac{u_r}{u}-\frac{u}{r}\right\}=0 \quad \text{in } \Omega, \tag{4.2}$$

and a differentiation with respect to θ yields that

$$\frac{\partial}{\partial\theta}\left(\frac{u_\theta}{u}\right)+ru_r=0 \quad \text{in } \Omega. \tag{4.3}$$

Hence, by (4.2), there exists a function g, of r alone such that

$$ru_r = g(r)\,u + u^2 \tag{4.4}$$

which, when substituted in (4.3) yields

$$\frac{\partial}{\partial\theta}\left(\frac{u_\theta}{u}\right) + g(r)\,u + u^2 = 0 \quad \text{in } \Omega. \tag{4.5}$$

This can be re-written

$$(u_\theta)^2 + 2g(r)\,u^3 + u^4 = h(r) \tag{4.6}$$

for some function h of r alone.

If u is not radially symmetric, differentiation with respect to θ then yields

$$u_{\theta\theta} + 3g(r)\,u^2 + 2u^3 = 0 \quad \text{on } \hat{\Omega}. \tag{4.7}$$

Now, from (4.1 b) we find that in (4.4) $g(1) = -1$. But u is harmonic in Ω and so

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} = 0 \quad \text{in } \Omega. \tag{4.8}$$

Combining (4.4), (4.7) and (4.8) we find that

$$r\frac{\partial}{\partial r}(g(r)\,u + u^2) = 3g(r)\,u^2 + 2u^3,$$

whence, using (4.4) again,

$$rg' + g^2 = 0. \tag{4.9}$$

Now, by (4.1) and (4.4), $g(1) = -1$ and so $g(r) = 1/\{(\log r) - 1\}$, from which it follows that $g(0) = 0$. However by the maximum principle $u > 0$ in Ω and it then follows from (4.4) that $u_r \to \infty$ as $r \to 0$. This is a contradiction. Hence u is radially symmetric. However, if u is radially symmetric, then (4.3) yields that u is constant. Clearly the only constant solution of (4.1) is $u = 1$.

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